Liu and Zhao *Boundary Value Problems* (2023) 2023:3
https://doi.org/10.1186/s13661-023-01692-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence of triple solutions for elliptic equations driven by p-Laplacian-like operators with Hardy potential under Dirichlet–Neumann boundary conditions

Jian Liu[1*] and Zengqin Zhao[2]

*Correspondence:
liujianmath@163.com
[1]School of Statistics and Mathematics, Shandong University of Finance and Economics, Jinan, 250014, China
Full list of author information is available at the end of the article

Abstract

In this article, we focus on triple weak solutions for some p-Laplacian-type elliptic equations with Hardy potential, two parameters, and mixed boundary conditions. We show the existence of at least three distinct weak solutions by using variational methods, the Hardy inequality, and the Bonanno–Marano-type three critical points theorem under suitable assumptions, and the existence of solutions to some particular cases of this type of elliptic equations are also obtained.

MSC: 35J15; 35J20; 35J25

Keywords: Elliptic equations; p-Laplacian type; Two parameters; Hardy potential; Mixed boundary conditions

1 Introduction

Elliptic differential equations in bounded domains with singular Hardy potential and Dirichlet–Neumann-type mixed boundary conditions are used to describe many engineering or physical phenomena and play a role in modeling in applied sciences such as the heat conduction in electrically conducting materials, singular minimal surfaces, and the non-Newtonian fluids, and the state of stress and strain on the elastic surface in mechanics and the solidification and melting of materials in industrial processes are only some examples involving mixed conditions. In particular, an intuitive example is that an iceberg is partially immersed in water, and mixed boundary conditions must be imposed on its boundary. In the underwater part, a Dirichlet boundary condition is required, while the Neumann condition is used in the remaining part of the boundary in contact with air.

Recently, researches on the numbers of the existence of weak solutions to nonlinear differential equations via variational methods have received wide attention (see, for example [1, 2, 6, 7, 9–12]). In particular, in this very interesting paper [6], the author studied the existence of two nontrivial solutions for a class of mixed elliptic problems with Dirichlet–Neumann mixed boundary conditions and concave–convex nonlinearity has been obtained. In the detailed literature [7], the existence of at least one positive solution of a class of perturbed equations with mixed boundary conditions was discussed. It is worth noting that in the papers cited, the boundary conditions are homogeneous. In this paper, we deal with the existence of at least three weak solutions to the following elliptic equations with homogeneous Neumann boundary conditions, and the results of some particular cases of this type elliptic problems are also obtained,

$$\begin{cases} -\operatorname{div}\mathbf{A}(x,\nabla u)+\frac{a(x)}{|x|^p}|u|^{p-2}u=\lambda f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \Gamma_1,\\ \mathbf{A}(x,\nabla u)\cdot\nu=\mu g(x,\gamma(u)) & \text{on } \Gamma_2, \end{cases} \tag{1.1}$$

where $\mathbf{A}:\Omega\times\mathbb{R}^N\to\mathbb{R}^N$, Ω is an open bounded subset in $\mathbb{R}^N$ ($N\geq 3$) with smooth boundary $\partial\Omega$, ν is the outward normal vector field on $\partial\Omega$, Γ_1 and Γ_2 are two smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ such that $\Gamma_1\cap\Gamma_2=\emptyset$, $\overline{\Gamma_1}\cup\overline{\Gamma_2}=\partial\Omega$, $\overline{\Gamma_1}\cap\overline{\Gamma_2}$ is a $(N-2)$-dimensional submanifold of $\partial\Omega$, $\lambda>0$ and $\mu\geq 0$ are real positive parameters, $a(x)\in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega a(x)>0$, $a_0=\operatorname{ess\,sup}_\Omega a(x)$, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying

$$(\mathbf{f}):\left|f(x,u)\right|\leq b(x)\left(M_1+M_2|u|^{s-1}\right),\quad \text{a.e. } x\in\Omega,\forall u\in\mathbb{R},$$

where $0<b\in L^\alpha(\Omega)$, $\alpha>\frac{N}{p}$, $1<s\leq p$, and M_1,M_2 are positive constants; with g satisfying

$$(\mathbf{g}):0\leq g(x,u)\leq h(x)|u|^{q-1},\quad \forall(x,u)\in\Gamma_2\times\mathbb{R},$$

where $q\in(1,p)$, $0<h(x)\in L^\beta(\Gamma_2)$, $\beta>\frac{N-1}{p-1}$ and $\gamma:W^{1,p}(\Omega)\to L^p(\partial\Omega)$ is called a trace map satisfying $\gamma(u)=u|_{\partial\Omega}$, $\forall u\in W^{1,p}(\Omega)\cap C^1(\overline{\Omega})$, that is, $\gamma(u)$ is the trace of u or the generalized boundary values of u.

2 Preliminaries and variational structure

Throughout the paper we denote the L^z-norm by $\|u\|_z$. Let Ω be an open, bounded subset in $\mathbb{R}^N$ ($N\geq 3$), $1<p<N$, $W_0^{1,p}(\Omega)=\{u\in W^{1,p}(\Omega):u|_{\Gamma_1}=0\}$ be the Sobolev space with the norm

$$\|u\|=\|\nabla u\|_p=\left(\int_\Omega|\nabla u|^p\,dx\right)^{\frac{1}{p}}.$$

Obviously, $W_0^{1,p}(\Omega)$ is a reflexive Banach space, and the embedding $W_0^{1,p}(\Omega)\hookrightarrow L^{p^*}(\Omega)$ is continuous, thus there exists a positive constant T such that $\|u\|_{p*}\leq T\|u\|$, $\forall u\in W_0^{1,p}(\Omega)$, where $p*=\frac{Np}{N-p}$. Furthermore, we can obtain $\|u\|_p\leq|\Omega|^{\frac{p^*-p}{p^*p}}T\|u\|$ by the Hölder inequality.

$\mathcal{A}:\overline{\Omega}\times\mathbb{R}^N\to\mathbb{R}$ is a continuous function with continuous derivative $\mathbf{A}(x,\xi)=\partial_\xi\mathcal{A}(x,\xi)$, satisfying $\mathbf{A}(x,u+v)\leq\bar{c}(\mathbf{A}(x,u)+\mathbf{A}(x,v))$, $\forall u,v\in W_0^{1,p}(\Omega)$, for some positive constant $\bar{c}$, and $\mathcal{A}$ satisfy the following assumptions,

(A1) $\mathcal{A}(x,0)=0$, $\mathcal{A}(x,\xi)=\mathcal{A}(x,-\xi)$ for all $x\in\Omega$, $\xi\in\mathbb{R}^N$.

(A2) $\mathcal{A}$ is strictly convex in $\mathbb{R}^N$ for all $x\in\Omega$.

(A3) There exist $a_1, a_2 > 0$ such that

$$\mathbf{A}(x,\xi)\cdot\xi > a_1|\xi|^p, \qquad \left|\mathbf{A}(x,\xi)\right| \le a_2|\xi|^{p-1}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$.

From (A1) and (A3), one has $a_1|\xi|^p \le p\mathcal{A}(x,\xi) \le a_2|\xi|^p$, see [5, Remark 2.3] for details.

Define the functional $\mathcal{I}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\mathcal{I}_{\lambda,\mu}(u) := \frac{1}{\lambda}\Phi(u) - \Psi(u),$$

where

$$\Phi(u) := \int_\Omega \mathcal{A}\big(x, \nabla u(x)\big)\,dx + \frac{1}{p}\int_\Omega \frac{a(x)|u(x)|^p}{|x|^p}\,dx,$$
$$\Psi(u) := \int_\Omega F\big(x, u(x)\big)\,dx + \frac{\mu}{\lambda}\int_{\partial\Gamma_2} G\big(x, \gamma(u)\big)\,d\sigma,$$

and $F(x,u) = \int_0^u f(x,\tau)\,d\tau$, $G(x,u) = \int_0^u g(x,\tau)\,d\tau$, $\forall (x,u) \in \Omega \times \mathbb{R}$.

We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of the problem (1.1) if

$$\mathcal{I}'_{\lambda,\mu}(u)[v] = 0, \quad \forall v \in W_0^{1,p}(\Omega).$$

The following Bonanno–Marano-type three critical points theorem is from the results contained in [3], which is the main tool used to obtain our results.

Theorem 2.1 ([3, Theorem 3.6]) *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous, continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$ such that

(i) $r_1 = \sup_{\Phi(x)\le r} \Psi(x) < r\Psi(\bar{x})/\Phi(\bar{x}) = r_2$;

(ii) *for each $\lambda \in \Lambda_r = (\frac{r}{r_2}, \frac{r}{r_1})$, the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Lemma 2.2 *The functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ is convex, sequentially weakly lower semicontinuous, and of class C^1 in $W_0^{1,p}(\Omega)$ with*

$$\Phi'(u)[v] = \int_\Omega \mathbf{A}(x, \nabla u)\cdot\nabla v\,dx + \int_\Omega \frac{a(x)|u(x)|^{p-2}u(x)}{|x|^p}v(x)\,dx,$$

and $\Phi' : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ admits a continuous inverse in $W^{-1,p'}(\Omega)$, where $W^{-1,p'}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$.

Proof By Lemma 2.5 in [5], one has that $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ is convex, sequentially weakly lower semicontinuous, and of class C^1 in $W_0^{1,p}(\Omega)$ with

$$\Phi'(u)[v] = \int_\Omega \mathbf{A}(x, \nabla u) \cdot \nabla v \, dx + \int_\Omega \frac{a(x)|u(x)|^{p-2}u(x)}{|x|^p} v(x) \, dx.$$

Next, we prove that Φ' admits a continuous inverse in $W^{-1,p'}(\Omega)$.

For any $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, one has

$$\begin{aligned}\Phi'(u)[u] &= \int_\Omega \mathbf{A}(x, \nabla u) \cdot \nabla u \, dx + \int_\Omega \frac{a(x)|u(x)|^{p-2}u(x)}{|x|^p} u(x) \, dx \\ &\geq \int_\Omega a_1 |\nabla u|^p \, dx \\ &= a_1 \|u\|^p,\end{aligned}$$

thus

$$\liminf_{\|u\| \to \infty} \frac{\Phi'(u)[u]}{\|u\|} \geq a_1 \liminf_{\|u\| \to \infty} \frac{\|u\|^p}{\|u\|} = +\infty,$$

then Φ' is coercive thanks to $p > 1$.

For any $u, v \in W_0^{1,p}(\Omega)$, in view of $\mathbf{A}(x, u + v) \leq \bar{c}(\mathbf{A}(x, u) + \mathbf{A}(x, v))$, for some $\bar{c} > 0$, and assumption (A3), one has

$$\begin{aligned}\big(\Phi'(u) - \Phi'(v)\big)[u - v] &= \int_\Omega \big(\mathbf{A}(x, \nabla u) - \mathbf{A}(x, \nabla v)\big) \cdot (\nabla u - \nabla v) \, dx \\ &\quad + \int_\Omega \left(\frac{a(x)|u|^{p-2}}{|x|^p} u(u - v) - \frac{a(x)|v|^{p-2}}{|x|^p} v(u - v) \right) dx \\ &\geq \int_\Omega \big(\mathbf{A}(x, \nabla u) + \mathbf{A}(x, -\nabla v)\big) \cdot (\nabla u - \nabla v) \, dx \\ &\geq \frac{1}{\bar{c}} \int_\Omega (\mathbf{A}(x, \nabla u - \nabla v) \cdot (\nabla u - \nabla v) \, dx \\ &\geq \frac{a_1}{\bar{c}} \int_\Omega |\nabla u - \nabla v|^p \, dx \\ &= \frac{a_1}{\bar{c}} \|u - v\|^p,\end{aligned}$$

thus we have that Φ' is uniformly monotone in $W_0^{1,p}(\Omega)$.

Taking into account Theorem 26.(A)d of [13], we obtain the conclusion. □

In view of f fulfilling (**f**), g fulfilling (**g**), according to [5, Lemma 3.2] and [5, Lemma 4.4], we can obtain the following lemma.

Lemma 2.3 *The functional Ψ is a sequentially weakly upper semicontinuous, continuously Gâteaux differentiable functional with*

$$\Psi'(u)[v] := \int_\Omega f(x, u) v \, dx + \frac{\mu}{\lambda} \int_{\Gamma_2} g\big(x, \gamma(u)\big) v \, d\sigma,$$

and $\Psi' : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is compact.

3 Main results

Noting that $\alpha > \frac{N}{p} > 1$, one has $W^{1,p}(\Omega)$ is embedded in $L^{\alpha'}(\Omega)$, where $\alpha' = \frac{\alpha}{\alpha-1}$ is the conjugate of α, thus $W_0^{1,p}(\Omega)$ is embedded in $L^{\alpha'}(\Omega)$. Similarly, $W_0^{1,p}(\Omega)$ is embedded in $L^{\alpha' s}(\Omega), 1 < s \leq p$. Let $c_{\alpha'}$ be the embedding constant of the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^{\alpha'}(\Omega)$, and $c_{\alpha' s}$ be the embedding constant of the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^{\alpha' s}(\Omega)$, $1 < s \leq p$.

Noting that $q \in (1,p)$, $\beta > \frac{(N-1)}{p-1}$, one has $1 < \beta' q < \frac{(N-1)p}{N-p}$, where $\beta' = \frac{\beta}{\beta-1}$ is the conjugate exponent of β, so $W_0^{1,p}(\Omega)$ is embedded in $L^{\beta' q}(\partial\Omega)$ (see [4], Theorem 2.79). Let $c_{\beta' q}$ be the continuous embedding constant of $W^{1,p}(\Omega) \hookrightarrow L^{\beta' q}(\partial\Omega)$.

Putting

$$l(x) = \sup\{l > 0 : B(x,l) \subseteq \Omega\}$$

for all $x \in \Omega$, we can show that there exists $x_0 \in \Omega$ such that $B(x_0,d) \subseteq \Omega$, where

$$d = \sup_{x\in\Omega} l(x).$$

Suppose there exist two positive constants δ, r, with $r = \frac{a_1}{p}(\frac{2}{d})^{p-N}\delta^p(2^N - 1)|B(0,1)|$, such that

$$\begin{aligned} &\frac{pd^p C}{2^{p-N}\delta^p(a_0 + a_2 C)d^N(2^N - 1)} \operatorname*{ess\,inf}_{B(x_0,\frac{d}{2})} F(x,\delta) \\ &:= \frac{1}{\check{\lambda}} > \frac{M_1\gamma}{r} c_{\alpha'}\|b\|_\alpha + \frac{M_2\gamma^s}{rs} c_{\alpha' s}^s \|b\|_\alpha := \frac{1}{\hat{\lambda}}, \end{aligned} \tag{3.1}$$

where $\gamma = (\frac{pr}{a_1})^{\frac{1}{p}}$, $C = (\frac{N-p}{p})^p$.

With the above notations we present the following results.

Theorem 3.1 *Assume conditions* (**f**), (**g**), *and*

(H1) $F(x,\xi) \geq 0, \forall (x,\xi) \in B(x_0,d) \times [0,\delta]$;

(H2) $\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in\Omega} F(x,\xi)}{\xi^p} \leq \frac{a_1}{pc}(\frac{M_1\gamma}{r} c_{\alpha'}\|b\|_\alpha + \frac{M_2\gamma^s}{rs} c_{\alpha' s}\|b\|_\alpha)$, *where* $c = |\Omega|^{\frac{p^*-p}{p^*}} T^p$, $r = \frac{a_1}{p}(\frac{2}{d})^{p-N}\delta^p(2^N - 1)|B(0,1)|$

hold, then for every $\lambda \in (\check{\lambda}, \hat{\lambda})$, *when* $\mu \in [0, \frac{q(rs-\lambda s\gamma M_1 c_{\alpha'}\|b\|_\alpha - \lambda\gamma^s M_2 c_{\alpha' s}\|b\|_\alpha)}{\gamma^q c_q h_0 s})$, *where* $h_0 = \operatorname{ess\,sup}_{\Gamma_2} h(x)$, *the problem* (1.1) *has at least three weak solutions.*

Proof Let $u_0(x) = 0$, δ be a constant, and

$$\bar{u}(x) = \begin{cases} 0, & x \in \Omega \setminus \bar{B}(x_0,d), \\ \frac{2\delta}{d}(d - |x - x_0|), & x \in B(x_0,d) \setminus \bar{B}(x_0,\frac{d}{2}), \\ \delta, & x \in B(x_0,\frac{d}{2}), \end{cases}$$

thus $u_0, \bar{u} \in W_0^{1,p}(\Omega)$, $\Phi(u_0) = \Psi(u_0) = 0$, and

$$\begin{aligned} \Phi(\bar{u}) &= \int_\Omega \mathcal{A}\big(x, \nabla \bar{u}(x)\big)\,dx + \frac{1}{p}\int_\Omega \frac{a(x)|\bar{u}(x)|^p}{|x|^p}\,dx \\ &\geq \frac{a_1}{p}\frac{(2\delta)^p}{d^p}\left|B(x_0,d) \setminus \bar{B}\left(x_0, \frac{d}{2}\right)\right| \\ &= \frac{a_1}{p}\frac{(2\delta)^p}{d^p}\big|B(0,1)\big|\left(d^N - \left(\frac{d}{2}\right)^N\right) \\ &= \frac{a_1}{p}\left(\frac{2}{d}\right)^{p-N}\delta^p\big(2^N - 1\big)\big|B(0,1)\big| = r. \end{aligned} \tag{3.2}$$

Taking account of the following Hardy inequality (see [8, Lemma 2.1] for more details),

$$\int_\Omega \frac{|\bar{u}(x)|^p}{|x|^p}\,dx \leq \frac{1}{C}\int_\Omega \big|\nabla \bar{u}(x)\big|^p\,dx, \quad \forall u \in W_0^{1,p}(\Omega),$$

where $C = (\frac{N-p}{p})^p$ is the optimal constant, one has

$$\begin{aligned} \Phi(\bar{u}) &= \int_\Omega \mathcal{A}\big(x, \nabla \bar{u}(x)\big)\,dx + \frac{1}{p}\int_\Omega \frac{a(x)|\bar{u}(x)|^p}{|x|^p}\,dx \\ &\leq \frac{a_2}{p}\frac{(2\delta)^p}{d^p}\left|B(x_0,d) \setminus \bar{B}\left(x_0, \frac{d}{2}\right)\right| + \frac{a_0}{p}\int_\Omega \frac{|\bar{u}(x)|^p}{|x|^p}\,dx \\ &\leq \frac{a_2}{p}\frac{(2\delta)^p}{d^p}\left|B(x_0,d) \setminus \bar{B}\left(x_0, \frac{d}{2}\right)\right| + \frac{a_0}{pC}\int_\Omega \big|\nabla \bar{u}(x)\big|^p\,dx \\ &\leq \frac{2^{p-N}\delta^p(a_0 + a_2 C)}{pd^p C} d^N\big(2^N - 1\big)\big|B(0,1)\big|. \end{aligned} \tag{3.3}$$

In view of the conditions (H1) and (**g**), one has

$$\begin{aligned} \Psi(\bar{u}) &= \int_\Omega F(x,\bar{u})\,dx + \frac{\mu}{\lambda}\int_{\partial\Gamma_2} G\big(x, \gamma(\bar{u})\big)\,d\sigma \\ &\geq \int_{B(x_0,\frac{d}{2})} F(x,\delta)\,dx \\ &\geq \left|B\left(x_0, \frac{d}{2}\right)\right| \operatorname*{ess\,inf}_{B(x_0,\frac{d}{2})} F(x,\delta). \end{aligned} \tag{3.4}$$

Noting that $0 < b \in L^\alpha(\Omega)$, $\alpha > N/p$, since $p < N$, $W^{1,p}(\Omega)$ is embedded in $L^{\alpha'}(\Omega)$, where $\alpha' = \frac{\alpha}{\alpha-1}$ is the conjugate exponent of α, so $W_0^{1,p}(\Omega)$ is embedded in $L^{\alpha'}(\Omega)$. Similarly, $W_0^{1,p}(\Omega)$ is embedded in $L^{\alpha' s}(\Omega), 1 < s \leq p$.

By the Hölder inequality, we can obtain

$$\int_\Omega b(x)u\,dx \leq \|b\|_\alpha \|u\|_{\alpha'}, \qquad \int_\Omega b(x)u^s\,dx \leq \|b\|_\alpha \|u\|_{\alpha' s}^s,$$
$$\int_{\partial\Omega} h|u|^q\,d\sigma \leq \|h\|_{\beta,\partial\Omega}\|u\|_{\beta' q,\partial\Omega}^q.$$

For every $u \in \Phi^{-1}(-\infty, r]$, one has $\Phi(u) \le r$, and $\|u\| \le (\frac{pr}{a_1})^{\frac{1}{p}} = \gamma$. Thus,

$$\begin{aligned}
&\sup_{u\in\Phi^{-1}(-\infty,r]} \Psi(u) \\
&\quad = \sup_{u\in\Phi^{-1}(-\infty,r]} \left(\int_\Omega F(x,u)\,dx + \frac{\mu}{\lambda}\int_{\Gamma_2} G\bigl(x,\gamma(u)\bigr)\,d\sigma \right) \\
&\quad \le \int_\Omega \sup_{\|u\|\le(\frac{pr}{a_1})^{\frac{1}{p}}} \left(F(x,u)\,dx + \frac{\mu}{\lambda}\int_{\Gamma_2} G\bigl(x,\gamma(u)\bigr)\,d\sigma \right) \\
&\quad \le \int_\Omega \sup_{\|u\|\le(\frac{pr}{a_1})^{\frac{1}{p}}} \left(b(x)\left(M_1|u| + \frac{M_2}{s}|u|^s \right) dx + \frac{\mu}{\lambda q}\int_{\partial\Omega} h|u|^q\,d\sigma \right) \\
&\quad \le \int_\Omega \sup_{\|u\|\le(\frac{pr}{a_1})^{\frac{1}{p}}} \left(M_1\|b\|_\alpha \|u\|_{\alpha'} + \frac{M_2}{s}\|b\|_\alpha \|u\|^s_{\alpha' s} + \frac{\mu}{\lambda q}\|h\|_{\beta,\partial\Omega}\|u\|^q_{\beta' q,\partial\Omega} \right) \\
&\quad \le \int_\Omega \sup_{\|u\|\le(\frac{pr}{a_1})^{\frac{1}{p}}} \left(M_1 c_{\alpha'}\|b\|_\alpha \|u\| + \frac{M_2}{s}c^s_{\alpha' s}\|b\|_\alpha \|u\|^s + \frac{c^q_{\beta' q}\mu}{\lambda q}\|h\|_{\beta,\partial\Omega}\|u\|^q \right) \\
&\quad \le M_1\gamma c_{\alpha'}\|b\|_\alpha + \frac{M_2\gamma^s}{s}c^s_{\alpha' s}\|b\|_\alpha + \frac{\mu\gamma^q c^q_{\beta' q}}{\lambda q}\|h\|_{\beta,\partial\Omega}.
\end{aligned}$$

Thus, taking account of $\mu \in [0, \frac{q(rs-\lambda s\gamma M_1 c_{\alpha'}\|b\|_\alpha - \lambda\gamma^s M_2 c^s_{\alpha' s}\|b\|_\alpha)}{s\gamma^q c^q_{\beta' q}\|h\|_{\beta,\partial\Omega}})$, one has

$$\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}(-\infty,r]}\Psi(u)}{r} &\le \frac{M_1\gamma}{r}c_{\alpha'}\|b\|_\alpha + \frac{M_2\gamma^s}{rs}c^s_{\alpha' s}\|b\|_\alpha + \frac{\mu\gamma^q c^q_{\beta' q}}{\lambda q r}\|h\|_{\beta,\partial\Omega} \\
&< \frac{1}{\lambda}.
\end{aligned} \tag{3.5}$$

Combining (3.1), (3.3) with (3.4), one has

$$\begin{aligned}
\frac{\Psi(\bar{u})}{\Phi(\bar{u})} &\ge \frac{|B(x_0,\frac{d}{2})|\operatorname{ess\,inf}_{B(x_0,\frac{d}{2})} F(x,\delta)}{\frac{2^{p-N}\delta^p(a_0+a_2C)}{pd^pC} d^N(2^N-1)|B(0,1)|} \\
&= \frac{pd^pC}{2^{p-N}\delta^p(a_0+a_2C)d^N(2^N-1)} \operatorname*{ess\,inf}_{B(x_0,\frac{d}{2})} F(x,\delta) = \frac{1}{\bar{\lambda}} > \frac{1}{\lambda}.
\end{aligned} \tag{3.6}$$

Therefore, thanks to (3.5) and (3.6), one has that assumption (i) of Theorem 2.1 is satisfied.

Now, we prove the coercivity of the functional $\mathcal{I}_{\lambda,\mu}(u)$.

In view of condition (H2), we can choose a constant θ satisfying

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in\Omega} F(x,\xi)}{\xi^p} < \theta < \frac{a_1}{pc}\left(\frac{M_1\gamma}{r}c_{\alpha'}\|b\|_\alpha + \frac{M_2\gamma^s}{rs}c_{\alpha' s}\|b\|_\alpha \right), \tag{3.7}$$

then, there exists a function $k_\theta(x) \in L^1(\Omega)$ such that

$$F(x,\xi) \le \theta|\xi|^p + k_\theta(x), \quad \forall x \in \Omega, \xi \in \mathbb{R}. \tag{3.8}$$

Combining (3.7), (3.8) with the Hölder inequality, we have

$$
\begin{aligned}
\mathcal{I}_{\lambda,\mu}(u) &= \frac{1}{\lambda}\Phi(u) - \Psi(u) \\
&= \frac{1}{\lambda}\int_\Omega \mathcal{A}(x,\nabla u)\,dx + \frac{1}{p}\int_\Omega a(x)|u|^p\,dx - \int_\Omega F(x,u)\,dx - \frac{\mu}{\lambda}\int_{\Gamma_2} G(x,u)\,d\sigma \\
&\geq \frac{a_1}{\lambda p}\|u\|^p - \theta\|u\|_p^p - \|k_\theta\|_1 - \frac{\mu}{\lambda q}\int_{\partial\Omega} h|u|^q\,d\sigma \\
&\geq \frac{a_1}{p}\left(\frac{M_1}{r}c_{\alpha'}\|b\|_\alpha\left(\frac{pr}{a_1}\right)^{\frac{1}{p}} + \frac{M_2}{rs}c_{\alpha's}\|b\|_\alpha\left(\frac{pr}{a_1}\right)^{\frac{s}{p}}\right)\|u\|^p - \theta T^p|\Omega|^{\frac{p^*-p}{p^*}}\|u\|^p \\
&\quad - \|k_\theta\|_1 - \frac{\mu}{\lambda q}\|h\|_{\beta,\partial\Omega}\|u\|^q_{\beta' q,\partial\Omega} \\
&\geq \frac{a_1}{p}\left(\frac{M_1}{r}c_{\alpha'}\|b\|_\alpha\left(\frac{pr}{a_1}\right)^{\frac{1}{p}} + \frac{M_2}{rs}c_{\alpha's}\|b\|_\alpha\left(\frac{pr}{a_1}\right)^{\frac{s}{p}}\right)\|u\|^p \\
&\quad - \theta c\|u\|^p - \|k_\theta\|_1 - \frac{c^q_{\beta' q}\mu}{\lambda q}\|h\|_{\beta,\partial\Omega}\|u\|^q \\
&= \left(\frac{a_1}{pc}\left(\frac{M_1\gamma}{r}c_{\alpha'}\|b\|_\alpha + \frac{M_2\gamma^s}{rs}c_{\alpha's}\|b\|_\alpha\right) - \theta\right)c\|u\|^p \\
&\quad - \|k_\theta\|_1 - \frac{c^q_{\beta' q}\mu}{\lambda q}\|h\|_{\beta,\partial\Omega}\|u\|^q.
\end{aligned}
$$

Thus, the coercivity of $\mathcal{I}_{\lambda,\mu}(u)$ is obtained according to (3.7) and $q < p$. Hence, combining Lemma 2.2 with Lemma 2.3, Theorem 2.1 ensures the conclusion. □

As special cases of Theorem 3.1, we can obtain the following results.

Theorem 3.2 *Assume conditions* (**f**), (**g**), *and*

(H_1) $F(x,\xi) \geq 0$, $\forall(x,\xi) \in B(x_0,d) \times [0,\delta]$;

$(H_2)'$ $\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in\Omega} F(x,\xi)}{\xi^p} = 0$

hold, then for every $\lambda \in (\check{\lambda},\hat{\lambda})$, *when* $\mu \in [0, \frac{q(rs-\lambda s\gamma M_1 c_{\alpha'}\|b\|_\alpha - \lambda\gamma^s M_2 c_{\alpha's}\|b\|_\alpha)}{\gamma^q c_q h_0 s})$, *where* $h_0 = \operatorname{ess\,sup}_{\Gamma_2} h(x)$, *the problem* (1.1) *has at least three weak solutions.*

Proof We only need to prove the coercivity of the functional $\mathcal{I}_{\lambda,\mu}(u)$.

Fix $0 \leq \varepsilon \leq \frac{a_1}{\lambda pc}$. In view of condition $(H_2)'$, there is a function $k_\varepsilon(x) \in L^1(\Omega)$ such that

$$
F(x,\xi) \leq \varepsilon|\xi|^p + k_\varepsilon(x), \quad \forall x \in \Omega, \xi \in \mathbb{R}. \tag{3.9}
$$

Combining (3.9) with the Hölder inequality, we have

$$
\begin{aligned}
\mathcal{I}_{\lambda,\mu}(u) &= \frac{1}{\lambda}\Phi(u) - \Psi(u) \\
&= \frac{1}{\lambda}\int_\Omega \mathcal{A}(x,\nabla u)\,dx + \frac{1}{p}\int_\Omega a(x)|u|^p\,dx - \int_\Omega F(x,u)\,dx - \frac{\mu}{\lambda}\int_{\Gamma_2} G(x,u)\,d\sigma
\end{aligned}
$$

$$\begin{aligned}&\ge \frac{a_1}{\lambda p}\|u\|^p - \varepsilon\|u\|_p^p - \|k_\varepsilon\|_1 - \frac{c_2\mu h_0}{\lambda q}\|u\|^q\\&\ge \frac{a_1}{\lambda p}\|u\|^p - \varepsilon c\|u\|^p - \|k_\varepsilon\|_1 - \frac{c_2\mu h_0}{\lambda q}\|u\|^q.\end{aligned}$$

Thus, the coercivity of $\mathcal{I}_{\lambda,\mu}(u)$ is obtained according to (3.7) and $q < p$, $0 < \varepsilon < \frac{a_1}{\lambda pc}$. Hence, combining Lemma 2.2 with Lemma 2.3, Theorem 2.1 ensures the conclusion. □

Suppose there exist two positive constants δ, r, with $r = \frac{a_1}{p}(\frac{2}{d})^{p-N}\delta^p(2^N - 1)|B(0,1)|$, such that

$$\begin{aligned}&\frac{pd^pC}{2^{p-N}\delta^p(a_0 + a_2C)d^N(2^N - 1)} \operatorname*{ess\,inf}_{B(x_0,\frac{d}{2})} F(x,\delta)\\&:= \frac{1}{\check{\alpha}} > \frac{M_1\gamma}{r}c_1 + \frac{M_2\gamma^s}{rs}c_s := \frac{1}{\hat{\alpha}},\end{aligned} \tag{3.10}$$

where $\gamma = (\frac{pr}{a_1})^{\frac{1}{p}}$, $C = (\frac{N-p}{p})^p$.

Let c_1 be the embedding constant of the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$; and c_s be the embedding constant of the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$, $1 < s \le p$.

Similarly, we can obtain the following two theorems as special cases of Theorem 3.1.

Theorem 3.3 *Assume the condition* (**g**), *and*

$(H_1)'$ $|f(u)| \le M_1 + M_2|u|^{s-1}$, $\forall u \in \mathbb{R}$, $1 < s \le p$, $F(\xi) \ge 0$, $\forall \xi \in [0,\delta]$;

$(H_2)''$

$$\limsup_{|\xi|\to+\infty} \frac{F(\xi)}{\xi^p} \le \frac{a_1}{pc}\left(\frac{M_1\gamma}{r}c_1 + \frac{M_2\gamma^s}{rs}c_s\right),$$

where $c = |\Omega|^{\frac{p^*-p}{p^*}} T^p$, $r = \frac{a_1}{p}(\frac{2}{d})^{p-N}\delta^p(2^N - 1)|B(0,1)|$

hold, then for every $\lambda \in (\check{\alpha}, \hat{\alpha})$, *when* $\mu \in [0, \frac{q(rs-\lambda s\gamma M_1c_1-\lambda\gamma^sM_2c_s)}{\gamma^q c_q h_0 s})$, *where* $h_0 = \operatorname{ess\,sup}_{\Gamma_2} h(x)$, *the following elliptic problem*

$$\begin{cases}-\operatorname{div}\mathbf{A}(x,\nabla u) + \frac{a(x)}{|x|^p}|u|^{p-2}u = \lambda f(u) & \text{in } \Omega,\\ u = 0 & \text{on } \Gamma_1,\\ \mathbf{A}(x,\nabla u)\cdot\nu = \mu g(x,\gamma(u)) & \text{on } \Gamma_2,\end{cases} \tag{3.11}$$

has at least three weak solutions.

Theorem 3.4 *Assume the condition* (**g**), *and*

$(H_1)'$ $|f(u)| \le M_1 + M_2|u|^{s-1}$, $\forall u \in \mathbb{R}$, $1 < s \le p$, $F(\xi) \ge 0$, $\forall \xi \in [0,\delta]$;

$(H_2)'''$

$$\limsup_{|\xi|\to+\infty} \frac{F(\xi)}{\xi^p} = 0$$

hold, then for every $\lambda \in (\breve{\alpha}, \hat{\alpha})$, *when* $\mu \in [0, \frac{q(rs-\lambda s\gamma M_1 c_1 - \lambda\gamma^s M_2 c_s)}{\gamma^q c_q h_0 s})$, *where* $h_0 = \operatorname{ess\,sup}_{\Gamma_2} h(x)$, *the following elliptic problem*

$$\begin{cases} -\operatorname{div}\mathbf{A}(x,\nabla u) + \frac{a(x)}{|x|^p}|u|^{p-2}u = \lambda f(u) & in\ \Omega, \\ u = 0 & on\ \Gamma_1, \\ \mathbf{A}(x,\nabla u)\cdot\nu = \mu g(x,\gamma(u)) & on\ \Gamma_2, \end{cases} \tag{3.12}$$

has at least three weak solutions.

Funding
Projects ZR2021MA070 and ZR2020MA012 supported by Shandong Provincial Natural Science Foundation.

Availability of data and materials
Not applicable.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
Investigation and formal analysis, Liu, J., Zhao, Z.; writing-original draft, Liu, J., Zhao, Z.; writing-review and editing, Liu, J.; All authors read and approved the final manuscript.

Author details
[1]School of Statistics and Mathematics, Shandong University of Finance and Economics, Jinan, 250014, China. [2]School of Mathematical Sciences, Qufu Normal University, Qufu, 273165, China.

Publisher's Note

Received: 8 October 2022 Accepted: 2 January 2023 Published online: 11 January 2023

References
1. Afrouzi, G.A., Hadjian, A., Rădulescu, V.D.: Variational approach to fourth-order impulsive differential equations with two control parameters. Results Math. **65**, 371–384 (2014)
2. Bonanno, G., Chinnì, A.: Existence of three solutions for a perturbed two-point boundary value problem. Appl. Math. Lett. **23**, 807–811 (2010)
3. Bonanno, G., Marano, S.: On the structure of the critical set of non-differentiable functions with a weak compactness condition. Appl. Anal. **89**, 1–10 (2010)
4. Carl, S., Le, V.K., Motreanu, D.: Nonsmooth Variational Problems and Their Inequalities. Springer, New York (2007)
5. Colasuonno, F., Pucci, P., Varga, C.: Multiple solutions for an eigenvalue problem involving *p*-Laplacian type operators. Nonlinear Anal. **75**, 4496–4512 (2012)
6. Colorado, E., Peral, I.: Semilinear elliptic problems with mixed Dirichlet–Neumann boundary conditions. J. Funct. Anal. **199**, 468–507 (2003)
7. Garcia Azorero, J., Malchiodi, A., Montoro, L., Peral, I.: Concentration of solutions for some singularly perturbed mixed problems: asymptotics of minimal energy solutions. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **27**, 37–56 (2010)
8. García Azorero, P.J., Peral Alonso, I.: Hardy inequalities and some critical elliptic and parabolic problems. J. Differ. Equ. **144**, 441–476 (1998)
9. Liu, J., Yu, W.: Two solutions to superlinear Hamiltonian systems with impulsive effects. Appl. Math. Lett. **102**, 106162 (2020)
10. Liu, J., Yu, W.: Triple solutions for elliptic Dirichlet problems with a parameter. Results Appl. Math. **10**, 100157 (2021)
11. Liu, J., Zhao, Z.: Multiple solutions for impulsive problems with non-autonomous perturbations. Appl. Math. Lett. **64**, 143–149 (2017)
12. Liu, J., Zhao, Z., Zhang, T.: Multiple solutions to damped Hamiltonian systems with impulsive effects. Appl. Math. Lett. **91**, 173–180 (2019)
13. Zeilder, E.: Nonlinear Functional Analysis and Its Applications, II/B: Nonlinear Monotone Operators. Springer, Berlin (1990)