Liu et al. *Boundary Value Problems* (2023) 2023:9
https://doi.org/10.1186/s13661-023-01696-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence of the positive solutions for boundary value problems of mixed differential equations involving the Caputo and Riemann–Liouville fractional derivatives

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Abstract

We prove the existence of the solutions for the new mixed differential equations, which is characteristic of the right-sided Caputo and the left-sided Riemann–Liouville fractional derivatives. There are four major ingredients. The first is composed of some basic definitions and lemmas. The second is the Green's function of the new mixed fractional differential equations. We calculate the corresponding Green's functions as well as their properties. The third, which is the main new ingredient of this paper, is demonstration of the existence of the solutions for fractional equations by the fixed-point theorem in cone expansion and compression of norm type. The fourth, as applications, is the example provided to illustrate our main results.

Keywords: Caputo fractional derivative; Riemann–Liouville fractional derivative; Green's function; Coupled system; Boundary value problem

1 Introduction

In recent years, due to applications in mathematics, physics, biology, neural networks, and so on, the theory of fractional calculus has become the main focus of many scholars. At the same time, the theory of fractional differential equations is becoming more and more extensive and systematic [1–7]. There have been some new definitions of fractional calculus, having found the connection with the classical definitions of Riemann–Liouville and Caputo fractional calculus [8–10].

Some authors studied the existence of solutions for a class of mixed fractional differential equations. In 2006, Agrawal presented the mixed differential equation involving both the Caputo and the Riemann–Liouville fractional derivatives, concentrated on the solutions in different cases [11]. Later, Blaszczyk presented the numerical solution of the mixed boundary value problems [12]. Furthermore, some authors [13–19] have discussed the existence of solutions for the mixed boundary value problems by different methods, such as the upper and lower solutions theorem, Krasnoselskii's fixed-point theorem, the Leray–Schauder alternation theorem, the coincidence degree theory, the mixed monotone operator theorem, and so on.

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In this paper, we investigate the mixed fractional boundary value problems defined by the left-sided Riemann–Liouville and the right-sided Caputo fractional derivatives

$$\begin{cases} D_{0+}^{\beta}({}^{C}D_{1-}^{\alpha}x(t)) = f(t,x(t),y(t)), & 0 \le t \le 1, \\ D_{0+}^{\gamma}({}^{C}D_{1-}^{\mu}y(t)) = g(t,x(t),y(t)), & 0 \le t \le 1, \\ {}^{C}D_{1-}^{\alpha}x(0) = x'(0) = 0, & x'(1) = x(1), \\ {}^{C}D_{1-}^{\mu}y(0) = y'(0) = 0, & y'(1) = y(1), \end{cases} \tag{1.1}$$

where D_{0+}^{β} is the left-sided Riemann–Liouville fractional derivative, ${}^{C}D_{1-}^{\mu}$ is the right-sided Caputo fractional derivative, $1 < \alpha, \mu \le 2$, $0 < \beta, \gamma \le 1$, $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0+\infty)$, and $g : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous.

In contrast to previous studies, the mixed fractional derivative in this paper is defined first by Caputo fractional derivatives and secondly by Riemann–Liouville fractional derivatives. We calculate the corresponding Green's functions and their properties. By the classic fixed-point theorem of cone expansion and compression of norm type [20–22], we obtain the existence of the solution for coupled systems of the mixed differential equations. One example is presented to demonstrate the applications of the main theorems.

2 Preliminaries

In this part, we present some related definitions, properties, and lemmas.

Definition 2.1 (see (2.1.1) and (2.1.2) in [4]) The left-sided and right-sided Riemann–Liouville fractional integrals of order α ($\alpha > 0$) of function $f(t) \in C[0,1]$ are defined, respectively, by

$$I_{0+}^{\alpha}f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,\mathrm{d}s \quad \text{and} \quad I_{1-}^{\alpha}f(t) = \int_t^1 \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,\mathrm{d}s,$$

where $\Gamma(\alpha)$ is the Gamma function.

Definition 2.2 (see (2.1.5) in [4]) The left-sided Riemann–Liouville fractional derivative of order α ($\alpha > 0$) of function $f \in C^n[0,1]$ is given by

$$D_{0+}^{\alpha}f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\big(I_{0+}^{n-\alpha}f\big)(t),$$

where $n-1 < \alpha < n$.

Definition 2.3 (see (2.4.16) in [4]) The right-sided Caputo fractional derivative of order α ($\alpha > 0$) of function $f \in C^n[0,1]$ is given by

$${}^{C}D_{1-}^{\alpha}f(t) = (-1)^n I_{1-}^{n-\alpha}f^{(n)}(t),$$

where $n-1 < \alpha < n$.

Property 2.1 (see (2.1.39) and (2.4.43) in [4]) Let $\alpha > 0$ and $n = [\alpha] + 1$. If $f(t) \in C^n[0,1]$, then

$$I_{0+}^{\alpha}\big(D_{0+}^{\alpha}f(t)\big) = f(t) - \sum_{j=1}^{n} C_j t^{\alpha-j},$$

$$I_{1-}^{\alpha}\big({}^{C}D_{1-}^{\alpha}f(t)\big) = f(t) - (-1)^j \sum_{j=0}^{n-1} C_j'(1-t)^j,$$

where $C_j, C_j' \in R$ are arbitrary constants.

Property 2.2 (see (2.1.33) in [4]) Let $\alpha > 0$, $k \in N$ and $\alpha > k$. If $f(t) \in C^n[0,1]$, then

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\big(I_{1-}^{\alpha}f(t)\big) = (-1)^k I_{1-}^{\alpha-k}f(x).$$

Lemma 2.1 (see Theorem 2.3.4 in [20]) *Suppose X is a Banach space and $P \subset X$ is a cone in X. Let Ω_1 and Ω_2 be two bounded, open subsets in X such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Let $S : P \to P$ be completely continuous. Suppose that one of the two conditions*

(1) $\|Sw\| \le \|w\|$, $w \in P \cap \partial\Omega_1$ *and* $\|Sw\| \ge \|w\|$, $w \in P \cap \partial\Omega_2$ *and*

(2) $\|Sw\| \ge \|w\|$, $w \in P \cap \partial\Omega_1$ *and* $\|Sw\| \le \|w\|$, $w \in P \cap \partial\Omega_2$ *is satisfied.*

Then, S has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Next, we derive the corresponding Green's function for boundary value problem (1.1) and build some properties of the Green's function.

Lemma 2.2 *Assume that $y(t) \in C[0,1]$, $1 < \alpha \le 2$ and $0 < \beta \le 1$ hold. Then, $x(t) \in C[0,1]$ is the solution of the fractional differential equation*

$$\begin{cases} D_{0+}^{\beta}({}^{C}D_{1-}^{\alpha}x(t)) = y(t), & 0 \le t \le 1, \\ {}^{C}D_{1-}^{\alpha}x(0) = x'(0) = 0, & x'(1) = x(1), \end{cases} \tag{2.1}$$

if and only if $x(t)$ satisfies the integral equation $x(t) = \int_0^1 G_1(t,s)y(s)\,\mathrm{d}s$, $t \in [0,1]$, where

$$G_1(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha)\Gamma(\beta)}[\int_t^1 (\tau-t)^{\alpha-1}(\tau-s)^{\beta-1}\,\mathrm{d}\tau + (\alpha-1)t\int_s^1 \tau^{\alpha-2}(\tau-s)^{\beta-1}\,\mathrm{d}\tau], \\ \quad 0 \le s \le t \le 1, \\ \frac{1}{\Gamma(\alpha)\Gamma(\beta)}[\int_s^1 (\tau-t)^{\alpha-1}(\tau-s)^{\beta-1}\,\mathrm{d}\tau + (\alpha-1)t\int_s^1 \tau^{\alpha-2}(\tau-s)^{\beta-1}\,\mathrm{d}\tau], \\ \quad 0 \le t \le s \le 1. \end{cases}$$

Proof Integrating the equation (2.1) by Property 2.1, we have

$${}^{C}D_{1-}^{\alpha}x(t) = I_{0+}^{\beta}y(t) + C_1 t^{\beta-1}.$$

By ${}^{C}D_{1-}^{\alpha}x(0) = 0$, we obtain $C_1 = 0$. Integrating the equation above, we obtain

$$x(t) = I_{1-}^{\alpha}\big(I_{0+}^{\beta}y(t)\big) + C_2 - C_3(1-t). \tag{2.2}$$

Letting $t = 1$, we know that $x(1) = C_2$.

Differentiating equation (2.2), by Property 2.2, we have

$$x'(t) = -I_{1-}^{\alpha-1}\big(I_{0+}^{\beta}y(t)\big) + C_3.$$

As $x'(0) = 0$, $x'(1) = x(1)$, we obtain

$$C_3 = I_{1-}^{\alpha-1}\big(I_{0+}^{\beta}y(0)\big) \quad \text{and} \quad x'(1) = C_3.$$

Hence,

$$C_2 = x(1) = x'(1) = C_3.$$

Substituting C_3 and C_2 into equation (2.2), we have

$$\begin{aligned}
x(t) &= I_{1-}^{\alpha}\big(I_{0+}^{\beta}y(t)\big) + t\cdot I_{1-}^{\alpha-1}\big(I_{0+}^{\beta}y(0)\big)\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_t^1(\tau-t)^{\alpha-1}\,\mathrm{d}\tau\int_0^{\tau}(\tau-s)^{\beta-1}y(s)\,\mathrm{d}s\\
&\quad + \frac{t}{\Gamma(\alpha-1)\Gamma(\beta)}\int_0^1\tau^{\alpha-2}\,\mathrm{d}\tau\int_0^{\tau}(\tau-s)^{\beta-1}y(s)\,\mathrm{d}s.
\end{aligned}$$

Changing the order of integration, we obtain

$$\begin{aligned}
x(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\biggl[\int_0^t y(s)\,\mathrm{d}s\int_t^1(\tau-t)^{\alpha-1}(\tau-s)^{\beta-1}\,\mathrm{d}\tau\\
&\quad + \int_t^1 y(s)\,\mathrm{d}s\int_s^1(\tau-t)^{\alpha-1}(\tau-s)^{\beta-1}\,\mathrm{d}\tau\biggr]\\
&\quad + \frac{t}{\Gamma(\alpha-1)\Gamma(\beta)}\int_0^1 y(s)\,\mathrm{d}s\int_s^1\tau^{\alpha-2}(\tau-s)^{\beta-1}\,\mathrm{d}\tau\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\biggl\{\int_0^t y(s)\,\mathrm{d}s\biggl[\int_t^1(\tau-t)^{\alpha-1}(\tau-s)^{\beta-1}\,\mathrm{d}\tau + (\alpha-1)t\int_s^1\tau^{\alpha-2}(\tau-s)^{\beta-1}\,\mathrm{d}\tau\biggr]\\
&\quad + \int_t^1 y(s)\,\mathrm{d}s\biggl[\int_s^1(\tau-t)^{\alpha-1}(\tau-s)^{\beta-1}\,\mathrm{d}\tau + (\alpha-1)t\int_s^1\tau^{\alpha-2}(\tau-s)^{\beta-1}\,\mathrm{d}\tau\biggr]\biggr\}\\
&= \int_0^1 G_1(t,s)y(s)\,\mathrm{d}s.
\end{aligned}$$

The proof is completed. □

It is simple to show that $G_1(t,s)\geq 0$ for any $s,t\in[0,1]$.

Lemma 2.3 *The Green's function $G_1(t,s)$ defined by Lemma 2.2 satisfies*

$$(\alpha-1)tA_1(s)\leq G_1(t,s)\leq A_1(s),$$

where $A_1(s) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_s^1\tau^{\alpha-2}(\tau-s)^{\beta-1}\,\mathrm{d}\tau$.

Proof First, for any $0 \leq t \leq s \leq 1$,

$$\begin{aligned}
G_1(t,s) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_s^1 (\tau - t)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau + (\alpha - 1)t\int_s^1 \tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau\right] \\
&\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(\alpha - 1)t\int_s^1 \tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau = (\alpha - 1)tA_1(s), \\
G_1(t,s) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_s^1 \tau^{\alpha-1}\left(1 - \frac{t}{\tau}\right)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau + (\alpha - 1)t\int_s^1 \tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau\right] \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_s^1 \left[\tau(1 - t)^{\alpha-1} + (\alpha - 1)t\right]\tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left[(1 - t)^{\alpha-1} + (\alpha - 1)t\right]\int_s^1 \tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau \leq A_1(s).
\end{aligned}$$

We show that $(1-t)^{\alpha-1} + (\alpha - 1)t \leq 1$ as follows. Let $\phi(t) = (1-t)^{\alpha-1} + (\alpha - 1)t$, we obtain $\phi'(t) = -(\alpha - 1)(1-t)^{\alpha-2} + (\alpha - 1) = -(\alpha - 1)[\frac{1}{(1-t)^{2-\alpha}} - 1] \leq 0$, for $t \in (0,1)$.

Hence, $\phi(t)$ is decreasing and $\phi(t) \leq \phi(0) = 1$ for any $t \in [0,1]$.

Next, for any $0 \leq s \leq t \leq 1$,

$$\begin{aligned}
G_1(t,s) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_t^1 (\tau - t)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau + (\alpha - 1)t\int_s^1 \tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau\right] \\
&\geq \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(\alpha - 1)t\int_s^1 \tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau = (\alpha - 1)tA_1(s), \\
G_1(t,s) &\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_s^1 (\tau - t)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau + (\alpha - 1)t\int_s^1 \tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau\right] \\
&\leq A_1(s).
\end{aligned}$$

The proof is completed. □

Remark 2.1 Changing the order of integrating, we have

$$\int_0^1 A_1(s)\,\mathrm{d}s = \frac{1}{\Gamma(\alpha)\Gamma(\beta+1)(\alpha+\beta-1)} := K_1.$$

3 Main results

Now, we consider the space $X = C[0,1]$ with the usual maximum norm

$$\|x\| = \max_{t\in[0,1]}\left|x(t)\right|.$$

For any $(x,y) \in X \times X$, the norm was defined as $\|(x,y)\| = \max\{\|x\|, \|y\|\}$. Hence, $(X \times X, \|\cdot\|)$ is a Banach space. We define set P by

$$P = \left\{x \in X : x(t) \geq 0, x(t) \geq \lambda t\|x\|, t \in [0,1]\right\},$$

where $\lambda = \min\{\alpha - 1, \mu - 1\} > 0$. Let $U = P \times P$. Obviously, U is a normal cone.

Let integral operator $T : U \to X \times X$ be defined by

$$T(x,y) = \big(T_1(x,y), T_2(x,y)\big),$$

where $T_1(x,y) = \int_0^1 G_1(t,s)f(s,x(s),y(s))\,\mathrm{d}s$, $T_2(x,y) = \int_0^1 G_2(t,s)g(s,x(s),y(s))\,\mathrm{d}s$.

According to the definition of $G_1(t,s)$ in Lemma 2.2, it follows that

$$G_2(t,s) = \begin{cases} \frac{1}{\Gamma(\mu)\Gamma(\gamma)}[\int_t^1 (\tau-t)^{\mu-1}(\tau-s)^{\gamma-1}\,\mathrm{d}\tau + (\mu-1)t\int_s^1 \tau^{\mu-2}(\tau-s)^{\gamma-1}\,\mathrm{d}\tau], \\ \quad 0 \le s \le t \le 1, \\ \frac{1}{\Gamma(\mu)\Gamma(\gamma)}[\int_s^1 (\tau-t)^{\mu-1}(\tau-s)^{\gamma-1}\,\mathrm{d}\tau + (\mu-1)t\int_s^1 \tau^{\mu-2}(\tau-s)^{\gamma-1}\,\mathrm{d}\tau], \\ \quad 0 \le t \le s \le 1. \end{cases}$$

Hence, the fixed point of the operator T is the solution of BVPs (1.1).

It is straightforward to show that

$$G_2(t,s) \ge 0 \quad \text{and} \quad (\mu-1)tA_2(s) \le G_2(t,s) \le A_2(s),$$

where $A_2(s) = \frac{1}{\Gamma(\mu)\Gamma(\gamma)}\int_s^1 \tau^{\mu-2}(\tau-s)^{\gamma-1}\,\mathrm{d}\tau$, then $\int_0^1 A_2(s)\,\mathrm{d}s = \frac{1}{\Gamma(\mu)\Gamma(\gamma+1)(\mu+\gamma-1)} := K_2$.

In order to prove the main results, we need the following conditions:

(H1) $\alpha + \beta > 2$, $\mu + \gamma > 2$;

(H2) $f(t,x,y) \in C([0,1] \times [0,+\infty) \times [0,+\infty), [0+\infty))$,
$g(t,x,y) \in C([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty))$.

Theorem 3.1 *If the conditions* (H1) *and* (H2) *hold, the operator* $T : U \to U$ *is completely continuous.*

Proof First, for any $(x,y) \in U$,

$$\|T_1(x,y)\| = \max_{t\in[0,1]}\left|\int_0^1 G_1(t,s)f\big(s,x(s),y(s)\big)\,\mathrm{d}s\right| \le \int_0^1 A_1(s)f\big(s,x(s),y(s)\big)\,\mathrm{d}s,$$

$$\begin{aligned} T_1(x,y) &= \int_0^1 G_1(t,s)f\big(s,x(s),y(s)\big)\,\mathrm{d}s \ge (\alpha-1)t\int_0^1 A_1(s)f\big(s,x(s),y(s)\big)\,\mathrm{d}s \\ &\ge (\alpha-1)t\|T_1(x,y)\| \ge \lambda t\|T_1(x,y)\|. \end{aligned}$$

Hence, $T_1(x,y) \in U$.

Similarly, we can prove that $T_2(x,y) \in U$ for any $(x,y) \in U$. Obviously, the operator $T : U \to U$.

Secondly, owing to the definition of $G_1(t,s)$ and $G_2(t,s)$ and (H2), the operator T is continuous on U.

Let $\Omega_L = \{(x,y) \in U : \|(x,y)\| \le L\}$ be a nonempty bounded closed set, where $L > 0$ is a constant. If $(x,y) \in \Omega_L$, there exists $M > 0$ such that $\|f(t,x,y)\| \le M$, $\|g(t,x,y)\| \le M$ for any $(x,y) \in \Omega_L$, $t \in [0,1]$. Hence, we have

$$\|T_1(x,y)\| = \max_{0\le t\le 1}\left|\int_0^1 G_1(t,s)f\big(s,x(s),y(s)\big)\,\mathrm{d}s\right| \le M\int_0^1 A_1(s)\,\mathrm{d}s = MK_1.$$

Similarly, $\|T_2(x,y)\| \le MK_2$. Then, $\|T(x,y)\| \le \max\{MK_1, MK_2\}$. Hence, the operator T is uniformly bounded.

Thirdly, for any $(x,y) \in U$ and $t_1, t_2 \in [0,1]$, without loss of generality, we suppose $t_1 < t_2$. Now, we prove the operator T is equicontinuous.

$$
\begin{aligned}
&\left|T_1(x,y)(t_2) - T_1(x,y)(t_1)\right| \\
&\quad = \left|\int_0^1 G_1(t_2,s) f\big(s,x(s),y(s)\big)\,\mathrm{d}s - \int_0^1 G_1(t_1,s) f\big(s,x(s),y(s)\big)\,\mathrm{d}s\right| \\
&\quad \le M \int_0^1 \left|G_1(t_2,s) - G_1(t_1,s)\right|\,\mathrm{d}s \\
&\quad = M\biggl[\int_0^{t_1} \left|G_1(t_2,s) - G_1(t_1,s)\right|\,\mathrm{d}s + \int_{t_1}^{t_2} \left|G_1(t_2,s) - G_1(t_1,s)\right|\,\mathrm{d}s \\
&\qquad + \int_{t_2}^1 \left|G_1(t_2,s) - G_1(t_1,s)\right|\,\mathrm{d}s\biggr] \\
&\quad = \frac{M}{\Gamma(\alpha)\Gamma(\beta)}\biggl\{\int_0^{t_1}\biggl|\biggl[\int_{t_2}^1 (\tau - t_2)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau + (\alpha-1)t_2 \int_s^1 \tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau\biggr] \\
&\qquad - \biggl[\int_{t_1}^1 (\tau - t_1)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau + (\alpha-1)t_1 \int_s^1 \tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau\biggr]\biggr|\,\mathrm{d}s \\
&\qquad + \int_{t_1}^{t_2}\biggl|\biggl[\int_{t_2}^1 (\tau - t_2)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau + (\alpha-1)t_2 \int_s^1 \tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau\biggr] \\
&\qquad - \biggl[\int_s^1 (\tau - t_1)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau + (\alpha-1)t_1 \int_s^1 \tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau\biggr]\biggr|\,\mathrm{d}s \\
&\qquad + \int_{t_2}^1\biggl|\biggl[\int_s^1 (\tau - t_2)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau + (\alpha-1)t_2 \int_s^1 \tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau\biggr] \\
&\qquad - \biggl[\int_s^1 (\tau - t_1)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau + (\alpha-1)t_1 \int_s^1 \tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau\biggr]\biggr|\,\mathrm{d}s\biggr\} \\
&\quad = \frac{M}{\Gamma(\alpha)\Gamma(\beta)}\biggl\{\int_0^1 (\alpha-1)(t_2 - t_1)\int_s^1 \tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau\,\mathrm{d}s \\
&\qquad + \int_0^{t_1}\biggl|\int_{t_2}^1 (\tau - t_2)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau - \int_{t_1}^1 (\tau - t_1)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau\biggr|\,\mathrm{d}s \\
&\qquad + \int_{t_1}^{t_2}\biggl|\int_{t_2}^1 (\tau - t_2)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau - \int_s^1 (\tau - t_1)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau\biggr|\,\mathrm{d}s \\
&\qquad + \int_{t_2}^1\biggl|\int_s^1 \bigl[(\tau - t_2)^{\alpha-1} - (\tau - t_1)^{\alpha-1}\bigr](\tau - s)^{\beta-1}\,\mathrm{d}\tau\biggr|\,\mathrm{d}s\biggr\} \\
&\quad \le \frac{M}{\Gamma(\alpha)\Gamma(\beta)}\biggl\{(\alpha-1)(t_2 - t_1)\int_0^1 \mathrm{d}s \int_s^1 \tau^{\alpha-2}(\tau - s)^{\beta-1}\,\mathrm{d}\tau \\
&\qquad + \int_0^{t_1}\biggl[\int_{t1}^{t_2} (\tau - t_1)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau \\
&\qquad + \int_{t_2}^1 \bigl((\tau - t_1)^{\alpha-1} - (\tau - t_2)^{\alpha-1}\bigr)(\tau - s)^{\beta-1}\,\mathrm{d}\tau\biggr]\,\mathrm{d}s \\
&\qquad + \int_{t_1}^{t_2}\biggl[\int_s^1 (\tau - t_1)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau - \int_{t_2}^1 (\tau - t_2)^{\alpha-1}(\tau - s)^{\beta-1}\,\mathrm{d}\tau\biggr]\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t_2}^{1}\mathrm{d}s\int_{s}^{1}\big[(\alpha-1)(\tau-\xi_1)^{\alpha-2}(t_2-t_1)\big](\tau-s)^{\beta-1}\,\mathrm{d}\tau\Bigg\}\\
&\big(\text{here } \xi_1\in(t_1,t_2)\big)\\
\le{}& \frac{M}{\Gamma(\alpha)\Gamma(\beta)}\Bigg\{(\alpha-1)(t_2-t_1)\int_0^1\mathrm{d}\tau\int_0^\tau \tau^{\alpha-2}(\tau-s)^{\beta-1}\,\mathrm{d}s\\
&+\int_0^{t_1}\mathrm{d}s\int_{t1}^{t_2}(\tau-s)^{\beta-1}\,\mathrm{d}\tau+\int_0^{t_1}\mathrm{d}s\int_{t_2}^{1}(\alpha-1)(t_2-t_1)(\tau-\xi_2)^{\alpha-2}(\tau-s)^{\beta-1}\,\mathrm{d}\tau\\
&+\int_{t_1}^{t_2}\mathrm{d}s\int_s^1(\tau-t_1)^{\alpha-1}(\tau-s)^{\beta-1}\,\mathrm{d}\tau+\int_{t_2}^{1}(\alpha-1)(t_2-t_1)\,\mathrm{d}s\int_s^1(\tau-s)^{\alpha+\beta-3}\,\mathrm{d}\tau\Bigg\}\\
&\big(\text{here } \xi_2\in(t_1,t_2)\big)\\
\le{}& \frac{M}{\Gamma(\alpha)\Gamma(\beta)}\Bigg\{(\alpha-1)(t_2-t_1)\int_0^1\tau^{\alpha-2}\frac{\tau^\beta}{\beta}\,\mathrm{d}\tau\\
&+\int_0^{t_1}(t_1-s)^{\beta-1}(t_2-t_1)\,\mathrm{d}s+(\alpha-1)(t_2-t_1)\int_0^{t_1}\mathrm{d}s\int_{t_2}^1(\tau-\xi_2)^{\alpha+\beta-3}\,\mathrm{d}\tau\\
&+\int_{t_1}^{t_2}\mathrm{d}s\int_s^1(\tau-s)^{\beta-1}\,\mathrm{d}\tau+(\alpha-1)(t_2-t_1)\int_{t_2}^1\frac{(1-s)^{\alpha+\beta-2}}{\alpha+\beta-2}\,\mathrm{d}s\Bigg\}\\
\le{}& \frac{M}{\Gamma(\alpha)\Gamma(\beta)}\Bigg\{(\alpha-1)(t_2-t_1)\frac{1}{\beta(\alpha+\beta-1)}+(t_2-t_1)\frac{1}{\beta}+(\alpha-1)(t_2-t_1)\frac{1}{\alpha+\beta-2}\\
&+\int_{t_1}^{t_2}\frac{(1-s)^\beta}{\beta}\,\mathrm{d}s+(t_2-t_1)\frac{1}{\alpha+\beta-2}\Bigg\}\\
\le{}& \frac{M}{\Gamma(\alpha)\Gamma(\beta)}(t_2-t_1)\left\{\frac{1}{\beta}+\frac{1}{\beta}+\frac{1}{\alpha+\beta-2}+\frac{1}{\beta}+\frac{1}{\alpha+\beta-2}\right\}\\
={}& K_3(t_2-t_1),\quad \text{where } K_3=\frac{M}{\Gamma(\alpha)\Gamma(\beta)}\left(\frac{3}{\beta}+\frac{2}{\alpha+\beta-2}\right).
\end{aligned}
$$

By (H1), K_3 is a positive constant.

Similarly, for any $(x,y)\in U$ and $t_1,t_2\in[0,1]$, we have

$$
\left|T_2(x,y)(t_2)-T_2(x,y)(t_1)\right|\le K_4(t_2-t_1),\quad \text{where } K_4=\frac{M}{\Gamma(\mu)\Gamma(\gamma)}\left(\frac{3}{\gamma}+\frac{2}{\mu+\gamma-2}\right).
$$

That is to say, the operator T is equicontinuous. According to Arzela–Ascoli's theorem, the operator T is completely continuous.

The proof is completed. □

For convenience, we introduce the following notations

$$
\begin{aligned}
f_0&=\lim_{x+y\to0+}\left\{\min_{t\in[0,1]}\frac{f(t,x,y)}{x+y}\right\}, & g_0&=\lim_{x+y\to0+}\left\{\min_{t\in[0,1]}\frac{g(t,x,y)}{x+y}\right\},\\
f^\infty&=\lim_{x+y\to+\infty}\left\{\max_{t\in[0,1]}\frac{f(t,x,y)}{x+y}\right\}, & g^\infty&=\lim_{x+y\to+\infty}\left\{\max_{t\in[0,1]}\frac{g(t,x,y)}{x+y}\right\},\\
f^0&=\lim_{x+y\to0+}\left\{\max_{t\in[0,1]}\frac{f(t,x,y)}{x+y}\right\}, & g^0&=\lim_{x+y\to+0}\left\{\max_{t\in[0,1]}\frac{g(t,x,y)}{x+y}\right\},\\
f_\infty&=\lim_{x+y\to+\infty}\left\{\min_{t\in[0,1]}\frac{f(t,x,y)}{x+y}\right\}, & g_\infty&=\lim_{x+y\to+\infty}\left\{\min_{t\in[0,1]}\frac{g(t,x,y)}{x+y}\right\}.
\end{aligned}
$$

Lemma 3.1 *Assume that* $[(\alpha-1)\lambda K_1]^{-1} < f_0 \le +\infty$ *and* $[(\mu-1)\lambda K_2]^{-1} < g_0 \le +\infty$ *hold, we have* $\|T(x,y)\| \ge \|(x,y)\|$.

Proof Case 1. If $[(\alpha-1)\lambda K_1]^{-1} < f_0 < +\infty$ and $[(\mu-1)\lambda K_2]^{-1} < g_0 < +\infty$ hold, set $0 < \epsilon_1 < f_0 - [(\alpha-1)\lambda K_1]^{-1}$ and $0 < \epsilon_2 < g_0 - [(\mu-1)\lambda K_2]^{-1}$, there exists $\delta_1 > 0$ such that

$$\frac{f(t,x,y)}{x+y} \ge f_0 - \epsilon_1, \qquad \frac{g(t,x,y)}{x+y} \ge g_0 - \epsilon_2, \quad \text{for all } 0 < x < \delta_1, 0 < y < \delta_1.$$

Then, we have

$$\begin{aligned} &f(t,x,y) \ge (f_0 - \epsilon_1)(x+y) > \left[(\alpha-1)\lambda K_1\right]^{-1}(x+y), \\ &g(t,x,y) \ge (g_0 - \epsilon_2)(x+y) > \left[(\mu-1)\lambda K_2\right]^{-1}(x+y). \end{aligned}$$

Case 2. If $f_0 = +\infty$ and $g_0 = +\infty$, for two given large numbers $N_1 \ge [(\alpha-1)\lambda K_1]^{-1}$ and $N_2 \ge [(\mu-1)\lambda K_2]^{-1}$, there exists $\delta_2 > 0$ such that

$$\frac{f(t,x,y)}{x+y} \ge N_1, \qquad \frac{g(t,x,y)}{x+y} \ge N_2, \quad \text{for all } 0 < x < \delta_2, 0 < y < \delta_2.$$

Then, we have

$$\begin{aligned} &f(t,x,y) \ge N_1(x+y) \ge \left[(\alpha-1)\lambda K_1\right]^{-1}(x+y), \\ &g(t,x,y) \ge N_2(x+y) \ge \left[(\mu-1)\lambda K_2\right]^{-1}(x+y). \end{aligned}$$

Letting $(x,y) \in U \cap \partial\Omega_{r_1}$, where $\Omega_{r_1} = \{(x,y) \in X \times X : \|(x,y)\| \le r_1\}$, $0 < r_1 \le \min\{\delta_1, \delta_2\}$, we have

$$\begin{aligned} \left\|T_1(x,y)\right\| &= \max_{t\in[0,1]}\left|\int_0^1 G_1(t,s) f\bigl(s,x(s),y(s)\bigr)\,\mathrm{d}s\right| \\ &\ge \max_{t\in[0,1]}\left|\int_0^1 (\alpha-1)tA_1(s)\left[(\alpha-1)\lambda K_1\right]^{-1}\bigl(x(s)+y(s)\bigr)\,\mathrm{d}s\right| \\ &\ge \max_{t\in[0,1]}\left|\int_0^1 tA_1(s)[\lambda K_1]^{-1}\lambda t\bigl(\|x\| + \|y\|\bigr)\,\mathrm{d}s\right| \\ &\ge K_1^{-1}\max_{t\in[0,1]}\left|t^2\int_0^1 A_1(s)\,\mathrm{d}s\right| \cdot \left\|(x,y)\right\| \\ &= \left\|(x,y)\right\|. \end{aligned}$$

Similarly, $\|T_2(x,y)\| \ge \|(x,y)\|$.

No matter which of the above cases holds, we have

$$\left\|T(x,y)\right\| = \max_{t\in[0,1]}\left\{\left\|T_1(x,y)\right\|, \left\|T_2(x,y)\right\|\right\} \ge \left\|(x,y)\right\|.$$

The proof is completed. □

Remark 3.1 Either $[(\alpha-1)\lambda K_1]^{-1} < f_0 < +\infty$ and $g_0 = +\infty$ hold or $f_0 = +\infty$ and $[(\mu-1)\lambda K_2]^{-1} < g_0 < +\infty$ hold, similar to Lemma 3.1, we also have $\|T(x,y)\| \ge \|(x,y)\|$.

Lemma 3.2 *Assume that* $0 \le f^\infty < (2K_1)^{-1}$ *and* $0 \le g^\infty < (2K_2)^{-1}$ *hold, we have* $\|T(x,y)\| \le \|(x,y)\|$.

Proof Case 1. If $0 < f^\infty < (2K_1)^{-1}$ and $0 < g^\infty < (2K_2)^{-1}$ hold, setting $0 < \delta_3 < (2K_1)^{-1} - f^\infty$ and $0 < \delta_4 < (2K_2)^{-1} - g^\infty$, there exists $N_3 > 0$ such that

$$\frac{f(t,x,y)}{x+y} \le f^\infty + \delta_3, \qquad \frac{g(t,x,y)}{x+y} \le g^\infty + \delta_4, \quad \text{for all } x > N_3, y > N_3.$$

Then, we have

$$\begin{aligned} f(t,x,y) &\le \big(f^\infty + \delta_3\big)(x+y) < (2K_1)^{-1}(x+y), \\ g(t,x,y) &\le \big(g^\infty + \delta_4\big)(x+y) < (2K_2)^{-1}(x+y). \end{aligned}$$

Case 2. If $f^\infty = 0$ and $g^\infty = 0$, for $\epsilon_3 \le (2K_1)^{-1}$ and $\epsilon_4 \le (2K_2)^{-1}$, there exists $N_4 > 0$ such that

$$\frac{f(t,x,y)}{x+y} \le \epsilon_3, \qquad \frac{g(t,x,y)}{x+y} \le \epsilon_4, \quad \text{for all } x \ge N_4, y \ge N_4.$$

Then, we have

$$\begin{aligned} f(t,x,y) &\le \epsilon_3(x+y) \le (2K_1)^{-1}(x+y), \\ g(t,x,y) &\le \epsilon_4(x+y) \le (2K_2)^{-1}(x+y). \end{aligned}$$

Letting $(x,y) \in U \cap \partial\Omega_{R_1}$, where $\Omega_{R_1} = \{(x,y) \in X \times X : \|(x,y)\| \le R_1\}$, $R_1 > \max\{r_1, N_3, N_4\}$, we have

$$\begin{aligned} \big\|T_1(x,y)\big\| &\le \max_{t\in[0,1]}\left|\int_0^1 A_1(s)(2K_1)^{-1}\big(x(s)+y(s)\big)\,\mathrm{d}s\right| \\ &\le K_1^{-1}\int_0^1 A_1(s)\,\mathrm{d}s \cdot \frac{\|x\|+\|y\|}{2} \le \big\|(x,y)\big\|. \end{aligned}$$

Similarly, $\|T_2(x,y)\| \le \|(x,y)\|$.

No matter which of the above cases holds, we have

$$\big\|T(x,y)\big\| = \max_{t\in[0,1]}\big\{\big\|T_1(x,y)\big\|, \big\|T_2(x,y)\big\|\big\} \le \big\|(x,y)\big\|.$$

The proof is completed. □

Remark 3.2 Either $0 < f^\infty < (2K_1)^{-1}$ and $g^\infty = 0$ hold or $f^\infty = 0$ and $0 < g^\infty < (2K_2)^{-1}$ hold, similar to Lemma 3.2, we also have $\|T(x,y)\| \le \|(x,y)\|$.

Lemma 3.3 *Assume that* $0 \le f^0 < (2K_1)^{-1}$ *and* $0 \le g^0 < (2K_2)^{-1}$ *hold, we have* $\|T(x,y)\| \le \|(x,y)\|$.

Proof Case 1. If $0 < f^0 < (2K_1)^{-1}$ and $0 < g^0 < (2K_2)^{-1}$ hold, setting $0 < \epsilon_5 < (2K_1)^{-1} - f^0$ and $0 < \epsilon_6 < (2K_2)^{-1} - g^0$, there exists $\delta_5 > 0$ such that

$$\frac{f(t,x,y)}{x+y} \le f^0 + \epsilon_5, \qquad \frac{g(t,x,y)}{x+y} \le g^0 + \epsilon_6, \quad \text{for all } 0 < x < \delta_5, 0 < y < \delta_5.$$

Then, we have

$$\begin{aligned} f(t,x,y) &\le \left(f^0 + \epsilon_5\right)(x+y) < (2K_1)^{-1}(x+y), \\ g(t,x,y) &\le \left(g^0 + \epsilon_6\right)(x+y) < (2K_2)^{-1}(x+y). \end{aligned}$$

Case 2. If $f^0 = 0$ and $g^0 = 0$, for $\epsilon_7 \le (2K_1)^{-1}$ and $\epsilon_8 \le (2K_2)^{-1}$, there exists $\delta_6 > 0$ such that

$$\frac{f(t,x,y)}{x+y} \le \epsilon_7, \qquad \frac{g(t,x,y)}{x+y} \le \epsilon_8, \quad \text{for all } 0 < x < \delta_6, 0 < y < \delta_6.$$

Then, we have

$$\begin{aligned} f(t,x,y) &\le \epsilon_7(x+y) < (2K_1)^{-1}(x+y), \\ g(t,x,y) &\le \epsilon_8(x+y) < (2K_2)^{-1}(x+y). \end{aligned}$$

Letting $(x,y) \in U \cap \partial\Omega_{r_2}$, where $\Omega_{r_2} = \{(x,y) \in X \times X : \|(x,y)\| \le r_2\}$, $0 < r_2 \le \min\{\delta_5, \delta_6\}$, we have

$$\begin{aligned} \left\|T_1(x,y)\right\| &\le \max_{t\in[0,1]} \left| \int_0^1 A_1(s)(2K_1)^{-1}\left(x(s) + y(s)\right) \mathrm{d}s \right| \\ &\le K_1^{-1} \int_0^1 A_1(s)\,\mathrm{d}s \cdot \frac{\|x\| + \|y\|}{2} \le \left\|(x,y)\right\|. \end{aligned}$$

Similarly, $\|T_2(x,y)\| \le \|(x,y)\|$.

No matter which of the above cases holds, we have

$$\left\|T(x,y)\right\| = \max_{t\in[0,1]} \left\{ \left\|T_1(x,y)\right\|, \left\|T_2(x,y)\right\| \right\} \le \left\|(x,y)\right\|.$$

The proof is completed. □

Remark 3.3 Either $0 < f^0 < (2K_1)^{-1}$ and $g^0 = 0$ hold or $f^0 = 0$ and $0 < g^0 < (2K_2)^{-1}$ hold, similar to Lemma 3.3, we also have $\|T(x,y)\| \le \|(x,y)\|$.

Lemma 3.4 *Assume that* $[(\alpha-1)\lambda K_1]^{-1} < f_\infty \le +\infty$ *and* $[(\mu-1)\lambda K_2]^{-1} < g_\infty \le +\infty$ *hold, we have* $\|T(x,y)\| \ge \|(x,y)\|$.

Proof Case 1. If $[(\alpha-1)\lambda K_1]^{-1} < f_\infty < +\infty$ and $[(\mu-1)\lambda K_2]^{-1} < g_\infty < +\infty$ hold, setting $0 < \delta_7 < f_\infty - [(\alpha-1)\lambda K_1]^{-1}$ and $0 < \delta_8 < g_\infty - [(\mu-1)\lambda K_2]^{-1}$, there exists a constant $N_5 > 0$ such that

$$\frac{f(t,x,y)}{x+y} \ge f_\infty - \delta_7, \qquad \frac{g(t,x,y)}{x+y} \ge g_\infty - \delta_8, \quad \text{for any } x > N_5, y > N_5.$$

Then, we have

$$f(t,x,y) \geq (f_\infty - \delta_7)(x+y) > \left[(\alpha-1)\lambda K_1\right]^{-1}(x+y),$$
$$g(t,x,y) \geq (g_\infty - \delta_8)(x+y) > \left[(\mu-1)\lambda K_2\right]^{-1}(x+y).$$

Case 2. If $f_0 = +\infty$ and $g_0 = +\infty$, for two constants $N_7 \geq [(\alpha-1)\lambda K_1]^{-1}$ and $N_8 \geq [(\mu-1)\lambda K_2]^{-1}$, there exists a constant $N_6 > 0$ such that

$$\frac{f(t,x,y)}{x+y} \geq N_7, \qquad \frac{g(t,x,y)}{x+y} \geq N_8, \quad \text{for any } x > N_6, y > N_6.$$

Then, we have

$$f(t,x,y) \geq N_7(x+y) \geq \left[(\alpha-1)\lambda K_1\right]^{-1}(x+y),$$
$$g(t,x,y) \geq N_8(x+y) \geq \left[(\mu-1)\lambda K_2\right]^{-1}(x+y).$$

Letting $(x,y) \in U \cap \partial\Omega_{R_2}$, where $\Omega_{R_2} = \{(x,y) \in X \times X : \|(x,y)\| \leq R_2\}$, $R_2 > \max\{r_2, N_5, N_6\}$, we have

$$\begin{aligned}\left\|T_1(x,y)\right\| &\geq \max_{t\in[0,1]}\left|\int_0^1 (\alpha-1)tA_1(s)\left[(\alpha-1)\lambda K_1\right]^{-1}\left(x(s)+y(s)\right)\mathrm{d}s\right| \\ &\geq \max_{t\in[0,1]}\left|\int_0^1 tA_1(s)[\lambda K_1]^{-1}\lambda t\left(\|x\|+\|y\|\right)\mathrm{d}s\right| \\ &\geq K_1^{-1}\max_{t\in[0,1]}\left|t^2\int_0^1 A_1(s)\,\mathrm{d}s\right| \cdot \left(\|x\|+\|y\|\right) \geq \left\|(x,y)\right\|.\end{aligned}$$

Similarly, $\|T_2(x,y)\| \geq \|(x,y)\|$.

No matter which of the above cases holds, we have

$$\left\|T(x,y)\right\| = \max_{t\in[0,1]}\left\{\left\|T_1(x,y)\right\|, \left\|T_2(x,y)\right\|\right\} \geq \left\|(x,y)\right\|.$$

The proof is completed. □

Remark 3.4 Either $[(\alpha-1)\lambda K_1]^{-1} < f_\infty < +\infty$ and $g_\infty = +\infty$ hold or $f_\infty = +\infty$ and $[(\mu-1)\lambda K_2]^{-1} < g_\infty < +\infty$ hold, similar to Lemma 3.4, we also have $\|T(x,y)\| \geq \|(x,y)\|$.

Theorem 3.2 *Supposing that* (H1) *and* (H2) *hold, and one of the two following conditions is satisfied*:

(1) $[(\alpha-1)\lambda K_1]^{-1} < f_0 \leq +\infty$, $[(\mu-1)\lambda K_2]^{-1} < g_0 \leq +\infty$ *and* $0 \leq f^\infty < (2K_1)^{-1}$, $0 \leq g^\infty < (2K_2)^{-1}$;

(2) $0 \leq f^0 < (2K_1)^{-1}$, $0 \leq g^0 < (2K_2)^{-1}$ *and* $[(\alpha-1)\lambda K_1]^{-1} < f_\infty \leq +\infty$, $[(\mu-1)\lambda K_2]^{-1} < g_\infty \leq +\infty$.

Then, the boundary value problem (1.1) *has at least one positive solution.*

Proof Case 1. By Lemma 3.1, for any $(x,y) \in U \cap \partial\Omega_{r_1}$, we have $\|T(x,y)\| \geq \|(x,y)\|$. By Lemma 3.2, for $(x,y) \in U \cap \partial\Omega_{R_1}$, and $r_1 < R_1$, we have $\|T(x,y)\| \leq \|(x,y)\|$. According to Lemma 2.1, the boundary value problem (1.1) has at least one positive solution for $(x,y) \in U \cap (\overline{\Omega}_{R_1} \setminus \Omega_{r_1})$.

Case 2. By Lemma 3.3, for any $(x,y) \in U \cap \partial\Omega_{r_2}$, we have $\|T(x,y)\| \leq \|(x,y)\|$. By Lemma 3.4, for $(x,y) \in U \cap \partial\Omega_{R_2}$, and $r_2 < R_2$, we have $\|T(x,y)\| \geq \|(x,y)\|$. According to Lemma 2.1, the boundary value problem (1.1) has at least one positive solution for $(x,y) \in U \cap (\overline{\Omega}_{R_2} \setminus \Omega_{r_2})$.

The proof is completed. □

4 Application

Now, we present the following example to illustrate our main theorems.

Example Consider the mixed fractional differential equations

$$\begin{cases} D_{0+}^{\frac{2}{3}}({}^{C}D_{1-}^{\frac{3}{2}}x(t)) = (x+y)^{\frac{1}{2}} + \ln(t(x+y)^2+1), \quad 0 \leq t \leq 1, \\ D_{0+}^{\frac{3}{4}}({}^{C}D_{1-}^{\frac{5}{3}}y(t)) = (x+y)^{\frac{1}{3}} + t\sin t, \quad 0 \leq t \leq 1, \\ {}^{C}D_{1-}^{\frac{3}{2}}x(0) = x'(0) = 0, \qquad x'(1) = x(1), \\ {}^{C}D_{1-}^{\frac{5}{3}}y(0) = y'(0) = 0, \qquad y'(1) = y(1). \end{cases} \tag{4.1}$$

Here, $\alpha = \frac{3}{2}$, $\beta = \frac{2}{3}$, $\mu = \frac{5}{3}$, $\gamma = \frac{3}{4}$.

Therefore,

$$\alpha + \beta = \frac{13}{6} > 2, \qquad \mu + \gamma = \frac{29}{12} > 2, \qquad \lambda = \frac{1}{2},$$

$$K1 = \frac{1}{\frac{7}{6}\Gamma(\frac{3}{2})\Gamma(\frac{5}{3})} = 1.0714, \qquad K2 = \frac{1}{\frac{17}{12}\Gamma(\frac{5}{3})\Gamma(\frac{7}{4})} = 0.8508.$$

$$\left(\frac{1}{2} \cdot \frac{1}{2}K1\right)^{-1} = 3.7335, \qquad \left(\frac{2}{3} \cdot \frac{1}{2}K2\right)^{-1} = 3.5261,$$

$$(2K1)^{-1} = 0.4667, \qquad (2K2)^{-1} = 0.5877,$$

$$f_0 = \lim_{(x+y)\to 0+} \left\{ \min_{t\in[0,1]} \frac{(x+y)^{\frac{1}{2}} + \ln(t(x+y)^2+1)}{x+y} \right\} = +\infty > 3.7335,$$

$$g_0 = \lim_{(x+y)\to 0+} \left\{ \min_{t\in[0,1]} \frac{(x+y)^{\frac{1}{3}} + t\sin t}{x+y} \right\} = +\infty > 3.5261,$$

$$f^{\infty} = \lim_{(x+y)\to +\infty} \left\{ \max_{t\in[0,1]} \frac{(x+y)^{\frac{1}{2}} + \ln(t(x+y)^2+1)}{x+y} \right\} = 0 < 0.4667,$$

$$g^{\infty} = \lim_{(x+y)\to +\infty} \left\{ \max_{t\in[0,1]} \frac{(x+y)^{\frac{1}{3}} + t\sin t}{x+y} \right\} = 0 < 0.5877.$$

Therefore, it follows from Theorem 3.2 that the fractional differential equation (4.1) has a nontrivial positive solution.

5 Conclusion

In our review of the literature, the equations of (1.1) were first studied in this paper. We discuss the coupled boundary value problem of mixed fractional differential equations defined first by the right-sided Caputo fractional derivatives and secondly by the left-sided

Riemann–Liouville fractional derivatives. These equations are different from the previous mixed fractional differential equations. We construct the Green's function, whose properties are described by a simple inequality. Furthermore, by using the fixed-point theorems of a cone, we obtain the existence of solution of equation (1.1). With this classic method, we improve the theory of the existence of solutions for mixed fractional equations.

Funding
The corresponding author was supported by the National Natural Science Foundation of China (Grant No. 11826035).

Availability of data and materials
Not applicable.

Declarations

Ethics approval and consent to participate
Ethics approval was not required for this research.

Competing interests
The authors declare no competing interests.

Author contributions
Each of the authors contributed to each part of this work equally. All the authors read and approved the final version of the manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 3 November 2022 Accepted: 18 January 2023 Published online: 02 February 2023

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