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The existence of discrete solitons for the discrete coupled nonlinear Schrödinger system

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Abstract

In this paper, we investigate the nonlinear coupled discrete Schrödinger equations with unbounded potentials. We find simple sufficient conditions for the existence of discrete soliton solution by using the Nehari manifold approach and the compact embedding theorem. Furthermore, by comparing the value of the action functional at the discrete soliton solution with those at nonzero solutions of one component zero, we demonstrate that both components of the discrete soliton solution are nontrivial.

Keywords: Nonlinear Schrödinger lattice; Discrete solitons; Nehari manifold approach; Compact embedding theorem

1 Introduction

The discrete nonlinear Schrödinger equation appears in a number of areas in physics such as nonlinear optics [1], biomolecular chains [2], and Bose–Einstein condensates [3]. In these areas and many others, it often emerges as a tight-binding approximation for the underlying continuum description (e.g., Bose–Einstein condensates trapped in optical lattices), or via an envelope wave expansion of the physical field (e.g., the electromagnetic wave in an optical system). In the past decades, due to wide applications of difference equations [4–7], extensive efforts have been devoted to the study of the existence of discrete solitons (also called discrete breathers) for the equation. Several sufficient conditions for the existence have been found by means of the continuation method [8–11], which is a powerful tool for both theoretical analysis and numerical computations [12–15], and the variational methods relying on the critical point techniques (Nehari manifold [16, 17], linking theorems [18–22]).

In this paper, we consider a system of two linearly coupled discrete nonlinear Schrödinger equations. Our interest in the two-component system has been motivated particularly by the experimental realization and control of mixtures of Bose–Einstein condensates composed either by two hyperfine states [23, 24] or by two species [25, 26]. For the binary Bose–Einstein condensate mixture confined in a one-dimensional lattice, the system of the two coupled discrete nonlinear Schrödinger equations takes the form

$$
\begin{aligned}
i\frac{du_n}{dt} &= -(\Delta u)_n + b_{1n}u_n - a_1|u_n|^2u_n - a_3|v_n|^2u_n - a_4v_n, \\
i\frac{dv_n}{dt} &= -(\Delta v)_n + b_{2n}v_n - a_2|v_n|^2v_n - a_3|u_n|^2v_n - a_4u_n,
\end{aligned}
\tag{1.1}
$$

where a_1, a_2, a_3, and a_4 are positive constants, $\{b_{1n}\}$ and $\{b_{2n}\}$ are real-valued sequences, and Δ is the discrete Laplacian operator defined as $(\Delta u)_n = u_{n+1} + u_{n-1} - 2u_n$. In applications to Bose–Einstein condensates (the condensation for a mixture of two interacting species), $u_n(v_n)$ denotes the wavefunction of condensate $u(v)$ on site n, a_i plays the role of intraspecies (a_1, a_2) and interspecies (a_3), respectively, a_4 is the strength of the electromagnetic field, and $\{b_{in}\}$ is the external potential. In this paper, the potential $V_i = \{b_{in}\}_{n\in\mathbb{Z}}$ is bounded below and satisfies

$$
\lim_{|n|\to\infty} b_{in} = +\infty, \quad i = 1, 2.
\tag{1.2}
$$

In this paper, we study the discrete solitons of (1.1), i.e., the solutions of (1.1) of the special form

$$
u_n = \exp(-i\omega t)\phi_n, \qquad v_n = \exp(-i\omega t)\psi_n, \quad n \in \mathbb{Z},
\tag{1.3}
$$

with real amplitudes ϕ_n and ψ_n, and

$$
\lim_{|n|\to\infty} u_n = 0, \qquad \lim_{|n|\to\infty} v_n = 0.
\tag{1.4}
$$

Inserting the ansatz of the discrete solitons (1.3) into (1.1), we obtain the following equivalent algebraic system:

$$
\begin{aligned}
&-(\Delta\phi)_n - \omega\phi_n + b_{1n}\phi_n - a_1|\phi_n|^2\phi_n - a_3|\psi_n|^2\phi_n - a_4\psi_n = 0, \\
&-(\Delta\psi)_n - \omega\psi_n + b_{2n}\psi_n - a_2|\psi_n|^2\psi_n - a_3|\phi_n|^2\psi_n - a_4\phi_n = 0,
\end{aligned}
\tag{1.5}
$$

and (1.4) becomes

$$
\lim_{|n|\to\infty} \phi_n = 0, \quad \text{and} \quad \lim_{|n|\to\infty} \psi_n = 0.
\tag{1.6}
$$

We are thus led to the study on the existence of solutions for system (1.5) with conditions (1.6). Indeed, our approach covers the problem for a slightly more general system

$$
\begin{aligned}
&(L_1\phi)_n - \omega_1\phi_n - a_1|\phi_n|^2\phi_n - a_3|\psi_n|^2\phi_n - a_4\psi_n = 0, \\
&(L_2\psi)_n - \omega_2\psi_n - a_2|\psi_n|^2\psi_n - a_3|\phi_n|^2\psi_n - a_4\phi_n = 0,
\end{aligned}
\tag{1.7}
$$

with the same boundary conditions as (1.6), where ϕ_n, ψ_n, ω_1, and ω_2 are the unknowns. Here L_1 and L_2 are the second-order difference operators defined by

$$
L_1\phi_n = -(\Delta\phi)_n + b_{1n}\phi_n \quad \text{and} \quad L_2\psi_n = -(\Delta\psi)_n + b_{2n}\phi_n.
$$

Hence (1.5) is a particular case of (1.7) with $\omega_1 = \omega_2$.

The soliton dynamics of (1.1) was studied in [27], where the authors performed a unitary transformation to remove the linear coupling. It helped them determine the soliton dynamics by using the known results for a single-component discrete nonlinear Schrödinger equation and showed that the soliton solutions oscillate from one species to the other. In [28] the authors claimed the existence of a standing wave of (1.1). However, they did not justify whether the solution has both nonzero components. In fact, if one of (u_n, v_n) is identically zero, then (1.1) reduces to a single equation. In this paper, we not only obtain the existence of discrete soliton solutions for system (1.7), but also show that both components of the discrete soliton solution are nontrivial. For the case $a_4 = 0$, the discrete soliton solutions of (1.1) were discussed in [29].

The main idea of this paper is as follows. First, by using the Nehari manifold approach we obtain that the sequence $\{(\phi^{(k)}, \psi^{(k)})\} \subset N$ is bounded in $E_1 \times E_2$. Then we prove that the limit of $\{(\phi^{(k)}, \psi^{(k)})\}$ exists and is the solution of (1.7) in $l^2 \times l^2$ by using the compact embedding theorem. Finally, we show that both components of the discrete soliton solution are not zero.

This paper is organized as follows. In Sect. 2, we introduce some preliminaries, including a discrete version of compact embedding theorem. In Sect. 3, we prove some key lemmas on the Nehari manifold. In Sect. 4, we present and prove our main result on the existence of a nontrivial discrete soliton solution that minimizes the corresponding action functional of (1.6) and (1.7). Moreover, we prove that both components of the discrete soliton solution are not zero.

2 Preliminaries

In this section, we describe the functional setting that will be used for our treatment of the infinite nonlinear system (1.7). We first introduce a compact embedding theorem.

Consider the real sequence spaces

$$l^p = \left\{\phi = \{\phi_n\}_{n\in\mathbb{Z}} : \|\phi\|_p = \left(\sum_{n\in\mathbb{Z}} |\phi_n|^p\right)^{\frac{1}{p}} < \infty, \phi_n \in \mathbb{R}, \forall n \in \mathbb{Z}\right\}.$$

Among l^p spaces, we have the following elementary embedding relation:

$$l^q \subset l^p, \qquad \|\phi\|_p \le \|\phi\|_q, \quad 1 \le q \le p \le \infty.$$

When $p = 2$, it becomes the usual Hilbert space l^2 endowed with the scalar product

$$(\phi, \psi) = \sum_{n\in\mathbb{Z}} \phi_n \psi_n, \quad \phi, \psi \in l^2.$$

The spectrum of $-\Delta$ in l^2 coincides with the interval $[0, 4]$. It is known that

$$0 \le (-\Delta\phi, \phi) \le 4\|\phi\|_2^2 \quad \forall \phi \in l^2.$$

Define

$$E_i = \left\{\phi \in l^2 : (L_i\phi, \phi) < \infty\right\}, \qquad \|\phi\|_{E_i} = (L_i\phi, \phi)^{\frac{1}{2}}, \quad i = 1, 2.$$

The following lemma can be found in [17].

Lemma 2.1 *If V_1 and V_2 satisfy (1.2), then E_i, $i = 1,2$, is compactly embedded into l^p for each $2 \leq p \leq \infty$, with the best embedding constant $\alpha_{ip} = \max_{\|\phi\|_p=1} 1/\|\phi\|_{E_i}$. Furthermore, the spectrum $\sigma(L_i)$ is discrete.*

We define the action functional on $E_1 \times E_2$ as

$$\begin{aligned} J(\phi,\psi) &= \frac{1}{2}\big((L_1-\omega_1)\phi,\phi\big) + \frac{1}{2}\big((L_2-\omega_2)\psi,\psi\big) \\ &\quad - a_4(\phi,\psi) - \frac{1}{4}\sum_{n\in\mathbb{Z}}\big(a_1\phi_n^4 + a_2\psi_n^4 + 2a_3\phi_n^2\psi_n^2\big). \end{aligned} \tag{2.1}$$

By Lemma 2.1 it follows that the action functional $J(\phi,\psi) \in C^1(E_1 \times E_2, \mathbb{R})$ and (1.7) corresponds to $J'(\phi,\psi) = 0$. So we define

$$\begin{aligned} I(\phi,\psi) &= \big(J'(\phi,\psi),(\phi,\psi)\big) \\ &= \big((L_1-\omega_1)\phi,\phi\big) + \big((L_2-\omega_2)\psi,\psi\big) \\ &\quad - 2a_4(\phi,\psi) - \sum_{n\in\mathbb{Z}}\big(a_1\phi_n^4 + a_2\psi_n^4 + 2a_3\phi_n^2\psi_n^2\big) \end{aligned} \tag{2.2}$$

and the Nehari manifold

$$N = \big\{(\phi,\psi) \in E_1 \times E_2 : I(\phi,\psi) = 0, (\phi,\psi) \neq 0\big\}. \tag{2.3}$$

3 Some lemmas on the Nehari manifold

Let

$$\lambda_i = \inf\big\{\sigma(L_i)\big\}, \quad i = 1,2.$$

Throughout the paper, we will assume the following hypothesis on ω_i:

(H) $\omega_i < \min\{\lambda_1 - a_4, \lambda_2 - a_4\}$, $i = 1,2$.

To prove the main result, we need the following lemmas on the Nehari manifold.

Lemma 3.1 *Assume that (H) and (1.2) hold. Then the Nehari manifold N is a nonempty closed C^1 submanifold in $E_1 \times E_2$. Moreover, for any $(\phi,\psi) \in E_1 \times E_2 - \{(0,0)\}$, there is a unique point $\tau(\phi,\psi) \in (0,\infty)$ such that $(\tau(\phi,\psi)\phi, \tau(\phi,\psi)\psi) \in N$, and $t = \tau(\phi,\psi)$ is the unique maximum value point of $J(t\phi,t\psi)$ in $t \in (0,\infty)$. We have the following equality:*

$$J\big(\tau(\phi,\psi)\phi, \tau(\phi,\psi)\psi\big) = \frac{I_1^2(\phi,\psi)}{4I_2(\phi,\psi)}, \tag{3.1}$$

where

$$\begin{aligned} I_1(\phi,\psi) &= \|\phi\|_{E_1}^2 - \omega_1\|\phi\|_2^2 + \|\psi\|_{E_2}^2 - \omega_2\|\psi\|_2^2 - 2a_4(\phi,\psi), \\ I_2(\phi,\psi) &= \sum_{n\in\mathbb{Z}}\big(a_1\phi_n^4 + a_2\psi_n^4 + 2a_3\phi_n^2\psi_n^2\big). \end{aligned}$$

Proof We first show that $N \neq \emptyset$. We rewrite (2.1) and (2.2) as

$$J(\phi,\psi) = \frac{1}{2}I_1(\phi,\psi) - \frac{1}{4}I_2(\phi,\psi) \tag{3.2}$$

and

$$I(\phi,\psi) = I_1(\phi,\psi) - I_2(\phi,\psi), \tag{3.3}$$

respectively. Then

$$I(t\phi,t\psi) = t^2 I_1(\phi,\psi) - t^4 I_2(\phi,\psi). \tag{3.4}$$

Since $a_4 < \min\{\lambda_1 - \omega_1, \lambda_2 - \omega_2\}$, we have

$$\begin{aligned} I_1(\phi,\psi) &= \|\phi\|_{E_1}^2 - \omega_1\|\phi\|_2^2 + \|\psi\|_{E_2}^2 - \omega_2\|\psi\|_2^2 - 2a_4(\phi,\psi) \\ &\geq \|\phi\|_{E_1}^2 - \omega_1\|\phi\|_2^2 + \|\psi\|_{E_2}^2 - \omega_2\|\psi\|_2^2 - a_4\|\phi\|_2^2 - a_4\|\psi\|_2^2 \\ &\geq (\lambda_1 - \omega_1 - a_4)\|\phi\|_2^2 + (\lambda_2 - \omega_2 - a_4)\|\psi\|_2^2 \\ &> 0. \end{aligned} \tag{3.5}$$

Clearly, $I_2(\phi,\psi) > 0$. Therefore by (3.4) and (3.5) we see that $I(t\phi,t\psi) > 0$ for $t > 0$ small enough and $I(t\phi,t\psi) < 0$ for $t > 0$ large enough. As a consequence, there exists $\tau(\phi,\psi) > 0$ such that $I(\tau(\phi,\psi)\phi, \tau(\phi,\psi)\psi) = 0$, that is, $(\tau(\phi,\psi)\phi, \tau(\phi,\psi)\psi) \in N$. In fact, from (3.4) we have

$$\tau(\phi,\psi) = \left(\frac{I_1(\phi,\psi)}{I_2(\phi,\psi)}\right)^{\frac{1}{2}}. \tag{3.6}$$

Moreover, in view of (2.1), we have

$$J\big(\tau(\phi,\psi)\phi, \tau(\phi,\psi)\psi\big) = \frac{1}{2}\tau^2(\phi,\psi)I_1(\phi,\psi) - \frac{1}{4}\tau^4(\phi,\psi)I_2(\phi,\psi) = \frac{I_1^2(\phi,\psi)}{4I_2(\phi,\psi)}.$$

Let $(\phi,\psi) \in N$. By (2.2) and the definition of N we obtain

$$\begin{aligned} \big(I'(\phi,\psi),(\phi,\psi)\big) &= \big(I'(\phi,\psi),(\phi,\psi)\big) - 2I(\phi,\psi) \\ &= -2\sum_{n\in\mathbb{Z}}\big(a_1(\phi_n)^4 + a_2(\psi_n)^4 + 2a_3(\phi_n)^2(\psi_n)^2\big) \\ &< 0. \end{aligned}$$

Hence $I' \neq 0$, and the implicit function theorem implies that N is a closed C^1 submanifold in $E_1 \times E_2$. □

Lemma 3.2 *Assume that* (H) *and* (1.2) *hold. Then there exists* $\eta > 0$ *such that* $J(\phi,\psi) \geq \eta$ *for all* $(\phi,\psi) \in N$.

Proof Since λ_1 is the smallest eigenvalue of L_1 and λ_2 is the smallest eigenvalue of L_2, from the definition of the constants α_{1p} and α_{2p} we get $\lambda_1 = 1/\alpha_{12}^2$ and $\lambda_2 = 1/\alpha_{22}^2$. For any $(\phi,\psi) \in N$, we have

$$\begin{aligned}
&\|\phi\|_{E_1}^2 - \omega_1\|\phi\|_2^2 + \|\psi\|_{E_2}^2 - \omega_2\|\psi\|_2^2 - 2a_4(\phi,\psi) \\
&\quad = \sum_{n\in\mathbb{Z}}\left(a_1\phi_n^4 + a_2\psi_n^4 + 2a_3\phi_n^2\psi_n^2\right) \\
&\quad \le a^*\left(\|\phi\|_4^2 + \|\psi\|_4^2\right)^2 \\
&\quad \le a^*\left(\|\phi\|_2^2 + \|\psi\|_2^2\right)^2 \\
&\quad \le a^*\left(\alpha_{12}^2\|\phi\|_{E_1}^2 + \alpha_{22}^2\|\psi\|_{E_2}^2\right)^2 \\
&\quad \le a^*\gamma_2^2\left(\|\phi\|_{E_1}^2 + \|\psi\|_{E_2}^2\right)^2,
\end{aligned} \tag{3.7}$$

where $a^* = \max\{a_1,a_2,a_3\}$ and $\gamma_2 = \max\{\alpha_{12}^2,\alpha_{22}^2\}$.

Let

$$\gamma_1 = \min\left\{1, 1 - \frac{\omega_1 + a_4}{\lambda_1}, 1 - \frac{\omega_2 + a_4}{\lambda_2}\right\}.$$

Then

$$\begin{aligned}
&\|\phi\|_{E_1}^2 - \omega_1\|\phi\|_2^2 + \|\psi\|_{E_2}^2 - \omega_2\|\psi\|_2^2 - 2a_4(\phi,\psi) \\
&\quad \ge \|\phi\|_{E_1}^2 - \omega_1\|\phi\|_2^2 + \|\psi\|_{E_2}^2 - \omega_2\|\psi\|_2^2 - a_4\|\phi\|_2^2 - a_4\|\psi\|_2^2 \\
&\quad \ge \gamma_1\left(\|\phi\|_{E_1}^2 + \|\psi\|_{E_2}^2\right).
\end{aligned} \tag{3.8}$$

By (3.7) and (3.8) we easily see that

$$\gamma_1\left(\|\phi\|_{E_1}^2 + \|\psi\|_{E_2}^2\right) \le a^*\gamma_2^2\left(\|\phi\|_{E_1}^2 + \|\psi\|_{E_2}^2\right)^2, \tag{3.9}$$

which implies that

$$\|\phi\|_{E_1}^2 + \|\psi\|_{E_2}^2 \ge \frac{\gamma_1}{a^*\gamma_2^2}. \tag{3.10}$$

Moreover, we have

$$\begin{aligned}
J(\phi,\psi) &= J(\phi,\psi) - \frac{1}{4}I(\phi,\psi) \\
&= \frac{1}{4}\left(\|\phi\|_{E_1}^2 - \omega_1\|\phi\|_2^2 + \|\psi\|_{E_2}^2 - \omega_2\|\psi\|_2^2 - 2a_4(\phi,\psi)\right) \\
&\ge \frac{\gamma_1}{4}\left(\|\phi\|_{E_1}^2 + \|\psi\|_{E_2}^2\right) \\
&\ge \frac{\gamma_1^2}{4a^*\gamma_2^2}.
\end{aligned} \tag{3.11}$$

Let $\eta = \gamma_1^2/(4a^*\gamma_2^2)$. Then we get $J(\phi,\psi) \ge \eta$ for all $(\phi,\psi) \in N$. □

4 Main results

Now we state the main result of this paper.

Theorem 4.1 *Assume that* (H) *and* (1.2) *hold. Then system* (1.7) *has at least one nontrivial discrete soliton solution* (ϕ^*, ψ^*) *in* $E_1 \times E_2$ *with* $\phi^* \neq 0$ *and* $\psi^* \neq 0$.

By a discrete soliton solution of system (1.7) we mean a minimizer of the following constrained minimization problem:

$$d \equiv \inf_{(\phi,\psi)\in N} J(\phi, \psi). \tag{4.1}$$

We will prove that minimizers of problem (4.1) are solutions of system (1.7) with (1.6). Such solutions give the lowest possible value of the action functional J among nontrivial solutions of system (1.7) in $E_1 \times E_2$.

Remark 4.1 If $(\phi, \psi) \in E_1 \times E_2$, then (1.6) holds naturally.

Remark 4.2 In system (1.7) the positive constant a_4 can be replaced by a negative one, say a_5. In this case, we replace ψ_n by $-\psi_n$ and $-a_5$ by a_4, and the conclusion of Theorem 4.1 still holds.

Now we are ready to prove Theorem 4.1.

Proof Let d be given by (4.1). By Lemma 3.1, N is nonempty, and there exists a sequence $\{(\phi^{(k)}, \psi^{(k)})\} \subset N$ such that

$$d = \lim_{k\to\infty} J\bigl(\phi^{(k)}, \psi^{(k)}\bigr). \tag{4.2}$$

By Lemma 3.2, $d > 0$ and $d \leq \tilde{d} = \max_k \{J(\phi^{(k)}, \psi^{(k)})\} < \infty$. By (3.11) we have

$$\bigl\|\phi^{(k)}\bigr\|_{E_1}^2 + \bigl\|\psi^{(k)}\bigr\|_{E_2}^2 \leq \frac{4}{\gamma_1} J\bigl(\phi^{(k)}, \psi^{(k)}\bigr) \leq \frac{4\tilde{d}}{\gamma_1} < \infty.$$

Thus the sequences $\{\phi^{(k)}\}$ and $\{\psi^{(k)}\}$ are bounded in the Hilbert spaces E_1 and E_2, respectively. Therefore there exist a subsequence of $\{\phi^{(k)}\}$ and a subsequence of $\{\psi^{(k)}\}$, denoted again by $\{\phi^{(k)}\}$ and $\{\psi^{(k)}\}$ for simplicity, that converge weakly to some $\phi^* \in E_1$ and $\psi^* \in E_2$, respectively. By Lemma 2.1 we get, for each $2 \leq p \leq \infty$,

$$\lim_{k\to\infty} \phi^{(k)} = \phi^*, \qquad \lim_{k\to\infty} \psi^{(k)} = \psi^*, \quad \text{in } l^p. \tag{4.3}$$

By (2.1) and (2.3) we know that

$$J\bigl(\phi^{(k)}, \psi^{(k)}\bigr) = \frac{1}{4} \sum_{n\in\mathbb{Z}} \bigl(a_1\bigl(\phi_n^{(k)}\bigr)^4 + a_2\bigl(\psi_n^{(k)}\bigr)^4 + 2a_3\bigl(\phi_n^{(k)}\bigr)^2\bigl(\psi_n^{(k)}\bigr)^2\bigr).$$

Now we claim that

$$\begin{aligned}&\lim_{k\to\infty}\sum_{n\in\mathbb{Z}}\left(a_1\left(\phi_n^{(k)}\right)^4+a_2\left(\psi_n^{(k)}\right)^4+2a_3\left(\phi_n^{(k)}\right)^2\left(\psi_n^{(k)}\right)^2\right)\\&\quad=\sum_{n\in\mathbb{Z}}\left(a_1\left(\phi_n^*\right)^4+a_2\left(\psi_n^*\right)^4+2a_3\left(\phi_n^*\right)^2\left(\psi_n^*\right)^2\right).\end{aligned}\tag{4.4}$$

According to (4.3), it suffices to show that

$$\lim_{k\to\infty}\sum_{n\in\mathbb{Z}}\left(\phi_n^{(k)}\right)^2\left(\psi_n^{(k)}\right)^2=\sum_{n\in\mathbb{Z}}\left(\phi_n^*\right)^2\left(\psi_n^*\right)^2.\tag{4.5}$$

Indeed,

$$\begin{aligned}&\left|\sum_{n\in\mathbb{Z}}\left(\phi_n^{(k)}\right)^2\left(\psi_n^{(k)}\right)^2-\sum_{n\in\mathbb{Z}}\left(\phi_n^*\right)^2\left(\psi_n^*\right)^2\right|\\&\quad\le\sum_{n\in\mathbb{Z}}\left|\phi_n^{(k)}-\phi_n^*\right|\left|\phi_n^{(k)}+\phi_n^*\right|\left(\psi_n^{(k)}\right)^2+\sum_{n\in\mathbb{Z}}\left|\psi_n^{(k)}-\psi_n^*\right|\left|\psi_n^{(k)}+\psi_n^*\right|\left(\phi_n^*\right)^2.\end{aligned}$$

Thus the Hölder inequality and (4.3) imply (4.5).

Next, we will show that $(\phi^*,\psi^*)\in N$ and $J(\phi^*,\psi^*)=d$. Since E_i is a Hilbert space for $i=1,2$, by (4.4) we have

$$\begin{aligned}&\|\phi^*\|_{E_1}^2+\|\psi^*\|_{E_2}^2\\&\quad=\left\|\text{weak-}\lim_{k\to\infty}\phi^{(k)}\right\|_{E_1}^2+\left\|\text{weak-}\lim_{k\to\infty}\psi^{(k)}\right\|_{E_2}^2\\&\quad\le\liminf_{k\to\infty}\|\phi^{(k)}\|_{E_1}^2+\liminf_{k\to\infty}\|\psi^{(k)}\|_{E_2}^2\\&\quad\le\liminf_{k\to\infty}\left(\|\phi^{(k)}\|_{E_1}^2+\|\psi^{(k)}\|_{E_2}^2\right)\\&\quad=\liminf_{k\to\infty}\sum_{n\in\mathbb{Z}}\left(a_1\left(\phi_n^{(k)}\right)^4+a_2\left(\psi_n^{(k)}\right)^4+2a_3\left(\phi_n^{(k)}\right)^2\left(\psi_n^{(k)}\right)^2\right)\\&\qquad+\liminf_{k\to\infty}\left(\omega_1\|\phi^{(k)}\|_2^2+\omega_2\|\psi^{(k)}\|_2^2+2a_4\left(\phi^{(k)},\psi^{(k)}\right)\right)\\&\quad=\sum_{n\in\mathbb{Z}}\left(a_1\left(\phi_n^*\right)^4+a_2\left(\psi_n^*\right)^4+2a_3\left(\phi_n^*\right)^2\left(\psi_n^*\right)^2\right)\\&\qquad+\omega_1\|\phi^*\|_2^2+\omega_2\|\psi^*\|_2^2+2a_4\left(\phi^*,\psi^*\right),\end{aligned}$$

which implies

$$\begin{aligned}I\left(\phi^*,\psi^*\right)&=\|\phi^*\|_{E_1}^2-\omega_1\|\phi^*\|_2^2+\|\psi^*\|_{E_2}^2-\omega_2\|\psi^*\|_2^2-2a_4\left(\phi^*,\psi^*\right)\\&\quad-\sum_{n\in\mathbb{Z}}\left(a_1\left(\phi_n^*\right)^4+a_2\left(\psi_n^*\right)^4+2a_3\left(\phi_n^*\right)^2\left(\psi_n^*\right)^2\right)\\&\le 0.\end{aligned}$$

Through an argument similar to the proof of Lemma 3.1, we know that $I(t\phi^*,t\psi^*)$ is positive as t is small enough. Therefore there exists $t^*\in(0,1]$ such that $I(t^*\phi^*,t^*\psi^*)=0$, which implies $(t^*\phi^*,t^*\psi^*)\in N$. Thus we have $J(t^*\phi^*,t^*\psi^*)=(1/4)W(t^*)$ and by (4.4), $W(1)=4d$, where

$$W(t) = t^4 \sum_{n\in\mathbb{Z}} \bigl(a_1(\phi_n^*)^4 + a_2(\psi_n^*)^4 + 2a_3(\phi_n^*)^2(\psi_n^*)^2\bigr).$$

Clearly, $W(t)$ is strictly increasing on $0 < t < \infty$. Therefore by (4.1)

$$d \le J(t^*\phi^*, t^*\psi^*) = \frac{1}{4}W(t^*) \le \frac{1}{4}W(1) = d.$$

This implies that $t^* = 1$ and $J(\phi^*, \psi^*) = d$.

Now let us prove that (ϕ^*, ψ^*) is a nontrivial solution to system (1.7). Since (ϕ^*, ψ^*) is an energy minimizer on the Nehari manifold N, there exists a Lagrange multiplier Λ such that

$$\bigl(J'(\phi^*, \psi^*) + \Lambda I'(\phi^*, \psi^*), (\phi, \psi)\bigr) = 0 \tag{4.6}$$

for all $(\phi, \psi) \in E_1 \times E_2$. Let $(\phi, \psi) = (\phi^*, \psi^*)$ in (4.6). Then $(J'(\phi^*, \psi^*), (\phi^*, \psi^*)) = I(\phi^*, \psi^*) = 0$ implies that

$$\Lambda\bigl(I'(\phi^*, \psi^*), (\phi^*, \psi^*)\bigr) = 0.$$

However,

$$\begin{aligned}
&\bigl(I'(\phi^*, \psi^*), (\phi^*, \psi^*)\bigr)\\
&\quad = 2\bigl((L_1 - \omega_1)\phi^*, \phi^*\bigr) + 2\bigl((L_2 - \omega_2)\psi^*, \psi^*\bigr) - 4a_4(\phi^*, \psi^*)\\
&\qquad - 4\sum_{n\in\mathbb{Z}} \bigl(a_1(\phi_n^*)^4 + a_2(\psi_n^*)^4 + 2a_3(\phi_n^*)^2(\psi_n^*)^2\bigr)\\
&\quad = -2\sum_{n\in\mathbb{Z}} \bigl(a_1(\phi_n^*)^4 + a_2(\psi_n^*)^4 + 2a_3(\phi_n^*)^2(\psi_n^*)^2\bigr) < 0.
\end{aligned} \tag{4.7}$$

Thus $\Lambda = 0$, and

$$\bigl(J'(\phi^*, \psi^*), (\phi, \psi)\bigr) = 0 \tag{4.8}$$

for all $(\phi, \psi) \in E_1 \times E_2$. Take $(\phi, \psi) = (e^{(k)}, 0)$ and $(\phi, \psi) = (0, e^{(k)})$ in (4.8) for $k \in \mathbb{Z}$, where

$$e_n^{(k)} = \begin{cases} 1, & n = k,\\ 0, & n \ne k. \end{cases}$$

We see that $J'(\phi^*, \psi^*) = 0$. Thus (ϕ^*, ψ^*) is a nontrivial solution to system (1.7).

Finally, we will show that $\phi^* \ne 0$ and $\psi^* \ne 0$. In fact, if one of the components of (ϕ^*, ψ^*), say $\psi^* = 0$, then $\phi^* \ne 0$. Since $\phi^* \ne 0$, there exists an integer K such that $\phi_K^* \ne 0$. Let $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$, where

$$\varphi_n = \begin{cases} \phi_n^*, & |n| \le |K|,\\ 0, & |n| \ge |K| + 1. \end{cases}$$

Obviously, $\varphi \in E_2 - \{0\}$.

For ϵ small enough, we consider $(\phi^*, \epsilon\varphi) \in (E_1 - \{0\}) \times (E_2 - \{0\})$.
For simplicity, let

$$B = \sum_{n\in\mathbb{Z}} \left(\phi_n^*\right)^4, \qquad B_K = \sum_{n=-K}^{K} \left(\phi_n^*\right)^4, \qquad D = \left((L_2 - \omega_2)\varphi, \varphi\right).$$

Then, for ϵ small enough, we have

$$\begin{aligned}
I_1^2\left(\phi^*, \epsilon\varphi\right) &= \left(\left((L_1 - \omega_1)\phi^*, \phi^*\right) + \left((L_2 - \omega_2)\epsilon\varphi, \epsilon\varphi\right) - 2a_4\left(\phi^*, \epsilon\varphi\right)\right)^2 \\
&= \left(a_1 B + D\epsilon^2 - 2a_4\|\varphi\|_2^2\epsilon\right)^2 \\
&= a_1^2 B^2 - 4a_1 a_4 B\|\varphi\|_2^2\epsilon + \left(2a_1 BD + 4a_4^2\|\varphi\|_2^4\right)\epsilon^2 \\
&\quad - 4a_4 D\|\varphi\|_2^2\epsilon^3 + D^2\epsilon^4 \\
&< a_1^2 B^2 + 2a_1 a_3 BB_K\epsilon^2 + a_1 a_2 BB_K\epsilon^4 \\
&= a_1 B\left(a_1 B + a_2 B_K\epsilon^4 + 2a_3 B_K\epsilon^2\right) \\
&= I_2\left(\phi^*, 0\right) I_2\left(\phi^*, \epsilon\varphi\right).
\end{aligned} \tag{4.9}$$

By Lemma 3.1, $(\tau(\phi^*, \epsilon\varphi)\phi^*, \tau(\phi^*, \epsilon\varphi)\epsilon\varphi) \in N$, and by (4.9) we have

$$\begin{aligned}
J\left(\tau\left(\phi^*, \epsilon\varphi\right)\phi^*, \tau\left(\phi^*, \epsilon\varphi\right)\epsilon\varphi\right) &= \frac{I_1^2(\phi^*, \epsilon\varphi)}{4I_2(\phi^*, \epsilon\varphi)} \\
&< \frac{1}{4} I_2\left(\phi^*, 0\right) \\
&= J\left(\phi^*, 0\right) \\
&= \inf_{(\phi,\psi)\in N} J(\phi, \psi).
\end{aligned}$$

This is a contradiction. So, $\psi^* \not\equiv 0$. □

5 Conclusion

We have studied the discrete solitons in the discrete coupled nonlinear Schrödinger equations with unbounded potentials. These solutions are obtained by using the Nehari manifold approach and the compact embedding theorem. However, we do not know if both components of the discrete soliton solution are nontrivial. Fortunately, we successfully found a way to prove that both components are nontrivial, namely, by comparing the value of the action functional at the discrete soliton solution with ones at nonzero solutions of one component zero. We think that our method can be applied to a variety of discrete models, especially to the N-component discrete coupled nonlinear Schrödinger equations. However, this method seems hard for the coupled discrete nonlinear Schrödinger equations with general nonlinear terms, because a formula of the action functional is not obtained and is left for our future work.

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Availability of data and materials
Data sharing is not applicable to this paper as no data sets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
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