Shan *Boundary Value Problems* (2023) 2023:7
https://doi.org/10.1186/s13661-023-01693-7

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

Open Access

Index theory and multiple solutions for asymptotically linear second-order delay differential equations

Yuan Shan[1*]

*Correspondence:
shannjnu@gmail.com
[1]School of Statistics and Data Science, Nanjing Audit University, Nanjing, Jiangsu 210029, P.R. China

Abstract

This paper is concerned with the existence of periodic solutions for asymptotically linear second-order delay differential equations. We will establish an index theory for the linear system directly in the sense that we do not need to change the problem of the original linear system into the problem of an associated Hamiltonian system. By using the critical point theory and the index theory, some new existence results are obtained.

MSC: 58E05; 34C25; 19K56

Keywords: Delay differential equation; Asymptotically linear nonlinearities; Index theory; Relative Morse index

1 Introduction

In the past few decades, many results on the existence of periodic solutions for delay differential equations were obtained by several different approaches, including various fixed point theorems, Hopf bifurcation theorems, coincidence degree theory, coupled system methods, Poincaré–Bendixson theorem, and so on. One can refer to [9, 12, 26, 27] for detailed discussions.

In [28], Kaplan and Yorke introduced a technique studying the existence of periodic solutions of the first-order delay differential equation

$$\dot{x}(t) = -f\bigl(x(t-\tau)\bigr) \tag{1}$$

that may reduce the existence problem of periodic solutions of (1) to a problem of finding periodic solutions of an associated plane ordinary differential system. Afterwards, Kaplan and York's original idea was used by many authors to study some general differential delay equations by transforming them into Hamiltonian systems, for example, [7, 8, 10, 20–22, 29–31, 33]. Especially, some known results involving various index theories in Hamiltonian systems were generalized to study delay differential equations. For example, in [29–31], by using Morse–Ekeland index theory for the associated Hamiltonian system, Li and He firstly studied the periodic solutions of delay differential equations with

© The Author(s) 2023. **Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

multiple delays via critical point theory. Some multiplicity results on periodic solutions in delay differential equations were given by Fei [20, 21] using S^1-pseudo index theory. In [32, 42], by relative Fredholm index and spectral flow, Liu and Wang constructed an index theory for the coupled Hamiltonian system and obtained some interesting existence and multiplicity results for delay differential equations.

In [24], Guo and Yu firstly applied the critical point theory to (1) directly in the sense that one does not need to transform the original existence problem of (1) to the existence problem for an associated Hamiltonian system while allowing a delay in the variational functional. By using pseudo-index theory introduced by Benci and Rabinowitz [4], they obtained multiple periodic solutions of (1) with odd nonlinearity that grows asymptotically linear both at the origin and at infinity. By establishing suitable variational frameworks for the second-order delay differential equation and high-dimensional case, Guo and Guo [23], Guo and Yu [25] obtained some sufficient conditions on the existence of periodic solutions.

Motivated by [23, 32, 42], we consider the existence of 2τ-periodic solutions for the following nonautonomous second-order delay system:

$$\ddot{x}(t) = -\nabla F\big(t, x(t-\tau)\big), \tag{2}$$

where $\tau > 0$ is a given constant.

Throughout this paper, we make use of the following hypotheses:

(F_1) $F \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, $F(t,0) = 0$.

Furthermore, we assume there exist continuous 2τ periodic symmetric matrix functions $A_0(t)$ and $A_\infty(t)$ such that

(F_0) $f(t,x) = A_0(t)x + o(x)$ as $|x| \to 0$,

(F_∞) $A_\infty > A_0$ and $f(t,x) = A_\infty(t)x + o(x)$ as $|x| \to \infty$,

where we denote $\nabla F = f$, and for two continuous $n \times n$ matrix-valued functions $B_1(t)$, $B_2(t)$, we say that $B_1(t) \le B_2(t)$ if and only if $\max_{\xi \in \mathbb{R}^n, |\xi|=1}(B_1(t) - B_2(t))\xi\xi \le 0$, and $B_1(t) > B_2(t)$ if and only if $B_1(t) \le B_2(t)$ does not hold. Furthermore, we write $B_1 \le B_2$ if $B_1(t) \le B_2(t)$ for a.e. $t \in S^1$ and denote by $B_1 < B_2$ if $B_1 \le B_2$ and $B_1(t) < B_2(t)$ on a subset of S^1 with nonzero measure.

Conditions (F_0) and (F_∞) are referred to in the literature as asymptotically linear quadratic for the nonlinearity F. This type of conditions was widely used in the problems of periodic solutions of Hamiltonian systems. It is well known that the existence and multiplicity of the periodic solutions are related to the difference between A_0 and A_∞, and a quantitative way to measure the difference is the index theory. Ekeland established an index theory for convex linear Hamiltonian systems in [17–19]. In [11, 34, 35, 37, 39], Conley et al. introduced an index theory for symplectic paths. In [38, 44], Long and Zhu defined spectral flows for paths of linear operators, the relative Morse index between two linear operators, and redefined Maslov index for symplectic paths. Liu [32] defined the $(\mathcal{J}, \mathcal{M})$-index and studied the delay Hamiltonian systems. Dong [15] developed an index theory for abstract operator equations with compact resolvent. We also mention that the authors in [6, 40, 43] established various index theories when essential spectrum emerges. We refer to the excellent books of Abbondandolo [1], Ekeland [18], and Long [36] for a more detailed account of the concept.

In this paper, we establish the existence and multiplicity of periodic solutions for (2) by combining the index theory with a generalized linking theorem developed by Ding and

Liu [14]. More precisely, for any 2τ-periodic continuous matrix-valued function $B(t)$, we consider the following linear delay differential equation:

$$\ddot{x}(t+\tau) - \lambda^2 B(t)x(t) = 0, \quad x(t) = x(t+2\tau). \tag{3}$$

Note that the spectrum of the operator $A : Ax = \ddot{x}(t+\tau)$ consists of a sequence of eigenvalues with finite multiplicity, which is unbounded from above and below. This implies that the Morse index of the energy functional corresponding to (3) is infinite. Thus, the main difficulty is to find the finite representation of the indefinite Morse index. By taking advantage of the spectral properties of A, we will modify the linear system (3) so that the Morse index of the modified linear system is finite. Then we will define a relative Morse index. Our approach follows closely that in [16]. Moreover, to combine the relative Morse index and the generalized linking theorem developed by Ding and Liu [14], we need to make a suitable choice of the reduction very carefully and overcome some involved issues.

To state our main results, we first introduce some notations. We denote by $\nu(B)$ the dimension of the kernel of (3) and call $\nu(B)$ the nullity of B. For any 2τ-periodic continuous matrix-valued functions B_1, B_2 with $B_2 > B_1$, we define

$$I(B_1,B_2) = \sum_{s=0}^{1} \nu\big((1-s)B_1 + sB_2\big).$$

We call $I(B_1,B_2)$ the relative Morse index between B_1 and B_2 (see Definition 2). For the definitions of the relative Morse index, we refer to [15, 18, 43].

Our main results are stated as follows.

Theorem 1 *Let (F_1), (F_0), (F_∞) be satisfied. Moreover, we assume that (F_0) holds with $\nu(A_0) = 0$ and (F_∞) holds with $\nu(A_\infty) = 0$. Then equation (2) has at least one nonconstant 2τ periodic solution.*

Theorem 2 *Under the assumption of Theorem 1, if furthermore f is odd, then (2) has at least $I(A_0,A_\infty)$ pairs of 2τ periodic solutions.*

Remark 1 The condition $\nu(A_\infty) = 0$ implies that the nonlinearity f is nonresonant at infinity. This assumption is crucial for the proof of the Palais–Smale condition (PS condition for short). We refer to [4, 41] and the references therein for results that allow resonance at infinity under some additional technical conditions such as Landsmann–Lazer condition, Rabinowitz resonant condition, strong resonant conditions, etc.

The paper is organized as follows. In Sect. 2, we formulate the variational setting and develop an index theory to classify the associated linear second-order delay differential equations and define the relative Morse index. After collection in Sect. 2.3 of the abstract critical point theorems, which we needed, we prove our theorems in Sect. 3.

2 Variational structure and classification theory

2.1 Variational setting

In this paper, we establish a variational structure that enables us to reduce the existence of 2τ-periodic solutions of (2) to the existence of critical points of corresponding functional defined on some appropriate function space.

Assume that $\lambda = \frac{\tau}{\pi}$, $s = \frac{\pi}{\tau}t$. Then equation (2) is transformed to

$$\ddot{x}(t) = -\lambda^2 f\big(x(t-\pi)\big), \tag{4}$$

and we seek 2π periodic solutions of (4), which, of course, correspond to the 2τ periodic solutions of (2). Define $Ax = -\ddot{x}(t+\pi)$. Then A is self-adjoint on $H^1(S^1,\mathbb{R}^n)$ with domain $D(A) = H^2(S^1,\mathbb{R}^n)$ (see Lemma 2.2 of [23]). Let $\sigma(A)$, $\sigma_d(A)$, $\sigma_e(A)$ denote, respectively, the spectrum, the discrete spectrum, and the essential spectrum of A. It is easy to calculate that

$$\sigma(A) = \sigma_d(A). \tag{5}$$

For any 2π-period continuous matrix-valued function $B_0(t)$, we denote B_0 as the multiplication operator by $B_0(t)$ in $L^2(S^1,\mathbb{R}^n)$. Let $A_{B_0} = A - B_0$. By (5), we also have

$$\sigma(A_{B_0}) = \sigma_d(A_{B_0}). \tag{6}$$

Throughout this paper, we require $0 \notin \sigma(A_{B_0})$, which means $\mathrm{Ker}(A - B_0) = \emptyset$. In fact, if $0 \in \sigma(A_{B_0})$, by property (6), there exists ϵ_0 such that, for any $\epsilon \in (0,\epsilon_0)$, we have $0 \notin \sigma(A_{B_0+\epsilon})$, and then we can replace B_0 by $B_0 + \epsilon$.

Moreover, (4) can be transformed to

$$-\ddot{x}(t+\pi) - \lambda^2 B_0(t)x(t) = \lambda^2 f\big(x(t-\pi)\big) - \lambda^2 B_0(t)x(t). \tag{7}$$

This modification is crucial for the establishment of the linking structure for the problem. More precisely, in Sect. 3, we may assume that $B_0 = A_0 - \epsilon$ for some ϵ small, where A_0 is defined as in condition (f_0).

Let $|A_{B_0}|$ be the absolute value of A_{B_0} and $E_{B_0} = D(|A_{B_0}|^{\frac{1}{2}})$ be the domain of the self-adjoint operator $|A_{B_0}|^{\frac{1}{2}}$, which is a Hilbert space equipped with the inner product

$$(z,w)_{B_0} = \big(|A_{B_0}|^{\frac{1}{2}}z, |A_{B_0}|^{\frac{1}{2}}w\big)_2$$

and the induced norm $\|z\|_{B_0} = (z,z)_{B_0}^{\frac{1}{2}}$. By the spectral properties (6), we have $|A_{B_0}|^{-\frac{1}{2}}$ is a compact operator and it follows that

Lemma 1 *E_{B_0} embeds continuously into $H^1(S^1,\mathbb{R}^n)$, and E_{B_0} embeds compactly into $L^p(S^1,\mathbb{R}^n)$ for all $p \geq 2$.*

Let $\{G_\lambda\}_{\lambda\in\mathbb{R}}$ denote the spectral family of A_{B_0}. We define the following projections:

$$P^+_{\beta,B_0} = \int_0^{+\infty} dG_\lambda, \qquad P^0_{\beta,B_0} = \int_{-\beta}^0 dG_\lambda, \qquad P^-_{\beta,B_0} = \int_{-\infty}^{-\beta} dG_\lambda.$$

Here $\beta > 0$ is large enough and $\pm\beta \notin \sigma(A_{B_0})$. This induces an orthogonal decomposition on E_{B_0}:

$$E_{B_0} = E^+_{\beta,B_0} \oplus E^0_{\beta,B_0} \oplus E^-_{\beta,B_0}, \quad \text{where } E^\pm_{\beta,B_0} = P^\pm_{\beta,B_0}E_{B_0} \text{ and } E^0_{\beta,B_0} = P^0_{\beta,B_0}E_{B_0}.$$

For any $x \in E_{B_0}$, we have $x = x^+ + x^0 + x^- \in E^+_{\beta,B_0} \oplus E^0_{\beta,B_0} \oplus E^-_{\beta,B_0}$ and $|A_{B_0}|x = A_{B_0}(x^+ - x^0 - x^-)$.

On E_{B_0} we define the functional

$$\begin{aligned} I_{\beta,B_0}(x) = {} & \frac{1}{2}\|x^+\|^2_{B_0} - \frac{1}{2}\|x^0\|^2_{B_0} - \frac{1}{2}\|x^-\|^2_{B_0} \\ & - \lambda^2 \int_0^{2\pi} F\bigl(x(t)\bigr) + \frac{1}{2}\lambda^2(B_0x, x)_2. \end{aligned} \tag{8}$$

By a standard argument as in [4, 23], the functional I_{β,B_0} is continuously differentiable on E_{B_0} and the existence of 2π-periodic solutions $x(t)$ for (2) is equivalent to the existence of critical points of functional $I_{\beta,B_0}(x)$.

2.2 Index theory

In this subsection, we investigate the linear second-order delay differential equations

$$\begin{aligned} & \ddot{x}(t+\pi) - \lambda^2 B_0(t)x(t) = \lambda^2\bigl(B(t) - B_0(t)\bigr)x(t), \\ & x(t+\pi) = x(t-\pi), \end{aligned} \tag{9}$$

where $B_0(t), B(t)$ are 2π-period continuous matrix-valued functions. Recall that we define $(A_{B_0}x)(t) = \ddot{x}(t+\pi) - \lambda^2 B_0(t)x(t)$ and $(Bx)(t) = B(t)x(t)$. Then (9) can be rewritten as

$$A_{B_0}x(t) - \lambda^2(B - B_0)x(t) = 0. \tag{10}$$

Let $x \in E_{B_0}$ be a solution of (10). Set $u = |A_{B_0}|^{\frac{1}{2}}x$. Then $u \in L^2(S^1, \mathbb{R}^n)$. The projections $P^+_{\beta,B_0}, P^0_{\beta,B_0}, P^-_{\beta,B_0}$ defined as in Sect. 2.1 also induce a decomposition on $L^2(S^1, \mathbb{R}^N)$:

$$L^2\bigl(S^1, \mathbb{R}^n\bigr) = L^+_{\beta,B_0} \oplus L^0_{\beta,B_0} \oplus L^-_{\beta,B_0},$$

where $L^{\pm}_{\beta,B_0} = P^{\pm}_{\beta,B_0}L^2(S^1, \mathbb{R}^n)$ and $L^0_{\beta,B_0} = P^0_{\beta,B_0}L^2(S^1, \mathbb{R}^n)$. For any $u \in L^2(S^1, \mathbb{R}^n)$, we have $u = u^+ + u^0 + u^- \in L^+_{\beta,B_0} \oplus L^0_{\beta,B_0} \oplus L^-_{\beta,B_0}$. Then (10) is equivalent to

$$u^+ - u^0 - u^- - |A_{B_0}|^{-\frac{1}{2}}\lambda^2(B - B_0)|A_{B_0}|^{-\frac{1}{2}}u = 0, \quad u \in L^2\bigl(S^1, \mathbb{R}^n\bigr). \tag{11}$$

Define the associated bilinear form

$$\begin{aligned} q_{\beta,B_0,B}(u,v) = {} & \frac{1}{2}\bigl(u^+, v^+\bigr)_2 - \frac{1}{2}\bigl(u^0, v^0\bigr)_2 - \frac{1}{2}\bigl(u^-, v^-\bigr)_2 \\ & - \frac{1}{2}\bigl(|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B - B_0)|A_{B_0}|^{-\frac{1}{2}}u, v\bigr)_2 \end{aligned}$$

for any $u, v \in L^2(S^1, \mathbb{R}^n)$. From (11), it is easy to get

$$-u^- - P^-_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B - B_0)|A_{B_0}|^{-\frac{1}{2}}u = 0$$

and

$$-\bigl(P^-_{\beta,B_0} + P^-_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B - B_0)|A_{B_0}|^{-\frac{1}{2}}P^-_{\beta,B_0}\bigr)u^-$$

$$= P^-_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}\left(u^+ + u^0\right).$$

Since $P^-_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}} = \int_{-\infty}^{-\beta}|\lambda|^{-\frac{1}{2}}\,dF_\lambda$, we have

$$\left\|P^-_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}\right\| \le \frac{1}{\sqrt{\beta}}. \tag{12}$$

Thus, by choosing $\beta > \lambda^2(\|B\| + \|B_0\|)$, we have

$$\left\|P^-_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}P^-_{\beta,B_0}\right\| \le \frac{\lambda^2(\|B\| + \|B_0\|)}{\beta} < 1.$$

It follows that $P^-_{\beta,B_0} + P^-_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}P^-_{\beta,B_0}$ is invertible. Let $L^*_{\beta,B_0} = L^+_{\beta,B_0} \oplus L^0_{\beta,B_0}$ and define $L_{\beta,B_0,B} : L^*_{\beta,B_0} \to L^-_{\beta,B_0}$:

$$\begin{aligned} L_{\beta,B_0,B} = &-\left(P^-_{\beta,B_0} + P^-_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}P^-_{\beta,B_0}\right)^{-1} \\ &\times P^-_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}\left(u^+ + u^0\right). \end{aligned}$$

Then it is easy to calculate that, for fixed $u^* \in L^*_{\beta,B_0}$, $L_{\beta,B_0,B}u^*$ is the unique solution for

$$-y - P^-_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}\left(u^* + y\right) = 0, \quad y \in L^-_{\beta,B_0}.$$

Moreover, for any $u^* \in L^*_{\beta,B_0}$, we define

$$\psi_{u^*}(y) = q_{\beta,B_0,B}\left(u^* + y, u^* + y\right), \quad \forall y \in L^-_{\beta,B_0}.$$

Then $\psi_{u^*}(y)$ is of class C^2 on L^-_{β,B_0}, and for any $w \in L^-_{\beta,B_0}$,

$$D^2\psi_{u^*}(y)(w,w) = -\|w\|_2^2 - \left(|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}w, w\right)_2.$$

And it follows from (12) that $D^2\psi_{u^*}(y)(w,w) < 0$. Thus $\psi_{u^*}(y)$ has unique maximum at the point $L_{\beta,B_0,B}u^*$. This yields

$$q_{\beta,B_0,B}\left(u^* + y, u^* + y\right) \le q_{\beta,B_0,B}\left(u^* + L_{\beta,B_0,B}u^*, u^* + L_{\beta,B_0,B}u^*\right). \tag{13}$$

Now we define a quadratic form $\tilde{q}_{\beta,B_0,B}$ on L^*_{β,B_0} as

$$\begin{aligned} \tilde{q}_{\beta,B_0,B}\left(u^*, v^*\right) &= q_{\beta,B_0,B}\left(u^* + L_{\beta,B_0,B}u^*, v^* + L_{\beta,B_0,B}v^*\right) \\ &= \frac{1}{2}\left(u^+, v^+\right)_2 - \frac{1}{2}\left(u^0, v^0\right)_2 - \frac{1}{2}\left(L_{\beta,B_0,B}u^*, L_{\beta,B_0,B}v^*\right)_2 \\ &\quad - \left(|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}\left(u^* + L_{\beta,B_0,B}u^*\right), v^* + L_{\beta,B_0,B}v^*\right)_2. \end{aligned}$$

By the definition of $L_{\beta,B_0,B}$, we have

$$L_{\beta,B_0,B}u^* = -P^-_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}\left(u^* + L_{\beta,B_0,B}u^*\right), \quad \forall u^* \in L^*_{\beta,B_0},$$

and for any $u^*, v^* \in L^*_{\beta,B_0}$,

$$\begin{aligned}&\left(L_{\beta,B_0,B}u^*, L_{\beta,B_0,B}v^*\right)_2\\&\quad = -\left(|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}\left(u^* + L_{\beta,B_0,B}u^*\right), L_{\beta,B_0,B}v^*\right)_2.\end{aligned}$$

This yields

$$\begin{aligned}\tilde{q}_{\beta,B_0,B}\left(u^*, v^*\right) = {}& \frac{1}{2}\left(u^+, v^+\right)_2 - \frac{1}{2}\left(u^0, v^0\right)_2\\&- \frac{1}{2}\left(|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}u^*, v^*\right)_2\\&- \frac{1}{2}\left(|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}L_{\beta,B_0,B}u^*, v^*\right)_2.\end{aligned} \tag{14}$$

Remark 2 In the spirit, the saddle-point reduction process was developed by Amann and Zehnder [2], Chang [5], and Long [36], although technically our constructions of reduction are different.

Lemma 2 *For any* 2π*-periodic continuous matrix-valued function* B*, there is a splitting*

$$L^*_{\beta,B_0} = L^+_{\beta,B_0}(B) \oplus L^0_{\beta,B_0}(B) \oplus L^-_{\beta,B_0}(B)$$

such that

(1) $L^+_{\beta,B_0}(B)$, $L^0_{\beta,B_0}(B)$, $L^-_{\beta,B_0}(B)$ *are* $\tilde{q}_{\beta,B_0,B}$*-orthogonal, and* $\tilde{q}_{\beta,B_0,B}$ *is positive definite, negative definite on* $L^+_{\beta,B_0}(B)$ *and* $L^-_{\beta,B_0}(B)$*, respectively. Moreover,* $\tilde{q}_{\beta,B_0,B}(u^*, u^*) = 0$, $\forall u^* \in L^0_{\beta,B_0}(B)$.

(2) $L^0_{\beta,B_0}(B)$, $L^-_{\beta,B_0}(B)$ *are two finite dimensional subspaces.*

Proof Define the self-adjoint operator $\Lambda_{\beta,B_0,B}$ on L^*_{β,B_0} as follows:

$$\begin{aligned}\Lambda_{\beta,B_0,B}u^* = {}& 2P^0_{\beta,B_0}u^* + |A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}u^*\\&+ |A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}L_{\beta,B_0,B}u^*.\end{aligned}$$

Then, by (14), we have

$$\tilde{q}_{\beta,B_0,B}\left(u^*, u^*\right) = \frac{1}{2}\left((\mathrm{Id} - \Lambda_{\beta,B_0,B})u^*, v^*\right)_2, \quad \forall u^*, v^* \in L^*_{\beta,B_0},$$

where Id denotes the identity map on L^*_{β,B_0}. Recalling that $|A_{B_0}|^{-\frac{1}{2}}$ is a compact operator, we have $\Lambda_{\beta,B_0,B} : L^*_{\beta,B_0} \to L^*_{\beta,B_0}$ is self-adjoint and compact. Then there is a basis $\{e_j\} \in L^*_{\beta,B_0}$ and a sequence $\mu_j \to 0$ in $\mathbb{R}$ such that

$$\Lambda_{\beta,B_0,B}e_j = \mu_j e_j; \quad (e_j, e_i)_2 = \delta_{i,j}. \tag{15}$$

Thus, for any $u^* \in L^*_{\beta,B_0}$, which can be expressed as $u^* = \sum_{j=1}^{\infty} c_j e_j$, we have

$$\tilde{q}_{\beta,B_0,B}\left(u^*, u^*\right) = \frac{1}{2}\left\|u^*\right\|_2^2 - \frac{1}{2}\left(\Lambda_{\beta,B_0,B}u^*, u^*\right)_2$$

$$= \frac{1}{2}\sum_{j=1}^{\infty}(1-\mu_j)c_j^2.$$

Since $\mu_j \to 0$, all the coefficients $(1-\mu_j)$ are positive except a finite number. Thus our lemma follows by

$$L^{+}_{\beta,B_0}(B) = \left\{\sum_{j=1}^{\infty} c_j e_j | c_j = 0, \text{ if } 1-\mu_j \leq 0\right\},$$

$$L^{0}_{\beta,B_0}(B) = \left\{\sum_{j=1}^{\infty} c_j e_j | c_j = 0, \text{ if } 1-\mu_j \neq 0\right\},$$

$$L^{-}_{\beta,B_0}(B) = \left\{\sum_{j=1}^{\infty} c_j e_j | c_j = 0, \text{ if } 1-\mu_j \geq 0\right\}.$$

□

Definition 1 For any 2τ-periodic continuous matrix-valued function B, we define

$$\nu_{\beta,B_0}(B) = \dim L^{0}_{\beta,B_0}(B); \qquad i_{\beta,B_0}(B) = \dim L^{-}_{\beta,B_0}(B).$$

We call $i_{\beta,B_0}(B)$ and $\nu_{\beta,B_0}(B)$ the index and nullity of B, respectively.

Lemma 3 $\nu_{\beta,B_0}(B) = \dim\ker(A-B)$.

Proof For any $u^* \in L^{0}_{\beta,B_0}(B)$, we have

$$\tilde{q}_{\beta,B_0,B}\big(u^*,v^*\big) = 0, \quad \forall v^* \in L^{*}_{\beta,B_0}.$$

This implies

$$\begin{aligned}&\big(P^{+}_{\beta,B_0} - P^{0}_{\beta,B_0}\big)u^* \\ &\quad - \big(P^{+}_{\beta,B_0} + P^{0}_{\beta,B_0}\big)|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}\big(u^* + L_{\beta,B_0,B}u^*\big) = 0.\end{aligned} \tag{16}$$

By the definition of $L_{\beta,B_0,B}$, we have

$$-L_{\beta,B_0,B}u^* - P^{-}_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}\big(u^* + L_{\beta,B_0,B}u^*\big) = 0. \tag{17}$$

Combining (16) and (17), we obtain

$$\begin{aligned}&\big(P^{+}_{\beta,B_0} - P^{0}_{\beta,B_0}\big)u^* - L_{\beta,B_0,B}u^* \\ &\quad - |A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}\big(u^* + L_{\beta,B_0,B}u^*\big) = 0.\end{aligned}$$

Let $x = |A_{B_0}|^{-\frac{1}{2}}(u^+ - u^0 - L_{\beta,B_0,B}u^*)$, where $u^+ = P^{+}_{\beta,B_0}u^*$, $u^0 = P^{0}_{\beta,B_0}u^*$. Then $x \in E_{B_0}$ and

$$Ax - Bx = 0.$$

Hence, $\dim L^{0}_{\beta,B_0}(B) = \dim\ker(A-B)$. □

Remark 3 From Lemma 3, we observe that $\nu_{\beta,B_0}(B)$ is independent of β and B_0. Thus, we will write $\nu(B)$ for convenience.

Recall that for a symmetric bilinear form φ defined on a Hilbert space X, its Morse index is defined as $m^-(\phi) = \max\{\dim X_1 | X_1$ is a subspace of X such that $\phi(x,x) < 0$ for any $x \in X_1\backslash\{0\}\}$.

Lemma 4 *$i_{\beta,B_0}(B)$ is the Morse index of $\tilde{q}_{\beta,B_0,B}$.*

Proof Let $E_1 \subset L^*_{\beta,B_0}$ with $\dim E_1 = k$ such that

$$\tilde{q}_{\beta,B_0,B}\bigl(u^*,u^*\bigr) < 0, \quad \forall u^* \in E_1\backslash\{0\}.$$

Let $\{e_j\}_{j=1}^k$ be linear independent in E_1. We have the following decomposition:

$$\begin{aligned} &e_j = e_j^+ + e_j^0 + e_j^-, \quad j = 1,\dots,k. \\ &e_j^+ \in L^+_{\beta,B_0}(B), \qquad e_j^0 \in L^0_{\beta,B_0}(B), \qquad e_j^- \in L^-_{\beta,B_0}(B). \end{aligned}$$

We claim that $e_1^-,\dots,e_k^-$ are linear independent. Arguing indirectly, we assume that there exist not all zero numbers $\alpha_j \in \mathbb{R}$ such that

$$\sum_{j=1}^k \alpha_j e_j^- = 0.$$

Denote $e = \sum_{j=1}^k \alpha_j e_j$. On the one hand, we have $e \in E_1$ and

$$\tilde{q}_{\beta,B_0,B}(e,e) < 0.$$

On the other hand, $e = \sum_{j=1}^k \alpha_j(e_j^0 + e_j^+) \in L^0_{\beta,B_0}(B) \oplus L^+_{\beta,B_0}(B)$, we have

$$\tilde{q}_{\beta,B_0,B}(e,e) \geq 0.$$

This is a contradiction. Thus, $e_1^-,\dots,e_k^-$ are linear independent, which implies $\dim L^-_{\beta,B_0} \geq k$ and $i_{\beta,B_0}(B) \geq m^-(\tilde{q}_{\beta,B_0,B})$.

In addition, by the definition of L^-_{β,B_0}, we have

$$\tilde{q}_{\beta,B_0,B}\bigl(u^*,u^*\bigr) < 0, \quad \forall u^* \in L^-_{\beta,B_0}\backslash\{0\}.$$

Thus, $m^-(\tilde{q}_{\beta,B_0,B}) \geq \dim L^-_{\beta,B_0} = i_{\beta,B_0}(B)$. This completes the proof. □

For any $B_1 < B_2$, let $B_s = (1-s)B_1 + sB_2$, $s \in (0,1]$, and let $i_{\beta,B_0}(s) = i_{\beta,B_0}(B_s)$, $\nu(s) = \nu(B_s)$.

Lemma 5 *For any $s_0 \in [0,1)$, there exists $\delta > 0$ such that*

$$i_{\beta,B_0}(s_0) + \nu(s_0) \leq i_{\beta,B_0}(s), \quad \forall s \in (s_0, s_0 + \delta).$$

Proof Let $C_{B_s} = (P^-_{\beta,B_0} + P^-_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B_s - B_0)|A_{B_0}|^{-\frac{1}{2}}P^-_{\beta,B_0})^{-1}$. By direct calculation, we have

$$C_{B_s} = C_{B_{s_0}} - (s - s_0)C_{B_{s_0}}P^-_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B_2 - B_1)|A_{B_0}|^{-\frac{1}{2}}P^-_{\beta,B_0}C_{B_{s_0}} + o(s - s_0)$$

and

$$\begin{aligned}
&\Lambda_{\beta,B_0,B_s} - \Lambda_{\beta,B_0,B_{s_0}} \\
&\quad = -(s - s_0)\bigl(P^*_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}} - P^*_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}B_{s_0}|A_{B_0}|^{-\frac{1}{2}}C_{B_{s_0}}P^-_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}\bigr) \\
&\qquad \times \lambda^2(B_2 - B_1)\bigl(P^*_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}} - |A_{B_0}|^{-\frac{1}{2}}C_{B_{s_0}}|A_{B_0}|^{-\frac{1}{2}}B_{s_0}P^*_{\beta,B_0}|A_{B_0}|^{-\frac{1}{2}}\bigr) + o(s - s_0),
\end{aligned}$$

where $P^*_{\beta,B_0} = P^+_{\beta,B_0} + P^0_{\beta,B_0}$. Thus, for any $u^* \in L^*_{\beta,B_0}(B_{s_0})$, there exists $\delta > 0$ such that if $0 < s - s_0 < \delta$, we have

$$\tilde{q}_{\beta,B_0,B_s}\bigl(u^*, u^*\bigr) < 0.$$

Note that $\dim L^*_{\beta,B_0}(B_{s_0}) < +\infty$, $L^*_{\beta,B_0}(B_{s_0})$ is compact. It is easy to deduce that there exists δ_0 independent of u^* such that

$$\tilde{q}_{\beta,B_0,B_s}\bigl(u^*, u^*\bigr) < 0, \quad u^* \in L^*_{\beta,B_0}(B_{s_0})$$

for all $s \in (s_0, s_0 + \delta)$. Thus, by Lemma 4, we have $i_{\beta,B_0}(s_0) + \nu(s_0) \le i_{\beta,B_0}(s)$. □

Let $i_{\beta,B_0}(s + 0) = \lim_{t\to s^+} i_{\beta,B_0}(t)$. We have

Lemma 6 $i_{\beta,B_0}(s + 0) = i_{\beta,B_0}(s) + \nu(s)$.

Proof From Lemma 5, we have $i_{\beta,B_0}(s+0) \ge i_{\beta,B_0}(s)+\nu(s)$. Thus, it suffices to prove $i_{\beta,B_0}(s+0) \le i_{\beta,B_0}(s) + \nu(s)$. Let $i_{\beta,B_0}(s + 0) = k$. Since i_{β,B_0} is a finite integer, there exists $s' > s$ such that

$$i_{\beta,B_0}(p) = i_{\beta,B_0}(s + 0), \qquad \nu(p) = 0, \quad \forall p \in \bigl(s, s'\bigr).$$

Similar to (15), for each p, there is a basis $\{e_{p,j}\}_{j=1}^k \subset L^-_{\beta,B_0}(B_p)$ such that

$$\Lambda_{\beta,B_0,B_p}e_{p,j} = \mu_{p,j}e_{p,j}; \quad (e_{p,j}, e_{p,i})_2 = \delta_{i,j}. \tag{18}$$

Here, $1 - \mu_{p,j} < 0$. Since Λ_{β,B_0,B_p} is bounded, we obtain that $\mu_{p,j} = (\Lambda_{\beta,B_0,B_p}e_{p,j}, e_{p,j})_2$ is bounded. Then, for any j, there exists $\{p_l\}_l \subseteq (s, s')$ with $p_l \to s + 0$ such that

$$\mu_{p_l,j} \to \mu_j, \qquad e_{p_l,j} \rightharpoonup e_j \quad \text{in } L^2\bigl(S^1, \mathbb{R}^n\bigr) \text{ as } l \to \infty.$$

Recall that $1 - \mu_{p_l,j} < 0$. We have $\{\frac{1}{\mu_{p_l,j}}\}$ is bounded. Taking the limit in (18), we have

$$1 - \mu_j \le 0, \quad \Lambda_{\beta,B_0,B_s}e_j = \mu_j e_j, \quad \forall j = 1, 2, \dots, k.$$

Moreover, for all $i,j = 1,2,\dots,k$,

$$e_{p_l,j} = \frac{1}{\mu_{p_l,j}}\Lambda_{\beta,B_0,B_{p_l}}e_{p_l,j} \to \frac{1}{\mu_j}\Lambda_{\beta,B_0,B_s}e_j = e_j \quad \text{and} \quad (e_j,e_i)_2 = \delta_{i,j}.$$

This means, by Definition 1, that $i_{\beta,B_0}(s) + \nu(s) \geq k = i_{\beta,B_0}(s+0)$. This completes the proof. □

By Lemma 5 and Lemma 6, we conclude that the index function $i_{\beta,B_0}(s)$ is integer-valued and nondecreasing on $[0,1)$. Its value at any point s must be equal to the sum of the jumps it incurred in $[0,1)$. Hence,

Lemma 7 *For any $B_1 < B_2$, we have*

$$i_{\beta,B_0}(B_2) - i_{\beta,B_0}(B_1) = \sum_{s\in[0,1)} \nu\bigl(B_1 + s(B_2 - B_1)\bigr).$$

By Lemma 7, we observe that the difference between $i_{\beta,B_0}(B_1)$ and $i_{\beta,B_0}(B_2)$ is independent of β and B_0. We define

Definition 2 For any 2π periodic continuous matrix-valued functions B_1 and B_2 with $B_1 < B_2$, we define

$$I(B_1,B_2) = \sum_{s\in[0,1)} \nu\bigl(B_1 + s(B_2 - B_1)\bigr),$$

and for any B_1, B_2 we define

$$I(B_1,B_2) = I(B_1,K\mathrm{id}) - I(B_2,K\mathrm{id}),$$

where K is a constant and id is the identity map on $L^*_{B_0}$, satisfying $K\mathrm{id} > B_1$ and $K\mathrm{id} > B_2$. We call $I(B_1,B_2)$ the relative Morse index between B_1 and B_2.

Remark 4 If we choose k_1, k_2 such that $k_1\mathrm{id}$, $k_2\mathrm{id} > B_1$ and B_2, we have

$$I(B_2,k_1\mathrm{id}) - I(B_1,k_1\mathrm{id}) = I(B_2,k_2\mathrm{id}) - I(B_1,k_2\mathrm{id}).$$

Thus, the relative Morse index $I(B_1,B_2)$ depends only on B_1, B_2 and the operator A and is well defined.

Lemma 8 *If $\nu(B) = 0$ for some β large enough, then $(\tilde{q}_{\beta,B_0,B}(u^*,u^*))^{\frac{1}{2}}$ and $(-\tilde{q}_{\beta,B_0,B}(u^*,u^*))^{\frac{1}{2}}$ are equivalent norms on $L^+_{\beta,B_0}(B)$ and $L^-_{\beta,B_0}(B)$, respectively.*

Proof It is sufficient to prove that, for sufficiently large β, there exists $\delta > 0$ independent of β such that

$$(1-\delta, 1+\delta) \cap \sigma(\Lambda_{\beta,B_0,B}) = \emptyset.$$

Arguing indirectly, there exist $\beta_k \to \infty$, $\mu_k \to 1$ such that

$$\Lambda_{\beta_k,B_0,B} e_k^* = \mu_k e_k^*,$$

where $e_k^* \in L^*_{\beta_k,B_0}$ with $\|e_k^*\|_2 = 1$. This yields

$$\begin{aligned}&\bigl(2P^0_{\beta_k,B_0} + |A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}} \\ &\quad + |A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}L_{\beta_k,B_0,B}\bigr)e_k^* = \mu_k e_k^*.\end{aligned} \tag{19}$$

Assume that $e_k^* = e_k^+ + e_k^0 \in L^+_{\beta_k,B_0}(B) \oplus L^0_{\beta_k,B_0}(B)$. Then, up to a subsequence,

$$e_k^* \rightharpoonup e^*, \qquad e_k^+ \rightharpoonup e^+, \qquad e_k^- \rightharpoonup e^-.$$

From (19) we have

$$\begin{aligned}\mu_k e_k^+ = {}&\bigl(P^+_{\beta_k,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}} \\ &+ P^+_{\beta_k,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}L_{\beta_k,B_0,B}\bigr)e_k^*\end{aligned}$$

and

$$\begin{aligned}(\mu_k - 2)e_k^0 = {}&\bigl(P^0_{\beta_k,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}} \\ &+ P^0_{\beta_k,B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}L_{\beta_k,B_0,B}\bigr)e_k^*.\end{aligned}$$

Recall that $|A_{B_0}|^{-\frac{1}{2}}$ is compact and $\|L_{\beta_k,B_0,B}\| \to 0$ as $\beta_k \to \infty$. Taking the limit as $k \to \infty$, we have

$$\begin{aligned}e^+ &= P^+_{B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}e^*, \\ e^0 &= P^-_{B_0}|A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}e^*,\end{aligned}$$

and

$$e^* = |A_{B_0}|^{-\frac{1}{2}}\lambda^2(B-B_0)|A_{B_0}|^{-\frac{1}{2}}e^*.$$

This yields

$$\bigl(A - \lambda^2 B\bigr)e^* = 0.$$

This is a contradiction to $\nu(B) = 0$. □

Note that, for any $x \in E_{B_0} = D(|A_{B_0}|^{\frac{1}{2}})$, we have $|A_{B_0}|^{\frac{1}{2}}x \in L^2(S^1,\mathbb{R}^n)$. Thus $L^2(S^1,\mathbb{R}^n)$ and E_{B_0} are isomorphic. Define $E^*_{\beta,B_0} = E^+_{\beta,B_0} \oplus E^0_{\beta,B_0}$ and $\tilde{L}_{\beta,B_0,B} : E^*_{\beta,B_0} \to E^-_{\beta,B_0}$

$$\tilde{L}_{\beta,B_0,B}x^* = |A_{B_0}|^{-\frac{1}{2}}L_{\beta,B_0,B}|A_{B_0}|^{\frac{1}{2}}x^*.$$

Definition 3 For any $x,y \in E_{B_0}$ and $x^* = P^*_{\beta,B_0}x$, $y^* = P^*_{\beta,B_0}y$, we define

- $Q_{\beta,B_0,B}(x,y) = q_{\beta,B_0,B}(|A_{B_0}|^{\frac{1}{2}}x, |A_\epsilon|^{\frac{1}{2}}x) = \frac{1}{2}(x^+,y^+)_{B_0} - \frac{1}{2}(x^0,y^0)_{B_0} - \frac{1}{2}(x^-,y^-)_{B_0} - \frac{1}{2}(\lambda^2(B-B_0)x,y)_2$.
- For the reduced functional, we define

$$\begin{aligned}\tilde{Q}_{\beta,B_0,B}\bigl(x^*,y^*\bigr) &= \tilde{q}_{\beta,B_0,B}\bigl(|A_{B_0}|^{\frac{1}{2}}x^*, |A_{B_0}|^{\frac{1}{2}}x^*\bigr) \\ &= \frac{1}{2}\bigl(x^+,y^+\bigr)_{B_0} - \frac{1}{2}\bigl(x^0,y^0\bigr)_{B_0} \\ &\quad - \frac{1}{2}\bigl(\lambda^2(B-B_0)x^*,y^*\bigr)_2 - \frac{1}{2}\bigl(\lambda^2(B-B_0)\tilde{L}_{\beta,B_0,B}x^*,y^*\bigr)_2.\end{aligned} \tag{20}$$

Also, by (13), for any $x^* \in E^*_{\beta,B_0}$, we have

$$Q_{\beta,B_0,B}\bigl(x^*+y,x^*+y\bigr) \le \tilde{Q}_{\beta,B_0,B}\bigl(x^*,x^*\bigr) \quad \text{for any } y \in E^-_{\beta,B_0}. \tag{21}$$

Applying Lemma 2, Lemma 7, and Lemma 8 to $E^*_{B_0}$ and $\tilde{Q}_{\beta,B}$, we conclude

Lemma 9 (1) *The $E^*_{B_0}$ has the following decomposition*:

$$E^*_{B_0} = E^+_{\beta,B_0}(B) \oplus E^0_{\beta,B_0}(B) \oplus E^-_{\beta,B_0}(B)$$

such that $\tilde{Q}_{\beta,B}$ is positive definite, zero, and negative definite on $E^+_{\beta,B_0}(B)$, $E^0_{\beta,B_0}(B)$, and $E^-_{\beta,B_0}(B)$, respectively. Furthermore, $E^0_{\beta,B_0}(B)$ and $E^-_{\beta,B_0}(B)$ are finitely dimensional with

$$\nu(B) = \dim E^0_{\beta,B_0}(B), \qquad i_{\beta,B_0}(B) = \dim E^-_{\beta,B_0}(B).$$

(2) *For any $x^* \in E^*_{\beta,B_0}$, $(\bar{Q}_{\beta,B}(x^*,x^*))^{\frac{1}{2}}$ and $(-\bar{Q}_{\beta,B}(x^*,x^*))^{\frac{1}{2}}$ are equivalent norms on $E^+_{\beta,B_0}(B)$ and $E^-_{\beta,B_0}(B)$.*

(3) *There exists $\epsilon_0 > 0$ such that, for any $\epsilon \in (0,\epsilon_0]$, we have*

$$\begin{aligned}&\nu(B+\epsilon) = 0 = \nu(B-\epsilon), \\ &i_{\beta,B_0}(B-\epsilon) = i_{\beta,B_0}(B) \\ &i_{\beta,B_0}(B+\epsilon) = i_{\beta,B_0}(B) + \nu(B).\end{aligned}$$

Proof (1) and (2) of Lemma 9 come from Lemma 2 and Lemma 8 directly. By Lemma 7, (3) follows by the fact that the index function and the relative Morse index are all integer-valued. □

2.3 Critical point theorem

To prove Theorem 1 and Theorem 2, we use the following critical point theorems.

Let E be a real Hilbert space with $E = X \oplus Y$. A sequence $(z_n) \subset E$ is said to be a $(PS)_c$-sequence if $\Phi(z_n) \to c$ and $\Phi'(z_n) \to 0$. Φ is said to satisfy the $(PS)_c$-condition if any $(PS)_c$-sequence has a convergent subsequence.

Theorem 3 ([14, Theorem 2.5]) *Let $e \in Y \setminus \{0\}$ and $\Omega = \{u = se + v : \|u\| < R, s > 0, v \in X\}$. Suppose that*

(I_1) $\Phi \in C^1(E,\mathbb{R})$ *satisfies the* $(PS)_c$*-condition for any* $c \in \mathbb{R}$;

(I_2) *there is* $r \in (0,R)$ *such that* $\rho := \inf \Phi(X \cap \partial B_r) > \omega := \sup \Phi(\partial\Omega)$, *where* $\partial\Omega$ *refers to the boundary of* Ω *relative to span*$\{e\} \oplus X$, *and* $B_r = \{u \in E : \|u\| < r\}$.

Then Φ *has a critical value* $c \geq \rho$ *with*

$$c = \inf_{h\in\Gamma} \sup_{u\in\Omega} \Phi\bigl(h(u)\bigr),$$

where

$$\Gamma = \bigl\{h \in C(E,E) : h|_{\partial\Omega} = \mathrm{id}, \Phi\bigl(h(u)\bigr) \leq \Phi(u)\ \text{for}\ u \in \bar{\Omega}\bigr\}.$$

Theorem 4 ([14, Theorem 2.8]) *Assume that* ϕ *is even and satisfies* (Φ_1). *If*

(I_3) *there exists* $r > 0$ *with* $\inf \Phi(S_r Y) > \Phi(0) = 0$, *where* $S_r = \partial B_r$;

(I_4) *there exists a finite dimensional subspace* $Y_0 \subset Y$ *and* $R > r$ *such that, for* $E_* = X \oplus Y_0$, $M_* = \sup \Phi(E_*) < +\infty$ *and* $\sigma := \sup \Phi(E_* \backslash B_R) < \rho$,

then Φ *possesses at least m distinct pairs of critical points, where* $m = \dim Y_0$.

Remark 5 (1) By using the abstract critical point theorems, we do not need to do the saddle point reduction procedure for the variational function I_{β,B_0}. Correspondingly, the nonlinearity f does not need to be C^1. For this kind of critical point theorems, we also refer to [3, 13].

(2) Recall that, in this paper, we only consider the nonresonance case ($\nu(A_\infty) = 0$). The PS-condition is sufficient to prove our theorems. Thus we use the PS-condition instead of the Cerami condition in the theorems.

3 Linking structure and proof of the main results

In view of condition (F_0), we assume $B_0 = A_0 - \epsilon$ for some $\epsilon > 0$ small enough. By (3) of Lemma 9, we have $\nu(A_0-\epsilon) = \nu(A_0) = 0$. Define $E_{A_0-\epsilon} = D(|A_{A_0-\epsilon}|^{\frac{1}{2}})$ with the inner product $(\cdot,\cdot)_{A_0-\epsilon}$ and the norm $\|\cdot\|_{A_0-\epsilon}$. In what follows, we will write $(\cdot,\cdot)$ and $\|\cdot\|$ for short. Define the following projections:

$$P^+_{\beta,A_0-\epsilon} = \int_0^{+\infty} dF_\lambda, \qquad P^0_{\beta,A_0-\epsilon} = \int_{-\beta}^0 dF_\lambda, \qquad P^-_{\beta,A_0-\epsilon} = \int_{-\infty}^{-\beta} dF_\lambda.$$

Here $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ denotes the spectral family of $A_{A_0-\epsilon}$. This deduces the decomposition on $E_{A_0-\epsilon}$:

$$E_{A_0-\epsilon} = E^+_{\beta,A_0-\epsilon} \oplus E^0_{\beta,A_0-\epsilon} \oplus E^-_{\beta,A_0-\epsilon}, \quad x = x^+ + x^0 + x^-. \tag{22}$$

Substituting $A_0 - \epsilon$ for B_0 in (8), we define

$$\begin{aligned} I_{\beta,A_0-\epsilon}(x) = {}& \frac{1}{2}\|x^+\|^2 - \frac{1}{2}\|x^0\|^2 - \frac{1}{2}\|x^-\|^2 \\ & - \lambda^2 \int_{S^1} F(x)\,dt + \frac{1}{2}\lambda^2\bigl((A_0-\epsilon)x,x\bigr)_2, \quad \forall x \in E_{A_0-\epsilon}. \end{aligned} \tag{23}$$

Applying Lemma 9 to $E^*_{A_0-\epsilon} = E^+_{\beta,A_0-\epsilon} \oplus E^0_{\beta,A_0-\epsilon}$ for any B, there is a splitting

$$E^*_{A_0-\epsilon} = E^+_{A_0-\epsilon}(B) \oplus E^0_{A_0-\epsilon}(B) \oplus E^-_{A_0-\epsilon}(B).$$

To apply the abstract Theorems 3 and 4 to $I_{\beta,A_0-\epsilon}$, we choose

$$X = E^-_{A_0-\epsilon} \oplus E^0_{A_0-\epsilon}, \qquad Y = E^+_{A_0-\epsilon}, \qquad Y_0 = E^-_{A_0-\epsilon}(A_\infty + 2\epsilon) \cap E^+_{A_0-\epsilon}.$$

Since $Y_0 \subseteq E^-_{A_0-\epsilon}(A_\infty + 2\epsilon)$, we have Y_0 is a finite dimensional subspace with $\dim Y_0 \leq i_{\beta,A_0-\epsilon}(A_\infty + 2\epsilon)$. Moreover

Lemma 10 $\dim Y_0 \geq I(A_0, A_\infty)$.

Proof By (20), for any B, we have

$$\begin{aligned}
&\tilde{Q}_{\beta,A_0-\epsilon,B}\big(x^*, x^*\big)\\
&\quad= \frac{1}{2}\big\|x^+\big\|^2 - \frac{1}{2}\big\|x^0\big\|^2\\
&\qquad - \frac{1}{2}\big(\lambda^2\big(B - (A_0 - \epsilon)\big)x^*, x^*\big)_2 - \frac{1}{2}\big(\lambda^2\big(B - (A_0 - \epsilon)\big)\tilde{L}_{\beta,(A_0-\epsilon),B}x^*, x^*\big)_2.
\end{aligned}$$

Then

$$\tilde{Q}_{\beta,A_0-\epsilon,A_0-\epsilon}\big(x^*, x^*\big) = \frac{1}{2}\big\|x^+\big\|^2 - \frac{1}{2}\big\|x^0\big\|^2, \quad \forall x^* \in E^*_{A_0-\epsilon}.$$

It is easy to get $E^+_{\beta,A_0-\epsilon}(A_0 - \epsilon) = E^+_{\beta,A_0-\epsilon}$, $E^-_{\beta,A_0-\epsilon}(A_0 - \epsilon) = E^0_{\beta,A_0-\epsilon}$. Recall that $Y_0 \subseteq E^*_{A_0-\epsilon} = E^0_{\beta,A_0-\epsilon} \oplus E^+_{\beta,A_0-\epsilon}$, we have

$$\begin{aligned}
Y_0 &= E^-_{A_0-\epsilon}(A_\infty + 2\epsilon) - E^0_{A_0-\epsilon}\\
&= E^-_{A_0-\epsilon}(A_\infty + 2\epsilon) - E^-_{\beta,A_0-\epsilon}(A_0 - \epsilon)
\end{aligned}$$

and

$$\begin{aligned}
\dim Y_0 &\geq \dim E^-_{\beta,A_0-\epsilon}(A_\infty + 2\epsilon) - \dim E^-_{\beta,A_0-\epsilon}(A_0 - \epsilon)\\
&= i_{\beta,A_0-\epsilon}(A_\infty + 2\epsilon) - i_{\beta,A_0-\epsilon}(A_0 - \epsilon)\\
&= i_{\beta,A_0-\epsilon}(A_\infty) - i_{\beta,A_0-\epsilon}(A_0)\\
&= I(A_0, A_\infty).
\end{aligned}$$

□

Now we are ready to verify the conditions of Theorems 3 and 4 for the functional $I_{\beta,A_0-\epsilon}$.

Lemma 11 *Let* (F_0) *and* (F_∞) *be satisfied. Then there exist* $r > 0$ *and* $\rho > 0$ *such that*

$$I_{\beta,A_0-\epsilon}(x) \geq \rho, \quad \forall x \in \partial B_r \cap Y.$$

Proof Assume that $f_1(t,x) = f(t,x) - A_0(t)x(t)$. By (F_0), we have $f_1(t,x) = o(x)$ as $|x| \to 0$ and $f_1(t,x) - (A_\infty - A_0)x = o(x)$ as $|x| \to \infty$. Then, for any $\epsilon > 0$, there exist $\delta, M_1, M_2 > 0$ such that

$$\left|f_1(t,x)\right| \le \epsilon|x|, \quad \forall |x| < \delta, \quad \text{and} \quad \left|f_1(t,x)\right| \le M_2|x|, \quad \forall |x| > M_1.$$

In conclusion, under (F_0) and (F_∞), given $p > 2$, for $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$\left|f_1(t,x)\right| \le \epsilon|x| + C_\epsilon|x|^{p-1}$$

and

$$\left|F_1(t,x)\right| \le \frac{1}{2}\epsilon|x|^2 + \frac{C_\epsilon}{p}|x|^p,$$

where $F_1(t,x) = \int_{S^1} f_1(t,\theta x)\,d\theta x$. This yields

$$\begin{aligned} I_{\beta,A_0-\epsilon}(x) &= \frac{1}{2}\left\|x^+\right\|^2 - \frac{1}{2}\left\|x^0\right\|^2 - \frac{1}{2}\left\|x^-\right\|^2 - \lambda^2\int_{S^1} F(t,x)\,dt + \frac{1}{2}\lambda^2\big((A_0-\epsilon)x,x\big)_2 \\ &\ge \frac{1}{2}\left\|x^+\right\|^2 - \frac{1}{2}\left\|x^0\right\|^2 - \frac{1}{2}\left\|x^-\right\|^2 - C\|x\|_p^p, \quad \forall x \in E_{A_0-\epsilon}. \end{aligned}$$

Thus, for $x \in E^+_{A_0-\epsilon}$ with $\|x\| = r$, we have $x = x^+$ and

$$\begin{aligned} I_{A_0-\epsilon}(x) &\ge \frac{1}{2}\|x\|^2 - C\|x\|_p^p \\ &\ge \frac{1}{2}\|x\|^2 - C\|x\|^p \\ &= \left(\frac{1}{2} - Cr^{p-2}\right)r^2. \end{aligned}$$

Thus, if we assume $\rho = \frac{1}{4}r^2$, then the lemma follows by choosing r small enough. □

Lemma 12 *Let (F_0) and (F_∞) be satisfied and $\epsilon > 0$ be given by Lemma 11. We have*

$$I_{\beta,A_0-\epsilon}(x) \to -\infty \quad \text{as } x \in X \oplus Y_0 \text{ and } \|x\| \to \infty.$$

Proof Arguing indirectly, we assume that for some sequence $\{x_j\} \in X \oplus Y_0$ with $\|x_j\| \to \infty$ there is $a > 0$ such that $I_{A_0-\epsilon}(x_j) > -a$ for all j. Setting $w_j = x_j/\|x_j\|$, we have $w_j \in X \oplus Y_0$ and $\|w_j\| = 1$. And according to (22), we have the decomposition

$$w_j = w_j^- + w_j^0 + w_j^+, \qquad w_j^- \in E^-_{A_0-\epsilon}, \qquad w_j^0 \in E^0_{A_0-\epsilon}, w_j^+ \in Y_0.$$

Since Y_0 is a finite dimensional subspace, we assume that $w_j \rightharpoonup w \in E_{A_0-\epsilon}$ and

$$w_j^- \rightharpoonup w^- \in E^-_{A_0-\epsilon}, \qquad w_j^0 \to w^0 \in E^0_{A_0-\epsilon}, \qquad w_j^+ \to w^+ \in Y_0.$$

Assume that $f_2(t,x) = f(t,x) - A_\infty(t)x(t)$. Under (F_0) and (F_∞), there is $M > 0$ such that

$$\left|f_2(t,x)\right| \le \epsilon|x| + M$$

and

$$\left|F_2(t,x)\right| \le \frac{1}{2}\epsilon|x|^2 + M|x| \le \epsilon|x|^2 + M_1,$$

where $F_2(t,x) = \int_{S^1} f_2(t,\theta x)\,d\theta x$. Thus,

$$\begin{aligned} \frac{-a}{\|x_j\|} &\le \frac{I_{A_0-\epsilon}(x_j)}{\|x_j\|} \\ &= \frac{1}{2}\left\|w_j^+\right\|^2 - \frac{1}{2}\left\|w_j^0\right\|^2 - \frac{1}{2}\left\|w_j^-\right\|^2 \\ &\quad - \frac{1}{2}\lambda^2\left(\left(A_\infty - (A_0-\epsilon)\right)w_j, w_j\right)_2 - \lambda^2\int_{S^1}\frac{F_2(t,x)}{\|x_j\|^2}\,dt \\ &\le \frac{1}{2}\left\|w_j^+\right\|^2 - \frac{1}{2}\left\|w_j^0\right\|^2 - \frac{1}{2}\left\|w_j^-\right\|^2 \\ &\quad - \frac{1}{2}\lambda^2\left(\left(A_\infty + 2\epsilon - (A_0-\epsilon)\right)w_j, w_j\right)_2 + \lambda^2\int_{S^1}\frac{M_1}{\|x_j\|^2}\,dt \\ &= Q_{\beta,A_0-\epsilon,A_\infty+2\epsilon}(w_j,w_j) + o(1). \end{aligned} \tag{24}$$

Firstly, by the definition of $\tilde{Q}_{\beta,A_0-\epsilon,A_\infty+2\epsilon}$ and taking the limit in (24), we have

$$\begin{aligned} o(1) &\le Q_{\beta,A_0-\epsilon,A_\infty+2\epsilon}(w_j,w_j) \\ &\le \tilde{Q}_{\beta,A_0-\epsilon,A_\infty+2\epsilon}\left(w_j^*,w_j^*\right) \\ &\to \tilde{Q}_{\beta,A_0-\epsilon,A_\infty+2\epsilon}\left(w^*,w^*\right). \end{aligned}$$

Here, $w_j^* = w_j^+ + w_j^0$ and $w^* = w^+ + w^0$. Besides, by the definition of $\tilde{Q}_{\beta,A_0-\epsilon,A_\infty+2\epsilon}$, we have

$$\begin{aligned} &\tilde{Q}_{\beta,A_0-\epsilon,A_\infty+2\epsilon}\left(w^0,w^0\right) \\ &\quad = -\frac{1}{2}\left\|w^0\right\|^2 - \frac{1}{2}\left\|\tilde{L}_{\beta,A_0-\epsilon,A_\infty+2\epsilon}w^0\right\|^2 \\ &\qquad - \frac{1}{2}\lambda^2\left((A_\infty - A_0 - 3\epsilon)\left(w^0 + \tilde{L}_{\beta,A_0-\epsilon,A_\infty+2\epsilon}w^0\right), w^0 + \tilde{L}_{\beta,A_0-\epsilon,A_\infty+2\epsilon}w^0\right) \end{aligned}$$

Since $A_\infty > A_0$ and $w^0 \ne 0$, we have $\tilde{Q}_{\beta,A_0-\epsilon,A_\infty+2\epsilon}(w^0,w^0) < 0$. Thus,

$$E^0_{\beta,A_0-\epsilon} \subset E^-_{\beta,A_0-\epsilon}(A_\infty + 2\epsilon),$$

and $w^* \in E^-_{\beta,A_0-\epsilon}(A_\infty + 2\epsilon)$. Moreover, since $\nu(A_\infty) = 0$, we have $\nu(A_\infty + 2\epsilon) = 0$ for ϵ small enough. By (2) of Lemma 9, we have $\sqrt{-\tilde{Q}_{A_0-\epsilon,A_\infty+2\epsilon}(w^*,w^*)}$ is an equivalent norm of w^* on $E^-_{A_0-\epsilon}(A_\infty + 2\epsilon)$. Thus, $w_j^* \to w^* = 0$.

Secondly, define the following projections:

$$P_1 = \int_{-\infty}^{-\beta} dF_\lambda, \qquad P_2 = \int_{-\beta}^{\beta} dF_\lambda, \qquad P_3 = \int_{\beta}^{+\infty} dF_\lambda.$$

We have $P_1 E_{A_0-\epsilon} = E^-_{A_0-\epsilon}$, $(P_2 + P_3)E_{A_0-\epsilon} = E^*_{A_0-\epsilon}$, and $P_2 E_{A_0-\epsilon}$ is a finite dimensional subspace. Recalling that $w^0_j \to 0$ and $w^+_j \to 0$, we obtain

$$P_2 w_j \to 0, \qquad P_3 w_j \to 0.$$

Moreover, by the definition of P_1 and P_3, we have

$$\|P_1 x\|_2 \le \frac{1}{\sqrt{\beta}}\|P_1 x\|, \qquad \|P_3 x\|_2 \le \frac{1}{\sqrt{\beta}}\|P_3 x\|, \quad \forall x \in E_{A_0-\epsilon}.$$

Suppose that $\lambda\|A_0 - A_\infty + \epsilon\| < M_3 < \beta$. By (24), we obtain

$$\begin{aligned} o(1) &\le \frac{1}{2}\|w^+_j\|^2 - \frac{1}{2}\|w^-_j\|^2 - \frac{1}{2}\lambda^2\bigl(\bigl(A_\infty + 2\epsilon - (A_0 - \epsilon)\bigr)w_j, w_j\bigr)_2 \\ &= o(1) - \frac{1}{2}\|w^-_j\|^2 - \frac{M_3}{2}\|w_j\|_2^2 \\ &\le o(1) - \frac{1}{2}\left(1 - \frac{M_3}{\beta}\right)\|P_1 w_j\|^2. \end{aligned}$$

This implies that $P_1 w_j \to 0$ in $E_{A_0-\epsilon}$. Hence, $w_j = (P_1 + P_2 + P_3)w_j \to 0$. This is a contradiction to $\|w_j\| = 1$. □

Lemma 13 *Let (F_0) and (F_∞) be satisfied. Then any $(PS)_c$-sequence of $I_{\beta,A_0-\epsilon}$ is bounded.*

Proof Let $\{x_j\} \subset E$ be such that $I_{\beta,A_0-\epsilon}(x_j) \to c$ and $I'_{\beta,A_0-\epsilon}(x_j) \to 0$. To prove that $\{x_j\}$ is bounded, we develop a contradiction argument. We assume that, up to a subsequence, $\|x_j\| \to \infty$ and set $v_j = \frac{x_j}{\|x_j\|}$. Then $\|v_j\| = 1$. Without loss of generality, we assume that

$$v_j \rightharpoonup v \quad \text{in } E_{A_0-\epsilon} \quad \text{and} \quad v_j \to v \quad \text{in } L^2\bigl(S^1, \mathbb{R}^n\bigr).$$

Then, up to a subsequence, $v_j(t) \to v(t)$ a.e. on S^1. Since, by (F_∞), $|f_2(t,x)| = |f(t,x) - A_\infty(t)x| = o(|x|)$ as $|x| \to \infty$ and $x_j(t) \to \infty$ if $v(t) \neq 0$, it is easy to see that

$$\int_{S^1} \frac{f(t,x_j)\varphi(t)}{\|x_j\|} \to \int_{S^1} A_\infty(t)v\varphi$$

for all $\varphi \in L^2(S^1, \mathbb{R}^n)$. From this we deduce that

$$A_{A_0-\epsilon}v - A_\infty v + (A_0 - \epsilon)v = 0$$

and

$$\ddot{v}(t+\tau) - A_\infty v(t) = 0.$$

Since $i(A_\infty) = 0$, we have $v = 0$. Thus, $v_j \rightharpoonup 0$ in $E_{A_0-\epsilon}$ and $v_j \to 0$ in $L^2(S^1, \mathbb{R}^n)$. Note that

$$o(1) = \frac{I'_{\beta,A_0-\epsilon}(x_j)}{\|x_j\|}\bigl(v^+_j - v^0_j - v^-_j\bigr)$$

$$
\begin{aligned}
&= 1 - \int_{S^1} \frac{f(t,x_j)}{|x_j|} |v_j| \big(v_j^+ - v_j^0 - v_j^-\big) + \big((A_0 - \epsilon)v_j, v_j^+ - v_j^0 - v_j^-\big)_2 \\
&\geq 1 - C \int_{S^1} |v_j| \big|v_j^+ - v_j^0 - v_j^-\big| \\
&\geq 1 - C\|v_j\|_2^2 = 1 - o(1).
\end{aligned}
$$

This is a contradiction. □

Lemma 14 *Any $(PS)_c$-sequence of $I_{A_0-\epsilon}$ has a convergence subsequence.*

Proof By Lemma 13, the $(PS)_c$-sequence is bounded. Assume that $v_j \rightharpoonup v$ and let $w_j = v_j - v$. Then we have $w_j^+ \rightharpoonup 0$ in $E_{A_0-\epsilon}$ and $w_j^+ \to 0$ in $L^2(S^1, \mathbb{R}^n)$. To establish strong convergence, it suffices to prove that $\|w_j\| \to 0$. Similar to Lemma 13, the proof follows from the procedure in Lemma 13 to $\frac{I'_{\beta,A_0-\epsilon}(x_j)}{\|x_j\|}(w_j^+ - w_j^0 - w_j^-)$. □

Proof of Theorems 1 and 2 (Existence) With $X = E^-_{A_0-\epsilon} \oplus E^0_{A_0-\epsilon}$, $Y = E^+_{A_0-\epsilon}$, $Y_0 = E^-_{A_0-\epsilon}(A_\infty + 2\epsilon) \cap E^+_{A_0-\epsilon}$, the condition (I_2) of Theorem 3 holds by Lemma 12. Lemma 14 shows that $I_{\beta,A_0-\epsilon}$ satisfies the $(PS)_c$-condition and (I_1) of Theorem 3 is verified. Therefore, $I_{\beta,A_0-\epsilon}$ has at least one critical point by Theorem 3.

(Multiplicity) Lemma 11 implies that $I_{\beta,A_0-\epsilon}$ satisfies (I_3) of Theorem 4. Lemma 10 and Lemma 12 say that $I_{\beta,A_0-\epsilon}$ satisfies (I_4) of Theorem 4 with $\dim Y_0 \geq I(A_0, A_\infty)$. Moreover, $I_{\beta,A_0-\epsilon}$ is even if f is odd. Therefore, $I_{\beta,A_0-\epsilon}$ has at least $I(A_0, A_\infty)$ pairs of nontrivial critical points by Theorem 4. □

Acknowledgements
The author is very grateful to the referees for helpful comments and suggestions.

Funding
This work has not received any funding.

Availability of data and materials
The data and materials in this paper are original.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
The author read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 19 October 2022 Accepted: 5 January 2023 Published online: 23 January 2023

References
1. Abbondandolo, A.: Morse Theory for Hamiltonian Systems. Chapman Hall, London (2001)
2. Amann, H., Zehnder, E.: Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **7**, 539–603 (1980)
3. Bartsch, T., Ding, Y.: Deformation theorems on non-metrizable vector spaces and applications to critical point theory. Math. Nachr. **279**, 1267–1288 (2006)
4. Benci, V., Rabinowitz, P.H.: Critical point theorems for indefinite functionals. Invent. Math. **53**, 241–273 (1979)
5. Chang, K.C.: Infinite-Dimensional Morse Theory and Multiple Solution Problems. Progr. Nonlinear Differential Equations Appl., vol. 6. Birkhuser, Boston (1993)

6. Chen, C., Hu, X.: Maslov index for homoclinic orbits of Hamiltonian systems. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **24**, 589–603 (2007)
7. Chen, Y.S.: The existence of periodic solutions of the equation $x'(t) = f(x(t), x(t-r))$. J. Math. Anal. Appl. **163**, 227–237 (1992)
8. Cheng, R., Xu, J.: Periodic solutions for a class of non-autonomous differential delay equation. Nonlinear Differ. Equ. Appl. **16**, 793–809 (2009)
9. Chow, S.N., Mallet-Paret, J.: The Fuller index and global Hopf bifurcation. J. Differ. Equ. **29**, 66–85 (1978)
10. Cima, A., Li, J., Llibre, J.: Periodic solutions of delay differential equations with two delays via bi-Hamiltonian systems. Ann. Differ. Equ. **17**, 205–214 (2001)
11. Conley, C., Zehnder, E.: Morse-type index theory for flows and periodic solutions for Hamiltonian equations. Commun. Pure Appl. Math. **37**, 207–253 (1984)
12. Diekmann, O., van Gils, S.A., Lunel, S.M.V., Walther, H.O.: Delay-Equations: Functional, Complex and Nonlinear Analysis. Springer, New York (1995)
13. Ding, Y.: Variational Methods for Strongly Indefinite Problems. Interdisciplinary Math. Sci, vol. 7. World Scientific, River Edge (2007)
14. Ding, Y., Liu, X.: Periodic solutions of an asymptotically linear Dirac equation. Ann. Mat. Pura Appl. **196**, 717–735 (2017)
15. Dong, Y.: Index theory for linear selfadjoint operator equations and nontrivial solutions for asymptotically linear operator equations. Calc. Var. Partial Differ. Equ. **38**, 75–109 (2010)
16. Dong, Y., Shan, Y.: Index theory for linear selfadjoint operator equations and nontrivial solutions for asymptotically linear operator equations (II). arXiv:1104.1670v1
17. Ekeland, I.: Une theorie de Morse pour les systemes Hamiltoniens convexes. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **1**, 19–78 (1984)
18. Ekeland, I.: Convexity Methods in Hamiltonian Mechanics. Springer, Berlin (1990)
19. Ekeland, I., Hofer, H.: Periodic solutions with prescribed period for convex autonomous Hamiltonian systems. Invent. Math. **81**, 155–188 (1985)
20. Fei, G.: Multiple periodic solutions of differential delay equations via Hamiltonian systems (I). Nonlinear Anal. **65**, 25–39 (2006)
21. Fei, G.: Multiple periodic solutions of differential delay equations via Hamiltonian systems (II). Nonlinear Anal. **65**, 40–58 (2006)
22. Ge, W.G.: Number of simple periodic solutions of differential difference equation $x'(t) = f(x(t-1))$. Chin. Ann. Math. **14A**, 480 (1993)
23. Guo, C.J., Guo, Z.M.: Existence of multiple periodic solutions for a class of second-order delay differential equations. Nonlinear Anal., Real World Appl. **10**, 3285–3297 (2009)
24. Guo, Z.M., Yu, J.S.: Multiplicity results for periodic solutions via critical point theory. J. Differ. Equ. **218**, 15–35 (2005)
25. Guo, Z.M., Yu, J.S.: Multiplicity results on periodic solutions to higher dimensional differential equations with multiple delays. J. Dyn. Differ. Equ. **23**, 1029–1052 (2011)
26. Hale, J.K.: Theory of Functional Differential Equations. Springer, New York (1977)
27. Hale, J.K., Lunel, S.M.V.: Introduction to Functional Differential Equations. Springer, New York (1993)
28. Kaplan, J.L., Yorke, J.A.: Ordinary differential equations which yield periodic solution of delay equations. J. Math. Anal. Appl. **48**, 317–324 (1974)
29. Li, J.B., He, X.Z.: Multiple periodic solutions of differential delay equations created by asymptotically linear Hamiltonian systems. Nonlinear Anal. **31**, 45–54 (1998)
30. Li, J.B., He, X.Z.: Proof and generalization of Kaplan-Yorke's conjecture on periodic solution of differential delay equations. Sci. China Ser. A **42**, 957–964 (1999)
31. Li, J.B., He, X.Z.: Periodic solutions of some differential delay equations created by Hamiltonian systems. Bull. Aust. Math. Soc. **60**, 377–390 (1999)
32. Liu, C.: Periodic solutions of asymptotically linear delay differential systems via Hamiltonian systems. J. Differ. Equ. **252**, 5712–5734 (2012)
33. Llibre, J., Tarta, A.A.: Periodic solutions of delay equations with three delays via bi-Hamiltonian systems. Nonlinear Anal. **64**, 2433–2441 (2006)
34. Long, Y.: Maslov-type index, degenerate critical points, and asymptotically linear Hamiltonian systems. Sci. China **33**, 1409–1419 (1990)
35. Long, Y.: A Maslov-type index theory for symplectic paths. Topol. Methods Nonlinear Anal. **10**, 47–78 (1997)
36. Long, Y.: Index Theory for Symplectic Paths with Applications. Progr. Math., vol. 207. Birkhäuser, Basel (2002)
37. Long, Y., Zehnder, E.: Morse theory for forced oscillations of asymptotically linear Hamiltonian systems. In: Alberverio, S., et al. (eds.) Stock, Process. Phys. Geom., pp. 528–563. World Scientific, Teaneck (1990)
38. Long, Y., Zhu, C.: Maslov type index theory for symplectic paths and spectral flow (II). Chin. Ann. Math. **21B**, 89–108 (2000)
39. Salamon, D.: Connected simple systems and the Conley index of isolated invariant sets. Trans. Am. Math. Soc. **291**, 1–41 (1985)
40. Shan, Y.: Morse index and multiple solutions for the asymptotically linear Schrodinger type equation. Nonlinear Anal. **89**, 170–178 (2013)
41. Szulkin, A.: Cohomology and Morse theory for strongly indefinite functionals. Math. Z. **209**, 375–418 (1992)
42. Wang, Q., Liu, C.: Periodic solutions of delay differential systems via Hamiltonian systems. Nonlinear Anal. **102**, 159–167 (2014)
43. Wang, Q., Liu, C.: A new index theory for linear self-adjoint operator equations and its applications. J. Differ. Equ. **260**, 3749–3784 (2015)
44. Zhu, C., Long, Y.: Maslov type index theory for symplectic paths and spectral flow (I). Chin. Ann. Math. **20B**, 413–424 (1999)