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An overdetermined problem of the biharmonic operator on Riemannian manifolds

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Abstract

Let (M,g) be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature. In this paper, we consider an overdetermined problem of the biharmonic operator on a bounded smooth domain Ω in M. We deduce that the overdetermined problem has a solution only if Ω is isometric to a ball in $\mathbb{R}^n$. Our method is based on using a P-function and the maximum principle argument. This result is a generalization of the overdetermined problem for the biharmonic equation in Euclidean space.

MSC: 31B30; 35J40; 35N25

Keywords: Biharmonic operator; Overdetermined problem; Maximum principle

1 Introduction and main results

Let (M,g) be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature. In this paper, we consider an overdetermined boundary value problem for the biharmonic operator. The overdetermined problem of partial differential equations usually contains too many prescribed boundary conditions. In general, these types of problems are not well posed, and the existence of solutions imposes strong constraints on the shape of the domain.

When $M = \mathbb{R}^n$, we know that the Ricci curvature is equal to zero. Let Ω be a bounded domain in $\mathbb{R}^n$ having a smooth boundary $\partial\Omega$. It is well known that there exists a solution of Poisson's equation with the Dirichlet condition or the Neumann condition. However, both the Dirichlet condition and the Neumann condition cannot be imposed simultaneously. In general, the resulting problem does not admit a solution unless the domain Ω has a special shape. Indeed, in a celebrated paper [14], Serrin proved that there exists a solution of the following overdetermined problem:

$$\begin{cases} \Delta u = -1 & \text{in } \Omega, \\ u = 0, \quad u_\nu = c & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

if and only if Ω is a ball and $u = \frac{n^2c^2-r^2}{2n}$ is a radial function, where ν denotes the unit outer normal of $\partial\Omega$, c is a constant, and r is the distance from the center of the ball. The main tool of Serrin's proof is a technique known as the moving-planes method. Later, Weinberger [15] used some integral identities and some basic inequalities to give a very simple proof. This result has an associated physical model: the associated Dirichlet problem describes a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of given cross-sectional form (see [13] for more detail). The literature about overdetermined problems of second-order elliptic equations is so wide that it is impossible to report it exhaustively.

Some authors have extended Serrin's result to higher-order elliptic equations. In [9], Payne and Schaefer considered a fourth-order overdetermied problem, that is

$$\begin{cases} \Delta^2 u = 1 & \text{in } \Omega, \\ u = \Delta u = 0, \quad u_\nu = c & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where ν and c denote a unit outer normal vector and a constant, respectively. They obtained the following result.

Theorem 1.1 *If u satisfies (1.2) for $\Omega \subset \mathbb{R}^2$, where Ω is star-shaped with respect to the origin and $\partial\Omega \in C^{2+\varepsilon}$, then Ω is a disk.*

Remark 1.1 They conjectured that Theorem 1.1 holds for $n > 2$ and the more general domain.

Later, Philippin and Ragoub [10] proved the above conjecture based on Serrin's reflection method. Recently, Goyal and Schaefer [5] gave a new proof of $n \geq 2$ by using the P-function and the maximum principle argument.

In 1986, Benett [2] considered another fourth-order overdetermined problem

$$\begin{cases} \Delta^2 u = 1, & \text{in } \Omega, \\ u = u_\nu = 0, \quad \Delta u = c & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Bennett modified Weinberger's method of the maximum principle to show that Ω must be an open ball. Later, Dalmasso [3] used Serrin's method of moving planes to show that Ω in (1.3) is a ball. In [1], Barkatou used the method of the shape derivative to provide another proof. An alternative method for determining the shape of the domain in fourth-order overdetermined problems was introduced in [8]. The technique in [8] did not use a maximum principle argument but depended on the establishment of an integral identity in the proof of the overdetermined problem. This method was applicable to some high-order elliptic overdetermined problems as well as to some problems involving the Green's function for the Laplacian and biharmonic operators.

In [9], Payne and Schaefer also studied the following fourth-order overdetermined problem

$$\begin{cases} \Delta^2 u = 1, & \text{in } \Omega, \\ u = \Delta u = 0, \quad u_\nu = cx \cdot \nu & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

They obtained that if the problem (1.4) has a classical solution, then Ω is a ball. In [4], Fromm and McDonald studied the boundary condition $u_\nu = c$ in (1.2), which is replaced by the other boundary condition of $u_\nu(\Delta u)_\nu = -c$, that is

$$\begin{cases} \Delta^2 u = 1, & \text{in } \Omega, \\ u = \Delta u = 0, \qquad u_\nu(\Delta u)_\nu = -c & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

They used the method of moving planes to obtain the same conclusion.

In this paper, we consider the overdetermined problem (1.2) on Riemannian manifolds. We obtain the following theorem:

Theorem 1.2 *Let (M, g) be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature and $\Omega \subset M$ be a smooth bounded domain. Suppose the mean curvature H of the boundary $\partial\Omega$ is positive. Then, the overdetermined problem (1.2) has a solution only if Ω is isometric to a ball in $\mathbb{R}^n$.*

There are results involving overdetermined boundary value problems in the context of constant-curvature space forms (see [6, 7] more detail).

In this paper, we use Weinberger's argument to solve the Riemannian analog of the overdetermined problem (1.2). Our method of the proof was mainly motivated by Goyal and Schaefer's [5] nice result.

Remark 1.2 It is worth noting that we believe that one can use Weinberger's argument to study the overdetermined problems (1.3), (1.4), and (1.5) on Riemannian manifolds. However, we cannot find some suitable P-functions to these fourth-order overdetermined problems.

Finally, we recall a well-known result of Reilly (see Lemma 3 of [12]), which is important in our proof. Our rigidity theorems are based on this result.

Lemma 1.1 ([12]) *Let (M, g) be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature and $\Omega \subset M$ be a smooth bounded domain. Suppose that Ω admits a function $f : \Omega \to \mathbb{R}$ and nonzero constant L such that*

$$\begin{cases} \nabla^2 f = L \cdot g, \\ f|_{\partial\Omega} \text{ is constant,} \end{cases}$$

where $\nabla^2 f$ denotes the Hessian matrix of f and g denotes the Riemannian metric on M. Then, Ω is isometric to a ball in $\mathbb{R}^n$.

2 Proof of Theorem 1.2

The fourth-order overdetermined problem (1.2) is equivalent to the following second-order overdetermined problem:

$$\begin{cases} \Delta u = -\varphi & \text{in } \overline{\Omega}, \\ u = 0, \qquad u_\nu = c & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where φ is the torsion function satisfying that

$$\begin{cases} \Delta\varphi = -1 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Using the maximum principle, we know that $\varphi \geq 0$.

Lemma 2.1 *If u satisfies (1.2), then $c = -\frac{1}{|\partial\Omega|}\int_\Omega \varphi\,dx$, where $|\partial\Omega|$ denotes the $(n-1)$-dimensional measure of $\partial\Omega$.*

Proof Using the formula of integration by parts, we have that

$$\int_\Omega \varphi\,dx = -\int_\Omega \Delta u\,dx = -\int_{\partial\Omega} u_\nu\,ds = -c\int_{\partial\Omega} ds = -c|\partial\Omega|. \tag{2.3}$$

This completes the proof of this lemma. □

We now consider the function ϕ defined by

$$\phi = \frac{n+2}{2n}u - \frac{\varphi^2}{4}. \tag{2.4}$$

A straightforward computation shows that

$$\Delta\phi = \frac{n+2}{2n}\Delta u - \frac{1}{2}|\nabla\varphi|^2 - \frac{1}{2}\varphi\Delta\varphi, \tag{2.5}$$

and

$$\phi_\nu = \frac{n+2}{2n}u_\nu - \frac{1}{2}\varphi\varphi_\nu. \tag{2.6}$$

From (2.4), (2.5), and (2.6), we note that ϕ satisfies the overdetermined problem

$$\begin{cases} \Delta\phi = -\frac{1}{2}|\nabla\varphi|^2 - \frac{\varphi}{n} & \text{in } \overline{\Omega}, \\ \phi = 0, \qquad \phi_\nu = \frac{n+2}{2n}c & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

By the maximum principle, we know that $\phi \geq 0$.

We define an auxiliary function P by

$$P = |\nabla\phi|^2 - \phi\Delta\phi - \frac{n-2}{n}m^2\varphi, \tag{2.8}$$

where $m^2 = \max_{\overline{\Omega}}(\Delta\phi)^2$. We obtain the following lemma.

Lemma 2.2 *If u satisfies (1.2), then*

$$\Delta P = 2\left[\|\nabla^2\phi\|^2 - \frac{1}{n}(\Delta\phi)^2\right] + \frac{n-2}{n}\left[m^2 - (\Delta\phi)^2\right] + \phi\left[\|\nabla^2\varphi\|^2 - \frac{1}{n}(\Delta\varphi)^2\right]$$
$$+ 2\,\mathrm{Ric}(\nabla\phi, \nabla\phi) + \phi\,\mathrm{Ric}(\nabla\varphi, \nabla\varphi),$$

where $\nabla^2\varphi$ *denotes the Hessian matrix of* φ *and* $\|\nabla^2\varphi\|$ *denotes its Hilbert–Schmidt norm defined to be*

$$\|\nabla^2\varphi\| = \left(\sum_{i,j} |\varphi_{ij}|^2\right)^{\frac{1}{2}}.$$

Proof From the Bochner formula, we have that

$$\frac{1}{2}\Delta|\nabla\varphi|^2 = \|\nabla^2\varphi\|^2 + \nabla\varphi\cdot\nabla\Delta\varphi + \mathrm{Ric}(\nabla\varphi,\nabla\varphi). \tag{2.9}$$

Then, we compute the Laplacian of $\Delta\phi$ as follows

$$\begin{aligned}\Delta^2\phi &= \frac{n+2}{2n}\Delta^2 u - \frac{1}{2}\Delta|\nabla\varphi|^2 - \frac{1}{2}\Delta(\varphi\Delta\varphi) \\ &= \frac{n+2}{2n}\Delta^2 u - \|\nabla^2\varphi\|^2 - \nabla\varphi\cdot\nabla\Delta\varphi - \mathrm{Ric}(\nabla\varphi,\nabla\varphi) \\ &\quad - \frac{1}{2}(\Delta\varphi)^2 - \nabla\varphi\cdot\nabla\Delta\varphi - \frac{1}{2}\varphi\Delta^2\varphi \\ &= \frac{1}{n} - \|\nabla^2\varphi\|^2 - \mathrm{Ric}(\nabla\varphi,\nabla\varphi).\end{aligned} \tag{2.10}$$

Thus, the Laplacian of P is

$$\begin{aligned}\Delta P &= \Delta|\nabla\phi|^2 - \Delta(\phi\Delta\phi) - \frac{n-2}{n}m^2\Delta\varphi \\ &= 2\|\nabla^2\phi\|^2 + 2\nabla\phi\cdot\nabla\Delta\phi + 2\,\mathrm{Ric}(\nabla\phi,\nabla\phi) - (\Delta\phi)^2 - 2\nabla\phi\cdot\nabla\Delta\phi - \phi\Delta^2\phi \\ &\quad - \frac{n-2}{n}m^2\Delta\varphi \\ &= 2\|\nabla^2\phi\|^2 + 2\,\mathrm{Ric}(\nabla\phi,\nabla\phi) - (\Delta\phi)^2 - \phi\Delta^2\phi + \frac{n-2}{n}m^2.\end{aligned} \tag{2.11}$$

Substituting (2.10) into (2.11), we obtain that

$$\begin{aligned}\Delta P &= 2\|\nabla^2\phi\|^2 - (\Delta\phi)^2 - \phi\left[\frac{1}{n}(\Delta\varphi)^2 - \|\nabla^2\varphi\|^2\right] + \frac{n-2}{n}m^2 \\ &\quad + 2\,\mathrm{Ric}(\nabla\phi,\nabla\phi) + \phi\,\mathrm{Ric}(\nabla\varphi,\nabla\varphi) \\ &= 2\left[\|\nabla^2\phi\|^2 - \frac{1}{n}(\Delta\phi)^2\right] + \frac{n-2}{n}\left[m^2 - (\Delta\phi)^2\right] + \phi\left[\|\nabla^2\varphi\|^2 - \frac{1}{n}(\Delta\varphi)^2\right] \\ &\quad + 2\,\mathrm{Ric}(\nabla\phi,\nabla\phi) + \phi\,\mathrm{Ric}(\nabla\varphi,\nabla\varphi).\end{aligned} \tag{2.12}$$

These complete the proof of this lemma. □

Now, we are ready to prove Theorem 1.2.

Proof Since the Ricci curvature is assumed to be nonnegative, then by the Cauchy–Schwarz inequality

$$\|\nabla^2\varphi\|^2 \geq \frac{1}{n}(\Delta\varphi)^2, \tag{2.13}$$

and Lemma 2.2, we obtain that P is subharmonic.

From the following boundary conditions

$$\phi = 0, \qquad \nabla\phi = \frac{n+2}{2n}\nabla u, \qquad \Delta\phi = -\frac{1}{2}|\nabla\varphi|^2 \quad \text{and} \quad |\nabla u|^2 = c^2 \quad \text{on } \partial\Omega,$$

it follows that P satisfies

$$P = \left(\frac{n+2}{2n}c\right)^2 \quad \text{on } \partial\Omega. \tag{2.14}$$

By the maximum principle, we have that

$$P \le \left(\frac{n+2}{2n}c\right)^2 \quad \text{in } \Omega. \tag{2.15}$$

By the Hopf Lemma [11], we know that either

$$(i)\ P = \left(\frac{n+2}{2n}c\right)^2 \quad \text{in } \Omega \quad \text{or} \quad (ii)\ P_\nu > 0 \quad \text{on } \partial\Omega. \tag{2.16}$$

Suppose case (i) holds, then the Laplacian of P vanishes, that is, all the equalities of the above inequalities hold. This implies that the Ricci curvature is equal to zero and $\nabla^2\varphi$ is proportional to the metric everywhere. As $\Delta\varphi = -1$, we conclude that

$$\nabla^2\varphi = \frac{1}{n}\Delta\varphi \cdot g = -\frac{1}{n}g. \tag{2.17}$$

From $\varphi = 0$ on $\partial\Omega$, and Lemma 1.1, then Ω is isometric to a ball in $\mathbb{R}^n$.

Suppose case (ii) holds. As $\phi|_{\partial\Omega} = 0$, we can choose the external unit normal on $\partial\Omega$ to be the following vector field

$$\nu = -\frac{\nabla\phi}{|\nabla\phi|}. \tag{2.18}$$

The mean curvature of the regular level sets of ϕ is given by $H = -\frac{1}{n-1}\operatorname{div}\frac{\nabla\phi}{|\nabla\phi|}$. The Laplacian of ϕ can be expressed in terms of H as follows

$$\Delta\phi = \phi_{\nu\nu} + (n-1)H\phi_\nu \quad \text{on } \partial\Omega, \tag{2.19}$$

where $\phi_{\nu\nu} = \nabla^2\phi(\nu,\nu)$. We deduce that

$$P_\nu = 2\phi_\nu\phi_{\nu\nu} + \frac{1}{2}|\nabla\varphi|^2\phi_\nu - \frac{n-2}{n}m^2\varphi_\nu \quad \text{on } \partial\Omega. \tag{2.20}$$

Substituting (2.5) into (2.19), we have that

$$\phi_{\nu\nu} = -\frac{1}{2}|\nabla\varphi|^2 - (n-1)H\phi_\nu \quad \text{on } \partial\Omega. \tag{2.21}$$

Combining (2.20) and (2.21), we obtain that

$$\frac{1}{2}\phi_\nu\varphi_\nu^2 + \left(\frac{n-2}{n}m^2\right)\varphi_\nu + 2H(n-1)\phi_\nu^2 < 0 \quad \text{on } \partial\Omega. \tag{2.22}$$

As a quadratic expression for φ_ν that holds at each point of the boundary, we have that

$$0 \leq \left(\frac{n-2}{n}m^2\right)^2 < 4H(n-1)\phi_\nu^3 = 4H(n-1)\left(\frac{n+2}{n}c\right)^3 \quad \text{on } \partial\Omega. \tag{2.23}$$

It follows from Lemma 2.1 that $c < 0$. Then, we reach a contradiction, which implies that case (ii) cannot hold. These complete the proof of Theorem 1.2. □

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The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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References
1. Barkatou, M.: A symmetry result for a fourth order overdetermined boundary value problem. Appl. Math. E-Notes **8**, 76–81 (2008)
2. Bennett, A.: Symmetry in an overdetermined fourth order elliptic boundary value problem. SIAM J. Math. Anal. **17**, 1354–1358 (1986)
3. Dalmasso, R.: A problem of symmetry for a biharmonic equation. Ann. Fac. Sci. Toulouse Math. **11**, 45–53 (1990)
4. Fromm, S., McDonald, P.: A symmetry problem from probability. Proc. Am. Math. Soc. **125**, 3293–3297 (1997)
5. Goyal, V., Schaefer, P.W.: On a conjecture for an overdetermined problem for the biharmonic operator. Appl. Math. Lett. **21**(4), 421–424 (2008)
6. Kumaresan, S., Prajapat, J.: Serrin's result for hyperbolic space and sphere. Duke Math. J. **91**, 17–28 (1998)
7. McDonald, P.: Isoperimetric conditions, Poisson problems, and diffusions in Riemannian manifolds. Potential Anal. **16**, 115–138 (2002)
8. Payne, L.E., Schaefer, P.W.: Duality theorems in some overdetermined boundary value problems. Math. Methods Appl. Sci. **11**, 805–819 (1989)
9. Payne, L.E., Schaefer, P.W.: On overdetermined boundary value problems for the biharmonic operator. J. Math. Anal. Appl. **187**(2), 598–616 (1994)
10. Philippin, G.A., Ragoub, L.: On some second order and fourth order elliptic overdetermined problems. Z. Angew. Math. Phys. **46**, 188–197 (1995)
11. Protter, M.H., Weinberger, H.F.: Maximum Principles in Differential Equations. Prentice Hall, Englewood Cliffs (1967)
12. Reilly, R.: Geometric applications of the solvability of Neumann problems on a Riemannian manifold. Arch. Ration. Mech. Anal. **75**(1), 23–29 (1980)
13. Ros, A., Sicbaldi, P.: Geometry and topology of some overdetermined elliptic problems. J. Differ. Equ. **255**, 951–977 (2013)
14. Serrin, J.: A symmetry problem in potential theory. Arch. Ration. Mech. Anal. **43**, 304–318 (1971)
15. Weinberger, H.F.: Remark on the preceding paper of Serrin. Arch. Ration. Mech. Anal. **43**, 319–320 (1971)