# RESEARCH

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Existence and multiplicity of solutions for three-point boundary value problems with instantaneous and noninstantaneous impulses

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## Abstract

In this paper, three-point boundary value problems for second-order *p*-Laplacian differential equations with instantaneous and noninstantaneous impulses are studied. The existence of at least one classical solution and infinitely many classical solutions is obtained by using variational methods and critical point theory. In addition, some examples are given to illustrate our main results.

**Keywords:** Three-point BVPs; Variational methods; Critical point theory; Instantaneous impulse; Noninstantaneous impulse

## **1** Introduction

In recent years, the study of differential equations with impulses has received much attention. Impulsive differential equations were used as mathematical models to describe the phenomena of sudden or discontinuous jumps. Depending on the duration of action, the impulses can be divided into instantaneous impulses and noninstantaneous impulses.

The instantaneous impulse was first presented in 1960 by Milman and Myshkis [1]. To date, the differential equations with instantaneous impulses have been studied by many authors. The existence and multiplicity of solutions for impulsive differential equations have been investigated by many different methods [2–8] such as fixed point theorem, topological degree theory, upper and lower solutions method, and variational approach. In [8], Tian and Ge first studied the existence of positive solutions for a second-order impulsive differential equations with Sturm–Liouville boundary conditions by using variational methods.

The noninstantaneous impulse was first introduced by Hernández and O'Regan [9]. The existence results for the differential equations with noninstantaneous impulses have been studied via some approaches [9–14] such as fixed point theory, theory of analytic semigroup, and variational methods. In [10], Bai and Nieto first studied the existence and uniqueness of weak solutions for second-order noninstantaneous impulsive differential equations by means of variational methods.

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However, some dynamical processes involve both instantaneous and noninstantaneous impulses in real life, such as intravenous injection. Therefore, the study of the differential equations with instantaneous and noninstantaneous impulses attracts widespread attention. Especially, in [15], Tian and Zhang first used variational methods to investigate the existence of solutions for second-order differential equations with instantaneous and non-instantaneous impulses. Based on [15], many authors studied different types of the differential equations with instantaneous and noninstantaneous impulses by using variational methods and obtained some excellent results [16–19].

On the other hand, boundary problems with nonlocal conditions arise in various fields of applied mathematics, physics, biology, and biotechnology. Recently, many different approaches, such as upper and lower solutions method, fixed point theory, and variational approach, have been used to investigate the solutions of nonlocal boundary value problems, see, for instance, [20–23]. In [20], Lian, Bai, and Du studied the existence of multiple solutions to the following three-point differential system boundary value problem via variational methods:

$$\begin{cases} (P(t)y'(t))' + f(t,y(t)) = 0, & \text{a.e. } 0 < t < 1, \\ y(0) = 0, & y(1) = \zeta y(\eta), \end{cases}$$
(1.1)

where  $P: [0,1] \to \mathbb{R}^{n \times n}$  is a continuously symmetric matrix,  $f: [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$  is a  $C^1$  function and locally Lipschitz continuous. The interesting point of this paper is that the authors chose an appropriate space instead of functional to contain the boundary conditions. Motivated by the study of [20], Wei and Bai [24] and Wei, Shang, and Bai [25] considered two different types of impulsive differential equations with nonlocal boundary conditions. In [24], Wei and Bai first used variational methods and critical point theory to study a class of nonlinear impulsive differential equations with three-point boundary conditions:

$$\begin{cases} -(J(t)y'(t))' = \nabla H(t, y(t)), & t \neq t_j, \text{ a.e. } t \in [0, 1], \\ -\Delta(J(t_j)y'(t_j)) = I_j(y(t_j)), & j = 1, 2, \dots, m, \\ y(0) = 0, & y(1) = \zeta y(\eta), \end{cases}$$
(1.2)

where  $0 = t_0 < t_1 < \cdots < t_{m_1} < t_{m_1+1} = \eta < t_{m_1+2} < \cdots < t_m < t_{m+1} = 1, \zeta > 0, 0 < \eta < 1$ , and  $\Delta(J(t_j)y'(t_j)) = J(t_j^+)y'(t_j^+) - J(t_j^-)y'(t_j^-)$  for  $y'(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} y'(t), j = 1, 2, \dots, m$ . The authors obtained that BVP (1.2) has at least one classical solution, at least two classical solutions, and infinitely many classical solutions.

In [25], Wei, Shang, and Bai studied a class of three-point boundary value problems with instantaneous and noninstantaneous impulses:

$$\begin{cases} -(\mu(t)\Phi_{p}(y'(t)))' + \lambda(t)\Phi_{p}(y(t)) = f_{j}(t,y(t)), & t \in (s_{j},t_{j+1}], j = 0,1,\ldots,m, \\ -\Delta(\mu(t_{j})\Phi_{p}(y'(t_{j}))) = I_{j}(y(t_{j})), & j = 1,2,\ldots,m, \\ \mu(t)\Phi_{p}(y'(t)) = \mu(t_{j}^{+})\Phi_{p}(y'(t_{j}^{+})), & t \in (t_{j},s_{j}], j = 1,2,\ldots,m, \\ \mu(s_{j}^{+})\Phi_{p}(y'(s_{j}^{-})) = \mu(s_{j}^{-})\Phi_{p}(y'(s_{j}^{-})), & j = 1,2,\ldots,m, \\ y(0) = 0, & y(1) = \zeta y(\eta), \end{cases}$$
(1.3)

where p > 1,  $\Phi_p(x) := |x|^{p-2}x, \mu(t), \lambda(t) \in L^p[0,1], 0 = s_0 < t_1 < s_1 < \cdots < s_{m_1} = \eta < t_{m_1+1} < \cdots < s_m < t_{m+1} = 1, \zeta > 0, 0 < \eta < 1, and <math>\Delta(\mu(t_j)\Phi_p(y'(t_j))) = \mu(t_j^+)\Phi_p(y'(t_j^+)) - \mu(t_j^-)\Phi_p(y'(t_j^-)))$  for  $y'(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} y'(t), j = 1, 2, \dots, m$ , and  $f_j \in C((s_j, t_{j+1}] \times \mathbb{R}, \mathbb{R}), I_j \in C(\mathbb{R}, \mathbb{R})$ . The instantaneous impulses occur at the points  $t_j$ , and the noninstantaneous impulses continue on the intervals  $(t_j, s_j]$ . The authors obtained the following results by applying variational methods and critical point theory.

### **Theorem 1.1** ([25, Theorem 1]) Assume that the following conditions hold:

- (A1)  $1 \le \lambda(t) \le c \text{ and } \mu(t) \ge 1 \text{ for } t \in (s_j, t_{j+1}], \ \mu(t), \lambda(t) \in L^p[0, 1], \ p > 1, \ j = 0, 1, ..., m,$ and c is a positive constant.
- (A2) There are constants  $\alpha_j$ ,  $\beta_j \in \mathbb{R}$  and  $M \ge 0$  such that for  $|x| \ge M$ ,
  - (*i*)  $0 < \alpha_j F_j(t, x) \le x f_j(t, x), t \in (s_j, t_{j+1}], j \in N_1;$
  - (*ii*)  $0 < \beta_j \int_0^{x(t)} I_j(s) ds \le xI_j(x), (t,x) \in [0,1] \times \mathbb{R}, j \in N_2, where <math>F_j(t,x) = \int_0^x f_j(t,s) ds$  for  $(t,x) \in (s_j, t_{j+1}] \times \mathbb{R}, j \in N_1, and 1$
- (A3) For p > 1, there are

$$\lim_{x \to 0} \frac{F_j(t,x)}{|x|^p} = 0, \quad (t,x) \in (s_j, t_{j+1}] \times \mathbb{R}, j \in N_1,$$
$$\lim_{x \to 0} \frac{\int_0^{x(t)} I_j(s) \, ds}{|x|^p} = 0, \quad (t,x) \in [0,1] \times \mathbb{R}, j \in N_2.$$

Then BVP(1.3) has at least two classical solutions.

**Proposition 1.2** ([25, Proposition 1]) Under the assumptions of Theorem 1.1, if  $F_j(t, x)$ , j = 0, 1, ..., m, and  $I_j(x)$ , j = 1, 2, ..., m, are odd functions with respect to x and p is an odd number, then BVP (1.3) has infinitely many classical solutions.

To obtain the multiple solutions for BVP (1.3) in [25], the nonlinearities  $f_j$  and the impulsive functions  $I_j$  are required to satisfy the superlinear growth conditions (*A*2). Motivated by the above fact, in this paper we will revisit BVP (1.3). Under the assumptions that the nonlinearities  $f_j$  and the impulsive functions  $I_j$  satisfy different growth conditions, we obtain the existence and multiplicity of solutions for BVP (1.3) by using variational methods and critical point theory. Our results are different from those above and extend the existing results in [25].

The paper is arranged as follows. Section 2 presents some preliminaries. Section 3 proves our main results via variational methods. Section 4 provides three examples to show our results.

### 2 Preliminaries

In this section, we introduce some important definitions, lemmas, and theorems used throughout this paper.

Let  $Z = \{y \in W^{1,p}([0,1], \mathbb{R}) : y(0) = 0, y(1) = \zeta y(\eta)\}$  with the norm

$$\|y\|_{Z} = \left(\int_{0}^{1} (\mu(t)|y'(t)|^{p} + \lambda(t)|y(t)|^{p}) dt\right)^{\frac{1}{p}}$$

We also consider the norm

$$||y|| = \left(\int_0^1 \mu(t) |y'(t)|^p dt\right)^{\frac{1}{p}}, \quad \forall y \in Z.$$

In fact, we can obtain that ||y|| is equivalent to the norm  $||y||_Z$ . For all  $y \in Z$ , there is  $y(t) = \int_0^t y'(s) ds$ , by Hölder's inequality, we have

$$\int_{0}^{1} |y'(t)| \, dt \leq \left( \int_{0}^{1} |y'(t)|^{p} \, dt \right)^{\frac{1}{p}}.$$

Let  $\left(\frac{c}{\mu(t)}+1\right)^{\frac{1}{p}} \leq c_0$ , we obtain

$$\|y\| \le \|y\|_Z \le c_0 \|y\|.$$

As shown in [26], Z is a separable and reflexive real Banach space. Consider the functional

$$J: Z \to \mathbb{R}$$

defined by

$$J(y) = \frac{1}{p} ||y||^{p} + \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} \lambda(t) \Phi_{p}(y(t)) y(t) dt - \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} F_{j}(t, y(t)) dt - \sum_{j=1}^{m} \int_{0}^{y(t_{j})} I_{j}(s) ds,$$

$$(2.1)$$

where  $F_j(t, y) = \int_0^y f_j(t, s) ds$ . By using the continuity of  $f_j$ , j = 0, 1, ..., m and  $I_j$ , j = 1, 2, ..., m, we can obtain that  $J \in C^1(Z, \mathbb{R})$  and

$$\langle J'(y), w \rangle = \int_0^1 \mu(t) \Phi_p(y'(t)) w'(t) dt - \sum_{j=0}^m \int_{s_j}^{t_{j+1}} (f_j(t, y(t)) - \lambda(t) \Phi_p(y(t))) w(t) dt$$

$$- \sum_{j=1}^m I_j(y(t_j)) w(t_j)$$
(2.2)

for any  $w \in Z$ .

**Lemma 2.1** ([20, Lemma 2.5] and [25, Lemma 1]) *The space Z is compactly embedded in*  $C([0,1],\mathbb{R})$ .

**Lemma 2.2** ([25, Lemma 2]) For each  $y \in Z$ , there is  $||y||_{\infty} \le ||y||$ .

**Lemma 2.3** ([25, Lemma 6]) *The weak solution*  $y \in Z$  *is the classical solution of problem* (1.3).

**Definition 2.4** Let *Z* be a Banach space and  $J : Z \to (-\infty, +\infty]$ . Functional *J* is said to be weakly lower semicontinuous if  $\liminf_{k\to\infty} J(y_k) \ge J(y)$  as  $y_k \rightharpoonup y$  in *Z*.

### **Lemma 2.5** The functional $J : Z \to \mathbb{R}$ is weakly lower semicontinuous.

*Proof* Assume that  $y_k \rightarrow y$  in Z as  $k \rightarrow \infty$ . The continuity and convexity of  $\frac{\|y\|^p}{p}$  imply that  $\frac{\|y\|^p}{p}$  is weakly lower semicontinuous. Furthermore, it follows from Lemma 2.1 that  $\{y_k\}$  is convergent uniformly to y in  $C([0, 1], \mathbb{R})$ . Thus, we have

$$\begin{split} \liminf_{k \to \infty} J(y_k) &= \frac{1}{p} \| y_k \|^p + \sum_{j=0}^m \int_{s_j}^{t_{j+1}} \lambda(t) \left| y_k(t) \right|^p dt - \sum_{j=0}^m \int_{s_j}^{t_{j+1}} F_j(t, y_k(t)) dt \\ &- \sum_{j=1}^m \int_0^{y_k(t_j)} I_j(s) ds \\ &\ge \frac{1}{p} \| y \|^p + \sum_{j=0}^m \int_{s_j}^{t_{j+1}} \lambda(t) \left| y(t) \right|^p dt - \sum_{j=0}^m \int_{s_j}^{t_{j+1}} F_j(t, y(t)) dt \\ &- \sum_{j=1}^m \int_0^{y(t_j)} I_j(s) ds \\ &= J(y). \end{split}$$

Therefore, *J* is weakly lower semicontinuous.

**Definition 2.6** ([27, (PS) condition]) Let *Z* be a real reflexive Banach space. For any sequence  $\{y_k\} \subset Z$ , if  $\{J(y_k)\}$  is bounded and  $J'(y_k) \to 0$  as  $k \to \infty$  possesses a convergent subsequence, then we say that *J* satisfies the Palais–Smale condition.

**Theorem 2.7** ([27]) Let Z be a reflexive Banach space. If  $J : Z \to (-\infty, +\infty]$  is coercive, then J has a bounded minimizing sequence.

**Theorem 2.8** ([27]) Let Z be a reflexive Banach space and let  $J : Z \to (-\infty, +\infty]$  be weakly lower semicontinuous on Z. If J has a bounded minimizing sequence, then J has a minimum on Z.

**Theorem 2.9** ([28, Mountain pass theorem]) Let Z be a real Banach space and suppose that  $J \in C^1(Z, \mathbb{R})$  satisfies the (PS) condition with J(0) = 0. If J satisfies the following conditions:

- (i) There exist constants  $\rho, \alpha > 0$  such that  $J|_{\partial B_{\rho}} \ge \alpha$ ;
- (ii) There exists  $e \in Z \setminus B_{\rho}$  such that  $J(e) \leq 0$ ,

then J possesses a critical value  $c \ge \alpha$ . Moreover, c is given by  $c = \inf_{g \in \Gamma} \max_{s \in [0,1]} J(g(s))$ , where

$$\Gamma = \{g \in C([0,1], Z) | g(0) = 0, g(1) = e\}.$$

**Theorem 2.10** ([28, Symmetric mountain pass theorem]) Let Z be an infinite-dimensional real Banach space. Let  $J \in C^1(Z, \mathbb{R})$  be an even functional which satisfies the (PS) condition, and J(0) = 0. Suppose that  $Z = V \oplus Y$ , where V is finite-dimensional, and J satisfies:

- (i) There exist  $\alpha > 0$  and  $\rho > 0$  such that  $J|_{\partial B_{\rho} \cap Y} \ge \alpha$ ;
- (ii) For each finite-dimensional subspace  $W \subset Z$ , there is R = R(W) such that  $J(y) \le 0$  on  $W \setminus B_{R(W)}$ ,

then J possesses an unbounded sequence of critical values.

## 3 Main results

In this paper, we assume the following condition:

(H1)  $1 \le \lambda(t) \le c$  and  $\mu(t) \ge 1$  for  $t \in (s_j, t_{j+1}]$ ,  $\mu(t), \lambda(t) \in L^p[0, 1]$ , p > 1, j = 0, 1, ..., m, and c is a positive constant.

Our main results are presented as follows.

## **Theorem 3.1** Assume that (H1) and the following conditions hold:

(H2) There exist  $a_j, b_j > 0$  and  $\gamma_j \in [0, p - 1), j = 0, 1, ..., m$ , such that

$$|f_j(t,x)| \leq a_j + b_j |x|^{\gamma_j}$$
 for every  $(t,x) \in (s_j,t_{j+1}] \times \mathbb{R}$ .

(H3) There exist  $c_i, d_i > 0$  and  $\sigma_i \in [0, p - 1), j = 1, 2, ..., m$ , such that

 $|I_j(x)| \leq c_j + d_j |x|^{\sigma_j}$  for every  $x \in \mathbb{R}$ .

Then BVP(1.3) has at least one classical solution.

Proof From (H2), we can get

$$\left|F_{j}(t,y)\right| \le a_{j}|y| + \frac{b_{j}}{\gamma_{j}+1}|y|^{\gamma_{j}+1}.$$
(3.1)

Then by (2.1), (3.1), (H1), (H3), and Lemma 2.2, we have

$$\begin{split} J(y) &= \frac{1}{p} \|y\|^p + \sum_{j=0}^m \int_{s_j}^{t_{j+1}} \lambda(t) \Phi_p(y(t)) y(t) \, dt - \sum_{j=0}^m \int_{s_j}^{t_{j+1}} F_j(t, y(t)) \, dt \\ &- \sum_{j=1}^m \int_0^{y(t_j)} I_j(s) \, ds \\ &\geq \frac{1}{p} \|y\|^p - \sum_{j=0}^m \int_{s_j}^{t_{j+1}} F_j(t, y(t)) \, dt - \sum_{j=1}^m \int_0^{y(t_j)} I_j(s) \, ds \\ &\geq \frac{1}{p} \|y\|^p - \sum_{j=0}^m \int_{s_j}^{t_{j+1}} \left(a_j \|y\|_{\infty} + \frac{b_j}{\gamma_j + 1} \|y\|_{\infty}^{\gamma_j + 1}\right) \, dt - \sum_{j=1}^m \left(c_j \|y\|_{\infty} + \frac{d_j}{\sigma_j + 1} \|y\|_{\infty}^{\sigma_j + 1}\right) \\ &\geq \frac{1}{p} \|y\|^p - \left(\sum_{j=0}^m a_j(t_{j+1} - s_j) + \sum_{j=1}^m c_j\right) \|y\| - \sum_{j=0}^m b_j(t_{j+1} - s_j) \|y\|^{\gamma_j + 1} \\ &- \sum_{j=1}^m d_j \|y\|^{\sigma_j + 1}. \end{split}$$

Since  $\gamma_j$ ,  $\sigma_j \in [0, p - 1)$ , we can obtain  $\lim_{\|y\| \to +\infty} J(y) = +\infty$ , i.e., *J* is coercive. Now, by Lemma 2.5 and Theorem 2.7, we know that *J* satisfies all the conditions of Theorem 2.8. So *J* has a minimum on *Z*, which is a critical point of *J*. Hence, BVP (1.3) has at least one classical solution.

**Corollary 3.2** Assume that  $f_j$ , j = 0, 1, ..., m, and  $I_j$ , j = 1, 2, ..., m, are bounded. Then BVP (1.3) has at least one classical solution.

**Theorem 3.3** Assume that (H1) and the following conditions hold:

(H4) There exist constants  $\alpha_i > p$  such that

 $0 < \alpha_j F_j(t, x) \leq x f_j(t, x)$ 

for every  $t \in (s_j, t_{j+1}]$  and  $x \in \mathbb{R} \setminus \{0\}$ , where  $F_j(t, x) = \int_0^x f_j(t, s) \, ds, \, j = 0, 1, ..., m$ . (H5) There exist  $\beta_j > p, j = 1, 2, ..., m$ , such that

$$0 < \beta_j \int_0^x I_j(s) \, ds \le x I_j(x) \quad for \ x \in \mathbb{R} \setminus \{0\}.$$

Then BVP(1.3) has at least one classical solution.

*Proof* Clearly,  $J \in C^1(Z, \mathbb{R})$  and J(0) = 0. In view of Theorem 2.9, we first show that J satisfies the (PS) condition. It follows from (H1), (H4), and (H5) that the (PS) condition holds. The proof of the (PS) condition is similar to [25], so we omit it here.

Next, we verify condition (i) in Theorem 2.9. In fact, by using the same methods as [29], it follows from (H4) that

$$F_j(t,x) \le M_j |x|^{\alpha_j} \quad \text{if } |x| \le 1, \tag{3.2}$$

and

$$F_j(t,x) \ge m_j |x|^{\alpha_j} - A_j, \quad \forall (t,x) \in (s_j, t_{j+1}] \times \mathbb{R},$$
(3.3)

where  $M_j = \max_{t \in (s_j, t_{j+1}], |x|=1} F_j(t, x)$ ,  $m_j = \min_{t \in (s_j, t_{j+1}], |x|=1} F_j(t, x)$ , and  $A_j > 0, j = 0, 1, ..., m$ . Similarly, it follows from (H5) that

$$\int_{0}^{x} I_{j}(s) \, ds \le Q_{j} |x|^{\beta_{j}} \quad \text{if } |x| \le 1,$$
(3.4)

and

$$\int_0^x I_j(s) \, ds \ge q_j |x|^{\beta_j} - B_j, \quad \forall x \in \mathbb{R},$$
(3.5)

where  $Q_j = \max_{|x|=1} \int_0^x I_j(s) \, ds$ ,  $q_j = \min_{|x|=1} \int_0^x I_j(s) \, ds$ , and  $B_j > 0$ , j = 1, 2, ..., m. For any  $y \in Z$ , we know that  $||y|| \le 1$  implies  $||y||_{\infty} \le 1$ . By (3.2) and (3.4), we have

$$J(y) = \frac{1}{p} \|y\|^p + \sum_{j=0}^m \int_{s_j}^{t_{j+1}} \lambda(t) \Phi_p(y(t)) y(t) \, dt - \sum_{j=0}^m \int_{s_j}^{t_{j+1}} F_j(t, y(t)) \, dt$$

$$\begin{split} &-\sum_{j=1}^{m} \int_{0}^{y(t_{j})} I_{j}(s) \, ds \\ &\geq \frac{1}{p} \|y\|^{p} - \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} M_{j} |y(t)|^{\alpha_{j}} \, dt - \sum_{j=1}^{m} Q_{j} |y(t_{j})|^{\beta_{j}} \\ &\geq \frac{1}{p} \|y\|^{p} - \sum_{j=0}^{m} \|y\|_{\infty}^{\alpha_{j}} \int_{s_{j}}^{t_{j+1}} M_{j} \, dt - \sum_{j=1}^{m} Q_{j} \|y\|_{\infty}^{\beta_{j}} \\ &\geq \frac{1}{p} \|y\|^{p} - \sum_{j=0}^{m} M_{j}(t_{j+1} - s_{j}) \|y\|^{\alpha_{j}} - \sum_{j=1}^{m} Q_{j} \|y\|^{\beta_{j}}, \quad \|y\| \leq 1. \end{split}$$

Since  $\alpha_j$ ,  $\beta_j > p$ , the above inequality implies that we can choose  $\rho > 0$  small enough such that  $J(y) \ge \alpha > 0$  with  $||y|| = \rho$ .

Finally, we verify condition (ii) in Theorem 2.9. From (3.3), (3.5), and (H1), we have

$$\begin{split} J(y) &= \frac{1}{p} \|\|y\|^p + \sum_{j=0}^m \int_{s_j}^{t_{j+1}} \lambda(t) \Phi_p(y(t)) y(t) \, dt - \sum_{j=0}^m \int_{s_j}^{t_{j+1}} F_j(t, y(t)) \, dt \\ &- \sum_{j=1}^m \int_0^{y(t_j)} I_j(s) \, ds \\ &\leq \frac{1}{p} \|y\|^p + c \sum_{j=0}^m \int_{s_j}^{t_{j+1}} |y(t)|^p \, dt - \sum_{j=0}^m \int_{s_j}^{t_{j+1}} (m_j |y(t)|^{\alpha_j} - A_j) \, dt \\ &- \sum_{j=1}^m (q_j |y(t)|^{\beta_j} - B_j). \end{split}$$

Now, for any given  $y \in Z$  with ||y|| = 1, we have

$$\begin{split} J(ry) &\leq \frac{1}{p} \|ry\|^{p} + c \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} |ry(t)|^{p} dt - \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} (m_{j} |ry(t)|^{\alpha_{j}} - A_{j}) dt \\ &- \sum_{j=1}^{m} (q_{j} |ry(t)|^{\beta_{j}} - B_{j}) \\ &\leq \frac{1}{p} \|ry\|^{p} + c \|ry\|^{p} \sum_{j=0}^{m} (t_{j+1} - s_{j}) - \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} m_{j} |ry(t)|^{\alpha_{j}} dt \\ &+ \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} A_{j} dt - \sum_{j=1}^{m} q_{j} |ry(t)|^{\beta_{j}} + \sum_{j=1}^{m} B_{j} \\ &\leq \left(\frac{1}{p} + c\right) |r|^{p} - \sum_{j=0}^{m} m_{j} |r|^{\alpha_{j}} \int_{s_{j}}^{t_{j+1}} |y(t)|^{\alpha_{j}} dt + \sum_{j=0}^{m} A_{j}(t_{j+1} - s_{j}) \\ &- \sum_{j=1}^{m} q_{j} |r|^{\beta_{j}} |y(t)|^{\beta_{j}} + \sum_{j=1}^{m} B_{j}. \end{split}$$

$$(3.6)$$

Noting that  $\alpha_j, \beta_j > p$ , (3.6) implies that  $J(ry) \to -\infty$  as  $r \to +\infty$ . Hence, there exists  $r_0 > \rho$  such that  $J(r_0y) \le 0$ .

According to Theorem 2.9, the functional *J* has at least one critical point, that is, BVP (1.3) has at least one classical solution.  $\Box$ 

Remark 3.4 In this theorem, condition (A3) of Theorem 1.1 is not needed.

**Theorem 3.5** Assume that (H1), (H4), (H5), and the following condition hold: (H6)  $f_j(t,x)$ , j = 0, 1, ..., m, and  $I_j(x)$ , j = 1, 2, ..., m, are odd functions of x. Then BVP (1.3) has infinitely many classical solutions.

*Proof* We will use Theorem 2.10 to prove the theorem. By (H6), we obtain *J* is even. From the proof of Theorem 3.3, we know that  $J \in C^1(Z, \mathbb{R})$  satisfies the (PS) condition and J(0) = 0. In the same way as in Theorem 3.3, we can easily prove that conditions (i) and (ii) of Theorem 2.10 are satisfied. According to Theorem 2.10, the functional *J* has infinitely many critical points, that is, BVP (1.3) has infinitely many classical solutions.  $\Box$ 

**Theorem 3.6** Assume that (H1), (H3), (H4), and (H6) hold. Moreover,  $I_j(x)$ , j = 1, 2, ..., m, are nonincreasing, then BVP (1.3) has infinitely many classical solutions.

*Proof* Obviously,  $J \in C^1(Z, \mathbb{R})$  is even and J(0) = 0. Firstly, we will prove that J satisfies the (PS) condition. Let  $\beta = \min\{\alpha_0, \alpha_1, ..., \alpha_m\}$ , from (2.1), (2.2), (H3), and (H4), we have

$$\begin{split} \beta J(y_{k}) &- \left\langle I'(y_{k}), y_{k} \right\rangle \\ &= \frac{\beta}{p} \| y_{k} \|^{p} + \beta \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} \lambda(t) \Phi_{p}(y_{k}(t)) y_{k}(t) \, dt - \beta \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} F_{j}(t, y_{k}(t)) \, dt \\ &- \beta \sum_{j=1}^{m} \int_{0}^{y_{k}(t_{j})} I_{j}(s) \, ds - \int_{0}^{1} \mu(t) \Phi_{p}(y'_{k}(t)) y'_{k}(t) \, dt \\ &+ \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} f_{j}(t, y_{k}(t)) y_{k}(t) \, dt - \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} \lambda(t) \Phi_{p}(y_{k}(t)) y_{k}(t) \, dt \\ &+ \sum_{j=1}^{m} I_{j}(y_{k}(t_{j})) y_{k}(t_{j}) \\ &\geq \left(\frac{\beta}{p} - 1\right) \| y_{k} \|^{p} + (\beta - 1) \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} \lambda(t) \Phi_{p}(y_{k}(t)) y_{k}(t) \, dt \\ &- \beta \sum_{j=1}^{m} (c_{j} \| y_{k} \|_{\infty} + d_{j} \| y_{k} \|_{\infty}^{\sigma_{j}+1}) - \sum_{j=1}^{m} (c_{j} \| y_{k} \|_{\infty} + d_{j} \| y_{k} \|_{\infty}^{\sigma_{j}+1}) \\ &\geq \left(\frac{\beta}{p} - 1\right) \| y_{k} \|^{p} - (\beta + 1) \sum_{j=1}^{m} c_{j} \| y_{k} \| - (\beta + 1) \sum_{j=1}^{m} d_{j} \| y_{k} \|^{\sigma_{j}+1}, \end{split}$$

which implies that  $\{y_k\}$  is bounded in *Z*. In the following, the proof of the (PS) condition is the same as that in [25], so we omit it.

Secondly, we will show that *J* satisfies condition (i) in Theorem 2.10. Since  $I_j(x)$  are non-increasing and odd about *x*, we obtain

$$\int_{0}^{y(t_j)} I_j(x) \, dx \le 0. \tag{3.7}$$

It is clear that  $||y|| \le 1$  implies  $||y||_{\infty} \le 1$ . Thanks to (3.2) and (3.7), one has

$$\begin{split} J(y) &= \frac{1}{p} \|y\|^p + \sum_{j=0}^m \int_{s_j}^{t_{j+1}} \lambda(t) \Phi_p(y(t)) y(t) \, dt - \sum_{j=0}^m \int_{s_j}^{t_{j+1}} F_j(t, y(t)) \, dt \\ &- \sum_{j=1}^m \int_0^{y(t_j)} I_j(s) \, ds \\ &\geq \frac{1}{p} \|y\|^p - \sum_{j=0}^m \int_{s_j}^{t_{j+1}} F_j(t, y(t)) \, dt \\ &\geq \frac{1}{p} \|y\|^p - \sum_{j=0}^m \int_{s_j}^{t_{j+1}} M_j |y(t)|^{\alpha_j} \, dt \\ &\geq \frac{1}{p} \|y\|^p - \sum_{j=0}^m M_j(t_{j+1} - s_j) \|y\|^{\alpha_j}, \quad \|y\| \le 1, \end{split}$$

which implies that we can choose *y* with ||y|| sufficiently small such that  $J(y) \ge \alpha > 0$ .

Finally, we will show that *J* satisfies condition (ii) in Theorem 2.10. For every  $r \in \mathbb{R} \setminus \{0\}$  and  $y \in W \setminus \{0\}$  with ||y|| = 1, it follows from (3.3), (H1), and (H3) that

$$\begin{split} J(ry) &= \frac{1}{p} \| ry \|^{p} + \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} \lambda(t) \Phi_{p}(ry(t)) ry(t) dt - \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} F_{j}(t, ry(t)) dt \\ &- \sum_{j=1}^{m} \int_{0}^{ry(t_{j})} I_{j}(s) ds \\ &\leq \frac{1}{p} \| ry \|^{p} + c \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} | ry(t) |^{p} dt - \sum_{j=0}^{m} \int_{s_{j}}^{t_{j+1}} (m_{j} | ry(t) |^{\alpha_{j}} - A_{j}) dt \\ &+ \sum_{j=1}^{m} (c_{j} \| ry \|_{\infty} + d_{j} \| ry \|_{\infty}^{\sigma_{j}+1}) \\ &\leq \frac{1}{p} \| ry \|^{p} + c \| ry \|^{p} - \sum_{j=0}^{m} m_{j} | r |^{\alpha_{j}} \int_{s_{j}}^{t_{j+1}} | y(t) |^{\alpha_{j}} dt + \sum_{j=0}^{m} A_{j}(t_{j+1} - s_{j}) \\ &+ \sum_{j=1}^{m} c_{j} | r | \| y \| + \sum_{j=1}^{m} d_{j} | r |^{\sigma_{j}+1} \| y \|^{\sigma_{j}+1} \\ &\leq \left( \frac{1}{p} + c \right) | r |^{p} - \sum_{j=0}^{m} m_{j} | r |^{\alpha_{j}} \int_{s_{j}}^{t_{j+1}} | y(t) |^{\alpha_{j}} dt + \sum_{j=0}^{m} A_{j}(t_{j+1} - s_{j}) \\ &+ \sum_{j=1}^{m} c_{j} | r | + \sum_{j=1}^{m} d_{j} | r |^{\alpha_{j}} \int_{s_{j}}^{t_{j+1}} | y(t) |^{\alpha_{j}} dt + \sum_{j=0}^{m} A_{j}(t_{j+1} - s_{j}) \\ &+ \sum_{j=1}^{m} c_{j} | r | + \sum_{j=1}^{m} d_{j} | r |^{\alpha_{j}} \int_{s_{j}}^{t_{j+1}} | y(t) |^{\alpha_{j}} dt + \sum_{j=0}^{m} A_{j}(t_{j+1} - s_{j}) \\ &+ \sum_{j=1}^{m} c_{j} | r | + \sum_{j=1}^{m} d_{j} | r |^{\alpha_{j}+1}. \end{split}$$

Noting that  $\sigma_j + 1 , the above inequality implies that there exists <math>r_1$  such that  $||ry|| > \rho$  and J(ry) < 0 for every  $r \ge r_1 > 0$ . Since W is a finite-dimensional subspace, there exists R = R(W) such that  $J(y) \le 0$  on  $W \setminus B_{R(W)}$ .

According to Theorem 2.10, the functional *J* has infinitely many critical points, that is, BVP (1.3) has infinitely many classical solutions.  $\Box$ 

*Remark* 3.7 In this theorem, the growth conditions of the nonlinearities  $f_j$  and the impulsive functions  $I_j$  are different from those of Proposition 1.2.

### 4 Examples

In this section, we give three examples to illustrate the application of our main results.

*Example* 4.1 Consider the following problem:

$$\begin{cases} -(|y'(t)|^2 y'(t))' + |y(t)|^2 y(t) = f_j(t, y(t)), & t \in (s_j, t_{j+1}], j = 0, 1, \\ -\Delta(|y'(t_1)|^2 y'(t_1)) = I_1(y(t_1)), \\ |y'(t)|^2 y'(t) = |y'(t_1^+)|^2 y'(t_1^+), & t \in (t_1, s_1], \\ |y'(s_1^+)|^2 y'(s_1^+) = |y'(s_1^-)|^2 y'(s_1^-), \\ y(0) = 0, & y(1) = 2y(\frac{7}{8}), \end{cases}$$

$$(4.1)$$

where p = 4, m = 1,  $0 = s_0 < t_1 = \frac{1}{8} < s_1 = \eta = \frac{7}{8} < t_2 = 1$ ,  $\zeta = 2$ ,  $\mu(t) = \lambda(t) = 1$ ,  $f_j(t, y) = \sin y + ty^2 \cos y$ , and  $I_1(y) = 2 \cos y + t^2 y$ . Obviously, conditions (H1)–(H3) are satisfied. Thus, by Theorem 3.1, BVP (4.1) has at least one classical solution.

*Example* 4.2 Consider the following problem:

$$\begin{cases} -((1+2t)y'(t))' + (1+t)y(t) = f_j(t,y(t)), & t \in (s_j, t_{j+1}], j = 0, 1, \\ -\Delta((1+2t_1)y'(t_1)) = I_1(y(t_1)), \\ (1+2t)y'(t) = (1+2t_1^+)y'(t_1^+), & t \in (t_1, s_1], \\ (1+2s_1^+)y'(s_1^+) = (1+2s_1^-)y'(s_1^-), \\ y(0) = 0, & y(1) = 3y(\frac{3}{4}), \end{cases}$$

$$(4.2)$$

where p = 2, m = 1,  $0 = s_0 < t_1 = \frac{1}{4} < s_1 = \eta = \frac{3}{4} < t_2 = 1$ ,  $\zeta = 3$ ,  $\mu(t) = 1 + 2t$ ,  $\lambda(t) = 1 + t$ ,  $\alpha_j = 5$ ,  $\beta_1 = 3$ ,  $\delta_1 = \frac{1}{8}$ ,  $f_j(t, y) = 3y^6$ , and  $I_1(y) = \frac{1}{2}y^3$ . Then it is easy to verify that all conditions in Theorem 3.3 are fulfilled. By Theorem 3.3, BVP (4.2) has at least one classical solution. Furthermore, if we choose  $f_j(t, y) = y^7$ , then condition (H6) is also satisfied. Applying Theorem 3.5, BVP (4.2) has infinitely many classical solutions.

*Example* 4.3 Consider the following problem:

$$\begin{cases} -((1+3t)|y'(t)|y'(t))' + (1+t^3)|y(t)|y(t) = f_j(t,y(t)), & t \in (s_j, t_{j+1}], j = 0, 1, \\ -\Delta((1+3t_1)|y'(t_1)|y'(t_1)) = I_1(y(t_1)), \\ (1+3t)|y'(t)|y'(t) = (1+3t_1^+)|y'(t_1^+)|y'(t_1^+), & t \in (t_1,s_1], \\ (1+3s_1^+)|y'(s_1^+)|y'(s_1^+) = (1+3s_1^-)|y'(s_1^-)|y'(s_1^-), \\ y(0) = 0, & y(1) = 4y(\frac{5}{6}), \end{cases}$$

$$(4.3)$$

where p = 3, m = 1,  $0 = s_0 < t_1 = \frac{1}{6} < s_1 = \eta = \frac{5}{6} < t_2 = 1$ ,  $\zeta = 4$ ,  $\mu(t) = 1 + 3t$ ,  $\lambda(t) = 1 + t^3$ ,  $\alpha_j = 8$ ,  $f_j(t, y) = y^9$ , and  $I_1(y) = -\frac{1}{5}y^{\frac{1}{3}}$ . It is easy to verify that conditions (H1), (H3), (H4), and (H6) are satisfied. So BVP (4.3) has infinitely many classical solutions by Theorem 3.6.

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