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On the boundary value conditions of evolutionary p_i-Laplacian equation

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Abstract

The initial boundary value problem of the evolutionary p_i-Laplacian equation

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}\right)$$

is considered, where $a_i(x)$ is nonnegative but is with 0 measure degeneracy. The weak solutions do not belong to $BV_x(Q_T)$, how to define the trace in a reasonable way? This is the main topic of this paper. A suitable new boundary value condition is quoted and the stability of weak solutions follows naturally.

MSC: 35K15; 35B35; 35K55

Keywords: Initial boundary value problem; 0 measure degeneracy; $BV_x(Q_T)$; Trace; Stability

1 Introduction

In this paper, we consider the initial boundary value problem of the evolutionary p_i-Laplacian equation of the form

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}\right), \quad (x,t) \in Q_T, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, $Q_T = \Omega \times (0,T)$, $p_i > 1$ is a constant, $0 \leq a_i(x) \in C(\overline{\Omega})$, $i = 1,2,\dots,N$. Moreover, we say $a_i(x)$ is with 0 measure degeneracy if the measure of $\{x \in \overline{\Omega} : a_i(x) = 0\}$ is zero, i.e., $a_i(x)$ is positive almost everywhere on $\overline{\Omega}$. The initial value condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

is imposed as usual. But, since $a_i(x)$ may be degenerate on the boundary $\partial\Omega$, how to impose the boundary value condition

$$u(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T) \tag{1.3}$$

becomes a new problem.

Equation (1.1) arises in fluid mechanics [1, 2] and biology [3, 4], whether it is suggested as a model to describe the spread of an epidemic disease in heterogeneous environments or it is used as the mathematical description for the dynamics of fluids with different conductivities in different directions. Naturally, the earliest work can be traced to the paper [7] by Ladyženskaja, Solonikov, and Ural'ceva, in which the solvability of the non-Newtonian fluid equation

$$u_t=\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr)+f(u,x,t), \quad (x,t)\in Q_T \tag{1.4}$$

was studied. Since then, there have been many papers about the existence and nonexistence of weak solution to equation (1.4), one can refer to [5, 8, 10, 18] and the references therein. In [11], the non-Newtonian fluid equation of the form

$$\frac{\partial u}{\partial t}-\operatorname{div}\bigl(a(x)|\nabla u|^{p-2}\nabla u\bigr)-\sum_{i=1}^{N}b_i(x)D_iu+c(x,t)u=f(x,t), \quad (x,t)\in Q_T, \tag{1.5}$$

was considered, where $p>1$, $D_i=\frac{\partial}{\partial x_i}$, $0\leq a(x)\in C(\overline{\Omega})$, $b_i(x)\in C^1(\overline{\Omega})$, $c(x,t)$, and $f(x,t)$ are continuous functions on $\overline{Q}_T$. The authors of [11] defined $\mathbf{B}$ as the closure of the set $C_0^\infty(Q_T)$ with respect to the norm

$$\|u\|_{\mathbf{B}}=\iint_{Q_T}a(x)\bigl(\bigl|u(x,t)\bigr|^p+\bigl|\nabla u(x,t)\bigr|^p\bigr)\,dx\,dt, \quad u\in\mathbf{B}.$$

For $u\in\mathbf{B}$, they found that the boundary value condition

$$u(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T) \tag{1.6}$$

can be imposed in the sense that

$$\limsup_{\lambda\to0}\int_0^T\int_{\{x\in\partial\Omega_\lambda:\sum_{i=1}^N b_i(x)n_i(x)<0\}}u^2\sum_{i=1}^{N}b_i(x)n_i(x)\,d\sigma\,dt=0, \tag{1.7}$$

where $\lambda>0$, $\limsup_{\lambda\to0}f(\lambda)=\inf_{\delta>0}\{\operatorname{ess\,sup}\{f(\lambda):|\lambda|<\delta\}\}$ is the super limit. In other words, for $u\in\mathbf{B}$, one can define its trace of u on the boundary $\partial\Omega$ in the way of (1.7). Obviously, the trace defined in this way is a generalization of the classical trace. Recently, we have made a progress on the well-posedness problem of equation (1.5) [13].

However, for a weak solution of equation (1.1), $u(x,t)$ is generally only with

$$\int_\Omega a_i(x)|u_{x_i}|^{p_i}\,dx<\infty, \tag{1.8}$$

and the boundary value condition (1.3) cannot be imposed in the way as that of [5, 8, 10, 18]. Moreover, since there is not a convective term $\sum_{i=1}^{N} b_i(x)D_i u$ in equation (1.1), we also cannot impose the boundary value condition as (1.7) in [11].

One of the duties of this paper is to explain how to impose the boundary value condition (1.3) provided that $u(x,t)$ is only with (1.8). For every i, $1 \le i \le N$, we denote that

$$\begin{aligned}
&\Sigma_{1i} = \left\{x \in \partial\Omega : a_i(x) > 0\right\},\\
&\Sigma_{2i} = \left\{x \in \partial\Omega : a_i(x) = 0, \text{there exists } r > 0, \text{such that } \int_{\Omega\cap B_r(x)} a_i(y)^{-\frac{1}{p_i-1}}\,dy < +\infty\right\},\\
&\Sigma_{3i} = \left\{x \in \partial\Omega : a_i(x) = 0 \text{ for any small } r > 0, \int_{\Omega\cap B_r(x)} a_i(y)^{-\frac{1}{p_i-1}}\,dy = +\infty\right\}.
\end{aligned}$$

Clearly, for every i, we have

$$\partial\Omega = \Sigma_{1i} \cup \Sigma_{2i} \cup \Sigma_{3i}.$$

In what follows, we denote that

$$BV_x(Q_T) = \left\{u(x,t) \in L^\infty(Q_T) : \int_\Omega |\nabla u|\,dx < \infty\right\}.$$

According to [11], we know

$$u(x,t) = 0, \quad (x,t) \in \Sigma_1 \times (0,T) \tag{1.9}$$

can be imposed as in $BV_x(Q_T)$, where $\Sigma_1 = \{(\bigcap_{i=1}^{N} \Sigma_{1i}) \cup (\bigcap_{i=1}^{N} \Sigma_{2i})\}$.

But there may be $\int_\Omega a_i(x)^{-\frac{1}{p_i(x)-1}}\,dx = +\infty$ for some $i \in \{1,2,\dots,N\}$, the space

$$\mathbb{B} = \left\{v \in L^\infty(Q_T) : \sum_{i=1}^{N} \iint_{Q_T} \left(\left|v(x,t)\right|^{p_i} + a_i(x)\left|v_{x_i}(x,t)\right|^{p_i}\right) dx\,dt < \infty\right\}$$

even is not a Banach space generally, how to define

$$u(x,t) = 0, \quad (x,t) \in \{\partial\Omega \setminus \Sigma_1\} \times (0,T) \tag{1.10}$$

becomes a new problem.

Now, we give a generalization of the classical trace of $u \in BV_x(Q_T)$ to $u \in \mathbb{B} \cap L^\infty(\Omega)$.

Let χ be a nonnegative function of Ω satisfying

$$\chi(x) > 0, \quad x \in \Omega \quad \text{and} \quad \chi(x) = 0, \quad x \in \partial\Omega \setminus \Sigma_1 \tag{1.11}$$

and $\chi_{x_i}(x)$ be a continuous function when x is near $\partial\Omega$.

Then, for any $u(x,t) \in \mathbb{B}$, we find that when $u \in \mathbb{B}$, besides (1.9), if there is a function $\chi(x)$ satisfying (1.11) such that

$$\limsup_{n\to\infty}\left[n \sup_{x\in D_n\setminus D_{\frac{n}{2}}} a_i(x)^{\frac{1}{p_i(x)}} |\chi_{x_i}||u|\right] = 0, \quad i = 1,2,\dots,N, \tag{1.12}$$

then the partial boundary value condition (1.10) can be defined in the sense of (1.12). Here $D_n = \{x \in \Omega : \chi(x) > \frac{1}{n}\}$ and sup represents the essential supremum.

The existence of such $\chi(x)$ is possible. For example, if $a_i(x) = d^{p_i}$, then $\int_\Omega a_i(x)^{-\frac{1}{p_i-1}}\,dx = +\infty$, and the trace cannot be defined in the classical sense. But, if we choose $\chi(x) = d(x)^2$, then, when $x \in D_n \setminus D_{\frac{n}{2}}$, we have

$$a_i(x)^{\frac{1}{p_i}}|\chi_{x_i}| = \chi(x)d(x)\frac{|\chi_{x_i}|}{\chi(x)} \le 2\chi(x)|d_{x_i}| \le 2\chi(x) \tag{1.13}$$

and

$$1 \le n\chi(x) \le 2.$$

Thus, from (1.12), the generalization of trace of $u \in \mathbb{B} \cap L^\infty(\Omega)$, $u = 0$ on $\partial\Omega \setminus \Sigma_1$ can be defined as

$$\limsup_{n\to\infty}\Big[n \sup_{x\in D_n\setminus D_{\frac{n}{2}}} d^2|d_{x_i}||u|\Big] = 0, \quad i = 1,2,\dots,N. \tag{1.14}$$

A sufficient condition for (1.14) is

$$\limsup_{n\to\infty} \sup_{x\in D_n\setminus D_{\frac{n}{2}}} |u| = 0. \tag{1.15}$$

When $u(x,t)$ is a continuous function, (1.15) is clearly true.

The above generalization of the trace may be applied to the double phase problems

$$u_t = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\big) + f(x,t,u), \quad q > p > 1,$$

and its stationary case

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\big) = f(x,t,u), \quad q > p > 1.$$

Remark 1 After I had completed this paper, my friends kindly reminded me that the concentration and multiplications of ground states for the perturbed double phase problem with competing potentials

$$-\varepsilon^p \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) - \varepsilon^p \operatorname{div}\big(|\nabla u|^{q-2}\nabla u\big) = f(x,t,u), \quad q > p > 1,$$

has been revealed in [14], where $u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, $u > 0$, and

$$f(x,t,u) = -V(x)\big(|u|^{p-2}u + |u|^{q-2}u\big) + K(x)g(u),$$

V, K are the potentials, ε is a small positive parameter, the nonlinear function $g(s)$ is a continuous function. It is well known that the concentration phenomena occur in the weak convergence of a bounded sequence of the functions in Banach spaces. Actually, the double phase problem with local nonlinear reaction has been considered recently by several authors, and the existence and multiplicity results, the concentration and multiplicity properties of semi-classical states have been studied in [15–17].

2 The definitions of weak solutions and the main results

Let

$$1 < p_0 = \min_{x\in\overline{\Omega}}\{p_1, p_2, \dots, p_{N-1}, p_N\}$$

and

$$p^0 = \max_{x\in\overline{\Omega}}\{p_1, p_2, \dots, p_{N-1}, p_N\}.$$

Definition 2 A function $u(x,t)$ is said to be a weak solution of equation (1.1) with the initial value condition (1.2) if

$$u \in L^\infty(Q_T), \qquad \frac{\partial u}{\partial t} \in L^2(Q_T), \qquad a_i(x)|u_{x_i}|^{p_i} \in L^1(Q_T), \tag{2.1}$$

and for any function $\varphi_1 \in C_0^1(Q_T)$, $\varphi_2 \in L^\infty(Q_T)$, and $\varphi_{2x_i} \in L^2(0,T; W^{1,p_i}_{\mathrm{loc}}(\Omega))$ such that

$$\iint_{Q_T}\left[\frac{\partial u}{\partial t}(\varphi_1\varphi_2) + \sum_{i=1}^N a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}(\varphi_1\varphi_2)_{x_i}\right]dx\,dt = 0. \tag{2.2}$$

The initial value condition (1.2) is satisfied in the sense of

$$\lim_{t\to 0}\int_\Omega \left|u(x,t) - u_0(x)\right| dx = 0. \tag{2.3}$$

Moreover, if the boundary value condition (1.3) is satisfied as, when $x \in \Sigma_1$, (1.9) is true in the sense of $BV_x(Q_T)$, while $x \in \partial\Omega \setminus \Sigma_1$, $u(x,t) = 0$ is true in the sense of (1.14), then we say $u(x,t)$ is a weak solution of the initial boundary value problem (1.1)–(1.3).

By this definition, we can prove the following existence theorem in the next section.

Theorem 3 *Suppose that for every i, $1 \leq i \leq N$, $a_i(x)$ is with 0 measure degeneracy. If $p_0 > 1$, $u_0(x) \in W^{1,p^0}(\Omega) \cap L^\infty(\Omega)$, then there is a weak solution of the initial boundary value problem* (1.1)–(1.3).

The main aim of this paper is to study the stability of weak solutions. We denote that $d(x) = \mathrm{dist}(x, \partial\Omega)$ is the distance function from the boundary $\partial\Omega$ and define

$$\Omega_n = \left\{x \in \Omega : d(x) > \frac{1}{n}\right\}.$$

Theorem 4 *Let $a_i(x) \in C(\overline{\Omega})$, $u(x,t)$ and $v(x,t)$ be two solutions of equation* (1.1), *$u_0(x)$ and $v_0(x)$ be the corresponding initial values. If for the sufficiently large n, $a_i(x) \geq 0$ satisfies*

$$n\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} a_i(x)\,dx\right)^{\frac{1}{p_i}} \leq c, \quad i = 1, 2, \dots, N, \tag{2.4}$$

then there is

$$\int_\Omega \left|u(x,t) - v(x,t)\right| dx \leq c\int_\Omega \left|u_0(x) - v_0(x)\right| dx. \tag{2.5}$$

One can see that condition (2.4) implies that $a_i(x)|_{x\in\partial\Omega} = 0$, $a_i(x)$ is infinitesimal and

$$a_i(x) \sim d^{1-\frac{1}{p_i}}$$

as $x \to \partial\Omega$. Moreover, in Theorem 4, there is nothing to do with the boundary value condition. We can understand the fact that condition (2.4) can replace the usual boundary value condition to ensure the stability of weak solutions. In fact, if $a_i(x)|_{x\in\partial\Omega} = 0$ is not always true, then the boundary value condition can still be replaced by the other conditions.

We denote by $U(u_0)$ all solutions to equation (1.1) with the initial value $u_0(x)$, then it is clear that

$$U(u_0) \subset \mathbb{B}.$$

We denote by $U_1(u_0)$ the set of u,

$$u \in U(u_0), \quad \lim_{n\to\infty} n \int_{\Omega_n \setminus \Omega_{\frac{n}{2}}} a_i(x)|u|^{p_i}\,dx = 0, \quad i = 1, 2, \dots, N. \tag{2.6}$$

Theorem 5 *Let $a_i(x) \in C(\overline{\Omega})$ be nonnegative, $u(x,t)$ and $v(x,t)$ be two solutions of equation (1.1) satisfying (2.6), and $u_0(x)$ and $v_0(x)$ be the corresponding initial values, i.e.,*

$$u(x,t) \in U_1(u_0), \qquad v(x,t) \in U_1(v_0).$$

Then the stability (2.5) is true.

Theorem 5 shows that the boundary value condition is replaced by condition (2.6). The inadequacy of the argument of Theorem 5 is how to verify condition (2.6)? So, the following stability theorem seems more important.

Theorem 6 *Let $a_i(x) = d^{p_i}$, $u(x,t)$ and $v(x,t)$ be two solutions of equation (1.1), and $u_0(x)$ and $v_0(x)$ be the corresponding initial values. The same homogeneous boundary value condition*

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \tag{2.7}$$

is clarified as follows:

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \Sigma_1 \times (0,T) \tag{2.8}$$

is true in the sense of $BV_x(Q_T)$, while

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \{\partial\Omega \setminus \Sigma_1\} \times (0,T) \tag{2.9}$$

is true in sense of (1.15), then

$$\int_\Omega \left|u(x,s) - v(x,s)\right|^2 dx \le \int_\Omega \left|u_0(x) - v_0(x)\right|^2 dx. \tag{2.10}$$

Actually, for $0 \le a_i(x) \in C^1(\overline{\Omega})$, we have a similar result.

Theorem 7 *Let $a_i(x) \in C(\overline{\Omega})$ be nonnegative, $u(x,t)$ and $v(x,t)$ be two solutions of equation (1.1), and $u_0(x)$ and $v_0(x)$ be the corresponding initial values. The same homogeneous boundary value condition (2.7) is clarified in two parts. One part is the same as (2.8), while for the other part (2.9) is true in sense of (1.12), in which χ is a nonnegative function satisfying (1.11). Then the stability (2.10) is also true.*

3 The proof of Theorem 3

In this section, we prove Theorem 3. We first introduce an embedding theorem related to an anisotropic exponent space.

Lemma 8 *Let $1 \le m < \frac{N\bar{q}}{N-\bar{q}}$ and $\frac{1}{\bar{q}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}$. Then $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^m(\Omega)$ and $\|u\|_m \le M(\prod_{i=1}^{N}\|u_{x_i}\|_{p_i})^{\frac{1}{N}}$ for all $u \in W_0^{1,\vec{p}}(\Omega)$, where M is a constant independent of u.*

This lemma is found in [9].

Proof of Theorem 3 Consider the regularized parabolic equation

$$u_t = \sum_{i=1}^{N}\big(a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}\big)_{x_i} + \varepsilon\Delta u, \quad (x,t) \in Q_T \tag{3.1}$$

with the usual initial boundary value conditions

$$u(x,0) = u_{0\varepsilon}(x), \quad x \in \Omega, \tag{3.2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3.3}$$

where $u_{0\varepsilon}(x) \in C_0^\infty(\Omega)$ is strongly convergent to $u_0(x)$ in $W_0^{1,p^0}(\Omega)$ and $u_{0\varepsilon}(x) \rightharpoonup u_0(x)$ weakly star in $L^\infty(Q_T)$.

According to the theory of classical parabolic partial differential equation [6, 7], we know there is a classical solution u_ε of the initial boundary value problem (3.1)–(3.3). Moreover, since $u_0(x) \in L^\infty(Q_T)$, similar to the theory of evolutionary p-Laplacian equation [6, 10], by the maximal value theory, we have

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \le c. \tag{3.4}$$

Also, $\|u_\varepsilon\|_{L^\infty(Q_T)} \le c$ can be proved by De Giorgi method, one can refer to [18] for the case of the evolutionary p-Laplacian equation.

Multiplying (3.1) by u_ε yields

$$\varepsilon\iint_{Q_T}|\nabla u_\varepsilon|^2\,dx \le c \tag{3.5}$$

and

$$\sum_{i=1}^{N}\iint_{Q_T}a_i(x)|u_{\varepsilon x_i}|^{p_i}\,dx\,dt \le c. \tag{3.6}$$

Since for every i, $a_i(x) \in C^1(\overline{\Omega})$ is positive almost everywhere in Ω, if we denote that

$$\Omega_{0i} = \{x \in \Omega : a_i(x) > 0\}, \quad i = 1, 2, \dots, N,$$

then

$$|\Omega_{0i}| = \operatorname{mes} \Omega_{0i} = \operatorname{mes} \Omega = |\Omega|, \quad i = 1, 2, \dots, N,$$

which implies

$$\operatorname{mes} \bigcap_{i=1}^{N} \Omega_{0i} = |\Omega|. \tag{3.7}$$

Thus, for every point $x \in \bigcap_{i=1}^{N} \Omega_{0i}$, there is a neighbourhood $U_x \in \bigcap_{i=1}^{N} \Omega_{0i}$, when $x \in U_x$, $a_i(x) > 0$ for every i. From (3.6), we have

$$\int_0^T \int_{U_x} |u_{\varepsilon x_i}|^{p_i} \,dx\,dt \le c, \quad i = 1, 2, \dots, N.$$

Combining with (3.5), by Lemma 8, we know there is a function $u \in L^m(U_x \times (0,T))$ and

$$u_\varepsilon \to u, \quad \text{in } L^m\big(U_x \times (0,T)\big),$$

and so

$$u_\varepsilon \to u, \quad \text{a.e. } (x,t) \in \big(U_x \times (0,T)\big).$$

By (3.7), we know

$$u_\varepsilon \to u, \quad \text{a.e. } (x,t) \in Q_T. \tag{3.8}$$

Hence, there exist a function u and an N-dimensional vector $\overrightarrow{\zeta} = (\zeta_1, \dots, \zeta_N)$ such that

$$u \in L^\infty(Q_T), \qquad \frac{\partial u}{\partial t} \in L^2(Q_T), \qquad \zeta_i \in L^1\big(0,T; L^{\frac{p_i}{p_i-1}}(\Omega)\big),$$

and

$$\begin{aligned}
&u_\varepsilon \rightharpoonup u, \quad \text{weakly star in } L^\infty(Q_T),\\
&\frac{\partial u_\varepsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^2(Q_T),\\
&\varepsilon \nabla u_\varepsilon \rightharpoonup 0, \quad \text{in } L^2(Q_T),\\
&a_i(x)|u_{\varepsilon x_i}|^{p_i-2} u_{\varepsilon x_i} \rightharpoonup \zeta_i \quad \text{in } L^1\big(0,T; L^{\frac{p_i}{p_i-1}}(\Omega)\big).
\end{aligned}$$

Similar to the proof of Theorem in [12], we can show that

$$\sum_{i=1}^{N} \iint_{Q_T} a_i(x)|u_{x_i}|^{p_i-2} u_{x_i} \varphi_{x_i} \,dx\,dt = \sum_{i=1}^{N} \iint_{Q_T} \zeta_i(x) \varphi_{x_i} \,dx\,dt \tag{3.9}$$

for any function $\varphi \in C_0^1(Q_T)$. By denoting that

$$\Omega_\varphi = \{x \in \Omega : \varphi \neq 0\},$$

we have

$$\sum_{i=1}^{N}\int_0^T\int_{\Omega_\varphi} a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}\varphi_{x_i}\,dx\,dt = \sum_{i=1}^{N}\int_0^T\int_{\Omega_\varphi}\zeta_i(x)\varphi_{x_i}\,dx\,dt. \tag{3.10}$$

For any function $\varphi_1 \in C_0^1(Q_T)$, $\varphi_2 \in L^\infty(Q_T)$, and $\varphi_{2x_i} \in L^2(0,T;W_{\mathrm{loc}}^{1,p_i}(\Omega))$, let J_ε be the usual mollifier. Since the mollified function $J_\varepsilon * \varphi_2$ satisfies

$$J_\varepsilon * \varphi_2 \to \varphi_2, \quad \text{in } L^2\bigl(0,T;W_{\mathrm{loc}}^{1,p_i}(\Omega_{\varphi_1})\bigr), \tag{3.11}$$

and by (3.10), we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_0^T\int_{\Omega_{\varphi_1}} a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}(\varphi_1 J_\varepsilon\varphi_2)_{x_i}\,dx\,dt\\ &\quad= \sum_{i=1}^{N}\int_0^T\int_{\Omega_{\varphi_1}}\zeta_i(x)(\varphi_1 J_\varepsilon * \varphi_2)_{x_i}\,dx\,dt.\end{aligned} \tag{3.12}$$

Let $\varepsilon \to 0$ in (3.12). Then, from (3.11), we have

$$\sum_{i=1}^{N}\int_0^T\int_{\Omega_{\varphi_1}} a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}(\varphi_1\varphi_2)_{x_i}\,dx\,dt = \sum_{i=1}^{N}\int_0^T\int_{\Omega_{\varphi_1}}\zeta_i(x)(\varphi_1\varphi_2)_{x_i}\,dx\,dt,$$

and so

$$\sum_{i=1}^{N}\int_0^T\int_{\Omega} a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}(\varphi_1\varphi_2)_{x_i}\,dx\,dt = \sum_{i=1}^{N}\int_0^T\int_{\Omega}\zeta_i(x)(\varphi_1\varphi_2)_{x_i}\,dx\,dt. \tag{3.13}$$

Now, similar to the general evolutionary p-Laplacian equation, we are able to prove that (the details are omitted here)

$$\lim_{t\to 0}\int_\Omega \bigl|u(x,t) - u_0(x)\bigr|\,dx = 0.$$

Then u satisfies equation (1.1) with the initial value (1.2) in the sense of Definition 2. □

4 Proofs of Theorem 4 and Theorem 5

For $n > 0$, let

$$h_n(s) = 2n\bigl(1 - n|s|\bigr)_+, \qquad g_n(s) = \int_0^s h_n(\tau)\,d\tau.$$

Obviously,

$$\lim_{n\to\infty} g_n(s) = \operatorname{sgn} s, \qquad \lim_{n\to\infty} s g_n'(s) = 0. \tag{4.1}$$

Proof of Theorem 4 Let $u(x,t)$ and $v(x,t)$ be two weak solutions of equation (1.1) with the initial values $u_0(x)$ and $v_0(x)$ respectively. We define

$$\phi_n(x)=\begin{cases}1, & \text{if } x\in\Omega_{\frac{n}{2}},\\ n(d(x)-\frac{1}{n}), & \text{if } x\in\Omega_n\setminus\Omega_{\frac{n}{2}},\\ 0, & \text{if } x\in\Omega\setminus\Omega_n,\end{cases} \tag{4.2}$$

where $\Omega_n=\{x\in\Omega: d(x)>\frac{1}{n}\}$. Let

$$\varphi_1=\chi_{[\tau,s]}\phi_n \quad \text{and} \quad \varphi_2=g_n(u-v).$$

Here, $\chi_{[\tau,s]}$ is the characteristic function of $[\tau,s]\subset(0,T)$. From Definition 2, we have

$$\begin{aligned}&\int_\tau^s\int_\Omega \phi_n g_n(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt\\ &\quad+\sum_{i=1}^N\int_\tau^s\int_\Omega a_i(x)\big(|u_{x_i}|^{p_i-2}u_{x_i}-|v_{x_i}|^{p_i-2}v_{x_i}\big)(u_{x_i}-v_{x_i})g_n'(u-v)\phi_n(x)\,dx\,dt\\ &\quad+\sum_{i=1}^N\int_\tau^s\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} a_i(x)\big(|u_{x_i}|^{p_i-2}u_{x_i}-|v_{x_i}|^{p_i-2}v_{x_i}\big)(u_{x_i}-v_{x_i})g_n(u-v)\phi_{nx_i}\,dx\,dt\\ &=0.\end{aligned} \tag{4.3}$$

In the first place, we have

$$\int_\Omega a_i(x)\big(|u_{x_i}|^{p_i-2}u_{x_i}-|v_{x_i}|^{p_i-2}v_{x_i}\big)(u_{x_i}-v_{x_i})g_n'(u-v)\phi_n(x)\,dx\ge 0, \tag{4.4}$$

and since $u_t\in L^2(Q_T)$, we have

$$\begin{aligned}&\lim_{n\to\infty}\int_\tau^s\int_\Omega \phi_n(x)g_n(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt\\ &=\int_\Omega |u-v|(x,s)\,dx-\int_\Omega|u-v|(x,\tau)\,dx.\end{aligned} \tag{4.5}$$

In the second place, since $\phi_{nx_i}=nd_{x_i}$ when $x\in\Omega_n\setminus\Omega_{\frac{n}{2}}$ and $|d_{x_i}|\le|\nabla d|=1$, by (2.4), we have

$$\begin{aligned}&\left|\int_\Omega a_i(x)\big(|u_{x_i}|^{p_i-2}u_{x_i}-|v_{x_i}|^{p_i-2}v_{x_i}\big)\phi_{nx_i}g_n(u-v)\,dx\right|\\ &=\left|\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} a_i(x)\big(|u_{x_i}|^{p_i-2}u_{x_i}-|v_{x_i}|^{p_i-2}v_{x_i}\big)\phi_{nx_i}g_n(u-v)\,dx\right|\\ &\le n\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} a_i(x)\big(|u_{x_i}|^{p_i-1}+|v_{x_i}|^{p_i-1}\big)\big|d_{x_i}g_n(u-v)\big|\,dx\\ &\le cn\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} a_i(x)\big(|u_{x_i}|^{p_i}+|v_{x_i}|^{p_i}\big)\,dx\right)^{\frac{1}{q_i}}\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} a_i(x)|d_{x_i}|^{p_i}\,dx\right)^{\frac{1}{p_i}}\end{aligned} \tag{4.6}$$

$$
\begin{aligned}
&\leq c\left[\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} a_i(x)|u_{x_i}|^{p_i}\,dx\right)^{\frac{1}{q_i}} + \left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} a_i(x)|v_{x_i}|^{p_i}\,dx\right)^{\frac{1}{q_i}}\right] \\
&\quad\cdot n\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} a_i(x)\,dx\right)^{\frac{1}{p_i}} \\
&\leq c\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} a_i(x)|u_{x_i}|^{p_i}\,dx\right)^{\frac{1}{q_i}} + c\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} a_i(x)|v_{x_i}|^{p_i}\,dx\right)^{\frac{1}{q_i}},
\end{aligned}
$$

here and in what follows, $q_i = \frac{p_i}{p_i-1}$.

Then we have

$$
\begin{aligned}
&\lim_{n\to\infty}\left|\int_\tau^s\int_\Omega a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\right)\phi_{nx_i}g_n(u-v)\,dx\,dt\right| \\
&\quad\leq c\lim_{n\to\infty}\left[\left(\int_{\Omega\setminus\Omega_n} a_i(x)|u_{x_i}|^{p_i}\,dx\right)^{\frac{1}{q_i}} + \left(\int_{\Omega\setminus\Omega_n} a_i(x)|v_{x_i}|^{p_i}\,dx\right)^{\frac{1}{q_i}}\right] \\
&\quad= 0.
\end{aligned}
\tag{4.7}
$$

Let $n \to \infty$ in (4.3). By the above discussion (4.4)–(4.7), we deduce that

$$
\int_\Omega\left|u(x,s) - v(x,s)\right|dx \leq \int_\Omega\left|u(x,\tau) - v(x,\tau)\right|dx.
$$

Let $\tau \to 0$. We have

$$
\int_\Omega\left|u(x,s) - v(x,s)\right|dx \leq c\int_\Omega\left|u_0(x) - v_0(x)\right|dx.
$$

The proof is complete. □

Proof of Theorem 5 If $u \in U_1(u_0)$ by (2.6), then we have

$$
\begin{aligned}
&\lim_{n\to\infty}\left|\int_\Omega a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\right)\phi_{nx_i}g_n(u-v)\,dx\right| \\
&\quad= \lim_{n\to\infty}\left|\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\right)\phi_{nx_i}g_n(u-v)\,dx\right| \\
&\quad\leq \lim_{n\to\infty}\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} na_i(x)\left(|u_{x_i}|^{p_i-1} + |v_{x_i}|^{p_i-1}\right)\left|d_{x_i}g_n(u-v)\right|dx \\
&\quad\leq c\lim_{n\to\infty}\left(n\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} a_i(x)\left(|u_{x_i}|^{p_i} + |v_{x_i}|^{p_i}\right)dx\right)^{\frac{1}{q_i}} \\
&\qquad\cdot\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} na_i(x)\left|d_{x_i}g_n(u-v)\right|^{p_i}dx\right)^{\frac{1}{p_i}} \\
&\quad\leq c\lim_{n\to\infty}\left(\int_{\Omega_n\setminus\Omega_{\frac{n}{2}}} na_i(x)\left|g_n(u-v)\right|^{p_i}dx\right)^{\frac{1}{p_i}} \\
&\quad= 0.
\end{aligned}
\tag{4.8}
$$

The remainder is the same as that of Theorem 4. □

5 Proof of Theorem 6

Proof of Theorem 6 Let $u(x,t)$ and $v(x,t)$ be two weak solutions of equation (1.1) with the initial values $u_0(x)$ and $v_0(x)$ respectively. Let

$$\varphi_n(x) = \begin{cases} 1, & \text{if } x \in D_{\frac{n}{2}}, \\ n(d^2 - \frac{1}{n}), & \text{if } x \in D_n \setminus D_{\frac{n}{2}}, \\ 0, & \text{if } x \in \Omega \setminus D_n, \end{cases} \tag{5.1}$$

where $D_n = \{x \in \Omega : d^2 > \frac{1}{n}\}$. Take

$$\varphi = \chi_{[\tau,s]}(u - v)\varphi_n(x),$$

where $\chi_{[\tau,s]}$ is the characteristic function on $[\tau, s]$. Then we have

$$\iint_{Q_T} \left[\left(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t} \right)\varphi + \sum_{i=1}^{N} a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\right)\varphi_{x_i} \right] dx\,dt = 0. \tag{5.2}$$

For the next term of the left-hand side of (5.2), we have

$$\begin{aligned} &\int_\tau^s \int_\Omega a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\right)\left[(u-v)\varphi_n\right]_{x_i}\,dx\,dt \\ &\quad = \int_\tau^s \int_\Omega a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\right)(u-v)_{x_i}\varphi_n\,dx\,dt \\ &\qquad + n\int_\tau^s \int_{D_n\setminus D_{\frac{n}{2}}} a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\right)(u-v)2dd_{x_i}\,dx\,dt \end{aligned} \tag{5.3}$$

and

$$\iint_{Q_{\tau s}} a_i(x)\varphi_n(x)\left(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\right)(u-v)_{x_i}\,dx\,dt \geq 0. \tag{5.4}$$

For the second term on the right-hand side of (5.3), by (1.12), i.e.,

$$\limsup_{n\to\infty}\left[n \sup_{x\in D_n\setminus D_{\frac{n}{2}}} a_i(x)^{\frac{1}{p_i}}|2dd_{x_i}||u-v|\right] = 0, \quad i = 1, 2, \dots, N,$$

we have

$$\begin{aligned} &\limsup_{n\to\infty}\left|\iint_{Q_{\tau s}} (u-v)a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\right)\varphi_{nx_i}\,dx\,dt\right| \\ &\quad \leq \limsup_{n\to\infty} n\int_\tau^s \int_{D_n\setminus D_{\frac{n}{2}}} |u-v|a_i(x)\left(|u_{x_i}|^{p_i-1} + |v_{x_i}|^{p_i-1}\right)|2dd_{x_i}|\,dx\,dt \\ &\quad \leq \limsup_{n\to\infty} n\int_\tau^s \int_{D_n\setminus D_{\frac{n}{2}}} a_i(x)\left(|u_{x_i}|^{p_i} + |v_{x_i}|^{p_i}\right)|u-v||2dd_{x_i}|\,dx\,dt \\ &\qquad + c\limsup_{n\to\infty} n\int_\tau^s \int_{D_n\setminus D_{\frac{n}{2}}} a_i(x)|u-v||2dd_{x_i}|\,dx\,dt \end{aligned} \tag{5.5}$$

$$\begin{aligned}&\leq c\limsup_{n\to\infty} n \sup_{x\in D_n\setminus D_{\frac{n}{2}}}\left[a_i(x)^{\frac{1}{p_i}}|2dd_{x_i}||u-v|\right]\\&\quad\times\int_\tau^s\int_{D_n\setminus D_{\frac{n}{2}}} a_i(x)\left(|u_{x_i}|^{p_i}+|v_{x_i}|^{p_i}\right)dx\,dt\\&\quad+c\limsup_{n\to\infty} n \sup_{x\in D_n\setminus D_{\frac{n}{2}}}\left[a_i(x)^{\frac{1}{p_i}}|2dd_{x_i}||u-v|\right].\end{aligned}$$

Inequality (5.5) yields

$$\limsup_{n\to\infty}\left|\iint_{Q_{\tau s}}(u-v)a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i}-|v_{x_i}|^{p_i-2}v_{x_i}\right)\varphi_{nx_i}\,dx\,dt\right|=0. \tag{5.6}$$

Meanwhile, we have

$$\begin{aligned}&\lim_{n\to\infty}\iint_{Q_{\tau s}}(u-v)\varphi_n(x)\frac{\partial(u-v)}{\partial t}\,dx\,dt\\&\quad=\int_\Omega\left[u(x,s)-v(x,s)\right]^2dx-\int_\Omega\left[u(x,\tau)-v(x,\tau)\right]^2dx.\end{aligned} \tag{5.7}$$

From (5.3)–(5.7), letting $n\to\infty$ in (5.2), we have

$$\int_\Omega\left|u(x,s)-v(x,s)\right|^2dx\leq\int_\Omega\left|u(x,\tau)-v(x,\tau)\right|^2dx.$$

Due to the arbitrariness of τ, we obtain

$$\int_\Omega\left|u(x,s)-v(x,s)\right|^2dx\leq\int_\Omega\left|u_0(x)-v_0(x)\right|^2dx.$$

Thus we have the stability (2.10). □

If we choose a general function $\chi(x)$, which satisfies (2.5), instead of the function $d(x)^2$ in the proof of Theorem 6, then we can prove Theorem 7, we omit the details here.

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