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Multipoint boundary value problems for higher-order Hadamard fractional neutral differential equations and inclusions

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Abstract

In this paper, we consider fractional neutral differential equations with multipoint boundary value conditions involving Hadamard derivatives and integrals. We obtain the existence and uniqueness of the solution of the equation by using several fixed point theorems, and we also consider the Ulam–Hyers stability of the solution. In addition, we study the differential inclusion problem with multipoint boundary value conditions and prove the existence of the solution of the boundary value problem when the multivalued map has convex values. We also give several examples to illustrate the feasibility of the results.

Keywords: Hadamard fractional derivative; Fractional differential inclusion; Existence; Boundary value problem; Fixed point

1 Introduction

Fractional calculus is a generalization of integer calculus. Its order can be any real or complex number. As an important branch of mathematics, fractional calculus has been widely used in many fields, such as engineering, biology, neural networks, economics, control theory, and so on. Compared with integer-order differential equations, fractional-order differential equations can describe some problems more accurately. For applications of fractional differential equations and details, we refer to [8, 11, 12, 14–16, 18] and references therein.

The Hadamard fractional derivative and integral are an important part of fractional calculus. For some recent results on the Hadamard fractional derivative, we refer the reader to [1, 2] and references therein. Many researchers consider the boundary value problems of fractional differential equations with Hadamard derivative. For example, in 2016, Tariboon [20] used the Leggett–Williams and Guo–Krasnoselskii fixed point theorems to study the existence of nonnegative multiple solutions of Hadamard fractional differential equations on infinite domain; in 2021, Zhang [22] used the generalized Avery–Henderson fixed point theorem to study nonlinear Hadamard fractional differential equations with nonlocal boundary conditions.

In recent years, the stability analysis of solutions of fractional differential equations has attracted extensive attention of researchers. The concept of Ulam–Hyers stability is very important in numerical analysis, economics, and other disciplines. For applications of Ulam–Hyers stability in fractional calculus, we refer to [3, 5]. Fractional differential inclusions are considered as a generalization of differential equations and inequalities. They are very useful in the study of dynamic systems, optimal control theory, and stochastic processes. Many results have been obtained about this kind of equations. In 2016, Ahmad [4] studied mixed initial value problems involving Hadamard and Riemann–Liouville fractional integro-differential inclusions. In 2020, Ntouyas [21] introduced the boundary value problem of Hilfer-type pantograph fractional inclusions and proved the existence of solutions when the multivalued map has convex and nonconvex values.

Based on the above research results, we consider the following multipoint boundary value problems for higher-order Hadamard fractional neutral differential equation:

$$\begin{cases} ({}_HD^{\alpha}Z)(t) = \chi(t,u(t)), \quad t \in [1,e^2], \\ Z(1) = Z'(1) = \cdots = Z^{(n-3)}(1) = 0, \\ Z(\sqrt{e}) = {}_HI^{\alpha}\chi(t,u(t))|_{t=\sqrt{e}}, \\ {}_HI^{p}u(t)|_{t=e^2} = \sum_{j=1}^{r}\omega_{j\,H}I^{q}u(t)|_{t=\xi_j}, \end{cases} \tag{1.1}$$

where ${}_HD^{\alpha}$ is the Hadamard derivative operator of order α, and ${}_HI^{\alpha}$, ${}_HI^{p}$, and ${}_HI^{q}$ represent the Hadamard integral operators of orders α, p, and q respectively, $\alpha \in (n-1,n]$, $n \in N$, $n \geq 3$, $p \in [1, n-2]$, $q \in [0,p]$, $\omega_j \in \mathbb{R}$ $(j = 1,2,\dots,r)$, $1 < \xi_1 < \cdots < \xi_r \leq e^2$, $\chi \in C([1,e^2] \times \mathbb{R},\mathbb{R})$, $Z(t) = u(t) - \sum_{k=1}^{m} g_k(t,u(t))$, $Z \in L^1(1,e^2)$, $g_k \in C([1,e^2] \times \mathbb{R},\mathbb{R})$, $({}_HI^{n-\alpha}Z)(t) \in AC_{\delta}^{n}[1,e^2] = \{{}_HI^{n-\alpha}Z : [1,e^2] \to \mathbb{R} : \delta^{n-1}({}_HI^{n-\alpha}Z) \in AC[1,e^2]\}$, $\delta = t\frac{d}{dt}$.

Using the Banach contraction mapping principle, Boyd and Wong fixed point theorem, and the Leray–Schauder nonlinear alternative, we obtain the existence and uniqueness of the solution of problem (1.1). Then we consider the Ulam–Hyers stability and generalized Ulam–Hyers stability of problem (1.1).

Next, we study the following fractional inclusion equation:

$$\begin{cases} ({}_HD^{\alpha}Z)(t) \in \mathfrak{G}(t,u(t)), \quad t \in [1,e^2], \\ Z(1) = Z'(1) = \cdots = Z^{(n-2)}(1) = 0, \\ {}_HI^{p}u(t)|_{t=e^2} = \sum_{j=1}^{r}\omega_{j\,H}I^{q}u(t)|_{t=\xi_j}, \end{cases} \tag{1.2}$$

where $Z(t) = u(t) - \sum_{k=1}^{m} g_k(t,u(t))$, $\mathfrak{G} : [1,e^2] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$, $\mathcal{P}(\mathbb{R})$ represents all families of nonempty subsets of $\mathbb{R}$, and $\mathfrak{G}$ has convex values.

When $\mathfrak{G}$ is a multivalued map with convex values, we obtain the existence of the solution of problem (1.2) by using several fixed point theorems.

The structure of this paper is as follows. In Sect. 2, we give some useful definitions, lemmas, and properties. In Sect. 3, we consider the existence and uniqueness of the solution of problem (1.1) by using three fixed point theorems and give examples illustrating the feasibility of the results. In addition, we prove that the solution of problem (1.1) is Ulam–Hyers stable and generalized Ulam–Hyers stable. Finally, in Sect. 4, we consider the existence of solutions of multivalued problem (1.2) and give an example.

2 Preliminaries

In this section, we introduce basic concepts of fractional calculus together with some important preliminary results.

Definition 2.1 ([13]) The Hadamard fractional integral of order $\alpha \in \mathbb{C}$ ($\Re(\alpha) > 0$) for a function $u : [a,\infty) \to \mathbb{R}$ ($a \geq 0$) is given by

$$\left({}_H I^\alpha u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\theta}\right)^{\alpha-1} \frac{u(\theta)}{\theta}\, d\theta.$$

Definition 2.2 ([13]) The Hadamard fractional derivative of order $\alpha \in \mathbb{C}$ ($\Re(\alpha) > 0$) for a function u is given by

$$\begin{aligned}\left({}_H D^\alpha u\right)(t) &= \delta^n \left({}_H I^{n-\alpha} u\right)(t) \\ &= \left(t \frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln \frac{t}{\theta}\right)^{n-\alpha-1} \frac{u(\theta)}{\theta}\, d\theta,\end{aligned}$$

where $n = [\Re(\alpha)] + 1$.

Lemma 2.3 ([13]) *Let* $\Re(\alpha) > 0$, $n = [\Re(\alpha)] + 1$, $u \in C[a,\infty) \cap L^1[a,\infty)$, *and* $({}_H I^{n-\alpha} u)(t) \in AC_\delta^n[a,\infty)$. *Then we have*

$$\left({}_H I^\alpha \left({}_H D^\alpha u\right)\right)(t) = u(t) - \sum_{i=1}^n a_i \left(\ln \frac{t}{a}\right)^{\alpha - i}, \quad i = 1, 2, \ldots, n, a_i \in \mathbb{R}.$$

Property 2.4 ([13]) Assume that $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $u \in C[a,\infty) \cap L^1[a,\infty)$. Then we have

$$\left({}_H I^\beta \left({}_H I^\alpha u\right)\right)(t) = \left({}_H I^{\alpha+\beta} u\right)(t).$$

Property 2.5 ([13]) Assume that $\Re(\alpha) > 0$ and $\Re(\beta) > 0$. Then we have

$$\left({}_H I^\alpha \left(\ln \frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\ln \frac{x}{a}\right)^{\alpha+\beta-1}.$$

Definition 2.6 ([7]) Let U be a Banach space. A mapping $\mathcal{A} : U \to U$ is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Theta(0) = 0$, and $\Theta(\iota) < \iota$ for all $\iota > 0$, and $\|\mathcal{A}u - \mathcal{A}v\| \leq \Theta(\|u - v\|)$ for all $u, v \in U$.

Lemma 2.7 (Boyd and Wong [7]) *Let U be a Banach space, and let $\mathcal{A} : U \to U$ be a nonlinear contraction. Then $\mathcal{A}$ has a unique fixed point in U.*

Lemma 2.8 (Leray–Schauder's nonlinear alternative [10]) *Let U be a Banach space, let E be a convex closed subset of U, and let $\mathcal{C}$ be an open subset of E such that $0 \in \mathcal{C}$. Suppose that $\mathcal{A} : \overline{\mathcal{C}} \to E$ is a continuous compact map. Then either*

(*i*) *$\mathcal{A}$ has a fixed point in $\overline{\mathcal{C}}$, or*

(*ii*) *there are $u \in \partial\mathcal{C}$ (the boundary of $\mathcal{C}$ in E) and $\epsilon \in (0,1)$ such that $u = \epsilon\mathcal{A}(u)$.*

3 Main results

First, we give a very important lemma for obtaining our results.

Lemma 3.1 *Let* $\widehat{\chi} \in C([1,e^2],\mathbb{R})$, $g_k \in C([1,e^2] \times \mathbb{R},\mathbb{R})$, $Z \in L^1(1,e^2)$, $({}_HI^{n-\alpha}Z)(t) \in AC_\delta^n[1,e^2]$, *and*

$$\begin{aligned}\Upsilon &= \frac{\Gamma(\alpha)}{\Gamma(\alpha+p)}2^{\alpha+p-1} - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha+p-1)}2^{\alpha+p-3} \\ &\quad - \frac{\Gamma(\alpha)}{\Gamma(\alpha+q)}\sum_{j=1}^{r}\omega_j(\ln\xi_j)^{\alpha+q-1} - \frac{\Gamma(\alpha-1)}{2\Gamma(\alpha+q-1)}\sum_{j=1}^{r}\omega_j(\ln\xi_j)^{\alpha+q-2} \\ &\neq 0.\end{aligned}$$

Then the function u is a solution of

$$\begin{cases}({}_HD^\alpha Z)(t) = \widehat{\chi}(t), t \in [1,e^2], \\ Z(1) = Z'(1) = \cdots = Z^{(n-3)}(1) = 0, \\ Z(\sqrt{e}) = {}_HI^\alpha\widehat{\chi}(t)|_{t=\sqrt{e}}, \\ {}_HI^p u(t)|_{t=e^2} = \sum_{j=1}^{r}\omega_{j\,H}I^q u(t)|_{t=\xi_j},\end{cases} \tag{3.1}$$

if and only if

$$\begin{aligned}u(t) &= {}_HI^\alpha\widehat{\chi}(t) + \sum_{k=1}^{m}g_k\big(t,u(t)\big) + \left(\frac{(\ln t)^{\alpha-1}}{\Upsilon} - \frac{(\ln t)^{\alpha-2}}{2\Upsilon}\right) \\ &\quad\cdot\left[\sum_{j=1}^{r}\omega_{j\,H}I^{\alpha+q}\widehat{\chi}(\xi_j) - {}_HI^{\alpha+p}\widehat{\chi}\big(e^2\big)\right. \\ &\quad\left.+\sum_{j=1}^{r}\omega_{j\,H}I^q\left(\sum_{k=1}^{m}g_k\big(\xi_j,u(\xi_j)\big)\right) - {}_HI^p\left(\sum_{k=1}^{m}g_k\big(e^2,u\big(e^2\big)\big)\right)\right],\end{aligned}$$

where

$$Z(t) = u(t) - \sum_{k=1}^{m}g_k\big(t,u(t)\big).$$

Proof Applying the Hadamard fractional integral of order α to both sides of $({}_HD^\alpha Z)(t) = \widehat{\chi}(t)$ and using Lemma 2.3, we obtain

$$Z(t) = {}_HI^\alpha\widehat{\chi}(t) + a_1(\ln t)^{\alpha-1} + a_2(\ln t)^{\alpha-2} + \cdots + a_n(\ln t)^{\alpha-n}.$$

From the boundary condition $Z(1) = Z'(1) = \cdots = Z^{(n-3)}(1) = 0$ we obtain $a_3 = a_4 = \cdots = a_{n-1} = a_n = 0$.

Then we get

$$Z(t) = {}_HI^\alpha\widehat{\chi}(t) + a_1(\ln t)^{\alpha-1} + a_2(\ln t)^{\alpha-2}$$

and

$$Z(\sqrt{e}) = {}_HI^{\alpha}\widehat{\chi}(\sqrt{e}) + a_1\left(\frac{1}{2}\right)^{\alpha-1} + a_2\left(\frac{1}{2}\right)^{\alpha-2}.$$

From $Z(\sqrt{e}) = {}_HI^{\alpha}\widehat{\chi}(t)|_{t=\sqrt{e}}$ we have $a_2 = -\frac{1}{2}a_1$ and

$$u(t) = {}_HI^{\alpha}\widehat{\chi}(t) + \sum_{k=1}^{m} g_k\bigl(t, u(t)\bigr) + a_1(\ln t)^{\alpha-1} - \frac{1}{2}a_1(\ln t)^{\alpha-2}. \tag{3.2}$$

Using equation (3.2) and Properties 2.4, and 2.5, we obtain

$$\begin{aligned}
{}_HI^{p}u(t)|_{t=e^2} &= {}_HI^{\alpha+p}\widehat{\chi}\bigl(e^2\bigr) + {}_HI^{p}\left(\sum_{k=1}^{m} g_k\bigl(e^2, u\bigl(e^2\bigr)\bigr)\right) \\
&\quad + \frac{a_1\Gamma(\alpha)2^{\alpha+p-1}}{\Gamma(\alpha+p)} - \frac{a_1\Gamma(\alpha-1)2^{\alpha+p-2}}{2\Gamma(\alpha+p-1)}, \\
\sum_{j=1}^{r} \omega_{j\,H}I^{q}u(t)\Big|_{t=\xi_j} &= \sum_{j=1}^{r} \omega_{j\,H}I^{\alpha+q}\widehat{\chi}(\xi_j) + \sum_{j=1}^{r} \omega_{j\,H}I^{q}\left(\sum_{k=1}^{m} g_k\bigl(t, u(t)\bigr)\right)\Big|_{t=\xi_j} \\
&\quad + \frac{a_1\Gamma(\alpha)}{\Gamma(\alpha+q)}\sum_{j=1}^{r}\omega_j(\ln\xi_j)^{\alpha+q-1} - \frac{a_1\Gamma(\alpha-1)}{2\Gamma(\alpha+q-1)}\sum_{j=1}^{r}\omega_j(\ln\xi_j)^{\alpha+q-2}.
\end{aligned}$$

From the boundary condition ${}_HI^{p}u(t)|_{t=e^2} = \sum_{j=1}^{r}\omega_{j\,H}I^{q}u(t)|_{t=\xi_j}$ we obtain

$$\begin{aligned}
a_1 &= \frac{1}{\Upsilon}\left[\sum_{j=1}^{r}\omega_{j\,H}I^{\alpha+q}\widehat{\chi}(\xi_j) - {}_HI^{\alpha+p}\widehat{\chi}\bigl(e^2\bigr)\right. \\
&\quad \left. + \sum_{j=1}^{r}\omega_{j\,H}I^{q}\left(\sum_{k=1}^{m} g_k\bigl(\xi_j, u(\xi_j)\bigr)\right) - {}_HI^{p}\left(\sum_{k=1}^{m} g_k\bigl(e^2, u\bigl(e^2\bigr)\bigr)\right)\right].
\end{aligned}$$

Substituting the value of a_1 into (3.2), we obtain

$$\begin{aligned}
u(t) &= {}_HI^{\alpha}\widehat{\chi}(t) + \sum_{k=1}^{m} g_k\bigl(t, u(t)\bigr) + \left(\frac{(\ln t)^{\alpha-1}}{\Upsilon} - \frac{(\ln t)^{\alpha-2}}{2\Upsilon}\right) \\
&\quad \cdot\left[\sum_{j=1}^{r}\omega_{j\,H}I^{\alpha+q}\widehat{\chi}(\xi_j) - {}_HI^{\alpha+p}\widehat{\chi}\bigl(e^2\bigr)\right. \\
&\quad \left. + \sum_{j=1}^{r}\omega_{j\,H}I^{q}\left(\sum_{k=1}^{m} g_k\bigl(\xi_j, u(\xi_j)\bigr)\right) - {}_HI^{p}\left(\sum_{k=1}^{m} g_k\bigl(e^2, u\bigl(e^2\bigr)\bigr)\right)\right].
\end{aligned}$$

This completes the proof. □

Let $U = C([1,e^2],\mathbb{R})$ denote the Banach space of all continuous functions with norm $\|u\| := \max_{t\in[1,e^2]}|u(t)|$. Define the operator $\mathcal{A} : U \to U$ by

$$\begin{aligned}
&\mathcal{A}(u)(t)\\
&\quad= {}_HI^{\alpha}\chi\bigl(t,u(t)\bigr) + \sum_{k=1}^{m} g_k\bigl(t,u(t)\bigr) + \left(\frac{(\ln t)^{\alpha-1}}{\Upsilon} - \frac{(\ln t)^{\alpha-2}}{2\Upsilon}\right)\\
&\qquad\cdot\Biggl[\sum_{j=1}^{r}\omega_{j\,H}I^{\alpha+q}\chi\bigl(\theta,u(\theta)\bigr)\Big|_{\theta=\xi_j} - {}_HI^{\alpha+p}\chi\bigl(\theta,u(\theta)\bigr)\Big|_{\theta=e^2}\\
&\qquad+\sum_{j=1}^{r}\omega_{j\,H}I^{q}\left(\sum_{k=1}^{m} g_k\bigl(\theta,u(\theta)\bigr)\right)\Bigg|_{\theta=\xi_j} - {}_HI^{p}\left(\sum_{k=1}^{m} g_k\bigl(\theta,u(\theta)\bigr)\right)\Bigg|_{\theta=e^2}\Biggr].
\end{aligned} \tag{3.3}$$

We notice that problem (1.1) has a solution which is equivalent to a fixed point of the operator $\mathcal{A}$.

For convenience, we introduce the constants

$$\begin{aligned}
\Delta_1 &= \frac{2^{\alpha}}{\Gamma(\alpha+1)} + \left(\frac{2^{\alpha-1}+2^{\alpha-3}}{|\Upsilon|}\right)\\
&\quad\cdot\left(\sum_{j=1}^{r}\frac{|\omega_j|}{\Gamma(\alpha+q+1)}(\ln\xi_j)^{\alpha+q} + \frac{2^{\alpha+p}}{\Gamma(\alpha+p+1)}\right),
\end{aligned} \tag{3.4}$$

$$\Delta_2 = m + \left(\frac{2^{\alpha-1}+2^{\alpha-3}}{|\Upsilon|}\right)\left(\sum_{j=1}^{r}\frac{m\cdot|\omega_j|}{\Gamma(q+1)}(\ln\xi_j)^{q} + \frac{m\cdot 2^{p}}{\Gamma(p+1)}\right). \tag{3.5}$$

3.1 Existence and uniqueness of solutions

Theorem 3.2 *Assume that*:

(H_1) *there exist constants* $l_1, l_2 > 0$ *such that for all* $t \in [1,e^2]$, $x_i \in \mathbb{R}$, $i = 1,2$,

$$\begin{aligned}
&\bigl|\chi(t,x_1) - \chi(t,x_2)\bigr| \le l_1|x_1 - x_2|,\\
&\bigl|g_k(t,x_1) - g_k(t,x_2)\bigr| \le l_2|x_1 - x_2|, \quad k = 1,2,\ldots,m.
\end{aligned}$$

Then problem (1.1) *has a unique solution on* $[1,e^2]$ *if* $\Delta_1 l_1 + \Delta_2 l_2 < 1$.

Proof Let $M_1 = \max_{t\in[1,e^2]}|\chi(t,0)| < \infty$ and $M_2 = \max_{1\le k\le m}\{\max_{t\in[1,e^2]}|g_k(t,0)|\} < \infty$.

Using condition (H_1), we obtain

$$\bigl|\chi\bigl(t,u(t)\bigr)\bigr| \le l_1\|u\| + M_1, \qquad \bigl|g_k\bigl(t,u(t)\bigr)\bigr| \le l_2\|u\| + M_2.$$

First, we prove that $\mathcal{A}(B_r) \subset B_r$, where $B_r = \{u \in U : \|u\| \le r\}$, $r \ge \frac{\Delta_1 M_1 + \Delta_2 M_2}{1-\Delta_1 l_1 - \Delta_2 l_2}$. Indeed, for $u \in B_r$, we have

$$\begin{aligned}
&\bigl|\mathcal{A}(u)(t)\bigr|\\
&\quad\le {}_HI^{\alpha}\bigl|\chi\bigl(t,u(t)\bigr)\bigr| + \sum_{k=1}^{m}\bigl|g_k\bigl(t,u(t)\bigr)\bigr| + \left|\frac{(\ln t)^{\alpha-1}}{\Upsilon} + \frac{(\ln t)^{\alpha-2}}{2\Upsilon}\right|
\end{aligned}$$

$$
\begin{aligned}
&\cdot \Bigg[\sum_{j=1}^{r}|\omega_j|_H I^{\alpha+q}\big(\big|\chi\big(\theta,u(\theta)\big)\big|\big)\Big|_{\theta=\xi_j} + {}_H I^{\alpha+p}\big(\big|\chi\big(\theta,u(\theta)\big)\big|\big)\Big|_{\theta=e^2} \\
&+ \sum_{j=1}^{r}|\omega_j|_H I^{q}\Bigg(\sum_{k=1}^{m}\big|g_k\big(\theta,u(\theta)\big)\big|\Bigg)\Bigg|_{\theta=\xi_j} + {}_H I^{p}\Bigg(\sum_{k=1}^{m}\big|g_k\big(\theta,u(\theta)\big)\big|\Bigg)\Bigg|_{\theta=e^2}\Bigg] \\
\le{}& \frac{1}{\Gamma(\alpha)}\big(l_1\|u\|+M_1\big)\int_1^t\left(\ln\frac{t}{\theta}\right)^{\alpha-1}\frac{1}{\theta}\,d\theta + m\big(l_2\|u\|+M_2\big) + \left(\frac{2^{\alpha-1}+2^{\alpha-3}}{|\Upsilon|}\right) \\
&\cdot \Bigg\{\big(l_1\|u\|+M_1\big) \\
&\cdot \left(\sum_{j=1}^{r}\frac{|\omega_j|}{\Gamma(\alpha+q)}\int_1^{\xi_j}\left(\ln\frac{\xi_j}{\theta}\right)^{\alpha+q-1}\frac{1}{\theta}\,d\theta + \frac{1}{\Gamma(\alpha+p)}\int_1^{e^2}\left(\ln\frac{e^2}{\theta}\right)^{\alpha+p-1}\frac{1}{\theta}\,d\theta\right) \\
&+ \big(l_2\|u\|+M_2\big) \\
&\cdot \left(\sum_{j=1}^{r}\frac{|\omega_j|}{\Gamma(q)}\left(\sum_{k=1}^{m}\int_1^{\xi_j}\left(\ln\frac{\xi_j}{\theta}\right)^{q-1}\frac{1}{\theta}\,d\theta\right) + \frac{1}{\Gamma(p)}\sum_{k=1}^{m}\int_1^{e^2}\left(\ln\frac{e^2}{\theta}\right)^{p-1}\frac{1}{\theta}\,d\theta\right)\Bigg\} \\
\le{}& \left\{\frac{2^{\alpha}}{\Gamma(\alpha+1)} + \left(\frac{2^{\alpha-1}+2^{\alpha-3}}{|\Upsilon|}\right)\left(\sum_{j=1}^{r}\frac{|\omega_j|(\ln\xi_j)^{\alpha+q}}{\Gamma(\alpha+q+1)} + \frac{2^{\alpha+p}}{\Gamma(\alpha+p+1)}\right)\right\} \\
&\cdot \big(l_1\|u\|+M_1\big) \\
&+ \left\{m + \left(\frac{2^{\alpha-1}+2^{\alpha-3}}{|\Upsilon|}\right)\left(\sum_{j=1}^{r}\frac{m\cdot|\omega_j|}{\Gamma(q+1)}(\ln\xi_j)^{q} + \frac{m\cdot 2^{p}}{\Gamma(p+1)}\right)\right\} \\
&\cdot \big(l_2\|u\|+M_2\big) \\
={}& \Delta_1\big(l_1\|u\|+M_1\big) + \Delta_2\big(l_2\|u\|+M_2\big) \\
\le{}& \Delta_1(l_1r+M_1) + \Delta_2(l_2r+M_2) \\
\le{}& r.
\end{aligned}
$$

This shows that $\|\mathcal{A}(u)\| = \max_{t\in[1,e^2]}|\mathcal{A}(u)(t)| \le r$. Thus $\mathcal{A}(B_r)\subset B_r$.

Now we prove that the operator $\mathcal{A}$ is a contraction. Let $u_1,u_2\in U$. Then, for each $t\in[1,e^2]$, we have

$$
\begin{aligned}
&\big|\mathcal{A}(u_2)(t)-\mathcal{A}(u_1)(t)\big| \\
\le{}& {}_H I^{\alpha}\big|\chi\big(t,u_2(t)\big)-\chi\big(t,u_1(t)\big)\big| + \sum_{k=1}^{m}\big|g_k\big(t,u_2(t)\big)-g_k\big(t,u_1(t)\big)\big| \\
&+ \left|\frac{(\ln t)^{\alpha-1}}{\Upsilon}+\frac{(\ln t)^{\alpha-2}}{2\Upsilon}\right|\cdot\Bigg[\sum_{j=1}^{r}|\omega_j|_H I^{\alpha+q}\big(\big|\chi\big(\theta,u_2(\theta)\big)-\chi\big(\theta,u_1(\theta)\big)\big|\big)\Big|_{\theta=\xi_j} \\
&+ {}_H I^{\alpha+p}\big(\big|\chi\big(\theta,u_2(\theta)\big)-\chi\big(\theta,u_1(\theta)\big)\big|\big)\Big|_{\theta=e^2} \\
&+ \sum_{j=1}^{r}|\omega_j|_H I^{q}\Bigg(\sum_{k=1}^{m}\big|g_k\big(\theta,u_2(\theta)\big)-g_k\big(\theta,u_1(\theta)\big)\big|\Bigg)\Bigg|_{\theta=\xi_j}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + {}_HI^p\left(\sum_{k=1}^{m}\left|g_k\big(\theta,u_2(\theta)\big)-g_k\big(\theta,u_1(\theta)\big)\right|\right)\Bigg|_{\theta=e^2}\Bigg] \\
&\le \left\{\frac{2^\alpha}{\Gamma(\alpha+1)}+\left(\frac{2^{\alpha-1}+2^{\alpha-3}}{|\Upsilon|}\right)\left(\sum_{j=1}^{r}\frac{|\omega_j|(\ln\xi_j)^{\alpha+q}}{\Gamma(\alpha+q+1)}+\frac{2^{\alpha+p}}{\Gamma(\alpha+p+1)}\right)\right\} \\
&\quad \cdot l_1\|u_2-u_1\| \\
&\quad + \left\{m+\left(\frac{2^{\alpha-1}+2^{\alpha-3}}{|\Upsilon|}\right)\left(\sum_{j=1}^{r}\frac{m\cdot|\omega_j|}{\Gamma(q+1)}(\ln\xi_j)^{q}+\frac{m\cdot 2^{p}}{\Gamma(p+1)}\right)\right\} \\
&\quad \cdot l_2\|u_2-u_1\| \\
&= \Delta_1 l_1\|u_2-u_1\|+\Delta_2 l_2\|u_2-u_1\| \\
&= (\Delta_1 l_1+\Delta_2 l_2)\|u_2-u_1\|.
\end{aligned}
$$

Thus

$$
\left\|\mathcal{A}(u_2)-\mathcal{A}(u_1)\right\| = \max_{t\in[1,e^2]}\left|\mathcal{A}(u_2)(t)-\mathcal{A}(u_1)(t)\right| \le (\Delta_1 l_1+\Delta_2 l_2)\|u_2-u_1\|.
$$

From the condition $\Delta_1 l_1+\Delta_2 l_2<1$ we get that the operator $\mathcal{A}$ is a contraction. By the principle of Banach contraction mapping, $\mathcal{A}$ has a unique fixed point, so problem (1.1) has a unique solution on $[1,e^2]$. This completes the proof. □

Theorem 3.3 *Assume that*:

(H_2)

$$
\begin{aligned}
&\left|\chi(t,x_1)-\chi(t,x_2)\right| \le \zeta_1(t)\frac{|x_1-x_2|}{W^*+|x_1-x_2|}, \\
&\left|g_k(t,x_1)-g_k(t,x_2)\right| \le \zeta_2(t)\frac{|x_1-x_2|}{W^*+|x_1-x_2|}, \quad k=1,2,\dots,m,
\end{aligned}
$$

for all $t\in[1,e^2]$, *where* $x_i\in\mathbb{R}$, $i=1,2$, $\zeta_i(t):[1,e^2]\to\mathbb{R}^+$, $i=1,2$, *are continuous nondecreasing functions, and the positive constant is defined as*

$$
\begin{aligned}
W^* &= {}_HI^\alpha\zeta_1\big(e^2\big)+m\cdot\zeta_2\big(e^2\big)+\left(\frac{2^{\alpha-1}+2^{\alpha-3}}{|\Upsilon|}\right) \\
&\quad \cdot\left(\sum_{j=1}^{r}|\omega_j|{}_HI^{\alpha+q}\zeta_1(\xi_j)+{}_HI^{\alpha+p}\zeta_1\big(e^2\big)+m\cdot\sum_{j=1}^{r}|\omega_j|{}_HI^{q}\zeta_2(\xi_j)+m\cdot{}_HI^{p}\zeta_2\big(e^2\big)\right).
\end{aligned}
$$

Then the boundary value problem (1.1) *has a unique solution on* $[1,e^2]$.

Proof Define the operator $\mathcal{A}:U\to U$ by (3.3) and the continuous nondecreasing function $\Theta:\mathbb{R}^+\to\mathbb{R}^+$ by $\Theta(\iota)=\frac{W^*\iota}{W^*+\iota}$.

Note that $\Theta(0)=0$ and $\Theta(\iota)<\iota$ for all $\iota>0$. For all $u_1,u_2\in U$, $t\in[1,e^2]$, we have

$$
\begin{aligned}
&\left|\mathcal{A}(u_1)(t)-\mathcal{A}(u_2)(t)\right| \\
&\quad \le {}_HI^\alpha\left|\chi\big(t,u_1(t)\big)-\chi\big(t,u_2(t)\big)\right|+\sum_{k=1}^{m}\left|g_k\big(t,u_1(t)\big)-g_k\big(t,u_2(t)\big)\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\frac{(\ln t)^{\alpha-1}}{\Upsilon}+\frac{(\ln t)^{\alpha-2}}{2\Upsilon}\right|\cdot\left[\sum_{j=1}^{r}|\omega_j|_H I^{\alpha+q}\left(\left|\chi\left(\theta,u_1(\theta)\right)-\chi\left(\theta,u_2(\theta)\right)\right|\right)\Big|_{\theta=\xi_j}\right.\\
&+{}_H I^{\alpha+p}\left(\left|\chi\left(\theta,u_1(\theta)\right)-\chi\left(\theta,u_2(\theta)\right)\right|\right)\Big|_{\theta=e^2}\\
&+\sum_{j=1}^{r}|\omega_j|_H I^{q}\left(\sum_{k=1}^{m}\left|g_k\left(\theta,u_1(\theta)\right)-g_k\left(\theta,u_2(\theta)\right)\right|\right)\Big|_{\theta=\xi_j}\\
&\left.+{}_H I^{p}\left(\sum_{k=1}^{m}\left|g_k\left(\theta,u_1(\theta)\right)-g_k\left(\theta,u_2(\theta)\right)\right|\right)\Big|_{\theta=e^2}\right]\\
\le{}& {}_H I^{\alpha}\left(\zeta_1(\theta)\frac{|u_1-u_2|}{W^*+|u_1-u_2|}\right)\Big|_{\theta=e^2}+\sum_{k=1}^{m}\left(\zeta_2(\theta)\frac{|u_1-u_2|}{W^*+|u_1-u_2|}\right)\Big|_{\theta=e^2}\\
&+\left(\frac{2^{\alpha-1}+2^{\alpha-3}}{|\Upsilon|}\right)\cdot\left[\sum_{j=1}^{r}|\omega_j|_H I^{\alpha+q}\left(\zeta_1(\theta)\frac{|u_1-u_2|}{W^*+|u_1-u_2|}\right)\Big|_{\theta=\xi_j}\right.\\
&+{}_H I^{\alpha+p}\left(\zeta_1(\theta)\frac{|u_1-u_2|}{W^*+|u_1-u_2|}\right)\Big|_{\theta=e^2}\\
&+\sum_{j=1}^{r}|\omega_j|_H I^{q}\left(\sum_{k=1}^{m}\left(\frac{\zeta_2(\theta)\cdot|u_1-u_2|}{W^*+|u_1-u_2|}\right)\right)\Big|_{\theta=\xi_j}\\
&\left.+{}_H I^{p}\left(\sum_{k=1}^{m}\left(\frac{\zeta_2(\theta)\cdot|u_1-u_2|}{W^*+|u_1-u_2|}\right)\right)\Big|_{\theta=e^2}\right]\\
\le{}&\frac{\Theta(\|u_1-u_2\|)}{W^*}\cdot\left[{}_H I^{\alpha}\zeta_1\left(e^2\right)+m\cdot\zeta_2\left(e^2\right)+\left(\frac{2^{\alpha-1}+2^{\alpha-3}}{|\Upsilon|}\right)\right.\\
&\cdot\left(\sum_{j=1}^{r}|\omega_j|_H I^{\alpha+q}\zeta_1(\xi_j)+{}_H I^{\alpha+p}\zeta_1\left(e^2\right)\right.\\
&\left.\left.+m\sum_{j=1}^{r}|\omega_j|_H I^{q}\zeta_2(\xi_j)+m\cdot{}_H I^{p}\zeta_2\left(e^2\right)\right)\right]\\
={}&\frac{\Theta(\|u_1-u_2\|)}{W^*}\cdot W^*\\
={}&\Theta\left(\|u_1-u_2\|\right),
\end{aligned}
$$

that is, $\|\mathcal{A}u_1-\mathcal{A}u_2\|\le\Theta(\|u_1-u_2\|)$. Therefore $\mathcal{A}$ is a nonlinear contraction. By Lemma 2.7 the operator $\mathcal{A}$ has a unique fixed point, which is the solution of boundary value problem (1.1), and hence the proof is completed. □

Theorem 3.4 *Assume that*:

(H_3) *there exist nondecreasing function* $\Phi\in C([0,\infty),\mathbb{R}^+)$ *and* $\kappa_i\in C([1,e^2],\mathbb{R}^+)$, $i=1,2$, *such that*

$$
\begin{aligned}
&\left|\chi(t,u)\right|\le\kappa_1(t)\Phi\left(\|u\|\right),\\
&\left|g_k(t,u)\right|\le\kappa_2(t)\Phi\left(\|u\|\right),\quad k=1,2,\dots,m,
\end{aligned}
$$

for all $(t,u)\in[1,e^2]\times\mathbb{R}$;

(H_4) *there exists a constant* $L > 0$, *such that* $\frac{L}{\Phi(L)(\|\kappa_1\|\Delta_1+\|\kappa_2\|\Delta_2)} > 1$.
Then the boundary value problem (1.1) *has a solution on* $[1, e^2]$.

Proof Let $\mathcal{A} : U \to U$ be the operator defined by (3.3). First, we prove that $\mathcal{A}$ maps bounded sets in U to bounded sets.

For $r > 0$, let $B_r = \{u \in U : \|u\| \le r\}$. Then for all $t \in [1, e^2]$, we have

$$\begin{aligned}
&\left|\mathcal{A}(u)(t)\right| \\
&\quad\le {}_H I^{\alpha}\left|\chi\left(t,u(t)\right)\right| + \sum_{k=1}^{m}\left|g_k\left(t,u(t)\right)\right| + \left|\frac{(\ln t)^{\alpha-1}}{\Upsilon} + \frac{(\ln t)^{\alpha-2}}{2\Upsilon}\right| \\
&\qquad\cdot\left[\sum_{j=1}^{r}|\omega_j|{}_H I^{\alpha+q}\left(\left|\chi\left(\theta,u(\theta)\right)\right|\right)\Big|_{\theta=\xi_j} + {}_H I^{\alpha+p}\left(\left|\chi\left(\theta,u(\theta)\right)\right|\right)\Big|_{\theta=e^2}\right. \\
&\qquad\left.+\sum_{j=1}^{r}|\omega_j|{}_H I^{q}\left(\sum_{k=1}^{m}\left|g_k\left(\theta,u(\theta)\right)\right|\right)\Big|_{\theta=\xi_j} + {}_H I^{p}\left(\sum_{k=1}^{m}\left|g_k\left(\theta,u(\theta)\right)\right|\right)\Big|_{\theta=e^2}\right] \\
&\quad\le \Phi\left(\|u\|\right){}_H I^{\alpha}\kappa_1\left(e^2\right) + \Phi\left(\|u\|\right)\sum_{k=1}^{m}\kappa_2(t) + \Phi\left(\|u\|\right)\left(\frac{2^{\alpha-1}+2^{\alpha-3}}{|\Upsilon|}\right) \\
&\qquad\cdot\left[\sum_{j=1}^{r}|\omega_j|{}_H I^{\alpha+q}\kappa_1(\xi_j) + {}_H I^{\alpha+p}\kappa_1\left(e^2\right)\right. \\
&\qquad\left.+\sum_{j=1}^{r}|\omega_j|{}_H I^{q}\left(\sum_{k=1}^{m}\kappa_2(\xi_j)\right) + {}_H I^{p}\left(\sum_{k=1}^{m}\kappa_2\left(e^2\right)\right)\right] \\
&\quad\le \Phi\left(\|u\|\right)\|\kappa_1\|\left\{\frac{2^{\alpha}}{\Gamma(\alpha+1)} + \left(\frac{2^{\alpha-1}+2^{\alpha-3}}{|\Upsilon|}\right)\left(\sum_{j=1}^{r}\frac{|\omega_j|(\ln\xi_j)^{\alpha+q}}{\Gamma(\alpha+q+1)} + \frac{2^{\alpha+p}}{\Gamma(\alpha+p+1)}\right)\right\} \\
&\qquad+\Phi\left(\|u\|\right)\|\kappa_2\|\left\{m + \left(\frac{2^{\alpha-1}+2^{\alpha-3}}{|\Upsilon|}\right)\left(\sum_{j=1}^{r}\frac{m\cdot|\omega_j|}{\Gamma(q+1)}(\ln\xi_j)^{q} + \frac{m\cdot 2^{p}}{\Gamma(p+1)}\right)\right\} \\
&\quad= \Phi\left(\|u\|\right)\|\kappa_1\|\Delta_1 + \Phi\left(\|u\|\right)\|\kappa_2\|\Delta_2.
\end{aligned}$$

In other words, $\|\mathcal{A}u\| \le \Phi(r)(\|\kappa_1\|\Delta_1 + \|\kappa_2\|\Delta_2)$.

Next, we prove that $\mathcal{A}$ is equicontinuous. Let $t_1, t_2 \in [1, e^2]$, $t_1 < t_2$, $u \in B_r$. Then

$$\begin{aligned}
&\left|\mathcal{A}(u)(t_2) - \mathcal{A}(u)(t_1)\right| \\
&\quad\le \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left(\left(\ln\frac{t_2}{\theta}\right)^{\alpha-1} - \left(\ln\frac{t_1}{\theta}\right)^{\alpha-1}\right)\frac{|\chi(\theta,u(\theta))|}{\theta}\,d\theta \\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{\theta}\right)^{\alpha-1}\frac{|\chi(\theta,u(\theta))|}{\theta}\,d\theta \\
&\qquad+\sum_{k=1}^{m}\left|g_k\left(t_2,u(t_2)\right) - g_k\left(t_1,u(t_1)\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{(\ln t_2)^{\alpha-1}}{\Upsilon}-\frac{(\ln t_2)^{\alpha-2}}{2\Upsilon}-\frac{(\ln t_1)^{\alpha-1}}{\Upsilon}+\frac{(\ln t_1)^{\alpha-2}}{2\Upsilon}\right| \\
&\cdot\left[\sum_{j=1}^{r}|\omega_j|_H I^{\alpha+q}\big(\big|\chi\big(\theta,u(\theta)\big)\big|\big)\Big|_{\theta=\xi_j}+{}_H I^{\alpha+p}\big(\big|\chi\big(\theta,u(\theta)\big)\big|\big)\Big|_{\theta=e^2}\right. \\
&\left.+\sum_{j=1}^{r}|\omega_j|_H I^{q}\left(\sum_{k=1}^{m}\big|g_k\big(\theta,u(\theta)\big)\big|\right)\Big|_{\theta=\xi_j}+{}_H I^{p}\left(\sum_{k=1}^{m}\big|g_k\big(\theta,u(\theta)\big)\big|\right)\Big|_{\theta=e^2}\right] \\
\le{}& \frac{\|\kappa_1\|\Phi(r)}{\Gamma(\alpha+1)}\big|(\ln t_2)^{\alpha}-(\ln t_1)^{\alpha}\big|+\sum_{k=1}^{m}\big|g_k\big(t_2,u(t_2)\big)-g_k\big(t_1,u(t_1)\big)\big| \\
&+\left(\frac{|(\ln t_2)^{\alpha-1}-(\ln t_1)^{\alpha-1}|}{|\Upsilon|}+\frac{|(\ln t_1)^{\alpha-2}-(\ln t_2)^{\alpha-2}|}{2|\Upsilon|}\right) \\
&\cdot\left[\|\kappa_1\|\Phi(r)\left(\sum_{j=1}^{r}\frac{|\omega_j|}{\Gamma(\alpha+q+1)}(\ln\xi_j)^{\alpha+q}+\frac{2^{\alpha+p}}{\Gamma(\alpha+p+1)}\right)\right. \\
&\left.+\|\kappa_2\|\Phi(r)\left(\sum_{j=1}^{r}\frac{m\cdot|\omega_j|}{\Gamma(q+1)}(\ln\xi_j)^{q}+\frac{m\cdot 2^{p}}{\Gamma(p+1)}\right)\right].
\end{aligned}
$$

By the continuity of g_k we get $|\mathcal{A}(u)(t_2)-\mathcal{A}(u)(t_1)|\to 0$ as $t_2\to t_1$. It follows that $\mathcal{A}$ is equicontinuous. So by the Arzelà–Ascoli theorem we get that $\mathcal{A}:U\to U$ is completely continuous.

For $\epsilon\in(0,1)$, let u satisfy $u=\epsilon\mathcal{A}(u)$. Then

$$\|u\|=\epsilon\big\|\mathcal{A}(u)\big\|\le\big\|\mathcal{A}(u)\big\|\le\Phi\big(\|u\|\big)\big(\|\kappa_1\|\Delta_1+\|\kappa_2\|\Delta_2\big),$$

i.e.,

$$\frac{\|u\|}{\Phi(\|u\|)(\|\kappa_1\|\Delta_1+\|\kappa_2\|\Delta_2)}\le 1.$$

By hypothesis (H_4) there exists L such that $\|u\|\ne L$. Let

$$\mathcal{C}=\big\{u\in C\big([1,e^2],\mathbb{R}\big):\|u\|<L\big\}.$$

The operator $\mathcal{A}:\overline{\mathcal{C}}\to U$ is a continuous compact map. By the choice of $\mathcal{C}$, for $\epsilon\in(0,1)$, there is no $u\in\partial\mathcal{C}$ such that $u=\epsilon\mathcal{A}(u)$. Therefore by Lemma 2.8 the operator $\mathcal{A}$ has a fixed point $u\in\overline{\mathcal{C}}$, which is a solution of boundary value problem (1.1). This completes the proof. □

Example 3.5 Consider

$$
\begin{cases}
({}_H D^{\frac{10}{3}}Z)(t)=\frac{e^{1-t}}{8}\sin u(t), \quad t\in[1,e^2],\\
Z(1)=Z'(1)=\cdots=Z^{(n-3)}(1)=0,\\
Z(\sqrt{e})={}_H I^{\frac{10}{3}}(\frac{e^{1-t}}{8}\sin u(t))|_{t=\sqrt{e}},\\
{}_H I^{\frac{4}{3}}u(e^2)=\frac{3}{5}{}_H I^{\frac{2}{3}}u(2)+\frac{1}{2}{}_H I^{\frac{2}{3}}u(\frac{10}{3}),
\end{cases}
\tag{3.6}
$$

where

$$Z(t) = u(t) - \left[\frac{e^{1-t}}{8(5t^2+3)^2}\left(\frac{2u^2(t)+2|u(t)|}{1+|u(t)|}\right) + e^{2t} + t^2 + 1\right],$$

$n = 4$, $\alpha = \frac{10}{3}$, $p = \frac{4}{3}$, $q = \frac{2}{3}$, $m = 1$, $r = 2$, $\omega_1 = \frac{3}{5}$, $\omega_2 = \frac{1}{2}$, $\xi_1 = 2$, $\xi_2 = \frac{10}{3}$.

By calculation we get $\Upsilon = 0.6579$, $\Delta_1 \approx 5.1052$, $\Delta_2 \approx 32.252$.

Let $\chi(t,u) = \frac{e^{1-t}}{8}\sin u$ and $g_1(t,u) = \frac{e^{1-t}}{8(5t^2+3)^2}(\frac{2u^2(t)+2|u(t)|}{1+|u(t)|}) + e^{2t} + t^2 + 1$. Then

$$\begin{aligned}
\left|\chi(t,x_1) - \chi(t,x_2)\right| &\le \frac{e^{1-t}}{8}|\sin x_1 - \sin x_2| \le \frac{1}{8}|x_1 - x_2|, \\
\left|g_1(t,x_1) - g_1(t,x_2)\right| &\le \frac{e^{1-t}}{4(5t^2+3)^2}|x_1 - x_2| \le \frac{1}{256}|x_1 - x_2|.
\end{aligned}$$

That is, we have found $l_1 = \frac{1}{8}$ and $l_2 = \frac{1}{256}$ such that $\chi(t,u)$ and $g_1(t,u)$ satisfy hypothesis (H_1) and $\Delta_1 l_1 + \Delta_2 l_2 \approx 0.7641343750 < 1$. Therefore by Theorem 3.2 boundary value problem (3.6) has a unique solution on $[1,e^2]$.

Example 3.6 Consider

$$\begin{cases}
({}_HD^{\frac{17}{4}}Z)(t) = (e^{2t} + t^2)(\frac{|u(t)|}{2100+|u(t)|}) + \frac{1}{2}t - 1, \quad t \in [1,e^2], \\
Z(1) = Z'(1) = \cdots = Z^{(n-3)}(1) = 0, \\
Z(\sqrt{e}) = {}_HI^{\frac{17}{4}}[(e^{2t} + t^2)(\frac{|u(t)|}{2100+|u(t)|}) + \frac{1}{2}t - 1]|_{t=\sqrt{e}}, \\
{}_HI^{\frac{9}{4}}u(e^2) = \frac{1}{10}{}_HI^{\frac{7}{4}}u(\frac{3}{2}) + \frac{3}{5}{}_HI^{\frac{7}{4}}u(\frac{7}{3}) + \frac{1}{2}{}_HI^{\frac{7}{4}}u(5),
\end{cases} \tag{3.7}$$

where

$$Z(t) = u(t) - g_1\bigl(t,u(t)\bigr) = u(t) - 2^t\left(\frac{|u(t)|}{2100+|u(t)|}\right),$$

$n = 5$, $\alpha = \frac{17}{4}$, $p = \frac{9}{4}$, $q = \frac{7}{4}$, $m = 1$, $r = 3$, $\omega_1 = \frac{1}{10}$, $\omega_2 = \frac{3}{5}$, $\omega_3 = \frac{1}{2}$, $\xi_1 = \frac{3}{2}$, $\xi_2 = \frac{7}{3}$, $\xi_3 = 5$.

We choose $\zeta_1(t) = e^{2t} + t^2$ and $\zeta_2(t) = 2^t$. By calculation we get $\Upsilon = 0.1139$, $W^* \approx 2098.586239$. Let $\chi(t,u) = (e^{2t} + t^2)(\frac{|u|}{2100+|u|}) + \frac{1}{2}t - 1$ and $g_1(t,u) = 2^t(\frac{|u|}{2100+|u|})$. Then

$$\begin{aligned}
\left|\chi(t,x_1) - \chi(t,x_2)\right| &\le \bigl(e^{2t} + t^2\bigr)\left(\frac{2100(|x_1| - |x_2|)}{2100^2 + 2100|x_1| + 2100|x_2| + |x_1||x_2|}\right) \\
&\le \bigl(e^{2t} + t^2\bigr)\frac{|x_1 - x_2|}{2098.586239 + |x_1 - x_2|}, \\
\left|g_1(t,x_1) - g_1(t,x_2)\right| &\le 2^t\left(\frac{2100(|x_1| - |x_2|)}{2100^2 + 2100|x_1| + 2100|x_2| + |x_1||x_2|}\right) \\
&\le 2^t\frac{|x_1 - x_2|}{2098.586239 + |x_1 - x_2|}.
\end{aligned}$$

Therefore by Theorem 3.3 the boundary value problem (3.7) has a unique solution on $[1,e^2]$.

Example 3.7 Consider

$$\begin{cases} ({}_HD^{\frac{7}{2}}Z)(t) = \frac{e^{1-t}}{\sqrt{t^2+5t+7}}(\frac{1}{100}\frac{|u^5(t)|}{1+|u^4(t)|} + \frac{1}{100}|u(t)|\cos(|u(t)|+1) + \frac{1}{2}), \quad t \in [1,e^2], \\ Z(1) = Z'(1) = \cdots = Z^{(n-3)}(1) = 0, \\ Z(\sqrt{e}) = {}_HI^{\frac{7}{2}}[\frac{e^{1-t}}{\sqrt{t^2+5t+7}}(\frac{1}{100}\frac{|u^5(t)|}{1+|u^4(t)|} + \frac{1}{100}|u(t)|\cos(|u(t)|+1) + \frac{1}{2})]|_{t=\sqrt{e}}, \\ {}_HI^{\frac{11}{6}}u(e^2) = 2{}_HI^{\frac{7}{6}}u(\frac{7}{3}), \end{cases} \tag{3.8}$$

where

$$Z(t) = u(t) - \left[\frac{1}{t^2+4t+3}\left(\frac{1}{100}\frac{|u^5(t)|}{1+|u^4(t)|} + \frac{1}{100}\left|u(t)\right|\cos\left(\left|u(t)\right|+1\right) + \frac{1}{2}\right)\right],$$

$n = 4$, $\alpha = \frac{7}{2}$, $p = \frac{11}{6}$, $q = \frac{7}{6}$, $m = 1$, $r = 1$, $\omega_1 = 2$, $\xi_1 = \frac{7}{3}$.

Let

$$\chi(t,u) = \frac{e^{1-t}}{\sqrt{t^2+5t+7}}\left(\frac{1}{100}\frac{|u^5|}{1+|u^4|} + \frac{1}{100}|u|\cos\left(|u|+1\right) + \frac{1}{2}\right),$$

$$g_1(t,u) = \frac{1}{t^2+4t+3}\left(\frac{1}{100}\frac{|u^5|}{1+|u^4|} + \frac{1}{100}|u|\cos\left(|u|+1\right) + \frac{1}{2}\right).$$

Obviously,

$$\left|\chi(t,u)\right| \le \frac{e^{1-t}}{\sqrt{t^2+5t+7}}\left(\frac{|u|}{50} + \frac{1}{2}\right), \qquad \left|g_1(t,u)\right| \le \frac{1}{t^2+4t+3}\left(\frac{|u|}{50} + \frac{1}{2}\right).$$

We choose $\kappa_1(t) = \frac{e^{1-t}}{\sqrt{t^2+5t+7}}$, $\kappa_2(t) = \frac{1}{t^2+4t+3}$, and $\Phi(u) = \frac{u}{50} + \frac{1}{2}$. By calculation we get $\|\kappa_1\| = \frac{1}{\sqrt{13}}$, $\|\kappa_2\| = \frac{1}{8}$, $\Upsilon \approx 0.487$, $\Delta_1 \approx 3.9031$, $\Delta_2 \approx 53.1174$, and there is $L > 4.566344512$ such that $\frac{L}{\Phi(L)(\|\kappa_1\|\Delta_1+\|\kappa_2\|\Delta_2)} > 1$. Therefore by Theorem 3.4 the boundary value problem (3.8) has at least one solution on $[1,e^2]$.

3.2 Ulam–Hyers stability

For $\varepsilon > 0$, consider the following inequality:

$$\left|{}_HD^\alpha\left(u_\pi(t) - \sum_{k=1}^m g_k\left(t,u_\pi(t)\right)\right) - \chi\left(t,u_\pi(t)\right)\right| \le \varepsilon, \quad t \in \left[1,e^2\right]. \tag{3.9}$$

Definition 3.8 ([19]) Problem (1.1) is said to be Ulam–Hyers stable if there exists a real number $c > 0$ such that for each $\varepsilon > 0$ and for each solution $u_\pi \in C([1,e^2],\mathbb{R})$ of inequality (3.9), there exists a solution $u \in C([1,e^2],\mathbb{R})$ of problem (1.1) such that $\|u_\pi - u\| \le c \cdot \varepsilon$.

Definition 3.9 ([19]) Problem (1.1) is said to be generalized Ulam–Hyers stable if there exists $\phi_{\chi,c} \in C(\mathbb{R}^+,\mathbb{R}^+)$, $\phi_{\chi,c}(0) = 0$, such that for each $\varepsilon > 0$ and for each solution $u_\pi \in C([1,e^2],\mathbb{R})$ of inequality (3.9), there exists a solution $u \in C([1,e^2],\mathbb{R})$ of problem (1.1) such that $\|u_\pi - u\| \le \phi_{\chi,c}(\varepsilon)$.

Theorem 3.10 *Assume that (H_1) holds. If for any $t \in [1,e^2]$ and $\varepsilon > 0$, a function $u_\pi : [1,e^2] \to \mathbb{R}$ satisfies inequality (3.9), then there exists a solution $u : [1,e^2] \to \mathbb{R}$ of problem (1.1) such that $\|u_\pi - u\| \le \frac{\Delta_1}{1-\Delta_1 l_1 - \Delta_2 l_2}\varepsilon$, that is, problem (1.1) is Ulam–Hyers stable.*

Proof Since u is a solution of the boundary value problem (1.1), we have

$$u(t) = {}_HI^{\alpha}\chi\big(t,u(t)\big) + \sum_{k=1}^{m} g_k\big(t,u(t)\big) + \left(\frac{(\ln t)^{\alpha-1}}{\Upsilon} - \frac{(\ln t)^{\alpha-2}}{2\Upsilon}\right) \cdot \Bigg[\sum_{j=1}^{r}\omega_j {}_HI^{\alpha+q}\chi\big(\theta,u(\theta)\big)\Big|_{\theta=\xi_j} - {}_HI^{\alpha+p}\chi\big(\theta,u(\theta)\big)\Big|_{\theta=e^2} + \sum_{j=1}^{r}\omega_j {}_HI^{q}\left(\sum_{k=1}^{m} g_k\big(\theta,u(\theta)\big)\right)\Bigg|_{\theta=\xi_j} - {}_HI^{p}\left(\sum_{k=1}^{m} g_k\big(\theta,u(\theta)\big)\right)\Bigg|_{\theta=e^2}\Bigg].$$

Suppose that there exists a function $\widetilde{\chi}$ satisfying the fractional equation

$${}_HD^{\alpha}\left(u_\pi(t) - \sum_{k=1}^{m} g_k\big(t,u_\pi(t)\big)\right) = \widetilde{\chi}\big(t,u_\pi(t)\big).$$

Then

$$u_\pi(t) = {}_HI^{\alpha}\widetilde{\chi}\big(t,u_\pi(t)\big) + \sum_{k=1}^{m} g_k\big(t,u_\pi(t)\big) + \left(\frac{(\ln t)^{\alpha-1}}{\Upsilon} - \frac{(\ln t)^{\alpha-2}}{2\Upsilon}\right) \cdot \Bigg[\sum_{j=1}^{r}\omega_j {}_HI^{\alpha+q}\widetilde{\chi}\big(\theta,u_\pi(\theta)\big)\Big|_{\theta=\xi_j} - {}_HI^{\alpha+p}\widetilde{\chi}\big(\theta,u_\pi(\theta)\big)\Big|_{\theta=e^2} + \sum_{j=1}^{r}\omega_j {}_HI^{q}\left(\sum_{k=1}^{m} g_k\big(\theta,u_\pi(\theta)\big)\right)\Bigg|_{\theta=\xi_j} - {}_HI^{p}\left(\sum_{k=1}^{m} g_k\big(\theta,u_\pi(\theta)\big)\right)\Bigg|_{\theta=e^2}\Bigg].$$

Let

$$v(t) = {}_HI^{\alpha}\chi\big(t,u_\pi(t)\big) + \sum_{k=1}^{m} g_k\big(t,u_\pi(t)\big) + \left(\frac{(\ln t)^{\alpha-1}}{\Upsilon} - \frac{(\ln t)^{\alpha-2}}{2\Upsilon}\right) \cdot \Bigg[\sum_{j=1}^{r}\omega_j {}_HI^{\alpha+q}\chi\big(\theta,u_\pi(\theta)\big)\Big|_{\theta=\xi_j} - {}_HI^{\alpha+p}\chi\big(\theta,u_\pi(\theta)\big)\Big|_{\theta=e^2} + \sum_{j=1}^{r}\omega_j {}_HI^{q}\left(\sum_{k=1}^{m} g_k\big(\theta,u_\pi(\theta)\big)\right)\Bigg|_{\theta=\xi_j} - {}_HI^{p}\left(\sum_{k=1}^{m} g_k\big(\theta,u_\pi(\theta)\big)\right)\Bigg|_{\theta=e^2}\Bigg].$$

By (3.9) we have

$$\begin{aligned} &\big|u_\pi(t) - v(t)\big| \\ &\quad\le {}_HI^{\alpha}\big|\widetilde{\chi}\big(t,u_\pi(t)\big) - \chi\big(t,u_\pi(t)\big)\big| + \left|\frac{(\ln t)^{\alpha-1}}{\Upsilon} + \frac{(\ln t)^{\alpha-2}}{2\Upsilon}\right| \\ &\qquad\cdot \Bigg[\sum_{j=1}^{r}|\omega_j| {}_HI^{\alpha+q}\big(\big|\widetilde{\chi}\big(\theta,u_\pi(\theta)\big) - \chi\big(\theta,u_\pi(\theta)\big)\big|\big)\Big|_{\theta=\xi_j} \\ &\qquad + {}_HI^{\alpha+p}\big(\big|\widetilde{\chi}\big(\theta,u_\pi(\theta)\big) - \chi\big(\theta,u_\pi(\theta)\big)\big|\big)\Big|_{\theta=e^2}\Bigg] \end{aligned}$$

$$
\begin{aligned}
&= {}_H I^{\alpha}\left|{}_H D^{\alpha}\left(u_{\pi}(t)-\sum_{k=1}^{m} g_k\big(t,u_{\pi}(t)\big)\right)-\chi\big(t,u_{\pi}(t)\big)\right| \\
&\quad + \left|\frac{(\ln t)^{\alpha-1}}{\Upsilon}+\frac{(\ln t)^{\alpha-2}}{2\Upsilon}\right| \\
&\quad \cdot \left[\sum_{j=1}^{r}|\omega_j|_H I^{\alpha+q}\left(\left|{}_H D^{\alpha}\left(u_{\pi}(\theta)-\sum_{k=1}^{m} g_k\big(\theta,u_{\pi}(\theta)\big)\right)-\chi\big(\theta,u_{\pi}(\theta)\big)\right|\right)\Bigg|_{\theta=\xi_j}\right. \\
&\quad \left. + {}_H I^{\alpha+p}\left(\left|{}_H D^{\alpha}\left(u_{\pi}(\theta)-\sum_{k=1}^{m} g_k\big(\theta,u_{\pi}(\theta)\big)\right)-\chi\big(\theta,u_{\pi}(\theta)\big)\right|\right)\Bigg|_{\theta=e^2}\right] \\
&\leq \left\{\frac{2^{\alpha}}{\Gamma(\alpha+1)}+\left(\frac{2^{\alpha-1}+2^{\alpha-3}}{|\Upsilon|}\right)\left(\sum_{j=1}^{r}\frac{|\omega_j|}{\Gamma(\alpha+q+1)}(\ln \xi_j)^{\alpha+q}+\frac{2^{\alpha+p}}{\Gamma(\alpha+p+1)}\right)\right\}\varepsilon. \\
&= \Delta_1\varepsilon.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\big|u_{\pi}(t)-u(t)\big| \\
&\quad \leq \big|u_{\pi}(t)-v(t)\big|+\big|v(t)-u(t)\big| \\
&\quad \leq \Delta_1\varepsilon + {}_H I^{\alpha}\big|\chi\big(t,u_{\pi}(t)\big)-\chi\big(t,u(t)\big)\big| + \sum_{k=1}^{m}\big|g_k\big(t,u_{\pi}(t)\big)-g_k\big(t,u(t)\big)\big| \\
&\qquad + \left|\frac{(\ln t)^{\alpha-1}}{\Upsilon}+\frac{(\ln t)^{\alpha-2}}{2\Upsilon}\right| \cdot \left[\sum_{j=1}^{r}|\omega_j|_H I^{\alpha+q}\big(\big|\chi\big(\theta,u_{\pi}(\theta)\big)-\chi\big(\theta,u(\theta)\big)\big|\big)\Big|_{\theta=\xi_j}\right. \\
&\qquad + {}_H I^{\alpha+p}\big(\big|\chi\big(\theta,u_{\pi}(\theta)\big)-\chi\big(\theta,u(\theta)\big)\big|\big)\Big|_{\theta=e^2} \\
&\qquad + \sum_{j=1}^{r}|\omega_j|_H I^{q}\left(\sum_{k=1}^{m}\big|g_k\big(\theta,u_{\pi}(\theta)\big)-g_k\big(\theta,u(\theta)\big)\big|\right)\Bigg|_{\theta=\xi_j} \\
&\qquad \left. + {}_H I^{p}\left(\sum_{k=1}^{m}\big|g_k\big(\theta,u_{\pi}(\theta)\big)-g_k\big(\theta,u(\theta)\big)\big|\right)\Bigg|_{\theta=e^2}\right] \\
&\quad \leq \Delta_1\varepsilon + \Delta_1 l_1\|u_{\pi}-u\| + \Delta_2 l_2\|u_{\pi}-u\|.
\end{aligned}
$$

Hence $\|u_{\pi}-u\| \leq \Delta_1\varepsilon + \Delta_1 l_1\|u_{\pi}-u\| + \Delta_2 l_2\|u_{\pi}-u\|$. As a result,

$$
\|u_{\pi}-u\| \leq \frac{\Delta_1}{1-\Delta_1 l_1-\Delta_2 l_2}\cdot\varepsilon := c\cdot\varepsilon. \tag{3.10}
$$

Therefore problem (1.1) is Ulam–Hyers stable. This completes the proof. □

Remark 3.11 By setting $\phi_{\chi,c}(\varepsilon) = c\cdot\varepsilon$, $\phi_{\chi,c}(0) = 0$ in (3.10). So by Definition 3.9 we get that problem (1.1) is generalized Ulam–Hyers stable.

4 Existence of solutions for multivalued problems

To obtain the existence of the solution of the multivalued problem (1.2), we first consider the equation

$$\begin{cases} ({}_HD^\alpha Z)(t) = \chi(t,u(t)), \quad t \in [1,e^2], \\ Z(1) = Z'(1) = \cdots = Z^{(n-2)}(1) = 0, \\ {}_HI^p u(t)|_{t=e^2} = \sum_{j=1}^r \omega_{jH} I^q u(t)|_{t=\xi_j}, \end{cases} \tag{4.1}$$

where $Z(t) = u(t) - \sum_{k=1}^m g_k(t,u(t))$, and other conditions are the same as in problem (1.1).

Similarly to Lemma 3.1, we can easily draw the following conclusions.

Lemma 4.1 *Let* $\overline{\chi} \in C([1,e^2],\mathbb{R})$, $g_k \in C([1,e^2] \times \mathbb{R},\mathbb{R})$, $Z \in L^1(1,e^2)$, $({}_HI^{n-\alpha}Z)(t) \in AC_\delta^n[1,e^2]$, *and*

$$\overline{\Upsilon} = \frac{\Gamma(\alpha)}{\Gamma(\alpha+p)} 2^{\alpha+p-1} - \frac{\Gamma(\alpha)}{\Gamma(\alpha+q)} \sum_{j=1}^r \omega_j (\ln \xi_j)^{\alpha+q-1} \neq 0.$$

Then the function u is a solution of

$$\begin{cases} ({}_HD^\alpha Z)(t) = \overline{\chi}(t), \quad t \in [1,e^2], \\ Z(1) = Z'(1) = \cdots = Z^{(n-2)}(1) = 0, \\ {}_HI^p u(t)|_{t=e^2} = \sum_{j=1}^r \omega_{jH} I^q u(t)|_{t=\xi_j}, \end{cases} \tag{4.2}$$

if and only if

$$\begin{aligned} u(t) = {} & {}_HI^\alpha \overline{\chi}(t) + \sum_{k=1}^m g_k\bigl(t,u(t)\bigr) + \frac{(\ln t)^{\alpha-1}}{\overline{\Upsilon}} \\ & \cdot \Biggl[\sum_{j=1}^r \omega_{jH} I^{\alpha+q} \overline{\chi}(\xi_j) - {}_HI^{\alpha+p} \overline{\chi}\bigl(e^2\bigr) \\ & + \sum_{j=1}^r \omega_{jH} I^q \Biggl(\sum_{k=1}^m g_k\bigl(\xi_j,u(\xi_j)\bigr) \Biggr) - {}_HI^p \Biggl(\sum_{k=1}^m g_k\bigl(e^2,u\bigl(e^2\bigr)\bigr) \Biggr) \Biggr], \end{aligned}$$

where

$$Z(t) = u(t) - \sum_{k=1}^m g_k\bigl(t,u(t)\bigr).$$

For convenience, we introduce the constants

$$\overline{\Delta}_1 = \frac{2^\alpha}{\Gamma(\alpha+1)} + \frac{2^{\alpha-1}}{|\overline{\Upsilon}|} \Biggl(\sum_{j=1}^r \frac{|\omega_j|}{\Gamma(\alpha+q+1)} (\ln \xi_j)^{\alpha+q} + \frac{2^{\alpha+p}}{\Gamma(\alpha+p+1)} \Biggr), \tag{4.3}$$

$$\overline{\Delta}_2 = m + \frac{2^{\alpha-1}}{|\overline{\Upsilon}|} \Biggl(\sum_{j=1}^r \frac{m \cdot |\omega_j|}{\Gamma(q+1)} (\ln \xi_j)^q + \frac{m \cdot 2^p}{\Gamma(p+1)} \Biggr). \tag{4.4}$$

For a normed space $(U, \|\cdot\|)$, define $\mathcal{P}(U) = \{Y \subset U : Y \neq \emptyset\}$, $\mathcal{P}_{c,cp}(U) = \{Y \subset \mathcal{P}(U) : Y \text{is compact and convex}\}$, and $\mathcal{P}_{b,cl,c}(U) = \{Y \subset \mathcal{P}(U) : Y \text{is bounded, closed and convex}\}$.

Take $S_{\mathfrak{G},u} := \{\omega \in L^1([1,e^2],\mathbb{R}) : \omega(t) \in \mathfrak{G}(t,u(t)),\ \text{a.e. } t \in [1,e^2]\}$ as the selection set of $\mathfrak{G}$.

We define the solution of problem (1.2) as follows.

Definition 4.2 A function $u \in C([1,e^2],\mathbb{R})$ is called a solution of multivalued problem (1.2) if there is a function $h \in L^1([1,e^2],\mathbb{R})$ with $h(t) \in \mathfrak{G}(t,u)$ almost everywhere on $[1,e^2]$ such that

$$u(t) = {}_HI^\alpha h(t) + \sum_{k=1}^m g_k\bigl(t,u(t)\bigr) + \frac{(\ln t)^{\alpha-1}}{\overline{\Upsilon}}\Biggl[\sum_{j=1}^r \omega_{jH}I^{\alpha+q}h(\xi_j) - {}_HI^{\alpha+p}h\bigl(e^2\bigr) + \sum_{j=1}^r \omega_{jH}I^q\Biggl(\sum_{k=1}^m g_k\bigl(\theta,u(\theta)\bigr)\Biggr)\Bigg|_{\theta=\xi_j} - {}_HI^p\Biggl(\sum_{k=1}^m g_k\bigl(\theta,u(\theta)\bigr)\Biggr)\Bigg|_{\theta=e^2}\Biggr].$$

Definition 4.3 ([9]) Let $\mathfrak{F} : U \to \mathcal{P}(U)$ be a multivalued map. Then:

(i) $\mathfrak{F}$ is called upper semicontinuous if for each $u_0 \in U$, the set $\mathfrak{F}(u_0)$ is a nonempty closed subset of U, and for each open set G of U containing $\mathfrak{F}(u_0)$, there exists an open neighborhood G_0 of u_0 such that $\mathfrak{F}(G_0) \subseteq G$;

(ii) If the multivalued map $\mathfrak{F}$ is completely continuous with nonempty compact values, then $\mathfrak{F}$ is upper semicontinuous if and only if $\mathfrak{F}$ has a closed graph, that is, $u_n \to u_*$, $v_n \to v_*$, $v_n \in \mathfrak{F}(u_n)$ imply that $v_* \in \mathfrak{F}(u_*)$.

Definition 4.4 ([9]) A multivalued map $\mathfrak{F} : [1,e^2] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) for each $u \in \mathbb{R}$, $t \mapsto \mathfrak{F}(t,u)$ is measurable;

(ii) for almost all $t \in [1,e^2]$, $u \mapsto \mathfrak{F}(t,u)$ is upper semicontinuous.

In addition, a Carathéodory multivalued map $\mathfrak{F}$ is called L^1-Carathéodory if

(iii) for all $z > 0$ and $u \in \mathbb{R}$ with $\|u\| \leq z$, there exists $\eta_z \in L^1([1,e^2],\mathbb{R}^+)$ such that $\|\mathfrak{F}(t,u)\| = \sup\{|\omega| : \omega \in \mathfrak{F}(t,u)\} \leq \eta_z(t)$ for a.e. $t \in [1,e^2]$.

Lemma 4.5 ([21]) *Let U be a separable Banach space, let $\mathfrak{F} : [1,e^2] \times U \to \mathcal{P}_{c,cp}(U)$ be an L^1-Carathéodory multivalued map, and let $\mathcal{B} : L^1([1,e^2],U) \to C([1,e^2],U)$ be a linear continuous operator. Then the operator $\mathcal{B} \circ S_{\mathfrak{F},u} : C([1,e^2],U) \to \mathcal{P}_{c,cp}(C([1,e^2],U))$ is a closed graph operator on $C([1,e^2],U) \times C([1,e^2],U)$.*

Lemma 4.6 (Bohnenblust–Karlin fixed point theorem [6]) *Let U be a Banach space, and let Q be a nonempty bounded closed convex subset of U. Suppose $\mathcal{T} : [1,e^2] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is upper semicontinuous with closed convex values, $\mathcal{T}(Q) \subset Q$, and $\overline{\mathcal{T}(Q)}$ is compact. Then $\mathcal{T}$ has a fixed point.*

Lemma 4.7 (Martelli's fixed point theorem [17]) *Let U be a Banach space, and let $\mathcal{T} : U \to \mathcal{P}_{b,cl,c}(U)$ be a completely continuous multivalued map. If the set $\Omega = \{u \in U : \varrho u \in \mathcal{T}(u), \varrho > 1\}$ is bounded, then $\mathcal{T}$ has a fixed point.*

Lemma 4.8 (Nonlinear alternative for Kakutani maps [10]) *Let U be a Banach space, let Q be a closed convex subset of U, and let H be an open subset of Q such that $0 \in H$. Suppose $\mathcal{T} : \overline{H} \to \mathcal{P}_{c,cp}(Q)$ is an upper semicontinuous compact map. Then either* *(i)* $\mathcal{T}$ *has a fixed point on* $\overline{H}$, *or*

(ii) there exist $u \in \partial H$ *and* $\lambda \in (0,1)$ *such that* $u \in \lambda\mathcal{T}(u)$.

Theorem 4.9 *Assume that*:

(O_1) $\mathfrak{G} : [1,e^2]\times\mathbb{R} \to \mathcal{P}_{c,cp}(\mathbb{R})$ *is* L^1*-Carathéodory, that is, for each* $z > 0$, *there exists* $\eta_z \in L^1([1,e^2],\mathbb{R}^+)$ *such that for all* $u \in \mathbb{R}$ *with* $\|u\| \le z$, $\|\mathfrak{G}(t,u)\| = \sup\{|\omega| : \omega \in \mathfrak{G}(t,u)\} \le \eta_z(t)$ *for a.e.* $t \in [1,e^2]$;

(O_2) *there exists a constant* $G > 0$ *such that* $\|g_k(t,u)\| = \sup\{|g_k(t,u)|\} \le G$ *for all* $(t,u) \in [1,e^2]\times\mathbb{R}$ $(k = 1,2,\dots,m)$.

Then the boundary value problem (1.2) *has a solution on* $[1,e^2]$.

Proof Define the operator $\mathcal{T} : C([1,e^2],\mathbb{R}) \to \mathcal{P}(C([1,e^2],\mathbb{R}))$ by

$$\mathcal{T}(u) = \begin{cases} \mu \in C([1,e^2],\mathbb{R}), \\ \mu(t) = \begin{cases} {}_HI^{\alpha}h(t) + \sum_{k=1}^m g_k(t,\mu(t)) + \frac{(\ln t)^{\alpha-1}}{\Upsilon} \\ \quad \cdot [\sum_{j=1}^r \omega_j {}_HI^{\alpha+q}h(\xi_j) - {}_HI^{\alpha+p}h(e^2) \\ \quad + \sum_{j=1}^r \omega_j {}_HI^{q}(\sum_{k=1}^m g_k(\theta,\mu(\theta)))|_{\theta=\xi_j} - {}_HI^{p}(\sum_{k=1}^m g_k(\theta,\mu(\theta)))|_{\theta=e^2}], \end{cases} \\ h \in S_{\mathfrak{G},u}. \end{cases}$$

Obviously, the fixed point of the operator $\mathcal{T}$ is the solution of the boundary value problem (1.2). Next, we prove that the operator $\mathcal{T}$ satisfies the conditions of Lemma 4.6 in several steps.

In the first step, $\mathcal{T}$ maps bounded sets into bounded sets in $C([1,e^2],\mathbb{R})$.

Let $B_z = \{u \in C([1,e^2],\mathbb{R}) : \|u\| \le z\}$ be a bounded set on $C([1,e^2],\mathbb{R})$. For all $\mu \in \mathcal{T}(u)$ and $u \in B_z$, there exists $h \in S_{\mathfrak{G},u}$ such that

$$\begin{aligned} &\mu(t) \\ &\quad = {}_HI^{\alpha}h(t) + \sum_{k=1}^m g_k\bigl(t,\mu(t)\bigr) + \frac{(\ln t)^{\alpha-1}}{\Upsilon}\Biggl[\sum_{j=1}^r \omega_j {}_HI^{\alpha+q}h(\xi_j) - {}_HI^{\alpha+p}h\bigl(e^2\bigr) \\ &\qquad + \sum_{j=1}^r \omega_j {}_HI^{q}\Biggl(\sum_{k=1}^m g_k\bigl(\theta,\mu(\theta)\bigr)\Biggr)\Bigg|_{\theta=\xi_j} - {}_HI^{p}\Biggl(\sum_{k=1}^m g_k\bigl(\theta,\mu(\theta)\bigr)\Biggr)\Bigg|_{\theta=e^2}\Biggr]. \end{aligned}$$

From assumptions (O_1) and (O_2) we get

$$\begin{aligned} &\bigl|\mu(t)\bigr| \\ &\quad \le {}_HI^{\alpha}\bigl|h(t)\bigr| + \sum_{k=1}^m \bigl|g_k\bigl(t,\mu(t)\bigr)\bigr| + \biggl|\frac{(\ln t)^{\alpha-1}}{\Upsilon}\biggr| \\ &\qquad \cdot \Biggl[\sum_{j=1}^r |\omega_j| {}_HI^{\alpha+q}\bigl|h(\xi_j)\bigr| + {}_HI^{\alpha+p}\bigl|h\bigl(e^2\bigr)\bigr| \\ &\qquad + \sum_{j=1}^r |\omega_j| {}_HI^{q}\Biggl(\sum_{k=1}^m \bigl|g_k\bigl(\theta,\mu(\theta)\bigr)\bigr|\Biggr)\Bigg|_{\theta=\xi_j} + {}_HI^{p}\Biggl(\sum_{k=1}^m \bigl|g_k\bigl(\theta,\mu(\theta)\bigr)\bigr|\Biggr)\Bigg|_{\theta=e^2}\Biggr] \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{2^{\alpha}}{\Gamma(\alpha+1)} + \frac{2^{\alpha-1}}{|\overline{\Upsilon}|}\left(\sum_{j=1}^{r}\frac{|\omega_j|}{\Gamma(\alpha+q+1)}(\ln\xi_j)^{\alpha+q} + \frac{2^{\alpha+p}}{\Gamma(\alpha+p+1)}\right)\right)\cdot\|\eta_z\| \\
&\quad + \left(m + \frac{2^{\alpha-1}}{|\overline{\Upsilon}|}\left(\sum_{j=1}^{r}\frac{m\cdot|\omega_j|}{\Gamma(q+1)}(\ln\xi_j)^{q} + \frac{m\cdot 2^{p}}{\Gamma(p+1)}\right)\right)\cdot G \\
&= \overline{\Delta}_1\|\eta_z\| + \overline{\Delta}_2 G.
\end{aligned}$$

Then

$$\|\mu\| \leq \overline{\Delta}_1\|\eta_z\| + \overline{\Delta}_2 G.$$

In the second step, we prove that $\mathcal{T}$ is equicontinuous.

Let $\mu \in \mathcal{T}(u)$, $u \in B_z$. Then there exists $h \in S_{\mathfrak{G},u}$ such that

$$\begin{aligned}
&\mu(t) \\
&= {}_HI^{\alpha}h(t) + \sum_{k=1}^{m} g_k\bigl(t,\mu(t)\bigr) + \frac{(\ln t)^{\alpha-1}}{\overline{\Upsilon}}\left[\sum_{j=1}^{r}\omega_{jH}I^{\alpha+q}h(\xi_j) - {}_HI^{\alpha+p}h\bigl(e^2\bigr)\right. \\
&\quad \left. + \sum_{j=1}^{r}\omega_{jH}I^{q}\left(\sum_{k=1}^{m} g_k\bigl(\theta,\mu(\theta)\bigr)\right)\bigg|_{\theta=\xi_j} - {}_HI^{p}\left(\sum_{k=1}^{m} g_k\bigl(\theta,\mu(\theta)\bigr)\right)\bigg|_{\theta=e^2}\right].
\end{aligned}$$

Let $t_1, t_2 \in [1, e^2]$, $t_1 < t_2$. We have

$$\begin{aligned}
&\bigl|\mu(t_2) - \mu(t_1)\bigr| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left(\left(\ln\frac{t_2}{\theta}\right)^{\alpha-1} - \left(\ln\frac{t_1}{\theta}\right)^{\alpha-1}\right)\frac{|h(\theta)|}{\theta}\,d\theta + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{\theta}\right)^{\alpha-1}\frac{|h(\theta)|}{\theta}\,d\theta \\
&\quad + \sum_{k=1}^{m}\bigl|g_k\bigl(t_2,\mu(t_2)\bigr) - g_k\bigl(t_1,\mu(t_1)\bigr)\bigr| + \left|\frac{(\ln t_2)^{\alpha-1}}{\overline{\Upsilon}} - \frac{(\ln t_1)^{\alpha-1}}{\overline{\Upsilon}}\right| \\
&\quad \cdot\left[\sum_{j=1}^{r}|\omega_j|_HI^{\alpha+q}\bigl|h(\xi_j)\bigr| + {}_HI^{\alpha+p}\bigl|h\bigl(e^2\bigr)\bigr|\right. \\
&\quad \left. + \sum_{j=1}^{r}|\omega_j|_HI^{q}\left(\sum_{k=1}^{m}\bigl|g_k\bigl(\theta,\mu(\theta)\bigr)\bigr|\right)\bigg|_{\theta=\xi_j} + {}_HI^{p}\left(\sum_{k=1}^{m}\bigl|g_k\bigl(\theta,\mu(\theta)\bigr)\bigr|\right)\bigg|_{\theta=e^2}\right] \\
&\leq \frac{\|\eta_z\|}{\Gamma(\alpha+1)}\bigl|(\ln t_2)^{\alpha} - (\ln t_1)^{\alpha}\bigr| + \sum_{k=1}^{m}\bigl|g_k\bigl(t_2,\mu(t_2)\bigr) - g_k\bigl(t_1,\mu(t_1)\bigr)\bigr| \\
&\quad + \frac{|(\ln t_2)^{\alpha-1} - (\ln t_1)^{\alpha-1}|}{|\overline{\Upsilon}|}\cdot\left[\|\eta_z\|\left(\sum_{j=1}^{r}\frac{|\omega_j|(\ln\xi_j)^{\alpha+q}}{\Gamma(\alpha+q+1)} + \frac{2^{\alpha+p}}{\Gamma(\alpha+p+1)}\right)\right. \\
&\quad \left. + G\left(\sum_{j=1}^{r}\frac{m\cdot|\omega_j|}{\Gamma(q+1)}(\ln\xi_j)^{q} + \frac{m\cdot 2^{p}}{\Gamma(p+1)}\right)\right].
\end{aligned}$$

By the continuity of g_k, $|\mu(t_2) - \mu(t_1)| \to 0$ as $t_2 \to t_1$. Therefore, by the Arzelà–Ascoli theorem, $\mathcal{T} : C([1,e^2],\mathbb{R}) \to \mathcal{P}(C([1,e^2],\mathbb{R}))$ is completely continuous.

In the third step, we prove that for each $u \in C([1,e^2],\mathbb{R})$, $\mathcal{T}(u)$ is closed.

In $C([1,e^2],\mathbb{R})$, let $\{\mu_n\}_{n\geq 0} \in \mathcal{T}(u)$ and $\mu_n \to \mu(n \to \infty)$. Let us prove that $\mu \in \mathcal{T}(u)$. It is easy to get that $\mu \in C([1,e^2],\mathbb{R})$, and there exist $h_n \in S_{\mathfrak{G},u_n}$ such that for each $t \in [1,e^2]$,

$$\begin{aligned}
&\mu_n(t) \\
&= {}_HI^\alpha h_n(t) + \sum_{k=1}^m g_k\bigl(t,\mu_n(t)\bigr) + \frac{(\ln t)^{\alpha-1}}{\overline{\Upsilon}}\Biggl[\sum_{j=1}^r \omega_{jH}I^{\alpha+q}h_n(\xi_j) - {}_HI^{\alpha+p}h_n\bigl(e^2\bigr) \\
&\quad + \sum_{j=1}^r \omega_{jH}I^q\Biggl(\sum_{k=1}^m g_k\bigl(\theta,\mu_n(\theta)\bigr)\Biggr)\Bigg|_{\theta=\xi_j} - {}_HI^p\Biggl(\sum_{k=1}^m g_k\bigl(\theta,\mu_n(\theta)\bigr)\Biggr)\Bigg|_{\theta=e^2}\Biggr].
\end{aligned}$$

Since $\mathfrak{G}$ has convex values, let $h_n \to h \in L^1([1,e^2],\mathbb{R})$ (otherwise, we could find a subsequence converging to h), so $h \in S_{\mathfrak{G},u}$, and for each $t \in [1,e^2]$, we have

$$\begin{aligned}
&\mu_n(t) \to \mu(t) \\
&\quad = {}_HI^\alpha h(t) + \sum_{k=1}^m g_k\bigl(t,\mu(t)\bigr) + \frac{(\ln t)^{\alpha-1}}{\overline{\Upsilon}}\Biggl[\sum_{j=1}^r \omega_{jH}I^{\alpha+q}h(\xi_j) - {}_HI^{\alpha+p}h\bigl(e^2\bigr) \\
&\qquad + \sum_{j=1}^r \omega_{jH}I^q\Biggl(\sum_{k=1}^m g_k\bigl(\theta,\mu(\theta)\bigr)\Biggr)\Bigg|_{\theta=\xi_j} - {}_HI^p\Biggl(\sum_{k=1}^m g_k\bigl(\theta,\mu(\theta)\bigr)\Biggr)\Bigg|_{\theta=e^2}\Biggr].
\end{aligned}$$

Hence $\mu \in \mathcal{T}(u)$.

In the fourth step, we prove that for each $u \in C([1,e^2],\mathbb{R})$, $\mathcal{T}(u)$ is convex.

If $\mu_1,\mu_2 \in \mathcal{T}(u)$, then there exist $h_1,h_2 \in S_{\mathfrak{G},u}$ such that for each $t \in [1,e^2]$,

$$\begin{aligned}
&\mu_\rho(t) \\
&= {}_HI^\alpha h_\rho(t) + \sum_{k=1}^m g_k\bigl(t,\mu_\rho(t)\bigr) + \frac{(\ln t)^{\alpha-1}}{\overline{\Upsilon}}\Biggl[\sum_{j=1}^r \omega_{jH}I^{\alpha+q}h_\rho(\xi_j) - {}_HI^{\alpha+p}h_\rho\bigl(e^2\bigr) \\
&\quad + \sum_{j=1}^r \omega_{jH}I^q\Biggl(\sum_{k=1}^m g_k\bigl(\theta,\mu_\rho(\theta)\bigr)\Biggr)\Bigg|_{\theta=\xi_j} - {}_HI^p\Biggl(\sum_{k=1}^m g_k\bigl(\theta,\mu_\rho(\theta)\bigr)\Biggr)\Bigg|_{\theta=e^2}\Biggr], \quad \rho = 1,2.
\end{aligned}$$

Let $0 \leq \beta \leq 1$. Then, for each $t \in [1,e^2]$, we have

$$\begin{aligned}
&\bigl[\beta\mu_1 + (1-\beta)\mu_2\bigr](t) \\
&= {}_HI^\alpha\bigl[\beta h_1 + (1-\beta)h_2\bigr](t) + \sum_{k=1}^m\bigl[\beta g_k\bigl(t,\mu_1(t)\bigr) + (1-\beta)g_k\bigl(t,\mu_2(t)\bigr)\bigr] \\
&\quad + \frac{(\ln t)^{\alpha-1}}{\overline{\Upsilon}}\cdot\Biggl[\sum_{j=1}^r \omega_{jH}I^{\alpha+q}\bigl(\bigl[\beta h_1(\theta) + (1-\beta)h_2(\theta)\bigr]\bigr)\Big|_{\theta=\xi_j} \\
&\quad - {}_HI^{\alpha+p}\bigl(\bigl[\beta h_1(\theta) + (1-\beta)h_2(\theta)\bigr]\bigr)\Big|_{\theta=e^2} \\
&\quad + \sum_{j=1}^r \omega_{jH}I^q\Biggl(\sum_{k=1}^m\bigl[\beta g_k\bigl(\theta,\mu_1(\theta)\bigr) + (1-\beta)g_k\bigl(\theta,\mu_2(\theta)\bigr)\bigr]\Biggr)\Bigg|_{\theta=\xi_j}
\end{aligned}$$

$$- {}_HI^p\left(\sum_{k=1}^{m}\left[\beta g_k\big(\theta,\mu_1(\theta)\big) + (1-\beta)g_k\big(\theta,\mu_2(\theta)\big)\right]\right)\bigg|_{\theta=e^2}\Bigg].$$

Because $\mathfrak{G}$ has convex values, $S_{\mathfrak{G},u}$ is convex, and $\beta\mu_1 + (1-\beta)\mu_2 \in \mathcal{T}(u)$. So $\mathcal{T}(u)$ is convex.

In the fifth step, we prove that the operator $\mathcal{T}$ is upper semicontinuous. By Definition 4.3 a completely continuous operator with a closed graph is upper semicontinuous. From the first and second steps we have that the operator $\mathcal{T}$ is completely continuous, so it is only necessary to prove that $\mathcal{T}$ has a closed graph.

Let $u_n \to u_*$, $\mu_n \to \mu_*$, and $\mu_n \in \mathcal{T}(u_n)$. Since $\mu_n \in \mathcal{T}(u_n)$, there exists $h_n \in S_{\mathfrak{G},u_n}$ such that for each $t \in [1,e^2]$,

$$\begin{aligned}
&\mu_n(t)\\
&\quad= {}_HI^\alpha h_n(t) + \sum_{k=1}^{m} g_k\big(t,\mu_n(t)\big) + \frac{(\ln t)^{\alpha-1}}{\overline{\Upsilon}}\Bigg[\sum_{j=1}^{r}\omega_{jH}I^{\alpha+q}h_n(\xi_j) - {}_HI^{\alpha+p}h_n\big(e^2\big)\\
&\qquad+ \sum_{j=1}^{r}\omega_{jH}I^q\left(\sum_{k=1}^{m} g_k\big(\theta,\mu_n(\theta)\big)\right)\bigg|_{\theta=\xi_j} - {}_HI^p\left(\sum_{k=1}^{m} g_k\big(\theta,\mu_n(\theta)\big)\right)\bigg|_{\theta=e^2}\Bigg].
\end{aligned}$$

We show that there is $h_* \in S_{\mathfrak{G},u_*}$ such that for each $t \in [1,e^2]$,

$$\begin{aligned}
&\mu_*(t)\\
&\quad= {}_HI^\alpha h_*(t) + \sum_{k=1}^{m} g_k\big(t,\mu_*(t)\big) + \frac{(\ln t)^{\alpha-1}}{\overline{\Upsilon}}\Bigg[\sum_{j=1}^{r}\omega_{jH}I^{\alpha+q}h_*(\xi_j) - {}_HI^{\alpha+p}h_*\big(e^2\big)\\
&\qquad+ \sum_{j=1}^{r}\omega_{jH}I^q\left(\sum_{k=1}^{m} g_k\big(\theta,\mu_*(\theta)\big)\right)\bigg|_{\theta=\xi_j} - {}_HI^p\left(\sum_{k=1}^{m} g_k\big(\theta,\mu_*(\theta)\big)\right)\bigg|_{\theta=e^2}\Bigg].
\end{aligned}$$

Consider the linear operator $\mathcal{B}: L^1([1,e^2],\mathbb{R}) \to C([1,e^2],\mathbb{R})$ defined by

$$\begin{aligned}
&h \mapsto \mathcal{B}(h)(t)\\
&\quad= {}_HI^\alpha h(t) + \sum_{k=1}^{m} g_k\big(t,h(t)\big) + \frac{(\ln t)^{\alpha-1}}{\overline{\Upsilon}}\Bigg[\sum_{j=1}^{r}\omega_{jH}I^{\alpha+q}h(\xi_j) - {}_HI^{\alpha+p}h\big(e^2\big)\\
&\qquad+ \sum_{j=1}^{r}\omega_{jH}I^q\left(\sum_{k=1}^{m} g_k\big(\theta,h(\theta)\big)\right)\bigg|_{\theta=\xi_j} - {}_HI^p\left(\sum_{k=1}^{m} g_k\big(\theta,h(\theta)\big)\right)\bigg|_{\theta=e^2}\Bigg].
\end{aligned}$$

Obviously, $\|\mu_n - \mu_*\| \to 0$ as $n \to \infty$.

Therefore, by Lemma 4.5, $\mathcal{B} \circ S_{\mathfrak{G},u}$ is a closed graph operator, and $\mu_n(t) \in \mathcal{B}(S_{\mathfrak{G},u_n})$. Since $u_n \to u_*$, there exists $h_* \in S_{\mathfrak{G},u_*}$ such that

$$\begin{aligned}
&\mu_*(t)\\
&\quad= {}_HI^\alpha h_*(t) + \sum_{k=1}^{m} g_k\big(t,\mu_*(t)\big) + \frac{(\ln t)^{\alpha-1}}{\overline{\Upsilon}}\Bigg[\sum_{j=1}^{r}\omega_{jH}I^{\alpha+q}h_*(\xi_j) - {}_HI^{\alpha+p}h_*\big(e^2\big)
\end{aligned}$$

$$+\sum_{j=1}^{r}\omega_{jH}I^{q}\left(\sum_{k=1}^{m}g_k\big(\theta,\mu_*(\theta)\big)\right)\bigg|_{\theta=\xi_j}-{}_HI^{p}\left(\sum_{k=1}^{m}g_k\big(\theta,\mu_*(\theta)\big)\right)\bigg|_{\theta=e^2}\Bigg].$$

From the above proof we can see that $\mu_*\in\mathcal{T}(u_*)$, that is, $\mathcal{T}$ has a closed graph, so $\mathcal{T}$ is upper semicontinuous.

To sum up, we obtain that the operator $\mathcal{T}$ is an upper semicontinuous compact multivalued map with convex closed values. It is known from Lemma 4.6 that the operator $\mathcal{T}$ has a fixed point, which is a solution of the boundary value problem (1.2). This completes the proof. □

Theorem 4.10 *On the basis of* (O_2), *we assume that*:

(O_3) $\mathfrak{G}:[1,e^2]\times\mathbb{R}\to\mathcal{P}_{b,c,cp}(\mathbb{R})$ *is* L^1*-Carathéodory*;

(O_4) *there exists function* $\phi(t)$ *such that for each* $u\in\mathbb{R}$, $\|\mathfrak{G}(t,u)\|\le\phi(t)$ *for a.e.* $t\in[1,e^2]$.

Then the boundary value problem (1.2) *has a solution on* $[1,e^2]$.

Proof Considering the operator $\mathcal{T}$ defined in Theorem 4.9, from the proof of the latter we obtain that the operator $\mathcal{T}$ is a convex completely continuous multivalued mapping. So we only need to prove that the set $\Omega=\{u\in U:\varrho u\in\mathcal{T}(u),\varrho>1\}$ is bounded.

Let $u\in\Omega$. Then $\varrho u\in\mathcal{T}(u)$, $\varrho>1$, and there exists a function $h\in S_{\mathfrak{G},u}$ such that

$$\begin{aligned}
&\varrho u(t)\\
&\quad={}_HI^{\alpha}h(t)+\sum_{k=1}^{m}g_k\big(t,\varrho u(t)\big)+\frac{(\ln t)^{\alpha-1}}{\overline{\Upsilon}}\Bigg[\sum_{j=1}^{r}\omega_{jH}I^{\alpha+q}h(\xi_j)-{}_HI^{\alpha+p}h\big(e^2\big)\\
&\qquad+\sum_{j=1}^{r}\omega_{jH}I^{q}\left(\sum_{k=1}^{m}g_k\big(\theta,\varrho u(\theta)\big)\right)\bigg|_{\theta=\xi_j}-{}_HI^{p}\left(\sum_{k=1}^{m}g_k\big(\theta,\varrho u(\theta)\big)\right)\bigg|_{\theta=e^2}\Bigg].
\end{aligned}$$

Because $\varrho>1$, we have

$$\begin{aligned}
\big|u(t)\big|&\le\big|\varrho u(t)\big|\\
&\le{}_HI^{\alpha}\big|h(t)\big|+\sum_{k=1}^{m}\big|g_k\big(t,\varrho u(t)\big)\big|+\left|\frac{(\ln t)^{\alpha-1}}{\overline{\Upsilon}}\right|\\
&\quad\cdot\Bigg[\sum_{j=1}^{r}|\omega_j|_HI^{\alpha+q}\big|h(\xi_j)\big|+{}_HI^{\alpha+p}\big|h\big(e^2\big)\big|\\
&\quad+\sum_{j=1}^{r}|\omega_j|_HI^{q}\left(\sum_{k=1}^{m}\big|g_k\big(\theta,\varrho u(\theta)\big)\big|\right)\bigg|_{\theta=\xi_j}+{}_HI^{p}\left(\sum_{k=1}^{m}\big|g_k\big(\theta,\varrho u(\theta)\big)\big|\right)\bigg|_{\theta=e^2}\Bigg]\\
&\le\left(\frac{2^{\alpha}}{\Gamma(\alpha+1)}+\frac{2^{\alpha-1}}{|\overline{\Upsilon}|}\left(\sum_{j=1}^{r}\frac{|\omega_j|}{\Gamma(\alpha+q+1)}(\ln\xi_j)^{\alpha+q}+\frac{2^{\alpha+p}}{\Gamma(\alpha+p+1)}\right)\right)\cdot\|\phi\|\\
&\quad+\left(m+\frac{2^{\alpha-1}}{|\overline{\Upsilon}|}\left(\sum_{j=1}^{r}\frac{m\cdot|\omega_j|}{\Gamma(q+1)}(\ln\xi_j)^{q}+\frac{m\cdot 2^{p}}{\Gamma(p+1)}\right)\right)\cdot G\\
&=\overline{\Delta}_1\|\phi\|+\overline{\Delta}_2G,
\end{aligned}$$

that is,

$$\|u\| \leq \overline{\Delta}_1 \|\phi\| + \overline{\Delta}_2 G < \infty.$$

Therefore the set $\Omega = \{u \in U : \varrho u \in \mathcal{T}(u), \varrho > 1\}$ is bounded. By Lemma 4.7 the operator $\mathcal{T}$ has at least one fixed point, so the boundary value problem (1.2) has at least one solution on $[1, e^2]$. This completes the proof. □

Theorem 4.11 *On the basis of* (O_1) *and* (O_2), *we assume that*:

(O_5) *there exist a continuous nondecreasing function* $\varphi : [0,\infty) \to (0,\infty)$ *and a function* $\tau \in C([1,e^2],\mathbb{R}^+)$ *such that for each* $(t,u) \in [1,e^2] \times \mathbb{R}$, $\|\mathfrak{G}(t,u)\| = \sup\{|\omega| : \omega \in \mathfrak{G}(t,u(t))\} \leq \tau(t)\varphi(\|u\|)$;

(O_6) *there exists a constant* $K > 0$ *satisfying* $\frac{K}{\overline{\Delta}_1\|\tau\|\varphi(K)+\overline{\Delta}_2 G} > 1$.

Then the boundary value problem (1.2) *has at least one solution on* $[1, e^2]$.

Proof Considering the operator $\mathcal{T}$ defined in Theorem 4.9, from the proof of the latter we just need to prove that there exists an open set $H \subseteq C([1,e^2],\mathbb{R})$ such that for all $\lambda \in (0,1)$ and $u \in \partial H$, $u \notin \lambda\mathcal{T}(u)$.

Let $u \in \lambda\mathcal{T}(u)$ and $\lambda \in (0,1)$. Then there exists $h \in L^1([1,e^2],\mathbb{R})$, $h \in S_{\mathfrak{G},u}$, such that for $t \in [1,e^2]$,

$$\begin{aligned} u(t) &= \lambda\big({}_HI^{\alpha}h(t)\big) + \lambda\sum_{k=1}^{m} g_k\big(t,u(t)\big) + \frac{\lambda(\ln t)^{\alpha-1}}{\overline{\Upsilon}}\Bigg[\sum_{j=1}^{r}\omega_{j\,H}I^{\alpha+q}h(\xi_j) - {}_HI^{\alpha+p}h\big(e^2\big) \\ &\quad + \sum_{j=1}^{r}\omega_{j\,H}I^{q}\Bigg(\sum_{k=1}^{m} g_k\big(\theta,u(\theta)\big)\Bigg)\Bigg|_{\theta=\xi_j} - {}_HI^{p}\Bigg(\sum_{k=1}^{m} g_k\big(\theta,u(\theta)\big)\Bigg)\Bigg|_{\theta=e^2}\Bigg]. \end{aligned}$$

Since $\lambda \in (0,1)$, for all $t \in [1,e^2]$, we have

$$\begin{aligned} \big|u(t)\big| &\leq {}_HI^{\alpha}\big|h(t)\big| + \sum_{k=1}^{m}\big|g_k\big(t,u(t)\big)\big| + \left|\frac{(\ln t)^{\alpha-1}}{\overline{\Upsilon}}\right| \\ &\quad \cdot \Bigg[\sum_{j=1}^{r}|\omega_j|_HI^{\alpha+q}\big|h(\xi_j)\big| + {}_HI^{\alpha+p}\big|h\big(e^2\big)\big| \\ &\quad + \sum_{j=1}^{r}|\omega_j|_HI^{q}\Bigg(\sum_{k=1}^{m}\big|g_k\big(\theta,u(\theta)\big)\big|\Bigg)\Bigg|_{\theta=\xi_j} + {}_HI^{p}\Bigg(\sum_{k=1}^{m}\big|g_k\big(\theta,u(\theta)\big)\big|\Bigg)\Bigg|_{\theta=e^2}\Bigg] \\ &\leq \frac{2^{\alpha}}{\Gamma(\alpha+1)}\|\tau\|\varphi\big(\|u\|\big) + \frac{2^{\alpha-1}}{|\overline{\Upsilon}|}\cdot\Bigg\{\|\tau\|\varphi\big(\|u\|\big)\cdot\Bigg(\sum_{j=1}^{r}\frac{|\omega_j|(\ln\xi_j)^{\alpha+q}}{\Gamma(\alpha+q+1)} + \frac{2^{\alpha+p}}{\Gamma(\alpha+p+1)}\Bigg) \\ &\quad + G\cdot\Bigg(\sum_{j=1}^{r}\frac{m\cdot|\omega_j|}{\Gamma(q+1)}(\ln\xi_j)^{q} + \frac{m\cdot 2^{p}}{\Gamma(p+1)}\Bigg)\Bigg\} + mG \\ &= \overline{\Delta}_1\|\tau\|\varphi\big(\|u\|\big) + \overline{\Delta}_2 G. \end{aligned}$$

Therefore

$$\|u\| \leq \overline{\Delta}_1 \|\tau\| \varphi\big(\|u\|\big) + \overline{\Delta}_2 G,$$

that is,

$$\frac{\|u\|}{\overline{\Delta}_1 \|\tau\| \varphi(\|u\|) + \overline{\Delta}_2 G} \leq 1.$$

By assuming (O_6) there exists K such that $\|u\| \neq K$. Let $H = \{u \in C(I, \mathbb{R}) : \|u\| < K\}$, and let $\mathcal{T} : \overline{H} \to \mathcal{P}(C([1, e^2], \mathbb{R}))$ be a compact multivalued mapping with convex closed values. By the choice of H, for $\lambda \in (0,1)$, there is no $u \in \partial H$ such that $u \in \lambda \mathcal{T}(u)$. Therefore, by Lemma 4.8, $\mathcal{T}$ has a fixed point $u \in \overline{H}$, which is also a solution of problem (1.2). This completes the proof. □

Example 4.12 Consider

$$\begin{cases} ({}_H D^{\frac{9}{2}} Z)(t) \in \mathfrak{G}(t, u(t)), \quad t \in [1, e^2], \\ Z(1) = Z'(1) = \cdots = Z^{(n-2)}(1) = 0, \\ {}_H I^{\frac{8}{3}} u(e^2) = \frac{3}{2} {}_H I^{\frac{7}{3}} u(\frac{4}{3}) + \frac{6}{7} {}_H I^{\frac{7}{3}} u(\frac{9}{4}) + \frac{5}{8} {}_H I^{\frac{7}{3}} u(\frac{11}{2}), \end{cases} \tag{4.5}$$

where

$$\begin{aligned} Z(t) &= u(t) - \big(g_1\big(t, u(t)\big) + g_2\big(t, u(t)\big)\big) \\ &= u(t) - \left[\big(2t \cos u(t) + 1\big) + \left(2e^{-(1-t)^2} \frac{|u(t)|}{1 + |u(t)|}\right)\right], \end{aligned}$$

$n = 5$, $\alpha = \frac{9}{2}$, $p = \frac{8}{3}$, $q = \frac{7}{3}$, $m = 2$, $r = 3$, $\omega_1 = \frac{3}{2}$, $\omega_2 = \frac{6}{7}$, $\omega_3 = \frac{5}{8}$, $\xi_1 = \frac{4}{3}$, $\xi_2 = \frac{9}{4}$, $\xi_3 = \frac{11}{2}$.

Define

$$\mathfrak{G}(t, u) = \left[\frac{\cos t}{(3+t)^2}\left(|u| + \frac{1}{2}\right), \frac{e^{1-t}}{150}\left(\frac{|u|^3}{1 + |u|^2} + \frac{1}{4}\right)\right].$$

So for all $u \in \mathbb{R}$ with $\|u\| \leq z$ and for a.e. $t \in [1, e^2]$, we obviously obtain

$$\begin{aligned} \big\|\mathfrak{G}(t, u)\big\| &\leq \frac{1}{(3+t)^2} \cdot \left(z + \frac{1}{2}\right), \\ \big\|g_1(t, u)\big\| &= \sup_{t \in [1, e^2]} \{2t \cos u + 1\} \leq 2e^2 + 1, \\ \big\|g_2(t, u)\big\| &= \sup_{t \in [1, e^2]} \left\{2e^{-(1-t)^2} \frac{|u|}{1 + |u|}\right\} \leq 2. \end{aligned}$$

Choose $\eta_z(t) = \frac{1}{(3+t)^2} \cdot (z + \frac{1}{2})$ and $G = 2e^2 + 1$. Then all conditions of Theorem 4.9 are satisfied. Therefore there is at least one solution of the boundary value problem (4.5) on $[1, e^2]$.

5 Conclusion

Using different types of fixed point theorems described in Sect. 2, we have established the existence and uniqueness of solutions of Hadamard fractional neutral differential equations with multipoint boundary value conditions and considered some suitable conditions for the system to be Ulam–Hyers stable and generalized Ulam–Hyers stable (see Theorem 3.10 and Remark 3.11, respectively). In addition, in Sect. 4, using the Bohnenblust–Karlin fixed point theorem, the Martelli fixed point theorem, and the nonlinear alternative for Kakutani maps, we have obtained the corresponding conditions for the existence of solutions to fractional differential inclusion problems when multivalued mappings have convex values. We also give some examples to show the applicability of the results. The mentioned existence of solutions is rarely investigated for Hadamard fractional differential inclusions and is very important.

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