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Existence results to a Leray–Lions type problem on the Heisenberg Lie groups

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Abstract

Here, the existence and multiplicity of weak solutions to a generalized $(p(\cdot), q(\cdot))$-Laplace equation involving Leray–Lions type operators with Hardy potential are studied under Dirichlet boundary conditions on the Heisenberg groups.

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1 Introduction

During the note, from start to finish, Ω is a Korányi ball in the Heisenberg group $\mathbb{H}^n$ $(n \geq 1)$ with boundary $\partial\Omega$; the functions $\mathfrak{p}, \mathfrak{q} \in C(\Omega)$ and the scalar $\mathfrak{s}$ satisfy the following inequalities:

$$1 < \mathfrak{q}^- \leq \mathfrak{q}(\xi) \leq \mathfrak{q}^+ < \mathfrak{s} < \mathfrak{p}^- \leq \mathfrak{p}(\xi) \leq \mathfrak{p}^+ < Q = 2n + 2.$$

We consider the following Leray–Lions type problem consisting of a Hardy potential term:

$$\begin{cases} -\operatorname{div}_{\mathbb{H}^n}(a(\xi, \nabla_{\mathbb{H}^n} u)) - \operatorname{div}(b(\xi, \nabla_{\mathbb{H}^n} u)) + \vartheta(\xi)\frac{|u|^{\mathfrak{s}-2}u}{|\xi|^{\mathfrak{s}}} = \lambda f(\xi, u) & \xi \in \Omega, \\ u = 0 & \xi \in \partial\Omega, \end{cases} \tag{$\mathcal{P}$}$$

where the potentials

$$a, b : \overline{\Omega} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

are Carathéodory functions satisfying some suitable supplementary conditions:

($\mathcal{H}1$) $a(\xi, -\tau) = -a(\xi, \tau)$ and $b(\xi, -\tau) = -b(\xi, \tau)$ for a.e. $\xi \in \Omega$ and all $\tau \in \mathbb{R}^{2n}$.

($\mathcal{H}2$) There exist nonnegative functions $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (L^\infty(\Omega), |\cdot|_\infty)$ such that

$$\left|a(\xi, \tau)\right| \leq \alpha_1(\xi) + \beta_1(\xi)|\tau|^{\mathfrak{p}(\xi)-1}$$

and

$$\left|b(\xi, \tau)\right| \leq \alpha_2(\xi) + \beta_2(\xi)|\tau|^{\mathfrak{q}(\xi)-1}$$

for a.e. $\xi \in \Omega$ and all $\tau \in \mathbb{R}^{2n}$, where $|\cdot|$ denotes the Euclidean norm.

($\mathcal{H}3$) For all $\tau, \tau' \in \mathbb{R}^{2n}$, where $\tau' \neq \tau$, the following inequalities hold:

$$\big(a(\xi,\tau) - a(\xi,\tau')\big) \cdot \big(\tau - \tau'\big) > 0$$

and

$$\big(b(\xi,\tau) - b(\xi,\tau')\big) \cdot \big(\tau - \tau'\big) > 0$$

for a.e. $\xi \in \Omega$, where dot is the inner product on $\mathbb{R}^{2n}$.

($\mathcal{H}4$) There exist constants $c_1, c_2 \geq 1$ such that

$$c_1|\tau|^{\mathfrak{p}(\xi)} \leq \min\big\{a(\xi,\tau)\cdot\tau, \mathfrak{p}\mathcal{A}(\xi,\tau)\big\}$$

and

$$c_2|\tau|^{\mathfrak{q}(\xi)} \leq \min\big\{b(\xi,\tau)\cdot\tau, \mathfrak{q}\mathcal{B}(\xi,\tau)\big\}$$

for a.e. $\xi \in \Omega$ and all $\tau \in \mathbb{R}^{2n}$, where

$$\mathcal{A}, \mathcal{B} : \overline{\Omega} \times \mathbb{R}^{2n} \to \mathbb{R}$$

are Carathéodory functions that are continuously differentiable with respect to its second argument, such that $\mathcal{A}(\xi,0) = \mathcal{B}(\xi,0) = 0$ for a.e. $\xi \in \Omega$ and

$$\nabla_{\mathbb{H}^n,\tau}\mathcal{A}(\xi,\tau) = a(\xi,\tau) \quad \& \quad \nabla_{\mathbb{H}^n,\tau}\mathcal{B}(\xi,\tau) = b(\xi,\tau)$$

for a.e. $\xi \in \Omega$ and all $\tau \in \mathbb{R}^{2n}$; in other words,

$$\mathcal{A}(\xi,\tau) := \int_0^1 a(\xi,s\tau)\cdot\tau\, ds \quad \& \quad \mathcal{B}(\xi,\tau) := \int_0^1 b(\xi,s\tau)\cdot\tau\, ds.$$

$\vartheta \in (L^\infty(\Omega), |\cdot|_\infty)$ is a nonnegative function and λ is a positive parameter; the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function for which the following growth condition holds:

$$\big|f(\xi,\zeta)\big| \leq \rho_1(\xi) + \rho_2(\xi)|\zeta|^{\gamma(\xi)-1} \tag{1.1}$$

for nonnegative functions $\rho_1, \rho_2 \in (L^\infty(\Omega), |\cdot|_\infty)$ and $\gamma \in C(\Omega)$ such that

$$2 \leq \gamma^- \leq \gamma(\xi) \leq \gamma^+ < \big(\mathfrak{p}^-\big)^* = \frac{Q\mathfrak{p}^-}{Q-\mathfrak{p}^-} \quad \text{a.e. in } \Omega.$$

Notice that if $a(\xi,\tau) = |\tau|^{\mathfrak{p}(\xi)-2}\tau$ and $b(\xi,\tau) = |\tau|^{\mathfrak{q}(\xi)-2}\tau$, in fact, problem ($\mathcal{P}$) is a Dirichlet $(\mathfrak{p}(\cdot), \mathfrak{q}(\cdot))$-Laplacian problem.

The existence and multiplicity of solutions to the following degenerate $p(x)$-Laplace equations with Leray–Lions type operators using direct methods and critical point theory were studied by Ho et al. [7]:

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) = \lambda f(x,u) & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

where U is a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary; $a : U \times \mathbb{R}^n \to \mathbb{R}^n$ and $f : U \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions that have suitable growth conditions. They proved the uniqueness and nonnegativeness of solutions when the principal operator is monotone and the nonlinearity is nonincreasing. Interested reader can see some special case of problem ($\mathcal{P}$) with distinct boundary conditions in [2, 10].

On the other hand, the existence of solutions for the problems on the Heisenberg groups has been intensively studied in the last decades. See some examples in [3, 4, 6, 11, 12, 17] and the references therein. We point out that the authors have probed some problems as special cases of problem ($\mathcal{P}$) (see [14, 15, 18–20]).

Taking inspiration from the mentioned works, our aim here is to prove the existence and multiplicity of weak solutions of problem ($\mathcal{P}$), applying the following theorem due to Ricceri [16].

Theorem 1.1 *Let X be a reflexive real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semicontinuous and coercive; and Ψ is sequentially weakly upper-semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty,r)} \frac{(\sup_{v \in \Phi^{-1}(-\infty,r)} \Psi(v)) - \Psi(u)}{r - \Phi(u)},$$

$$\eta := \liminf_{r \to +\infty} \varphi(r), \qquad \nu := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) *For every $r > \inf_X \Phi$ and every $\eta \in (0, \frac{1}{\varphi(r)})$, the restriction of the functional $I_\lambda : \Phi - \lambda\Psi$ to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point* (*local minimum*) *of I_λ in X.*

(b) *If $\eta < +\infty$, then for each $\lambda \in (0, \frac{1}{\eta})$, the following alternative holds*: *either*

(b1) *I_λ possesses a global minimum or*

(b2) *There is a sequence $\{u_n\}$ of critical points* (*local minima*) *of I_λ such that $\lim_{k\to+\infty} \Phi(u_k) = +\infty$.*

(c) *If $\nu < +\infty$, then for each $\lambda \in (0, \frac{1}{\nu})$, the following alternative holds*: *either*

(c1) *there is a global minimum of Φ that is a local minimum of I_λ or*

(c2) *there is a sequence $\{u_k\}$ of pairwise distinct critical points of I_λ that weakly converges to a global minimum of Φ with $\lim_{k\to+\infty} \Phi(u_k) = \inf_X \Phi$.*

This work is divided into four sections: In Sect. 2, we present some preliminaries of Heisenberg groups and notations of the note. Section 3 deals with some remarks that we make use of to prove our claims. The last section focuses on the proof of the existence and multiplicity of weak solutions.

2 Initial facts and notations

Throughout the note, $\mathbb{H}^n$ is the Heisenberg Lie group that has $\mathbb{R}^{2n+1}$ as a background manifold and is endowed with the following noncommutative law of product:

$$(x,y,t)\circ\left(x',y',t'\right)=\left(x+x',y+y',t+t'+2\left(\langle y|x'\rangle-\langle x|y'\rangle\right)\right),$$

where $x,x',y,y'\in\mathbb{R}^n$, $t,t'\in\mathbb{R}$, and $\langle|\rangle$ denotes the standard inner product in $\mathbb{R}^n$. With respect to this operation, the neutral element is $1=(0,0,\dots,0)$ and the inverse is given by

$$(x,y,t)^{(-1)}=(-x,-y,-t).$$

We denote by $|\cdot|_{\mathbb{H}^n}$ the Korányi norm with respect to the parabolic dilation $\delta_\lambda\xi=(\lambda x,\lambda y,\lambda^2 t)$, i.e.,

$$|\xi|_{\mathbb{H}^n}=\left(|z|^4+t^2\right)^{\frac{1}{4}}=\left(\left(x^2+y^2\right)^2+t^2\right)^{\frac{1}{4}}$$

for $z=(x,y)\in\mathbb{R}^{2n}$ and $\xi=(z,t)\in\mathbb{H}^n$. The Korányi distance between $\xi=(z,t)$ and $\xi'=(z',t')$ in $\mathbb{H}^n$ is as follows:

$$\rho\left(z,t;z',t'\right):=\left|\left(z',t'\right)^{-1}\circ(z,t)\right|_{\mathbb{H}^n}.$$

The Heisenberg group is an example of a sub-Riemannian manifold homeomorphic, but not bi-Lipschitz equivalent to the Euclidean space. The Heisenberg group (and more generally, stratified groups) is a special case of metric measure spaces with doubling measures. Its metric is derived from curves that are only allowed to move in so-called horizontal directions. A Korányi ball of center ξ_0 and radius r is defined by

$$B_{\mathbb{H}^n}(\xi_0,r):=\left\{\xi:\left|\xi^{-1}\circ\xi_0\right|_{\mathbb{H}^n}\le r\right\},$$

and it satisfies the following equalities:

$$\left|B_{\mathbb{H}^n}(\xi_0,r)\right|=\left|B_{\mathbb{H}^n}(0,r)\right|=r^Q\left|B_{\mathbb{H}^n}(0,1)\right|,$$

where $|U|$ denotes the $(2n+1)$-dimensional Lebesgue measure of U and $Q=2n+2$ is homogeneous dimension of $\mathbb{H}^n$. The Heisenberg gradient is given by

$$\nabla_{\mathbb{H}^n}=(X_1,\dots,X_n,Y_1,\dots,Y_n),$$

where

$$X_i=\frac{\partial}{\partial x_i}+2y_i\frac{\partial}{\partial t},\qquad Y_i=\frac{\partial}{\partial y_i}-2x_i\frac{\partial}{\partial t},\quad i=1,2,3,\dots,n,$$

are vector fields that constitute a basis for the real Lie algebra of left-invariant vector fields on $\mathbb{H}^n$. The key point is that the family

$$\left\{X_1,\dots,X_n,Y_1,\dots,Y_n,[X_1,Y_1]\right\}$$

satisfies the Hörmander condition, which means it spans the whole tangent space $T\mathbb{R}^{2n+1}$ (by definition, the tangent space to a manifold at a point is the vector space of derivations at the point).

A left invariant vector field X, which is in the span of $\{X_i, Y_i\}_{i=1}^n$, is called horizontal. In the span of $\{X_i, Y_i\}_{i=1}^n \simeq \mathbb{R}^{2n}$, we consider the natural inner product given by

$$(X,Y)_{\mathbb{H}^n} = \sum_{i=1}^{n} \left(x_i y_i + x_i' y_i'\right)$$

for each $X = \{x_i X_i + x_i' Y_i\}_{i=1}^n$ and $Y = \{y_i X_i + y_i' Y_i\}_{i=1}^n$. The inner product $(\cdot,\cdot)_{\mathbb{H}^n}$ produces a Hilbertian norm:

$$|X|_{\mathbb{H}^n} = \sqrt{(X,X)_{\mathbb{H}^n}}$$

for horizontal vector field X. Moreover, the Cauchy–Schwarz inequality

$$\left|(X,Y)_{\mathbb{H}^n}\right| \le |X|_{\mathbb{H}^n} |Y|_{\mathbb{H}^n}$$

holds for any horizontal vector fields X and Y. For any horizontal vector field function $X = X(\xi)$, $X = \{x_i X_i + x_i' Y_i\}_{i=1}^n$, of the class $C^1(\mathbb{H}^n, \mathbb{R}^{2n})$, we define the horizontal divergence of X by

$$\operatorname{div}_{\mathbb{H}^n} X := \sum_{i=1}^{n} \left[X_i(x_i) + Y_i\left(x_i'\right)\right].$$

The following is the Poincaré inequality.

Theorem 2.1 ([4]) *Let $U \subset \mathbb{H}^n$ $(n \ge 1)$ be a measurable bounded set and $u \in C_0^\infty(U)$. Then we have*

$$\int_U \left|u(\xi)\right|^p d\xi \le C \int_U |\nabla_{H^n} u|^p \, d\xi.$$

Definition 2.1 (Horizontal curve) A piecewise smooth curve $y : [0,1] \to \mathbb{H}^n$ is called a horizontal curve if $\dot{y}(t)$ belongs to the span of $\{X_i, Y_i\}_{i=1}^n$ a.e. in $[0,1]$. The horizontal length of y is defined as follows:

$$L_{\mathbb{H}^n}(y) = \int_0^1 \sqrt{\left(\dot{y}(t), \dot{y}(t)\right)_{\mathbb{H}^n}} \, dt = \int_0^1 \left|\dot{y}(t)\right|_{\mathbb{H}^n} dt,$$

where

$$(X,Y)_{\mathbb{H}^n} = \sum_{i=1}^{n} \left(x_i y_i + x_i' y_i'\right)$$

for each $X = \{x_i X_i + x_i' Y_i\}_{i=1}^n$ and $Y = \{y_i X_i + y_i' Y_i\}_{i=1}^n$.

The Carnot–Carathéodory distance of two points $\xi_1, \xi_2 \in \mathbb{H}^n$ is defined by

$$d_{cc}(\xi_1, \xi_2) := \inf\left\{L_{\mathbb{H}^n}(y) : y \text{ is a horizontal curve joining } \xi_1, \xi_2 \text{ in } \mathbb{H}^n\right\}.$$

Remark 2.1 The absolutely continuous curve

$$\Gamma = (x_1, \dots, x_n, y_1, \dots, y_n, t) : [0, T] \to \mathbb{H}^n$$

is horizontal if and only if

$$\dot{t}(s) = 2\sum_{j=1}^{n} \bigl(\dot{x}_j(s)y_j(s) - \dot{y}_j(s)x_j(s)\bigr)$$

for a.e. $s \in [0, T]$. Then we can conclude that

$$t(T) - t(0) = 2\sum_{j=1}^{n} \int_0^T \bigl(\dot{x}_j(s)y_j(s) - \dot{y}_j(s)x_j(s)\bigr)\, ds.$$

That is, if we are given a curve $\gamma : [0, T] \to \mathbb{R}^{2n}$, the horizontal curve $\Gamma = (x_1, \dots, x_n, y_1, \dots, y_n, t)$ is uniquely determined (up to its starting height $t(0)$); then the above definition is well defined.

d_{cc} is left invariant metric on $\mathbb{H}^n$ and homogeneous of degree 1 with respect to dilations δ_λ [8], that is,

$$d_{cc}\bigl(\delta_\lambda(\xi_1), \delta_\lambda(\xi_2)\bigr) = \lambda d_{cc}(\xi_1, \xi_2)$$

for all $\xi_1, \xi_2 \in \mathbb{H}^n$. In the case of the Heisenberg group, it is easy to check that the Lebesgue measure on $\mathbb{R}^{2n+1}$ is invariant under left translations; more precisely, the Heisenberg group is unimodular with the Haar measure $dx\,dy\,dt$ coinciding with the Lebesgue measure $\mathbb{R}^{2n+1}$. Thus, from here on, we denote by $d\xi$ the Haar measure on $\mathbb{H}^n$ since the Haar measures on Lie groups are unique up to constant multipliers [11].

Here, we recall Hardy's inequality on the Heisenberg groups established in [21, Theorem 1.1].

Lemma 2.1 *Let* $1 < s < Q$ *and* $u \in C_0^\infty(\mathbb{H}^n)$. *Then*

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^s\, d\xi \geq \left(\frac{Q-s}{s}\right)^s \int_{\mathbb{H}^n} \frac{|u(\xi)|^s}{d_{cc}^s}\, d\xi.$$

As usual, for any measurable set $U \subset \mathbb{H}^n$ $(n \geq 1)$, the Lebesgue space $L^p(U)$ is defined as

$$L^p(U) := \left\{ u : \Omega \longrightarrow \mathbb{R} : u \text{ is measurable and } \int_U |u|^p\, d\xi < \infty \right\},$$

which has the norm

$$|u|_p := \left(\int_U \bigl|u(\xi)\bigr|^p\, d\xi \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty,$$

while

$$|u|_\infty := \operatorname*{ess\,sup}_U u = \inf\bigl\{ M : \bigl|u(\xi)\bigr| \leq M \text{ for a.e. } \xi \in U \bigr\}.$$

Also, the first-order Heisenberg Sobolev space on U is defined as follows:

$$HW^{1,p}(U) := \{u \in L^p(U) : |\nabla_{\mathbb{H}^n} u| \in L^p(U)\},$$

endowed with the norm

$$\|u\|_{1,p} = |u|_p + |\nabla_{\mathbb{H}^n} u|_p;$$

and we set $HW_0^{1,p}(U) = \overline{(C_0^\infty(U), \|u\|_{1,p})}$ equipped with the norm

$$\|u\|_p = |\nabla_{\mathbb{H}^n} u|_p.$$

It is well known that $L^p(U)$, $HW^{1,p}(U)$, and $HW_0^{1,p}(U)$ are separable, reflexive Banach spaces.

Definition 2.2 (Poincaré–Sobolev domain) An open set U of $\mathbb{H}^n$ is said to be a Poincaré–Sobolev domain if there exist a bounded open set $V \subset \mathbb{H}^n$ with $U \subset \overline{U} \subset V$, a covering $\{B\}_{B\in\mathfrak{F}}$ of U by Carnot–Carathéodory balls B, and the numbers $N > 0$, $\alpha \geq 1$, and $\nu \geq 1$ such that

(i) $\sum_{B\in\mathfrak{F}} \mathbf{1}_{(\alpha+1)B} \leq N\mathbf{1}_U$ in U, where $\mathbf{1}_D$ is the characteristic function of a Lebesgue measurable subset D.

(ii) There exists a (central) ball $B_0 \in \mathfrak{F}$ such that for all $B \in \mathfrak{F}$ there is a finite chain $B_0, B_1, \dots, B_{s(B)}$ with $B_i \cap B_{i+1} \neq \emptyset$ and

$$|B_i \cap B_{i+1}| \geq \frac{\max\{|B_i|, |B_{i+1}|\}}{N}, \quad i = 0, 1, \dots, s(B) - 1;$$

moreover, $B \subset \nu B_i$ for $i = 0, 1, \dots, s(B)$.

This definition is purely metric. There is a multiplicity of Poincaré–Sobolev domains in $\mathbb{H}^n$, as explained in detail in [6]. The next result is a special case of Theorem 1.3.1 in [8].

Theorem 2.2

(i) *Let U be a bounded Poincaré–Sobolev domain in $\mathbb{H}^n$, and let $1 \leq p \leq Q$. Then the compact embedding*

$$HW_0^{1,p}(U) \hookrightarrow\hookrightarrow L^\sigma(U)$$

holds for all σ with $1 \leq \sigma < p^*$, where $p^* = \frac{pQ}{Q-p}$ is the critical Sobolev exponent related to p.

(ii) *The Carnot–Carathéodory balls are Poincaré–Sobolev domains.*

Remark 2.2 Combining Theorem 2.2 with the fact that the Carnot–Carathéodory distance and the Korányi distance are equivalent on $\mathbb{H}^n$, we get that the following embedding is compact:

$$HW_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^\sigma(\Omega) \quad \text{for } 1 \leq \sigma < p^*,$$

when $1 \le p \le Q$ and Ω is a Korányi ball. Furthermore, there exists $C_\sigma > 0$ such that

$$|u|_\sigma \le C_\sigma \|u\|_p \quad \text{for } 1 \le \sigma \le p^*,$$

for all $u \in HW_0^{1,p}(\Omega)$.

Remark 2.3 From Hardy's inequality (Lemma 2.1), because the Carnot–Carathéodory distance and the Korányi distance are equivalent on $\mathbb{H}^n$, we gain the following inequality:

$$\int_\Omega |\nabla_{\mathbb{H}^n} u|^s \, d\xi \ge \left(\frac{Q-s}{s}\right)^s \int_\Omega \frac{|u(\xi)|^s}{|\xi|^s_{\mathbb{H}^n}} \, d\xi$$

for $1 < s < Q$ and $u \in HW_0^{1,s}(\Omega)$. For convenience, we set $H = (\frac{Q-s}{s})^s$, and so we have

$$\int_\Omega \frac{|u(\xi)|^s}{|\xi|^s_{\mathbb{H}^n}} \, d\xi \le \frac{1}{H} \int_\Omega |\nabla_{\mathbb{H}^n} u|^s \, d\xi.$$

We put

$$p^- = \inf_{\xi \in \Omega} p(\xi) \quad \& \quad p^+ = \sup_{\xi \in \Omega} p(\xi)$$

for $p \in C_+(\overline{\Omega}) = \{g \in C(\overline{\Omega}) : g^- > 1\}$. The generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ is the collection of all measurable functions u on Ω for which there exists $\zeta > 0$ such that

$$\int_\Omega \left(\frac{u(\xi)}{\zeta}\right)^{p(\xi)} d\xi < \infty,$$

and it has the norm

$$|u|_{p(\cdot)} = \inf\left\{\zeta > 0 : \int_\Omega \left|\frac{u(\xi)}{\zeta}\right|^{p(\xi)} d\xi \le 1\right\}.$$

We know that for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, i.e., the conjugate space of $L^{p(\cdot)}(\Omega)$, the Hölder type inequality

$$\left|\int_\Omega uv \, d\xi\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{2.1}$$

holds true. Following the authors of [13], for any $\iota > 0$, we put

$$\iota^{\check{r}} := \begin{cases} \iota^{r^+} & \iota < 1, \\ \iota^{r^-} & \iota \ge 1, \end{cases}$$

and

$$\iota^{\hat{r}} := \begin{cases} \iota^{r^-} & \iota < 1, \\ \iota^{r^+} & \iota \ge 1 \end{cases}$$

for each $r \in C_+(\Omega)$. Then the well-known Proposition 2.7 of [9] will be rewritten as follows.

Proposition 2.1 *For each $u \in L^{p(\cdot)}(\Omega)$ and $p \in C_+(\Omega)$, we have*

$$|u|_{p(\cdot)}^{\check{p}} \leq \int_\Omega \big|u(\xi)\big|^{p(\xi)}\, d\xi \leq |u|_{p(\cdot)}^{\hat{p}}.$$

The next lemma was established in [5].

Lemma 2.2 *Assume that $p, q \in C_+(\overline{\Omega})$. If $q(\xi) \leq p(\xi)$ for all $\xi \in \overline{\Omega}$, then*

$$L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega);$$

moreover, there exists $\kappa_q > 0$ such that

$$|u|_{q(\cdot)} \leq \kappa_{q(\cdot)} |u|_{p(\cdot)}.$$

We denote the Heisenberg–Sobolev space with a variable exponent by

$$HW^{1,p(\cdot)}(\Omega) := \big\{u \in L^{p(\cdot)}(\Omega) : |\nabla_{\mathbb{H}^n} u| \in L^{p(\cdot)}(\Omega)\big\},$$

equipped with the norm

$$\|u\|_{1,p(\cdot)} = |u|_{p(\cdot)} + |\nabla_{\mathbb{H}^n} u|_{p(\cdot)}.$$

We also denote by $HW_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $HW^{1,p(\cdot)}(\Omega)$, which, by the Poincaré inequality, has the norm

$$\|u\| = |\nabla_{\mathbb{H}^n} u|_{p(\cdot)}.$$

Here-in-after, for the functions $\mathfrak{p}, \mathfrak{q} \in C(\Omega)$ and the constant $\mathfrak{s}$ satisfying the following inequalities:

$$1 < \mathfrak{q}^- \leq \mathfrak{q}(\xi) \leq \mathfrak{q}^+ < \mathfrak{s} < \mathfrak{p}^- \leq \mathfrak{p}(\xi) \leq \mathfrak{p}^+ < Q \quad \text{a.e. in } \Omega,$$

we set

$$X := HW_0^{1,\mathfrak{p}(\cdot)}(\Omega)$$

endowed with the norm $\|u\|$.

3 Auxiliary remarks

Remark 3.1

(i) Let $p, q \in C_+(\overline{\Omega})$. If $q(\xi) \leq p(\xi)$ for all $\xi \in \overline{\Omega}$, then from Lemma 2.2 one has

$$HW^{1,p(\cdot)}(\Omega) \hookrightarrow HW^{1,q(\cdot)}(\Omega).$$

In the special case,

$$X \hookrightarrow HW_0^{1,\mathfrak{q}(\cdot)}(\Omega) \quad \& \quad X \hookrightarrow HW_0^{1,\mathfrak{s}}(\Omega);$$

besides, there exist $\kappa_{\mathfrak{q}(\cdot)}, \kappa_{\mathfrak{s}} > 0$ such that

$$|\nabla_{\mathbb{H}^n} u|_{\mathfrak{q}(\cdot)} \leq \kappa_{\mathfrak{q}(\cdot)} \|u\| \quad \& \quad |\nabla_{\mathbb{H}^n} u|_{\mathfrak{s}} \leq \kappa_{\mathfrak{s}} \|u\|$$

for $u \in X$.

(ii) Thanks to part (i) and Remark 2.2, we have the following embedding:

$$X \hookrightarrow HW^{1,\mathfrak{p}^-}(\Omega) \hookrightarrow L^{(\mathfrak{p}^-)^*}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$$

for $r \in C(\Omega)$ with $1 \leq r^- \leq r(\xi) \leq r^+ \leq (\mathfrak{p}^-)^*$ a.e. in Ω; and for every bounded sequence $\{u_k\}$ in X, up to the subsequence, $\{u_k\}$ converges to some $\bar{u}$ in $L^{r(\cdot)}(\Omega)$ as $1 \leq r^- \leq r(\xi) \leq r^+ < (\mathfrak{p}^-)^*$ a.e. in Ω. In the special case,

$$X \hookrightarrow L^1(\Omega) \quad \& \quad X \hookrightarrow L^{\gamma(\cdot)-1}(\Omega) \quad \& \quad X \hookrightarrow L^{\gamma(\cdot)}(\Omega),$$

where $\gamma \in C(\Omega)$ is as (1.1); moreover, there exist constants $K_1, K_{\gamma-1} = K(\gamma(\cdot) - 1)$, $K_\gamma = K(\gamma(\cdot)) > 0$ such that

$$\int_\Omega |u|\, d\xi \leq K_1 \|u\| \quad \& \quad |u|_{\gamma(\cdot)-1} \leq K_{\gamma-1} \|u\| \quad \& \quad |u|_{\gamma(\cdot)} \leq K_\gamma \|u\|$$

for $u \in X$.

Remark 3.2 Suppose that conditions $(\mathcal{H}1)$–$(\mathcal{H}4)$ hold, then we have:

- The aforementioned functions $\mathcal{A}(\xi, t)$ and $\mathcal{B}(\xi, t)$ are C^1-Carathéodory functions.
- From $(\mathcal{H}3)$, thanks to [1, Proposition 1.5.10], the functionals $\mathcal{A}$ and $\mathcal{B}$ are strictly convex.
- There exist constants

$$\underline{c} \leq \min\{c_1, c_2\} \quad \& \quad \bar{c} \geq \max\left\{ |\alpha_1|_\infty, |\alpha_2|_\infty, \frac{|\beta_1|_\infty}{\mathfrak{p}^-}, \frac{|\beta_2|_\infty}{\mathfrak{q}^-} \right\}$$

such that

$$\frac{\underline{c}}{\mathfrak{p}^+} |\tau|^{\mathfrak{p}(\xi)} \leq \big|\mathcal{A}(\xi, \tau)\big| \leq \bar{c}\big(|\tau| + |\tau|^{\mathfrak{p}(\xi)}\big),$$

and, in the same way,

$$\frac{\underline{c}}{\mathfrak{q}^+} |\tau|^{\mathfrak{q}(\xi)} \leq \big|\mathcal{B}(\xi, \tau)\big| \leq \bar{c}\big(|\tau| + |\tau|^{\mathfrak{q}(\xi)}\big)$$

for a.e. $\xi \in \Omega$ and all $\tau \in \mathbb{R}$. So, one has the following estimate:

$$\begin{aligned}
\frac{\underline{c}}{\mathfrak{p}^+} \|u\|^{\check{\mathfrak{p}}} &\leq \int_\Omega \mathcal{A}(\xi, \nabla_{\mathbb{H}^n} u)\, d\xi + \int_\Omega \mathcal{B}(\xi, \nabla_{\mathbb{H}^n} u)\, d\xi \\
&\leq \int_\Omega \big|\mathcal{A}(\xi, \nabla_{\mathbb{H}^n} u)\big| d\xi + \int_\Omega \big|\mathcal{B}(\xi, \nabla_{\mathbb{H}^n} u)\big| d\xi \\
&\leq \bar{c}\left(2 \int_\Omega \big(|\nabla_{\mathbb{H}^n} u| + |\nabla_{\mathbb{H}^n} u|^{\mathfrak{p}(\xi)} \big)\, d\xi + |\nabla_{\mathbb{H}^n} u|^{\mathfrak{q}(\xi)} \right) d\xi)
\end{aligned}$$

$$
\begin{aligned}
&\leq \bar{c}\big(2\kappa_1\|u\| + \|u\|^{\hat{\mathfrak{p}}} + \kappa_{\mathfrak{q}(\cdot)}^{\hat{\mathfrak{q}}}\|u\|^{\hat{\mathfrak{q}}}\big) \\
&\leq \bar{C}\big(\|u\| + \|u\|^{\hat{\mathfrak{p}}}\big),
\end{aligned}
\tag{3.1}
$$

where $\bar{C} = 2\bar{c}(\kappa_1 + 1 + \kappa_{\mathfrak{q}(\cdot)}^{\hat{\mathfrak{q}}})$.

We mean that a weak solution to problem ($\mathcal{P}$) is as follows.

Definition 3.1 (Weak solution) We say that $u \in X \setminus \{0\}$ is a weak solution of problem ($\mathcal{P}$) if $u = 0$ on Ω and

$$
\begin{aligned}
&\int_\Omega a(\xi, \nabla_{\mathbb{H}^n} u) \cdot \nabla_{\mathbb{H}^n} v \, d\xi + \int_\Omega b(\xi, \nabla_{\mathbb{H}^n} u) \cdot \nabla_{\mathbb{H}^n} v \, d\xi \\
&\quad + \int_\Omega \vartheta(\xi) \frac{|u|^{\mathfrak{s}-2} uv}{|\xi|^{\mathfrak{s}}} \, d\xi - \lambda \int_\Omega f(\xi, u) v \, d\xi = 0
\end{aligned}
$$

for each $v \in X$.

We define the functional $\Phi : X \to \mathbb{R}$ by

$$
\Phi(u) := \int_\Omega \mathcal{A}(\xi, \nabla_{\mathbb{H}^n} u) \, d\xi + \int_\Omega \mathcal{B}(\xi, \nabla_{\mathbb{H}^n} u) \, d\xi + \frac{1}{\mathfrak{s}} \int_\Omega \vartheta(\xi) \frac{|u(\xi)|^{\mathfrak{s}}}{|\xi|^{\mathfrak{s}}} \, d\xi.
$$

Remark 3.3 The following assertions hold:

(i) There exists $C > 0$ such that

$$
\epsilon_0 \|u\|^{\check{p}} \leq \Phi(u) \leq C\big(\|u\| + \|u\|^{\hat{\mathfrak{p}}}\big). \tag{3.2}
$$

Proof Applying Remark 3.2 and Hardy's inequality, one has the following estimates:

$$
\begin{aligned}
\epsilon_0 \|u\|^{\check{p}} &\leq \int_\Omega \mathcal{A}(\xi, \nabla_{\mathbb{H}^n} u) \, d\xi + \int_\Omega \mathcal{B}(\xi, \nabla_{\mathbb{H}^n} u) \, d\xi \\
&\leq \Phi(u) \\
&= \int_\Omega \mathcal{A}(\xi, \nabla_{\mathbb{H}^n} u) \, d\xi + \int_\Omega \mathcal{B}(\xi, \nabla_{\mathbb{H}^n} u) \, d\xi + \frac{1}{\mathfrak{s}} \int_\Omega \vartheta(\xi) \frac{|u(\xi)|^{\mathfrak{s}}}{|\xi|^{s}} \, d\xi \\
&\leq \bar{C}\big(\|u\| + \|u\|^{\hat{\mathfrak{p}}}\big) + \frac{|\vartheta|_\infty}{\mathfrak{s}H} \int_\Omega |\nabla_{\mathbb{H}^n} u|^{\mathfrak{s}} \, d\xi \\
&\leq \bar{C}\big(\|u\| + \|u\|^{\hat{\mathfrak{p}}}\big) + \frac{|\vartheta|_\infty}{\mathfrak{s}H} \kappa_{\mathfrak{s}}^{\mathfrak{s}} \|u\|^{\mathfrak{s}} \\
&\leq C\big(\|u\| + \|u\|^{\hat{\mathfrak{p}}}\big),
\end{aligned}
$$

where $C = \max\{\bar{C}, \frac{\kappa_{\mathfrak{s}}^{\mathfrak{s}} |\vartheta|_\infty}{sH}\}$. So, the proof is completed. □

(ii) Inequality (3.2) ensures that Φ is coercive.

(iv) Φ is sequentially weakly lower semicontinuous: if $u_k \rightharpoonup u$ in X, so, up to a subsequence, $u_k(\xi) \to u(\xi)$ a.e. in Ω; by applying Fatou's lemma, one has

$$
\liminf_{k \to \infty} \int_\Omega |\nabla_{\mathbb{H}^n} u_k|^{\mathfrak{s}} \, d\xi \geq \int_\Omega |\nabla_{\mathbb{H}^n} u|^{\mathfrak{s}} \, d\xi;
$$

and also we have

$$\lim_{k\to\infty} \mathcal{A}(\xi,\nabla_{\mathbb{H}^n}u_k) = \lim_{k\to\infty}\int_0^{\nabla_{\mathbb{H}^n}u_k} a(\xi,\eta)\,d\eta$$
$$= \int_0^{\nabla_{\mathbb{H}^n}u} a(\xi,\eta)\,d\eta = \mathcal{A}(\xi,\nabla_{\mathbb{H}^n}u) \quad \text{a.e. in } \Omega;$$

in the same way, we have

$$\lim_{k\to\infty} \mathcal{B}(\xi,\nabla_{\mathbb{H}^n}u_k) = \mathcal{B}(\xi,\nabla_{\mathbb{H}^n}u) \quad \text{a.e. in } \Omega.$$

On the other side, using Remark 3.2, $\mathcal{A},\mathcal{B} \in L^1(\Omega)$. Then, in accordance with Lebesgue's dominated convergence theorem, we gain that

$$\liminf_{k\to\infty} \Phi(u_k) \geq \Phi(u).$$

(iii) It is known that Φ is continuously Gâteaux differentiable functional; moreover,

$$\langle \Phi'(u),v\rangle = \int_\Omega a(\xi,\nabla_{\mathbb{H}^n}u)\cdot\nabla_{\mathbb{H}^n}v\,d\xi + \int_\Omega b(\xi,\nabla_{\mathbb{H}^n}u)\cdot\nabla_{\mathbb{H}^n}v\,d\xi$$
$$+ \int_\Omega \vartheta(\xi)\frac{|u|^{\mathfrak{s}-2}uv}{|\xi|^{\mathfrak{s}}}\,d\xi$$

for each $v \in X$.

Now, corresponding to the function f with the growth condition (1.1), we introduce the functional $F : \Omega\times\mathbb{R}\to\mathbb{R}$ by

$$F(\xi,\zeta) := \int_0^{|\zeta|} f(\xi,s)\,ds.$$

And we define the functional $\Psi : X\to\mathbb{R}$ as follows:

$$\Psi(u) := \int_\Omega F(\xi,u)\,d\xi.$$

Remark 3.4

(i) By applying (1.1), F is well defined and one has

$$\bigl|F(\xi,\zeta)\bigr| \leq |\rho_1|_\infty|\zeta| + \frac{|\rho_2|_\infty}{\gamma^-}|\zeta|^{\gamma(\xi)}.$$

Then, by simple calculations, we gain

$$\Psi(u) \leq \int_\Omega \bigl|F(\xi,u)\bigr|\,d\xi \leq \tilde{C}\bigl(\|u\| + \|u\|^{\hat{\gamma}}\bigr),$$

where $\tilde{C} = \max\{K_1|\rho_1|_\infty, \frac{K_\gamma^{\hat{\gamma}}}{\gamma^-}|\rho_2|_\infty\}$.

(ii) Let $u_k \rightharpoonup u$ in X. So, up to a subsequence, $u_k(\xi) \to u(\xi)$ a.e. in Ω; since $||x| - |y|| \le |x - y|$ for every $x, y \in \mathbb{R}$, then $|u_k(\xi)| \to |u(\xi)|$ a.e. in Ω as $k \to \infty$; then

$$\begin{aligned}\lim_{k\to\infty} F(\xi, u_k) &= \lim_{k\to\infty} \int_0^{|u_k|} f(\xi, s)\, ds \\ &= \int_0^{|u|} f(\xi, s)\, ds = F(\xi, u) \quad \text{a.e. in } \Omega;\end{aligned}$$

thus,

$$\begin{aligned}\limsup_{k\to\infty} \Psi(u_k) &\le \int_\Omega \limsup_{k\to\infty} F(\xi, u_k)\, d\xi \\ &= \int_\Omega F(\xi, u)\, d\xi = \Psi(u),\end{aligned}$$

which shows that Ψ is weakly upper semicontinuous.

(iii) As everyone knows, Ψ is Gâteaux derivative and its derivative is given by

$$\langle \Psi'(u), v \rangle = \int_\Omega f\big(\xi, u(\xi)\big) v(\xi)\, d\xi$$

for each $v \in X$.

We set

$$I_\lambda(u) = \Phi(u) - \lambda \Psi(u) \quad \text{for all } u \in X.$$

Clearly, I_λ is the energy functional corresponding to problem ($\mathcal{P}$); and so, its critical points are weak solutions of problem ($\mathcal{P}$).

4 Existence results

The following is one of the main results of this note.

Theorem 4.1 *For given $r > 0$ and every $\lambda \in (0, \lambda^*)$ with*

$$\lambda^* = \frac{r}{C(k_1(\frac{r}{\epsilon_0})^{\frac{1}{\check{\mathfrak{p}}}} + (\frac{r}{\epsilon_0})^{\frac{\hat{s}}{\check{\mathfrak{p}}}})},$$

problem ($\mathcal{P}$) admits at least one weak solution $u_\lambda \in X$.

Proof For given $r > 0$, from (3.2), we have

$$\begin{aligned}\Phi^{-1}\big(]-\infty, r[\big) &:= \big\{u \in X : \Phi(u) < r\big\} \\ &\subseteq \left\{u \in X : \int_\Omega \mathcal{A}(\xi, \nabla_{\mathbb{H}^n} u)\, d\xi < r\right\} \\ &\subseteq \big\{u \in X : \epsilon_0 \|u\|^{\check{\mathfrak{p}}} < r\big\}.\end{aligned}$$

So, for $r > 0$ and all $u \in X$ with $\Phi(u) < r$, one has

$$\|u\| \leq \left(\frac{r}{\epsilon_0}\right)^{\frac{1}{\mathfrak{p}^-}}.$$

Then, from Remark 3.4(i) and because $\Phi(0) = 0$ and $\Psi(0) = 0$, one has

$$\begin{aligned}\varphi(r) &= \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)) - \Psi(u)}{r - \Phi(u)} \\ &\leq \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)}{r} \\ &\leq \frac{C}{r}\left(k_1\left(\frac{r}{\epsilon_0}\right)^{\frac{1}{\mathfrak{p}^-}} + \left(\frac{r}{\epsilon_0}\right)^{\frac{\hat{s}}{\mathfrak{p}^-}}\right).\end{aligned}$$

So, using the assumption,

$$\lambda \in \left(0, \lambda^*\right) \subset \left(0, \frac{1}{\varphi(r)}\right).$$

Thus, applying part (a) of Theorem 1.1, the restriction of the functional $I_\lambda : \Phi - \lambda\Psi$ to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X. □

In the next theorem, we present enough conditions for having infinitely many solutions of the problem.

Theorem 4.2 *Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is the aforementioned Carathéodory function satisfying the following Ambrosetti–Rabinowitz (AR) type condition: there are constants $\mu > \mathfrak{p}^+$, $\mathcal{R} > 0$ such that*

$$0 < \mu F(\xi, \tau) \leq \tau f(\xi, \tau)$$

for all $\xi \in \Omega$ and $|\tau| > \mathcal{R}$. Then there exists an unbounded sequence of weak solutions to problem ($\mathcal{P}$).

Proof Let $\{r_k\}$ be a sequence such that $\lim_{k\to\infty} r_k = +\infty$. For each $u \in \Phi^{-1}(-\infty, r_k)$, we have

$$\frac{(\sup_{v\in\Phi^{-1}(-\infty,r_k)} \Psi(v)) - \Psi(u)}{r_k - \Phi(u)} \geq \frac{\sup_{v\in\Phi^{-1}(-\infty,r_k)}(\Psi(v) - \Psi(u))}{r_k}. \tag{4.1}$$

But Ψ is differentiable; so, by definition, we have

$$\begin{aligned}\Psi(v) - \Psi(u) &= \Psi'(u)(v-u) + o\big(\|u-v\|\big) \\ &\leq \big|\Psi'(u)(v-u)\big| + o\big(\|u-v\|\big) \\ &\leq \big\|\Psi'(u)\big\|\big(\|v\| + \|u\|\big) + o\big(\|u-v\|\big) < +\infty,\end{aligned}$$

as $\|u-v\| \to 0$. Then the right-hand side of (4.1) converges to zero. So,

$$\eta = \liminf_{k\to\infty} \varphi(r_k) < \infty.$$

Now, we show that I_λ is unbounded from below. To this end, we set

$$\delta(\xi) = \sup\{\delta > 0 : B(\xi,\delta) \subseteq \Omega\} \quad \& \quad R := \sup_{x\in\Omega} \delta(\xi).$$

Obviously, there exists $\xi_0 \in \Omega$ such that

$$B(\xi_0, R) \subseteq \Omega.$$

Assume that $\delta > 0$ and consider the following function:

$$w(\xi) = \begin{cases} 0 & x \in \Omega \setminus B(\xi_0, R), \\ \delta & x \in B(\xi_0, \frac{R}{2}), \\ \frac{2\delta}{R}(R - |\xi^{-1} \circ \xi_0|_{\mathbb{H}^n}) & x \in B(\xi_0, R) \setminus B(\xi_0, \frac{R}{2}). \end{cases} \tag{4.2}$$

On the one hand, for $t > 1$, one has

$$\Psi(tw) = \int_\Omega F(\xi, tw)\, d\xi \geq \int_{B(\xi_0,\frac{R}{2})} F(\xi, t\delta)\, d\xi.$$

On the other hand, according to the (AR) condition, there exist $D_1, D_2 > 0$ such that

$$F(\xi, t\delta) \geq D_1 t^\mu |\delta|^\mu - D_2.$$

Then

$$\Psi(tw) \geq D_1 t^\mu \int_{B(\xi_0,\frac{R}{2})} |\delta|^\mu\, d\xi - D_2|\Omega|.$$

Therefore, for $t > 1$, we deduce that

$$\begin{aligned} I_\lambda(tw) &= \Phi(tw) - \lambda\Psi(tw) \\ &= \int_\Omega \mathcal{A}(\xi, tw)\, d\xi + \int_\Omega \mathcal{B}(\xi, tw)\, d\xi + \frac{t^s}{s}\int_\Omega \vartheta(\xi)\frac{|w(\xi)|^s}{|\xi|^s}\, d\xi - \lambda\Psi(tw) \\ &\leq Ct^{\mathfrak{p}^+}\left(\left(\frac{2\delta}{R}\right) + \left(\frac{2\delta}{R}\right)^{\hat{\mathfrak{p}}}\right) - \lambda\int_{B(\xi_0,\frac{R}{2})} F(\xi, t\delta)\, d\xi \\ &\leq Ct^{\mathfrak{p}^+}\left(\left(\frac{2\delta}{R}\right) + \left(\frac{2\delta}{R}\right)^{\hat{\mathfrak{p}}}\right) - D_1 t^\mu \int_{B(\xi_0,\frac{R}{2})} |\delta|^\mu\, d\xi + D_2|\Omega|. \end{aligned}$$

Because $\mu > \mathfrak{p}^+$, so $\lim_{t\to+\infty} I_\lambda(tw) = -\infty$, and hence the claim follows.

The alternative of Theorem 1.1 case (b) assures the existence of an unbounded sequence $\{w_i\}_{i\in\mathbb{N}}$ of critical points of the functional I_λ. This completes the proof in view of the relation between the critical points of I_λ and the weak solutions of problem $(\mathcal{P})$. □

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