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Nondegeneracy of the bubble solutions for critical equations involving the polyharmonic operator

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Abstract

We reprove a result by Bartsch, Weth, and Willem (Calc. Var. Partial Differ. Equ. 18(3):253–268, 2003) concerning the nondegeneracy of bubble solutions for a critical semilinear elliptic equation involving the polyharmonic operator. The merit of our proof is that it does not rely on the comparison theorem. The argument of our proof mainly uses the stereographic projection with the Funk–Hecke formula, which works for general critical semilinear elliptic equations.

Keywords: Nondegeneracy; Polyharmonic operator; Bubble solution; Spherical harmonics

1 Introduction

We consider the nondegeneracy property of the bubble solutions for the critical semilinear equation involving the polyharmonic operator

$$\begin{cases} (-\Delta)^m u = |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}^{m,2}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $m < \frac{N}{2}$ is a positive integer, $2^* = \frac{2N}{N-2m}$, and $\mathcal{D}^{m,2}(\mathbb{R}^N)$ is the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \begin{cases} (\int_{\mathbb{R}^N} |\Delta^{\frac{m}{2}} u(x)|^2 \,\mathrm{d}x)^{\frac{1}{2}} & \text{if } m \text{ is even}, \\ (\int_{\mathbb{R}^N} |\nabla \Delta^{\frac{m}{2}} u(x)|^2 \,\mathrm{d}x)^{\frac{1}{2}} & \text{if } m \text{ is odd}. \end{cases}$$

For $m = 1$, the equation

$$\begin{cases} -\Delta u = |u|^{\frac{4}{N-2}}u & \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.2}$$

arises in the study of blowup solutions of the H^1 critical focusing nonlinear Schrödinger equation

$$iu_t - \Delta u = |u|^{\frac{4}{N-2}}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

and the H^1 critical focusing nonlinear wave equation

$$u_{tt} - \Delta u = |u|^{\frac{4}{N-2}}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

which have attracted the attention of a lot of scholars; see, for instance, [1–3].

For $m > 1$, (1.1) arises in the Q-curvature problem. Indeed, if u is a positive solution of (1.1), then the Q-curvature of the conformal metric $g = u^{\frac{4}{N-2m}}|dx|^2$ ($|dx|^2$ is the standard Euclidean metric on $\mathbb{R}^N$) is constant (see [4–6]). Swanson [7] showed that the function

$$\omega(x) = \left(\frac{\Gamma(\frac{N}{2}+m)}{\Gamma(\frac{N}{2}-m)}\right)^{\frac{N-2m}{4m}} \left(\frac{2}{1+|x|^2}\right)^{\frac{N-2m}{2}} \tag{1.3}$$

is a bubble solution of (1.1), which is also one extremal of the sharp Sobolev inequality

$$\pi^{\frac{m}{2}} 2^m \left(\frac{\Gamma(\frac{N}{2}+m)}{\Gamma(\frac{N}{2}-m)}\right)^{\frac{1}{2}} \left(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right)^{\frac{m}{N}} \|u\|_{2^*} \le \|u\| \quad \text{for all } u \in \mathcal{D}^{m,2}(\mathbb{R}^N) \setminus \{0\},$$

where $\|u\|_{2^*} = (\int_{\mathbb{R}^N} |u(x)|^{2^*}\,\mathrm{d}x)^{\frac{1}{2^*}}$.

Now noticing that (1.1) is invariant under scaling and translations, we observe that

$$(-\Delta)^m \omega_{\mu,z}(x) = \omega_{\mu,z}^{2^*-1}(x), \quad x \in \mathbb{R}^N, \tag{1.4}$$

where

$$\omega_{\mu,z}(x) = \mu^{\frac{N-2m}{2}} \omega(\mu x + z), \quad \mu \in (0,+\infty), z \in \mathbb{R}^N,$$

and ω is defined by (1.3). By differentiating (1.4) with respect to the parameters (μ, z) at $(1,0)$, we obtain that the $N+1$ linear independent functions

$$\frac{\partial \omega}{\partial x_j}(x), 1 \le j \le N, \qquad \Lambda\omega(x) = \frac{N-2}{2}\omega(x) + x \cdot \nabla\omega(x)$$

satisfy

$$(-\Delta)^m \varphi(x) - (2^*-1)\omega^{2^*-2}(x)\varphi(x) = 0. \tag{1.5}$$

A natural problem, arising in the study of bubbling phenomena of (1.1), is the nondegeneracy property of solution (1.3) for (1.1). More precisely, if φ is bounded and satisfies (1.5), then does φ belong to $\operatorname{span}\{\frac{\partial\omega}{\partial x_1}, \frac{\partial\omega}{\partial x_2}, \dots, \frac{\partial\omega}{\partial x_N}, \Lambda\omega\}$?

Bartsch, Weth, and Willem [8] obtained the nondegeneracy property of solution (1.3) for (1.1). More precisely, we have the following:

Theorem 1.1 ([8]) *For any bounded φ satisfying*

$$(-\Delta)^m\varphi(x) - \left(2^* - 1\right)\omega^{2^*-2}(x)\varphi(x) = 0,$$

we have

$$\varphi \in \operatorname{span}\left\{\frac{\partial\omega}{\partial x_1}, \frac{\partial\omega}{\partial x_2}, \dots, \frac{\partial\omega}{\partial x_N}, \Lambda\omega\right\}.$$

Remark 1.2 We give several remarks on Theorem 1.1.

1. The argument used in our proof is different from those appeared in [8]. Our argument is inspired by Frank and Lieb [9, 10]; see also Dávilla, Del Pino, and Sire [11]. In this paper, we mainly use the stereographic projection argument combined with the Funk–Hecke formula, whereas the argument in [8] relies on the ODE technique with comparison theorem.
2. The nondegeneracy property of the bubble solutions for (1.1) plays a crucial role in the construction of multibubble solutions to (1.1); see, for instance, [12–14].

2 Proof of Theorem 1.1

For simplicity of notation, let us denote

$$\rho(x) = \left(\frac{2}{1 + |x|^2}\right)^{\frac{1}{2}}. \tag{2.1}$$

Then using (1.3), we can rewrite (1.5) as follows:

$$(-\Delta)^m\varphi(x) - \frac{N+2m}{N-2m}\frac{\Gamma(\frac{N}{2}+m)}{\Gamma(\frac{N}{2}-m)}\rho^{4m}(x)\varphi(x) = 0. \tag{2.2}$$

First of all, since $\omega \in C^\infty(\mathbb{R}^N)$, for any bounded φ satisfying (2.2), using the standard elliptic regularity theory, we have $\varphi \in C^\infty(\mathbb{R}^N)$. Using the fact that $\omega^{2^*-2}(x) \lesssim \frac{1}{1+|x|^{4m}}$, we deduce that φ satisfies the integral equation

$$\varphi(x) = \alpha(N, m)\int_{\mathbb{R}^N}\frac{\rho^{4m}(y)}{|x-y|^{N-2m}}\varphi(y)\,\mathrm{d}y, \tag{2.3}$$

where

$$\alpha(N, m) = \frac{2\Gamma(\frac{N}{2}+m+1)}{2^{2m}\pi^{\frac{N}{2}}\Gamma(m)(N-2m)}. \tag{2.4}$$

Moreover, since φ is bounded, using the fact that (see [15] for instance),

$$\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N-2m}}\frac{1}{1+|y|^\theta}\,\mathrm{d}y \lesssim \begin{cases} \frac{1}{1+|x|^{\theta-2m}} & \text{if } 2m < \theta < N, \\ \frac{1+\log(1+|x|)}{1+|x|^{N-2m}} & \text{if } \theta = N, \\ \frac{1}{1+|x|^{N-2m}} & \text{if } \theta > N, \end{cases}$$

by a bootstrap argument we deduce that

$$\left|\varphi(x)\right| \lesssim \frac{1}{1+|x|^{N-2m}}. \tag{2.5}$$

Next, to transform the integral equation (2.3) on $\mathbb{R}^N$ into the corresponding integral equation on $\mathbb{S}^N$, let us introduce the stereographic projection

$$\begin{aligned} &\mathcal{S}:\mathbb{R}^N \mapsto \mathbb{S}^N \setminus \left\{(0,0,\dots,0,-1)\right\},\\ &x \mapsto \left(\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}\right), \end{aligned}$$

where $\mathbb{S}^N = \{\xi = (\xi_1, \xi_2, \dots, \xi_{N+1}) \in \mathbb{R}^{N+1} | \sum_{j=1}^{N+1} \xi_j^2 = 1\}$. A direct computation implies that (see also [10, 16])

$$|\mathcal{S}x - \mathcal{S}y| = |x-y|\rho(x)\rho(y). \tag{2.6}$$

For any $f:\mathbb{R}^N \mapsto \mathbb{R}$, let us denote

$$\mathcal{S}_* f(\xi) = \frac{f(\mathcal{S}^{-1}\xi)}{\rho^{N-2m}(\mathcal{S}^{-1}\xi)}, \tag{2.7}$$

where $\mathcal{S}^{-1}: \mathbb{S}^N \setminus \{(0,0,\dots,0,-1)\} \mapsto \mathbb{R}^N$ is the inverse of the stereographic projection:

$$\mathcal{S}^{-1}(\xi_1, \xi_2, \dots, \xi_{N+1}) = \left(\frac{\xi_1}{1+\xi_{N+1}}, \frac{\xi_2}{1+\xi_{N+1}}, \dots, \frac{\xi_N}{1+\xi_{N+1}}\right). \tag{2.8}$$

Moreover, for any $F \in L^1(\mathbb{S}^N)$, we have the identity[1]

$$\int_{\mathbb{S}^N} f\left(\mathcal{S}^{-1}\xi\right) \mathrm{d}\xi = \int_{\mathbb{R}^N} f(x)\rho^{2N}(x)\,\mathrm{d}x. \tag{2.9}$$

By $\mathscr{H}_k^{N+1} (k \geq 0)$ we denote the mutually orthogonal space of the restriction on $\mathbb{S}^N$ of real harmonic polynomials, homogeneous of degree of k on $\mathbb{R}^{N+1}$. Moreover, we have the following orthogonal decomposition:

$$L^2\left(\mathbb{S}^N\right) = \bigoplus_{k=0}^{\infty} \mathscr{H}_k^{N+1}. \tag{2.10}$$

Especially, we have

$$\mathscr{H}_1^{N+1} = \operatorname{span}\{\xi_j | 1 \leq j \leq N+1\}. \tag{2.11}$$

For further analysis, we need the following Funk–Hecke lemma.

[1] The Jacobian of the stereographic projection is $\rho^{2N}(x)$.

Lemma 2.1 ([17, 18]) *Let* $\lambda \in (0,N)$. *For any* $Y \in \mathscr{H}_k^{N+1}$,

$$\int_{\mathbb{S}^N} \frac{1}{|\xi - \eta|^\lambda} Y(\eta)\,\mathrm{d}\eta = \mu_k(\lambda) Y(\xi), \tag{2.12}$$

where

$$\mu_k(\lambda) = 2^{N-\lambda}\pi^{\frac{N}{2}} \frac{\Gamma(k+\frac{\lambda}{2})\Gamma(\frac{N-\lambda}{2})}{\Gamma(\frac{\lambda}{2})\Gamma(k+N-\frac{\lambda}{2})}. \tag{2.13}$$

Now let us turn our attention to the integral equation (2.3). On the one hand, using (2.6), we have

$$\frac{1}{|x-y|^{N-2m}} = \frac{\rho^{N-2m}(x)\rho^{N-2m}(y)}{|\mathcal{S}x - \mathcal{S}y|^{N-2m}}. \tag{2.14}$$

By inserting (2.14) into (2.3) we immediately get

$$\varphi(x) = \alpha(N,m)\rho^{N-2m}(x) \int_{\mathbb{R}^N} \frac{1}{|\mathcal{S}x - \mathcal{S}y|^{N-2m}} \frac{\varphi(y)}{\rho^{N-2m}(y)} \rho^{2N}(y)\,\mathrm{d}y. \tag{2.15}$$

On the other hand, by (2.5) we have

$$\int_{\mathbb{R}^N} \left|\varphi(x)\right|^2 \rho^{4m}(x)\,\mathrm{d}x \lesssim \int_{\mathbb{R}^N} \frac{1}{1+|x|^{2N}}\,\mathrm{d}x < +\infty,$$

which, together with (2.7), implies that

$$\int_{\mathbb{S}^N} \left|\mathcal{S}_*\varphi(\xi)\right|^2 \mathrm{d}\xi < +\infty. \tag{2.16}$$

Hence inserting (2.7) into (2.15) and using (2.16), we deduce that $\mathcal{S}_*\varphi \in L^2(\mathbb{S}^N)$ satisfies

$$\mathcal{S}_*\varphi(\xi) = \alpha(N,m) \int_{\mathbb{R}^N} \frac{\mathcal{S}_*\varphi(\eta)}{|\xi - \eta|^{N-2m}}\,\mathrm{d}\eta. \tag{2.17}$$

Now observe that

$$\alpha(N,m) = \frac{1}{\mu_1(N-2m)},$$

where $\mu_1(N-2m)$ is defined by (2.13) with $k = 1$ and $\lambda = N - 2m$. Therefore using Equation (2.12), from (2.17) we obtain

$$\mathcal{S}_*\varphi \in \mathscr{H}_1^{N+1},$$

which, together with (2.11), implies that

$$\mathcal{S}_*\varphi \in \operatorname{span}\{\xi_j | 1 \le j \le N+1\}. \tag{2.18}$$

Therefore using the definition of $\mathcal{S}_*\varphi$ (see (2.7)), we deduce from (2.18) that

$$\varphi(x) \in \operatorname{span}\left\{\rho^{N-2m+2}(x)x_1, \rho^{N-2m+2}(x)x_2, \dots, \rho^{N-2m+2}(x)x_N, \rho^{N-2m+1}(x)\left(1-|x|^2\right)\right\},$$

which, together with (1.3) and (2.1), implies that

$$\varphi \in \operatorname{span}\left\{\frac{\partial\omega}{\partial x_1}, \frac{\partial\omega}{\partial x_2}, \dots, \frac{\partial\omega}{\partial x_N}, \Lambda\omega\right\}.$$

This ends the proof of Theorem 1.1.

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