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Asymptotic behavior of plate equations driven by colored noise on unbounded domains

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Abstract

This paper investigates mainly the asymptotic behavior of the nonautonomous random dynamical systems generated by the plate equations driven by colored noise defined on $\mathbb{R}^n$. First, we prove the well-posedness of the equation in the natural energy space. Secondly, we define a continuous cocycle associated with the solution operator. Finally, we establish the existence and uniqueness of random attractors of the equation by the uniform tail-ends estimates methods and the splitting technique.

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Keywords: Plate equation; Colored noise; Asymptotic compactness; Random attractor; Unbounded domain

1 Introduction

Colored noise was first introduced in [17, 21] in order to obtain information on the velocity of randomly moving particles, which cannot be obtained from white noise since the Wiener process is nowhere differentiable. Moreover, for many physical systems, the stochastic fluctuations are correlated and should be modeled by colored noise rather than white noise, see [14].

This paper is concerned with the asymptotic behavior of the plate equation driven by nonlinear colored noise in unbounded domains:

$$\begin{cases} u_{tt} + \alpha u_t + \Delta^2 u + \nu u + f(x,u) = g(x,t) + h(t,x,u)\zeta_\delta(\theta_t\omega), & t > \tau, x \in \mathbb{R}^n, \\ u(x,\tau) = u_0(x), \qquad u_t(x,\tau) = u_{1,0}(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\tau \in \mathbb{R}$, α, ν are positive constants, f and h are given nonlinearity, $g \in L^2_{\mathrm{loc}}(\mathbb{R}, H^1(\mathbb{R}^n))$, and ζ_δ is a colored noise with correlation time $\delta > 0$.

The existence and uniqueness of pathwise random attractors of stochastic plate equations have been studied in [12, 13, 15, 16] in the case of bounded domains; and in [30–35] in the case of unbounded domains. We also mention that the global attractors of deterministic plate equations have been investigated in [2, 7, 9, 10, 24, 26–29, 37] in bounded domains, and in [5, 6, 11, 25, 36] on unbounded domains.

In all these publications ([30–35]), only the additive white noise and linear multiplicative white noise were considered. Note that the random equation (1.1) is driven by colored noise rather than white noise. In general, it is very difficult to study the asymptotic dynamics of differential equations driven by nonlinear white noise, including the random attractors. Indeed, only when the white noise is linear can the stochastic equations be transformed into a deterministic equations, then one can obtain the existence of random attractors of the plate equation (1.1). However, this transformation does not apply to stochastic equations driven by nonlinear white noise, and that is why we are currently unable to prove the existence of random attractors for systems with nonlinear white noise.

For the colored noise, even if it is nonlinear, we are able to show system (1.1) has a random attractor in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, which is quite different from the nonlinear white noise. The reader is referred to [3, 4, 22, 23] for more details on random attractors of differential equations driven by colored noise. In this paper, instead of using white noise, we will consider the random equation (1.1) driven by nonlinear colored noise. The main aim of this paper is to obtain the existence and uniqueness of random attractors for (1.1) when the diffusion term h is a nonlinear continuous function.

Note that system (1.1) is defined in the unbounded domain $\mathbb{R}^n$ where the noncompactness of Sobolev embeddings on unbounded domains gives rise to difficulty in showing the pullback asymptotic compactness of solutions; to overcome this we use the tail-estimates method (as in [18]) and the splitting technique to obtain the pullback asymptotic compactness.

The rest of this article consists of four sections. In the next section, we define some functions sets and recall some useful results. In Sect. 3, we first establish the existence, uniqueness, and continuity of solutions in initial data of (1.1) in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, then define a nonautonomous random dynamical system based on the solution operator of problem (1.1). The last two sections are devoted to deriving necessary estimates of solutions of (1.1) and the existence of random attractors.

Throughout the paper, the inner product and the norm of $L^2(\mathbb{R}^n)$ will be denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. The letters c and c_i $(i = 1, 2, \dots)$ are generic positive constants that may depend on some parameters in the contexts.

2 Asymptotic compactness of cocycles

In this section, we define some functions sets and recall some useful results, see [19, 20]. These results will be used to establish the asymptotic compactness of the solutions and attractor for the random plate equation defined on the entire space $\mathbb{R}^n$.

From now on, we assume $(\Omega, \mathcal{F}, P)$ is the canonical probability space where $\Omega = \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}$ with compact-open topology, $\mathcal{F}$ is the Borel σ-algebra of Ω, and P is the Wiener measure on $(\Omega, \mathcal{F})$. Recall the standard group of transformations $\{\theta_t\}_{t\in\mathbb{R}}$ on Ω:

$$\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t), \quad \forall t \in \mathbb{R} \text{ and } \forall \omega \in \Omega.$$

Let X be a Banach space with norm $\|\cdot\|_X$. Suppose $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is a continuous cocycle on X over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. Let $\mathcal{D}$ be a collection of some families of the nonempty subset of X:

$$\mathcal{D} = \big\{D = \big\{D(\tau, \omega) \subseteq X : D(\tau, \omega) \neq \emptyset, \tau \in \mathbb{R}, \omega \in \Omega\big\}\big\}.$$

Suppose Φ has a $\mathcal{D}$-pullback absorbing set $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$; that is, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D \in \mathcal{D}$ there exists $T = T(\tau,\omega,D) > 0$ such that for all $t \geq T$,

$$\Phi\big(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)\big) \subseteq K(\tau,\omega). \tag{2.1}$$

Assume that

$$\Phi(t,\tau,\omega,x) = \Phi_1(t,\tau,\omega,x) + \Phi_2(t,\tau,\omega,x), \quad \forall t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega, x \in X, \tag{2.2}$$

where both Φ_1 and Φ_2 are mappings from $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times X$ to X.

Given $k \in \mathbb{N}$, denote by $\mathcal{O}_k = \{x \in \mathbb{R}^n : |x| < k\}$ and $\tilde{\mathcal{O}}_k = \{x \in \mathbb{R}^n : |x| > k\}$. Let X be a Banach space with norm $\|\cdot\|_X$ that consists of some functions defined on $\mathbb{R}^n$. Given a function $u : \mathbb{R}^n \to \mathbb{R}$, the restrictions of u to $\mathcal{O}_k$ and $\tilde{\mathcal{O}}_k$ are written as $u|_{\mathcal{O}_k}$ and $u|_{\tilde{\mathcal{O}}_k}$, respectively. Denote by

$$X_{\mathcal{O}_k} = \{u|_{\mathcal{O}_k} : u \in X\} \quad \text{and} \quad X_{\tilde{\mathcal{O}}_k} = \{u|_{\tilde{\mathcal{O}}_k} : u \in X\}.$$

Suppose $X_{\mathcal{O}_k}$ and $X_{\tilde{\mathcal{O}}_k}$ are Banach spaces with norm $\|\cdot\|_{\mathcal{O}_k}$ and $\|\cdot\|_{\tilde{\mathcal{O}}_k}$, respectively, and

$$\|u\|_X \leq \|u|_{\mathcal{O}_k}\|_{\mathcal{O}_k} + \|u|_{\tilde{\mathcal{O}}_k}\|_{\tilde{\mathcal{O}}_k}, \quad \forall u \in X. \tag{2.3}$$

We further assume that for every $\delta > 0$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$, there exists $t_0 = t_0(\delta,\tau,\omega,K) > 0$ and $k_0 = k_0(\delta,\tau,\omega) \geq 1$ such that

$$\big\|\Phi(t_0, \tau - t_0, \theta_{-t_0}\omega, x)|_{\tilde{\mathcal{O}}_{k_0}}\big\|_{\tilde{\mathcal{O}}_{k_0}} < \delta, \quad \forall x \in K(\tau - t_0, \theta_{-t_0}\omega), \tag{2.4}$$

and

$$\begin{aligned} &\Phi_1\big(t_0, \tau - t_0, \theta_{-t_0}\omega, K(\tau - t_0, \theta_{-t_0}\omega)\big)|_{\mathcal{O}_{k_0}} \\ &\quad \text{has a finite cover of balls of radius } \delta \text{ in } X|_{\mathcal{O}_{k_0}}. \end{aligned} \tag{2.5}$$

In addition, we assume that for every $k \in \mathbb{N}$, $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$, the set

$$\Phi_2\big(t, \tau - t, \theta_{-t}\omega, K(\tau - t, \theta_{-t}\omega)\big) \quad \text{is precompact in } X|_{\mathcal{O}_k}. \tag{2.6}$$

Theorem 2.1 *If* (2.1)–(2.6) *hold, then the cocycle* Φ *is* $\mathcal{D}$*-pullback asymptotically compact in* X; *that is, the sequence* $\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ *is precompact in* X *for any* $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D \in \mathcal{D}$, $t_n \to \infty$ *monotonically, and* $x_n \in D(\tau - t_n, \theta_{-t_n}\omega)$.

Theorem 2.2 *Let* $\mathcal{D}$ *be an inclusion closed collection of some families of nonempty bounded subsets of* X, *and* Φ *be a continuous cocycle on* X *over* $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. *Then,* Φ *has a unique* $\mathcal{D}$*-pullback random attractor* $\mathcal{A}$ *in* $\mathcal{D}$ *if* Φ *is* $\mathcal{D}$*-pullback asymptotically compact in* X *and* Φ *has a closed measurable* $\mathcal{D}$*-pullback absorbing set* K *in* $\mathcal{D}$.

3 Cocycles of random plate equations

In this section, we first establish the existence of a solution for problem (1.1), then we define a nonautonomous cocycle of (1.1).

Given $\delta > 0$, let $\zeta_\delta(\theta_t\omega)$ be the unique stationary solution of the stochastic equation:

$$d\zeta_\delta + \frac{1}{\delta}\zeta_\delta\, dt = \frac{1}{\delta}\, dW, \tag{3.1}$$

where W is a two-sided real-valued Wiener process on $(\Omega, \mathcal{F}, P)$. The process $\zeta_\delta(\theta_t\omega)$ is called one-dimensional colored noise. Recall that there exists a θ_t-invariant subset of full measure (see [1]), which is still denoted by Ω, such that for all $\omega \in \Omega$, $\zeta_\delta(\theta_t\omega)$ is continuous in $t \in \mathbb{R}$ and

$$\lim_{t\to\pm\infty} \frac{|\zeta_\delta(\theta_t\omega)|}{t} = 0, \quad \text{for } 0 < \delta \le 1.$$

Let $-\Delta$ denote the Laplace operator in $\mathbb{R}^n$, $A = \Delta^2$ with the domain $D(A) = H^4(\mathbb{R}^n)$. We can also define the powers A^ν of A for $\nu \in \mathbb{R}$. The space $V_\nu = D(A^{\frac{\nu}{4}})$ is a Hilbert space with the following inner product and norm

$$(u, v)_\nu = \bigl(A^{\frac{\nu}{4}}u, A^{\frac{\nu}{4}}v\bigr), \qquad \|\cdot\|_\nu = \bigl\|A^{\frac{\nu}{4}}\cdot\bigr\|.$$

We introduce the following hypotheses to complete the uniform estimates.

Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $F(x,r) = \int_0^r f(x,s)\, ds$ for all $x \in \mathbb{R}^n$, $r \in \mathbb{R}$ and $s, s_1, s_2 \in \mathbb{R}$,

$$\liminf_{|s|\to\infty} \inf_{x\in\mathbb{R}^n} \bigl(f(x,s)s\bigr) > 0, \tag{3.2}$$

$$f(x,0) = 0, \qquad \bigl|f(x,s_1) - f(x,s_2)\bigr| \le \alpha_1\bigl(\varphi(x) + |s_1|^p + |s_2|^p\bigr)|s_1 - s_2|, \tag{3.3}$$

$$F(x,s) + \varphi_1(x) \ge 0, \tag{3.4}$$

where $p > 0$ for $1 \le n \le 4$ and $0 < p \le \frac{4}{n-4}$ for $n \ge 5$, α_1 is a positive constant, $\varphi_1 \in L^1(\mathbb{R}^n)$, and $\varphi \in L^\infty(\mathbb{R}^n)$.

Let $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \times\mathbb{R}$ be continuous such that for all $t, s, s_1, s_2 \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$\bigl|h(t,x,s)\bigr| \le \alpha_2|s| + \varphi_2(t,x), \tag{3.5}$$

$$\bigl|h(t,x,s_1) - h(t,x,s_2)\bigr| \le \alpha_3|s_1 - s_2|, \tag{3.6}$$

where α_2 and α_3 are positive constants, and $\varphi_2 \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))$.

Definition 3.1 Given $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$, $u_0 \in H^2(\mathbb{R}^n)$, and $u_{1,0} \in L^2(\mathbb{R}^n)$, a function $u(\cdot, \tau, \omega, u_0, u_{1,0}) : [\tau, \tau + T] \to H^2(\mathbb{R}^n)$ is called a (weak) solution of (1.1) if the following conditions are fulfilled:

(i) $u(\cdot, \tau, \omega, u_0, u_{1,0}) \in L^\infty(\tau, \tau + T; H^2(\mathbb{R}^n)) \cap C([\tau, \tau + T], L^2(\mathbb{R}^n))$ with $u(\tau, \tau, \omega, u_0, u_{1,0}) = u_0$, $u_t(\cdot, \tau, \omega, u_0, u_{1,0}) \in L^\infty(\tau, \tau + T; L^2(\mathbb{R}^n)) \cap C([\tau, \tau + T], L^2(\mathbb{R}^n))$ with $u_t(\tau, \tau, \omega, u_0, u_{1,0}) = u_{1,0}$.

(ii) $u(t, \tau, \cdot, u_0, u_{1,0}) : \Omega \to H^2(\mathbb{R}^n)$ is $(\mathcal{F}, \mathcal{B}(H^2(\mathbb{R}^n))$-measurable, and $u_t(t, \tau, \cdot, u_0, u_{1,0}) : \Omega \to L^2(\mathbb{R}^n)$ is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^n))$-measurable.

(iii) For all $\xi \in C_0^\infty((\tau,\tau+T)\times\mathbb{R}^n)$,

$$\begin{aligned}
&-\int_\tau^{\tau+T}(u_t,\xi_t)\,dt+\alpha\int_\tau^{\tau+T}(u_t,\xi)\,dt+\int_\tau^{\tau+T}(\Delta u,\Delta\xi)\,dt\\
&\quad+\nu\int_\tau^{\tau+T}(u,\xi)\,dt+\int_\tau^{\tau+T}\int_{\mathbb{R}^n}f\bigl(x,u(t,x)\bigr)\xi(t,x)\,dx\,dt\\
&=\int_\tau^{\tau+T}\bigl(g(t,x),\xi\bigr)\,dt+\int_\tau^{\tau+T}\int_{\mathbb{R}^n}h\bigl(t,x,u(t,x)\bigr)\zeta_\delta(\theta_t\omega)\xi(t,x)\,dx\,dt.
\end{aligned}$$

In order to investigate the long-time dynamics, we are now ready to prove the existence and uniqueness of solutions of (1.1). We first recall the following well-known existence and uniqueness of solutions for the corresponding linear plate equations of (1.1).

Lemma 3.1 *Let $u_0\in H^2(\mathbb{R}^n)$, $u_{1,0}\in L^2(\mathbb{R}^n)$ and $g\in L^1(\tau,\tau+T;L^2(\mathbb{R}^n))$ with $\tau\in\mathbb{R}$ and $T>0$. Then, the linear plate equation*

$$u_{tt}+\alpha u_t+\Delta^2u+\nu u=g(t),\qquad \tau<t\le\tau+T,$$

with the initial conditions

$$u(\tau)=u_0,\quad \text{and}\quad u_t(\tau)=u_{1,0},$$

possesses a unique solution u in the sense of Definition 3.1. In addition,

$$u\in C\bigl([\tau,\tau+T],H^2\bigl(\mathbb{R}^n\bigr)\bigr)\quad \text{and}\quad u_t\in C\bigl([\tau,\tau+T],L^2\bigl(\mathbb{R}^n\bigr)\bigr),$$

and there exists a positive number C depending only on ν (but independent of τ, T, u_0, $u_{1,0}$, and g) such that for all $t\in[\tau,\tau+T]$,

$$\bigl\|u(t)\bigr\|_{H^2(\mathbb{R}^n)}+\bigl\|u_t(t)\bigr\|\le C\biggl(\|u_0\|_{H^2(\mathbb{R}^n)}+\|u_{1,0}\|+\int_\tau^{\tau+T}\bigl\|g(t)\bigr\|\,dt\biggr). \tag{3.7}$$

Furthermore, the solution u satisfies the energy equation

$$\frac{d}{dt}\bigl(\|u_t\|^2+\|\Delta u\|^2+\nu\|u\|^2\bigr)=-2\alpha\|u_t\|^2+2\bigl(g(t),u_t\bigr), \tag{3.8}$$

and

$$\frac{d}{dt}\bigl(u(t),u_t(t)\bigr)+\alpha\bigl(u(t),u_t(t)\bigr)+\bigl\|\Delta u(t)\bigr\|^2+\nu\bigl\|u(t)\bigr\|^2=\bigl\|u_t(t)\bigr\|^2+\bigl(g(t),u(t)\bigr), \tag{3.9}$$

for almost all $t\in[\tau,\tau+T]$.

Theorem 3.1 *Let $\tau\in\mathbb{R}$, $u_0\in H^2(\mathbb{R}^n)$, $u_{1,0}\in L^2(\mathbb{R}^n)$. Suppose (3.2)–(3.6) hold, then:*

(a) *Problem (1.1) possesses a solution u in the sense of Definition 3.1;*

(b) *The solution u to problem (1.1) is unique, continuous with initial data in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, and*

$$u\in C\bigl([\tau,\tau+T],H^2\bigl(\mathbb{R}^n\bigr)\bigr)\quad \text{and}\quad u_t\in C\bigl([\tau,\tau+T],L^2\bigl(\mathbb{R}^n\bigr)\bigr). \tag{3.10}$$

Moreover, the solution u to problem (1.1) *satisfies the energy equation*:

$$\frac{d}{dt}\left(\|u_t\|^2 + \nu\|u\|^2 + \|\Delta u\|^2 + 2\int_{\mathbb{R}^n} F\big(x, u(t,x)\big)\,dx\right) + 2\alpha\|u_t\|^2 = 2\big(g(t), u_t\big) + 2\zeta_\delta(\theta_t\omega)\int_{\mathbb{R}^n} h\big(t,x,u(t,x)\big)u_t(t,x)\,dx \tag{3.11}$$

for almost all $t \in [\tau, \tau + T]$.

Proof The proof will be divided into four steps. We first construct a sequence of approximate solutions, and then derive uniform estimates, in the last two steps we take the limit of those approximate solutions to prove the uniqueness of the solutions.

Step (i): Approximate solutions Given $k \in \mathbb{N}$, define a function $\eta_k : \mathbb{R} \to \mathbb{R}$ by

$$\eta_k(s) = \begin{cases} s, & \text{if } -k \le s \le k, \\ k, & \text{if } s > k, \\ -k, & \text{if } s < -k. \end{cases} \tag{3.12}$$

Then, for every fixed $k \in \mathbb{N}$, the function η_k as defined by (3.12) is bounded and Lipschitz continuous; more precisely, for all $s, s_1, s_2 \in \mathbb{R}$

$$\eta_k(0) = 0, \qquad \big|\eta_k(s)\big| \le |s| \quad \text{and} \quad \big|\eta_k(s_1) - \eta_k(s_2)\big| \le |s_1 - s_2|. \tag{3.13}$$

For all $x \in \mathbb{R}^n$ and $t, s \in \mathbb{R}$, denote

$$f_k(x,s) = f\big(x, \eta_k(s)\big), \qquad F_k(x,s) = \int_0^s f_k(x,r)\,dr \quad \text{and} \\ h_k(t,x,s) = h\big(t,x,\eta_k(s)\big). \tag{3.14}$$

By (3.2) we know that there exists $k_0 \in \mathbb{N}$ such that for all $|s| \ge k_0$ and $x \in \mathbb{R}^n$,

$$f(x,s)s > 0, \tag{3.15}$$

thus, for all $k \ge k_0$ and $x \in \mathbb{R}^n$,

$$f_k(x,k) > 0, \qquad f_k(x,-k) < 0. \tag{3.16}$$

By (3.3), (3.4), (3.13), (3.14), and (3.16) we know that for all $s, s_1, s_2 \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$\big|f_k(x,s_1) - f_k(x,s_2)\big| \le \alpha_1\big(\varphi(x) + |s_1|^p + |s_2|^p\big)|s_1 - s_2|, \quad \forall k \ge 1, \tag{3.17}$$

and

$$F_k(x,s) + \varphi_1(x) \ge 0, \quad \forall k \ge k_0. \tag{3.18}$$

By (3.17) we obtain that for all $s \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$\big|F_k(x,s)\big| \le \alpha_1\big(\varphi(x)|s|^2 + |s|^{p+2}\big), \quad \forall k \ge 1. \tag{3.19}$$

By (3.5), (3.6), (3.13), and (3.14) we obtain that for all $k \geq 1$, $t, s, s_1, s_2 \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$\left|h_k(t,x,s)\right| \leq \alpha_2 |s| + \varphi_2(t,x), \tag{3.20}$$

$$\left|h_k(t,x,s_1) - h_k(t,x,s_2)\right| \leq \alpha_3 |s_1 - s_2|. \tag{3.21}$$

By (3.3), (3.13), and (3.14), we find that for all $k \in \mathbb{N}$, $s, s_1, s_2 \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$\left|f_k(x,s)\right| \leq \alpha_1 k\big(\varphi(x) + k^p\big), \tag{3.22}$$

$$\left|f_k(x,s_1) - f_k(x,s_2)\right| \leq \alpha_1 \big(\varphi(x) + 2k^p\big)|s_1 - s_2|. \tag{3.23}$$

For every $k \in \mathbb{N}$, consider the following approximate system for u_k:

$$\begin{cases} \frac{\partial^2}{\partial t^2} u_k + \alpha \frac{\partial}{\partial t} u_k + \Delta^2 u_k + \nu u_k + f_k(\cdot, u_k) = g(\cdot, t) + h_k(t, \cdot, u_k)\zeta_\delta(\theta_t \omega), \quad t > \tau, \\ u_k(\tau) = u_0, \qquad \frac{\partial}{\partial t} u_k(\tau) = u_{1,0}. \end{cases} \tag{3.24}$$

From (3.21), (3.23), $\varphi \in L^\infty(\mathbb{R}^n)$, and the standard method (see, e.g., [8]), it follows that for each $\tau \in \mathbb{R}$, $\omega \in \Omega$, $u_0 \in H^2(\mathbb{R}^n)$, $u_{1,0} \in L^2(\mathbb{R}^n)$, problem (3.24) has a unique global solution u_k defined on $[\tau, \tau + T]$ for every $T > 0$ in the sense of Definition 3.1. In particular, $u_k(\cdot, \tau, \omega, u_0) \in C([\tau, \tau + T], H^2(\mathbb{R}^n))$ and $u_k(t, \tau, \omega, u_0)$ is measurable with respect to $\omega \in \Omega$ in $H^2(\mathbb{R}^n)$ for every $t \in [\tau, \tau + T]$. Similarly, $\partial_t u_k(\cdot, \tau, \omega, u_0) \in C([\tau, \tau + T], L^2(\mathbb{R}^n))$ and $\partial_t u_k(t, \tau, \omega, u_0)$ is measurable with respect to $\omega \in \Omega$ in $L^2(\mathbb{R}^n)$ for every $t \in [\tau, \tau + T]$. Furthermore, the solution u_k satisfies the energy equation:

$$\begin{aligned} &\frac{d}{dt}\left(\|\partial_t u_k\|^2 + \nu\|u_k\|^2 + \|\Delta u_k\|^2 + 2\int_{\mathbb{R}^n} F_k\big(x, u_k(t,x)\big)\,dx\right) + 2\alpha\|\partial_t u_k\|^2 \\ &\quad = 2\big(g(t), \partial_t u_k\big) + 2\zeta_\delta(\theta_t\omega)\int_{\mathbb{R}^n} h_k\big(t, x, u_k(t,x)\big)\partial_t u_k(t,x)\,dx \end{aligned} \tag{3.25}$$

for almost all $t \in [\tau, \tau + T]$. Next, we use the energy equation (3.25) to derive a uniform estimate on the sequence $\{u_k\}_{k=1}^\infty$.

Step (ii): Uniform estimates

For the last term on the right-hand side of (3.25), by (3.21) we have

$$\begin{aligned} &2\zeta_\delta(\theta_t\omega)\int_{\mathbb{R}^n} h_k\big(t, x, u_k(t,x)\big)\partial_t u_k(t,x)\,dx \\ &\quad \leq 2\left|\zeta_\delta(\theta_t\omega)\right|\left(\alpha_2\int_{\mathbb{R}^n}\left|u_k(t,x)\right| \cdot \left|\partial_t u_k(t,x)\right|dx + \int_{\mathbb{R}^n}\left|\varphi_2(t,x)\right| \cdot \left|\partial_t u_k(t,x)\right|dx\right) \\ &\quad \leq \left|\zeta_\delta(\theta_t\omega)\right|\big(\alpha_2\left\|u_k(t)\right\|^2 + (1+\alpha_2)\left\|\partial_t u_k(t)\right\|^2 + \left\|\varphi_2(t)\right\|^2\big). \end{aligned} \tag{3.26}$$

By Young's inequality, we obtain

$$2\big(g(t), \partial_t u_k\big) \leq \left\|\partial_t u_k(t)\right\|^2 + \left\|g(t)\right\|^2. \tag{3.27}$$

By (3.25)–(3.27), it follows that for almost all $t \in [\tau, \tau + T]$,

$$\frac{d}{dt}\left(\|\partial_t u_k\|^2 + \nu\|u_k\|^2 + \|\Delta u_k\|^2 + 2\int_{\mathbb{R}^n} F_k\big(x, u_k(t,x)\big)\,dx\right) + 2\alpha\|\partial_t u_k\|^2$$

$$\leq c_1\bigl(1+\bigl|\zeta_\delta(\theta_t\omega)\bigr|\bigr)\bigl(\bigl\|u_k(t)\bigr\|^2+\bigl\|\partial_t u_k(t)\bigr\|^2\bigr)+\bigl|\zeta_\delta(\theta_t\omega)\bigr|\cdot\bigl\|\varphi_2(t)\bigr\|^2+\bigl\|g(t)\bigr\|^2, \tag{3.28}$$

where $c_1>0$ depends only on α_2, but independent of k.

By (3.18) and (3.28) we obtain

$$\begin{aligned}&\frac{d}{dt}\biggl(\|\partial_t u_k\|^2+\nu\|u_k\|^2+\|\Delta u_k\|^2+2\int_{\mathbb{R}^n}F_k\bigl(x,u_k(t,x)\bigr)\,dx\biggr)\\&\quad\leq c_2\bigl(1+\bigl|\zeta_\delta(\theta_t\omega)\bigr|\bigr)\biggl(\bigl\|\partial_t u_k(t)\bigr\|^2+\nu\bigl\|u_k(t)\bigr\|^2+\|\Delta u_k\|^2+2\int_{\mathbb{R}^n}F_k\bigl(x,u_k(t,x)\bigr)\,dx\biggr)\\&\qquad+\bigl|\zeta_\delta(\theta_t\omega)\bigr|\cdot\bigl\|\varphi_2(t)\bigr\|^2+2c_1\bigl(1+\bigl|\zeta_\delta(\theta_t\omega)\bigr|\bigr)\|\varphi_1\|_{L^1(\mathbb{R}^n)}+\bigl\|g(t)\bigr\|^2,\end{aligned}\tag{3.29}$$

where $c_2>0$ depends only on ν and α_2, but is independent of k.

Multiplying (3.29) by $e^{-c_2\int_0^t(1+|\zeta_\delta(\theta_r\omega)|)\,dr}$, and then integrating the inequality on (τ,t), we have

$$\begin{aligned}&\|\partial_t u_k\|^2+\nu\|u_k\|^2+\|\Delta u_k\|^2+2\int_{\mathbb{R}^n}F_k\bigl(x,u_k(t,x)\bigr)\,dx\\&\quad\leq e^{c_2\int_\tau^t(1+|\zeta_\delta(\theta_r\omega)|)\,dr}\biggl(\|u_{1,0}\|^2+\nu\|u_0\|^2+\|\Delta u_0\|^2+2\int_{\mathbb{R}^n}F_k\bigl(x,u_0(x)\bigr)\,dx\biggr)\\&\qquad+\int_\tau^t e^{c_2\int_s^t(1+|\zeta_\delta(\theta_r\omega)|)\,dr}\bigl(\bigl|\zeta_\delta(\theta_s\omega)\bigr|\cdot\bigl\|\varphi_2(s)\bigr\|^2\\&\qquad+2c_1\bigl(1+\bigl|\zeta_\delta(\theta_s\omega)\bigr|\bigr)\|\varphi_1\|_{L^1(\mathbb{R}^n)}+\bigl\|g(s)\bigr\|^2\bigr)\,ds.\end{aligned}\tag{3.30}$$

By (3.19) we obtain, for all $k\geq 1$,

$$\begin{aligned}2\int_{\mathbb{R}^n}\bigl|F_k\bigl(x,u_0(x)\bigr)\bigr|\,dx&\leq 2\alpha_1\bigl(\|\varphi\|_{L^\infty(\mathbb{R}^n)}\|u_0\|^2+\|u_0\|^{p+2}_{L^{p+2}(\mathbb{R}^n)}\bigr)\\&\leq 2\alpha_1\bigl(\|\varphi\|_{L^\infty(\mathbb{R}^n)}\|u_0\|^2+\|u_0\|^{p+2}_{H^2(\mathbb{R}^n)}\bigr).\end{aligned}\tag{3.31}$$

Equations (3.30) and (3.31) imply that there exists a positive constant $c_3=c_3(\tau,T,\varphi,\varphi_1,\varphi_2,g,\omega,\delta,\alpha_1,\nu)$ (but independent of k, u_0, and $u_{1,0}$) such that for all $t\in[\tau,\tau+T]$ and $k\geq 1$,

$$\begin{aligned}&\|\partial_t u_k\|^2+\nu\|u_k\|^2+\|\Delta u_k\|^2+2\int_{\mathbb{R}^n}F_k\bigl(x,u_k(t,x)\bigr)\,dx\\&\quad\leq c_3+c_3\bigl(1+\|u_{1,0}\|^2+\|u_0\|^{p+2}_{H^2(\mathbb{R}^n)}\bigr),\end{aligned}$$

which along with (3.18) show that for all $t\in[\tau,\tau+T]$ and $k\geq k_0$,

$$\begin{aligned}&\|\partial_t u_k\|^2+\nu\|u_k\|^2+\|\Delta u_k\|^2+2\int_{\mathbb{R}^n}F_k\bigl(x,u_k(t,x)\bigr)\,dx\\&\quad\leq c_3+2\|\varphi_1\|_{L^1(\mathbb{R}^n)}+c_3\bigl(1+\|u_{1,0}\|^2+\|u_0\|^{p+2}_{H^2(\mathbb{R}^n)}\bigr),\end{aligned}\tag{3.32}$$

thus,

$$\{u_k\}_{k=1}^\infty\quad\text{is bounded in } L^\infty\bigl(\tau,\tau+T;H^2\bigl(\mathbb{R}^n\bigr)\bigr)\tag{3.33}$$

and

$$\{\partial_t u_k\}_{k=1}^{\infty} \quad \text{is bounded in } L^{\infty}\big(\tau,\tau+T;L^2\big(\mathbb{R}^n\big)\big). \tag{3.34}$$

By (3.17), there exists a positive constant $c_4 = c_4(p,n,\alpha_1)$ such that

$$\int_{\mathbb{R}^n} \big|f_k\big(x,u_k(t,x)\big)\big|^2\,dx \le c_4\bigg(\int_{\mathbb{R}^n}\big|\varphi(x)\big|^2\,dx + \int_{\mathbb{R}^n}\big|u_k(t,x)\big|^{2(p+1)}\,dx\bigg),$$

which along with the embedding $H^2(\mathbb{R}^n) \hookrightarrow L^{2(p+1)}(\mathbb{R}^n)$ and the assumption $\varphi \in L^{\infty}(\mathbb{R}^n)$ implies that there exists $c_5 = c_5(p,n,\alpha_1,\varphi) > 0$ (independent of k) such that

$$\int_{\mathbb{R}^n} \big|f_k\big(x,u_k(t,x)\big)\big|^2\,dx \le c_5\big(1+\big\|u_k(t)\big\|_{H^2(\mathbb{R}^n)}^{2(p+1)}\big). \tag{3.35}$$

By (3.33) and (3.35) we see that

$$\big\{f_k(\cdot,u_k)\big\}_{k=1}^{\infty} \quad \text{is bounded in } L^2\big(\tau,\tau+T;L^2\big(\mathbb{R}^n\big)\big). \tag{3.36}$$

By (3.20) we obtain

$$\int_{\mathbb{R}^n} \big|h_k\big(t,x,u_k(t,x)\big)\big|^2\,dx \le 2\alpha_2\|u_k\|^2 + 2\big\|\varphi_2(t)\big\|^2,$$

which together with (3.33) shows that

$$\big\{h_k(\cdot,\cdot,u_k)\big\}_{k=1}^{\infty} \quad \text{is bounded in } L^2\big(\tau,\tau+T;L^2\big(\mathbb{R}^n\big)\big). \tag{3.37}$$

By (3.33), (3.34), (3.36), and (3.37), it follows that there exists $u \in L^{\infty}(\tau,\tau+T;H^2(\mathbb{R}^n))$ with $\partial_t u \in L^{\infty}(\tau,\tau+T;L^2(\mathbb{R}^n))$, $\kappa_1 \in L^2(\tau,\tau+T;L^2(\mathbb{R}^n))$, $\kappa_2 \in L^2(\tau,\tau+T;L^2(\mathbb{R}^n))$, $v^{\tau+T} \in H^2(\mathbb{R}^n)$ and $v_1^{\tau+T} \in L^2(\mathbb{R}^n)$ such that

$$u_k \to u \quad \text{weak-star in } L^{\infty}\big(\tau,\tau+T;H^2\big(\mathbb{R}^n\big)\big), \tag{3.38}$$

$$\partial_t u_k \to \partial_t u \quad \text{weak-star in } L^{\infty}\big(\tau,\tau+T;L^2\big(\mathbb{R}^n\big)\big), \tag{3.39}$$

$$f_k(\cdot,u_k) \to \kappa_1 \quad \text{weakly in } L^2\big(\tau,\tau+T;L^2\big(\mathbb{R}^n\big)\big), \tag{3.40}$$

$$h_k(\cdot,\cdot,u_k) \to \kappa_2 \quad \text{weakly in } L^2\big(\tau,\tau+T;L^2\big(\mathbb{R}^n\big)\big), \tag{3.41}$$

$$u_k(\tau+T) \to v^{\tau+T} \quad \text{weakly in } H^2\big(\mathbb{R}^n\big), \tag{3.42}$$

$$\partial_t u_k(\tau+T) \to v_1^{\tau+T} \quad \text{weakly in } L^2\big(\mathbb{R}^n\big). \tag{3.43}$$

It follows from (3.38) and (3.39) that there exists a subsequence that is still denoted u_k, such that

$$u_k(t,x) \to u(t,x) \quad \text{for almost all } (t,x) \in [\tau,\tau+T]\times\mathbb{R}^n. \tag{3.44}$$

By (3.13) and (3.44) we obtain that for almost all $(t,x) \in [\tau,\tau+T]\times\mathbb{R}^n$,

$$\begin{aligned}
&\big|\eta_k\big(u_k(t,x)\big) - u(t,x)\big| \\
&\quad\le \big|\eta_k\big(u_k(t,x)\big) - \eta_k\big(u(t,x)\big)\big| + \big|\eta_k\big(u(t,x)\big) - u(t,x)\big|
\end{aligned}$$

$$\leq \left|u_k(t,x) - u(t,x)\right| + \left|\eta_k\big(u(t,x)\big) - u(t,x)\right| \to 0, \quad \text{as } k \to \infty. \tag{3.45}$$

By (3.45), we have

$$f_k\big(x, u_k(t,x)\big) \to f\big(x, u(t,x)\big) \quad \text{for almost all } (t,x) \in [\tau, \tau + T] \times \mathbb{R}^n, \tag{3.46}$$

$$h_k\big(t, x, u_k(t,x)\big) \to h\big(t, x, u(t,x)\big) \quad \text{for almost all } (t,x) \in [\tau, \tau + T] \times \mathbb{R}^n. \tag{3.47}$$

It follows from (3.40), (3.41), (3.46), and (3.47) that

$$f_k(\cdot, u_k) \to f(\cdot, u) \quad \text{weakly in } L^2\big(\tau, \tau + T; L^2(\mathbb{R}^n)\big), \tag{3.48}$$

$$h_k(\cdot,\cdot, u_k) \to h(\cdot,\cdot, u) \quad \text{weakly in } L^2\big(\tau, \tau + T; L^2(\mathbb{R}^n)\big). \tag{3.49}$$

Step (iii): Existence of solutions

Choosing an arbitrary $\xi \in C_0^\infty((\tau, \tau + T) \times \mathbb{R}^n)$. By (3.24) we obtain

$$\begin{aligned}
&-\int_\tau^{\tau+T} (\partial_t u_k, \xi_t)\, dt + \alpha \int_\tau^{\tau+T} (\partial_t u_k, \xi)\, dt \\
&\quad + \int_\tau^{\tau+T} (\Delta u_k, \Delta \xi)\, dt + \nu \int_\tau^{\tau+T} (u_k, \xi)\, dt \\
&\quad + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} f_k\big(x, u_k(t,x)\big)\xi(t,x)\, dx\, dt \\
&= \int_\tau^{\tau+T} \big(g(t), \xi\big)\, dt + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} h_k\big(t, x, u_k(t,x)\big)\zeta_\delta(\theta_t\omega)\xi(t,x)\, dx\, dt.
\end{aligned} \tag{3.50}$$

Letting $k \to \infty$ in (3.50), it follows from (3.38), (3.39), (3.48), and (3.49) that for any $\xi \in C_0^\infty((\tau, \tau + T) \times \mathbb{R}^n)$,

$$\begin{aligned}
&-\int_\tau^{\tau+T} (u_t, \xi_t)\, dt + \alpha \int_\tau^{\tau+T} (u_t, \xi)\, dt + \int_\tau^{\tau+T} (\Delta u, \Delta \xi)\, dt + \nu \int_\tau^{\tau+T} (u, \xi)\, dt \\
&\quad + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} f\big(x, u(t,x)\big)\xi(t,x)\, dx\, dt \\
&= \int_\tau^{\tau+T} \big(g(t), \xi\big)\, dt + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} h\big(t, x, u(t,x)\big)\zeta_\delta(\theta_t\omega)\xi(t,x)\, dx\, dt.
\end{aligned} \tag{3.51}$$

Note that

$$u \in L^\infty\big(\tau, \tau + T; H^2(\mathbb{R}^n)\big) \quad \text{and} \quad \partial_t u \in L^\infty\big(\tau, \tau + T; L^2(\mathbb{R}^n)\big). \tag{3.52}$$

By (3.52) we obtain

$$h(\cdot,\cdot, u) \in L^2\big(\tau, \tau + T; L^2(\mathbb{R}^n)\big). \tag{3.53}$$

We claim that

$$f(\cdot, u) \quad \text{belongs to } L^\infty\big(\tau, \tau + T; L^2(\mathbb{R}^n)\big). \tag{3.54}$$

In fact, by (3.3) we obtain that there exists some $c_6 = c_6(p, n, \alpha_1, \varphi) > 0$ such that

$$\begin{aligned} \|f(\cdot, u(t))\|^2 &\le 2\alpha_1^2 \big(\|\varphi\|^2_{L^\infty(\mathbb{R}^n)} \|u(t)\|^2 + \|u(t)\|^{2(p+1)}_{L^{2(p+1)}(\mathbb{R}^n)}\big) \\ &\le c_6 \big(\|u(t)\|^2 + \|u(t)\|^{2(p+1)}_{H^2(\mathbb{R}^n)}\big), \end{aligned}$$

which along with (3.52) leads to (3.54).

By (3.51)–(3.54), we can obtain

$$u_{tt} \quad \text{belongs to } L^2\big(\tau, \tau + T; H^{-2}(\mathbb{R}^n)\big), \tag{3.55}$$

where $H^{-2}(\mathbb{R}^n)$ is the dual space of $H^2(\mathbb{R}^n)$.

Next, we prove u and u_t satisfy the initial conditions $(1.1)_2$.

By (3.24), we obtain that for any $v \in C_0^\infty(\mathbb{R}^n)$ and $\psi \in C^2([\tau, \tau + T])$,

$$\begin{aligned} &\int_\tau^{\tau+T} \big(u_k(t), v\big)\psi''(t)\,dt + \big(\partial_t u_k(\tau + T), v\big)\psi(\tau + T) - \big(u_k(\tau + T), v\big)\psi'(\tau + T) \\ &\quad + (u_0, v)\psi'(\tau) - (u_{1,0}, v)\psi(\tau) + \alpha \int_\tau^{\tau+T} \big(\partial_t u_k(t), v\big)\psi(t)\,dt \\ &\quad + \int_\tau^{\tau+T} \big(\Delta u_k(t), \Delta v\big)\psi(t)\,dt \\ &\quad + \nu \int_\tau^{\tau+T} \big(u_k(t), v\big)\psi(t)\,dt + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} f_k\big(x, u_k(t,x)\big)v(x)\psi(t)\,dx\,dt \\ &= \int_\tau^{\tau+T} \big(g(t), v\big)\psi(t)\,dt + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} h_k\big(t, x, u_k(t,x)\big)\zeta_\delta(\theta_t\omega)v(x)\psi(t)\,dx\,dt. \end{aligned} \tag{3.56}$$

Letting $k \to \infty$ in (3.56), by (3.38), (3.39), (3.42), (3.43), (3.48), and (3.49) we obtain, for any $v \in C_0^\infty(\mathbb{R}^n)$ and $\psi \in C^2([\tau, \tau + T])$,

$$\begin{aligned} &\int_\tau^{\tau+T} \big(u(t), v\big)\psi''(t)\,dt + \big(v_1^{\tau+T}, v\big)\psi(\tau + T) - \big(v^{\tau+T}, v\big)\psi'(\tau + T) \\ &\quad + (u_0, v)\psi'(\tau) - (u_{1,0}, v)\psi(\tau) + \alpha \int_\tau^{\tau+T} \big(\partial_t u(t), v\big)\psi(t)\,dt \\ &\quad + \int_\tau^{\tau+T} \big(\Delta u(t), \Delta v\big)\psi(t)\,dt \\ &\quad + \nu \int_\tau^{\tau+T} \big(u(t), v\big)\psi(t)\,dt + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} f\big(x, u(t,x)\big)v(x)\psi(t)\,dx\,dt \\ &= \int_\tau^{\tau+T} \big(g(t), v\big)\psi(t)\,dt + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} h\big(t, x, u(t,x)\big)\zeta_\delta(\theta_t\omega)v(x)\psi(t)\,dx\,dt. \end{aligned} \tag{3.57}$$

By (3.51) we obtain that for any $v \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned} &\frac{d}{dt}(u_t, v) + \alpha(u_t, v) + (\Delta u, \Delta v) + \nu(u, v) + \int_{\mathbb{R}^n} f\big(x, u(t,x)\big)v(x)\,dx \\ &= \big(g(t), v\big) + \int_{\mathbb{R}^n} h\big(t, x, u(t,x)\big)\zeta_\delta(\theta_t\omega)v(x)\,dx. \end{aligned} \tag{3.58}$$

By (3.58) we find that for any $v \in C_0^\infty(\mathbb{R}^n)$ and $\psi \in C^2([\tau, \tau + T])$,

$$\begin{aligned}
&\int_\tau^{\tau+T} \big(u(t), v\big)\psi''(t)\,dt + \big(\partial_t u(\tau + T), v\big)\psi(\tau + T) - \big(u(\tau + T), v\big)\psi'(\tau + T) \\
&\quad + \big(u(\tau), v\big)\psi'(\tau) - \big(\partial_t u(\tau), v\big)\psi(\tau) + \alpha \int_\tau^{\tau+T} \big(\partial_t u(t), v\big)\psi(t)\,dt \\
&\quad + \int_\tau^{\tau+T} \big(\Delta u(t), \Delta v\big)\psi(t)\,dt \\
&\quad + \nu \int_\tau^{\tau+T} \big(u(t), v\big)\psi(t)\,dt + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} f\big(x, u(t,x)\big)v(x)\psi(t)\,dx\,dt \\
&= \int_\tau^{\tau+T} \big(g(t,\cdot), v\big)\psi(t)\,dt + \int_\tau^{\tau+T} \int_{\mathbb{R}^n} h\big(t, x, u(t,x)\big)\zeta_\delta(\theta_t\omega)v(x)\psi(t)\,dx\,dt,
\end{aligned} \tag{3.59}$$

together with (3.57) to obtain, for $v \in C_0^\infty(\mathbb{R}^n)$ and $\psi \in C^2([\tau, \tau + T])$,

$$\begin{aligned}
&\big(v_1^{\tau+T}, v\big)\psi(\tau + T) - \big(v^{\tau+T}, v\big)\psi'(\tau + T) + (u_0, v)\psi'(\tau) - (u_{1,0}, v)\psi(\tau) \\
&\quad = \big(\partial_t u(\tau + T), v\big)\psi(\tau + T) - \big(u(\tau + T), v\big)\psi'(\tau + T) + \big(u(\tau), v\big)\psi'(\tau) \\
&\qquad - \big(\partial_t u(\tau), v\big)\psi(\tau).
\end{aligned} \tag{3.60}$$

Let $\psi \in C^2([\tau, \tau + T])$ such that $\psi(\tau + T) = \psi'(\tau + T) = \psi'(\tau) = 0$ and $\psi(\tau) = 1$, then by (3.60) we have

$$\big(\partial_t u(\tau), v\big) = (u_{1,0}, v), \quad \forall v \in C_0^\infty\big(\mathbb{R}^n\big). \tag{3.61}$$

Let $\psi \in C^2([\tau, \tau + T])$ such that $\psi(\tau + T) = \psi'(\tau + T) = \psi(\tau) = 0$ and $\psi'(\tau) = 1$, then by (3.60) we have

$$\big(u(\tau), v\big) = (u_0, v), \quad \forall v \in C_0^\infty\big(\mathbb{R}^n\big), \tag{3.62}$$

which together with (3.61) shows that u satisfies the initial conditions $(1.1)_2$.

Through choosing proper $\psi \in C^2([\tau, \tau + T])$, we can also obtain from (3.60) that

$$u(\tau + T) = v^{\tau+T}, \quad \text{and} \quad \partial_t u(\tau + T) = v_1^{\tau+T},$$

which along with (3.42) and (3.43) implies that

$$u_k(\tau + T) \to u(\tau + T) \quad \text{weakly in } H^2\big(\mathbb{R}^n\big), \tag{3.63}$$

$$\partial_t u_k(\tau + T) \to \partial_t u(\tau + T) \quad \text{weakly in } L^2\big(\mathbb{R}^n\big). \tag{3.64}$$

Similar to (3.63) and (3.64), one can verify that for any $t \in [\tau, \tau + T]$,

$$u_k(t) \to u(t) \quad \text{weakly in } H^2\big(\mathbb{R}^n\big), \tag{3.65}$$

$$\partial_t u_k(t) \to \partial_t u(t) \quad \text{weakly in } L^2\big(\mathbb{R}^n\big). \tag{3.66}$$

Thus, we obtain the claim. By (3.65) and (3.66), we obtain that u is a solution of (1.1) in the sense of Definition 3.1.

Step (iv): Uniqueness of solutions

Let u_1 and u_2 be solutions to (1.1), denote $v = u_1 - u_2$. Then, we have

$$\begin{cases} v_{tt} + \alpha v_t + \Delta^2 v + \nu v = f(\cdot, u_2) - f(\cdot, u_1) + (h(t,\cdot,u_1) - h(t,\cdot,u_2))\zeta_\delta(\theta_t\omega), \\ v(\tau) = 0, \qquad v_t(\tau) = 0. \end{cases} \tag{3.67}$$

By (3.8), we obtain

$$\begin{aligned} &\frac{d}{dt}\big(\|v_t\|^2 + \|\Delta v\|^2 + \nu\|v\|^2\big) \\ &\quad = -2\alpha\|v_t\|^2 + 2\big(f(\cdot,u_2) - f(\cdot,u_1), v_t\big) + 2\big(h(t,\cdot,u_1) - h(t,\cdot,u_2), v_t\big)\zeta_\delta(\theta_t\omega). \end{aligned} \tag{3.68}$$

Since $H^2(\mathbb{R}^n) \hookrightarrow L^{2(p+1)}(\mathbb{R}^n)$ for $0 < p \le \frac{4}{n-4}$, by (3.3), we obtain

$$\big\|f(\cdot,u_2) - f(\cdot,u_1)\big\| \le \alpha_1\|\varphi\|_{L^\infty(\mathbb{R}^n)}\|v\| + \alpha_1\big(\|u_1\|^p_{H^2(\mathbb{R}^n)} + \|u_2\|^p_{H^2(\mathbb{R}^n)}\big)\|v\|_{H^2(\mathbb{R}^n)}$$

and hence

$$\begin{aligned} &2\big(f(\cdot,u_2) - f(\cdot,u_1), v_t\big) \\ &\quad \le 2\big\|f(\cdot,u_2) - f(\cdot,u_1)\big\|\|v_t\| \\ &\quad \le \alpha_1\big(\|\varphi\|_{L^\infty(\mathbb{R}^n)} + \|u_1\|^p_{H^2(\mathbb{R}^n)} + \|u_2\|^p_{H^2(\mathbb{R}^n)}\big)\big(\|v\|^2_{H^2(\mathbb{R}^n)} + \|v_t\|^2\big). \end{aligned} \tag{3.69}$$

By (3.6) we obtain

$$\begin{aligned} &2\big(h(t,\cdot,u_1) - h(t,\cdot,u_2), v_t\big)\zeta_\delta(\theta_t\omega) \\ &\quad \le \big\|h(t,\cdot,u_1) - h(t,\cdot,u_2)\big\|\|v_t\|\big|\zeta_\delta(\theta_t\omega)\big| \\ &\quad \le 2\alpha_3\|v\|\|v_t\|\big|\zeta_\delta(\theta_t\omega)\big| \\ &\quad \le \alpha_3\big(\|v\|^2 + \|v_t\|^2\big)\big|\zeta_\delta(\theta_t\omega)\big|. \end{aligned} \tag{3.70}$$

It follows from (3.68)–(3.70) that

$$\begin{aligned} &\frac{d}{dt}\big(\|v_t\|^2 + \|\Delta v\|^2 + \nu\|v\|^2\big) \\ &\quad \le c_7\big(1 + \|u_1\|^p_{H^2(\mathbb{R}^n)} + \|u_2\|^p_{H^2(\mathbb{R}^n)}\big)\big(\|v_t\|^2 + \|\Delta v\|^2 + \nu\|v\|^2\big), \end{aligned} \tag{3.71}$$

where $c_7 > 0$ depends on τ and T. Since $u_1, u_2 \in L^\infty(\tau, \tau + T; H^2(\mathbb{R}^n))$, then applying Gronwall's lemma on $[\tau, \tau + T]$, we can obtain the uniqueness of solution as well as the continuous dependence property of the solution with initial data. □

We now define a mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ such that for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $(u_0, u_{1,0}) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$,

$$\Phi\big(t, \tau, \omega, (u_0, u_{1,0})\big) = \big(u(t + \tau, \tau, \theta_{-\tau}\omega, u_0), u_t(t + \tau, \tau, \theta_{-\tau}\omega, u_{1,0})\big), \tag{3.72}$$

where u is the solution of (1.1). Then, Φ is a continuous cocycle on $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$.

4 Uniform estimates of solutions

In this section, we derive necessary estimates of solutions of (1.1) under stronger conditions than (3.2)–(3.6) on the nonlinear functions f and h. These estimates are useful for proving the asymptotic compactness of the solutions and the existence of pullback random attractors.

From now on, we assume f satisfies: for all $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$,

$$f(x,s)s - \gamma F(x,s) \geq \varphi_3(x), \tag{4.1}$$

$$F(x,s) + \varphi_1(x) \geq \alpha_4 |s|^{p+2}, \tag{4.2}$$

$$\left|\partial_s f(x,s)\right| \leq \iota |s|^p + \varsigma, \qquad \left|\partial_x f(x,s)\right| \leq \varphi_4(x), \tag{4.3}$$

where $p > 0$ for $1 \leq n \leq 4$ and $0 < p \leq \frac{4}{n-4}$ for $n \geq 5$, $\gamma \in (0,1]$, α_4, ς are positive constants, $\varphi_3 \in L^1(\mathbb{R}^n)$, and $\varphi_4 \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\iota > 0$ will be denoted later.

By (3.3) and (4.1) we obtain that for all $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$,

$$\gamma F(x,s) \leq \alpha_1 s^2 \varphi(x) + \alpha_1 |s|^{p+2} - \varphi_3(x). \tag{4.4}$$

Assume the nonlinearity h satisfies: for all $x \in \mathbb{R}^n$ and $t, s \in \mathbb{R}$,

$$\left|h(t,x,s)\right| \leq \varphi_5(x)|s| + \varphi_6(x), \tag{4.5}$$

$$\left|\partial_x h(t,x,s)\right| + \left|\partial_s h(t,x,s)\right| \leq \varphi_7(x), \tag{4.6}$$

where $\varphi_5 \in L^\infty(\mathbb{R}^n) \cap L^{2+\frac{4}{p}}(\mathbb{R}^n)$, $\varphi_6 \in L^2(\mathbb{R}^n)$, and $\varphi_7 \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Let $\mathcal{D}$ be the set of all tempered families of nonempty bounded subsets of $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. $D = \{D(\tau, \Omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called tempered if for any $c > 0$,

$$\lim_{t\to+\infty} e^{-ct} \left\|D(\tau - t, \theta_{-t}\omega)\right\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} = 0,$$

where $\|D\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} = \sup_{\xi\in D} \|\xi\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}$.

Under $\alpha > 0$, $\nu > 0$, and $\gamma \in (0,1]$, we can choose a sufficiently small positive constant ε such that

$$\begin{aligned} &\varepsilon < \min\left\{1, \nu, \frac{2\alpha}{5}\right\}, \qquad \frac{1}{2}\alpha - 2\varepsilon - \frac{1}{8}\varepsilon\gamma > 0, \\ &\nu - \frac{1}{2}\nu\gamma - \varepsilon\alpha + \frac{1}{8}\varepsilon^2\gamma > 0, \qquad \nu - \varepsilon - \varepsilon\alpha + \frac{1}{2}\varepsilon^2 > 0. \end{aligned} \tag{4.7}$$

We also assume

$$\int_{-\infty}^{\tau} e^{\frac{1}{4}\varepsilon\gamma s} \left\|g(s)\right\|_1^2 ds < \infty, \quad \forall \tau \in \mathbb{R}, \tag{4.8}$$

$$\lim_{t\to+\infty} e^{-ct} \int_{-\infty}^{0} e^{\frac{1}{4}\varepsilon\gamma s} \left\|g(s-t)\right\|_1^2 ds = 0, \quad \text{for } \forall c > 0. \tag{4.9}$$

Lemma 4.1 *Let* (3.2), (3.3), (3.6), (4.1), (4.2), *and* (4.5)–(4.8) *hold. Then, for any* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D \in \mathcal{D}$, *there exists* $T = T(\tau, \omega, D) > 0$ *such that for all* $t \geq T$, *the solution of* (1.1) *satisfies*

$$\begin{aligned}
&\|u_t(\tau, \tau - t, \theta_{-\tau}\omega, u_{1,0})\|^2 + \|u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)\|^2_{H^2(\mathbb{R}^n)} \\
&\quad + \int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\big(\|u_t(s, \tau - t, \theta_{-\tau}\omega, u_{1,0})\|^2 + \|u(s, \tau - t, \theta_{-\tau}\omega, u_0)\|^2_{H^2(\mathbb{R}^n)}\big)\,ds \\
&\leq M_1 + M_1 \int_{-\infty}^{0} e^{\frac{1}{4}\varepsilon\gamma s}\big(1 + \|g(s+\tau)\|^2 + |\zeta_\delta(\theta_s\omega)|^{2+\frac{4}{p}}\big)\,ds,
\end{aligned}$$

where $(u_0, u_{1,0}) \in D(\tau - t, \theta_{-t}\omega)$ *and* M_1 *is a positive constant independent of* τ, ω, *and* D.

Proof By (3.9), (3.11), (4.1), and (4.10) we obtain, for almost all $t \in [\tau, \tau + T]$,

$$\begin{aligned}
&\frac{d}{dt}\left(\|u_t\|^2 + \nu\|u\|^2 + \|\Delta u\|^2 + 2\int_{\mathbb{R}^n} F(x, u(t,x))\,dx + \varepsilon(u, u_t)\right) \\
&\quad + (2\alpha - \varepsilon)\|u_t\|^2 + \varepsilon\alpha(u, u_t) + \varepsilon\|\Delta u\|^2 + \varepsilon\nu\|u\|^2 + \varepsilon\gamma\int_{\mathbb{R}} F(x, u(t,x))\,dx \\
&\leq \varepsilon\|\varphi_3\|_{L^1(\mathbb{R}^n)} + \big(g(t) + h(t, \cdot, u(t))\zeta_\delta(\theta_t\omega), \varepsilon u + 2u_t\big).
\end{aligned} \tag{4.10}$$

For the second term on the right-hand side of (4.10), using (4.2) and (4.5) we have

$$\begin{aligned}
&\big(g(t) + h(t, \cdot, u(t))\zeta_\delta(\theta_t\omega), \varepsilon u + 2u_t\big) \\
&\leq \big(\|g(t)\| + \|h(t, \cdot, u(t))\zeta_\delta(\theta_t\omega)\|\big)\big(\varepsilon\|u\| + 2\|u_t\|\big) \\
&\leq \frac{1}{2}\varepsilon\nu\|u\|^2 + \alpha\|u_t\|^2 + \left(\alpha^{-1} + \frac{1}{2}\varepsilon\nu^{-1}\right)\big(\|g(t)\| + \|h(t, \cdot, u(t))\zeta_\delta(\theta_t\omega)\|\big)^2 \\
&\leq \frac{1}{2}\varepsilon\nu\|u\|^2 + \alpha\|u_t\|^2 + \big(2\alpha^{-1} + \varepsilon\nu^{-1}\big)\|g(t)\|^2 + \big(2\alpha^{-1} + \varepsilon\nu^{-1}\big)\|h(t, \cdot, u(t))\zeta_\delta(\theta_t\omega)\|^2 \\
&\leq \frac{1}{2}\varepsilon\nu\|u\|^2 + \alpha\|u_t\|^2 + \big(2\alpha^{-1} + \varepsilon\nu^{-1}\big)\|g(t)\|^2 + 2\big(2\alpha^{-1} + \varepsilon\nu^{-1}\big)|\zeta_\delta(\theta_t\omega)|^2\|\varphi_6\|^2 \\
&\quad + 2\big(2\alpha^{-1} + \varepsilon\nu^{-1}\big)|\zeta_\delta(\theta_t\omega)|^2\int_{\mathbb{R}^n}|\varphi_5(x)|^2|u(t,x)|^2\,dx \\
&\leq \frac{1}{2}\varepsilon\nu\|u\|^2 + \alpha\|u_t\|^2 + \big(2\alpha^{-1} + \varepsilon\nu^{-1}\big)\|g(t)\|^2 + c_1|\zeta_\delta(\theta_t\omega)|^2 \\
&\quad + \frac{1}{2}\varepsilon\gamma\alpha_4\int_{\mathbb{R}^n}|u(t,x)|^{p+2}\,dx \\
&\quad + c_2|\zeta_\delta(\theta_t\omega)|^{2+\frac{4}{p}}\int_{\mathbb{R}^n}|\varphi_5(x)|^{2+\frac{4}{p}}\,dx \\
&\leq \frac{1}{2}\varepsilon\nu\|u\|^2 + \alpha\|u_t\|^2 + \big(2\alpha^{-1} + \varepsilon\nu^{-1}\big)\|g(t)\|^2 + c_1|\zeta_\delta(\theta_t\omega)|^2 \\
&\quad + \frac{1}{2}\varepsilon\gamma\int_{\mathbb{R}} F(x, u(t,x))\,dx \\
&\quad + \frac{1}{2}\varepsilon\gamma\|\varphi_1\|_{L^1(\mathbb{R}^n)} + c_3|\zeta_\delta(\theta_t\omega)|^{2+\frac{4}{p}} \\
&\leq \frac{1}{2}\varepsilon\nu\|u\|^2 + \alpha\|u_t\|^2 + \big(2\alpha^{-1} + \varepsilon\nu^{-1}\big)\|g(t)\|^2
\end{aligned}$$

$$+\frac{1}{2}\varepsilon\gamma\int_{\mathbb{R}}F\big(x,u(t,x)\big)\,dx+c_4\big(1+\big|\zeta_\delta(\theta_t\omega)\big|^{2+\frac{4}{p}}\big),\tag{4.11}$$

where $c_4>0$ depends on $\alpha,\nu,\gamma,\varepsilon$.

It follows from (4.10) and (4.11) and rewriting the result obtained, we have

$$\begin{aligned}&\frac{d}{dt}\Big(\|u_t\|^2+\nu\|u\|^2+\|\Delta u\|^2+2\int_{\mathbb{R}^n}F\big(x,u(t,x)\big)\,dx+\varepsilon(u,u_t)\Big)\\&\quad+\frac{1}{4}\varepsilon\gamma\Big(\|u_t\|^2+\nu\|u\|^2+\|\Delta u\|^2+2\int_{\mathbb{R}^n}F\big(x,u(t,x)\big)\,dx+\varepsilon(u,u_t)\Big)\\&\quad+\Big(\alpha-\varepsilon-\frac{1}{4}\varepsilon\gamma\Big)\|u_t\|^2+\varepsilon\Big(1-\frac{1}{4}\gamma\Big)\|\Delta u\|^2+\frac{1}{2}\varepsilon\nu\Big(1-\frac{1}{2}\gamma\Big)\|u\|^2\\&\leq c_5\big(1+\big\|g(t)\big\|^2+\big|\zeta_\delta(\theta_t\omega)\big|^{2+\frac{4}{p}}\big)-\varepsilon\Big(\alpha-\frac{1}{4}\varepsilon\gamma\Big)(u,u_t)),\end{aligned}\tag{4.12}$$

where $c_5>0$ depends on $\alpha,\nu,\gamma,\varepsilon$.

For the second term on the right-hand side of (4.12) we obtain

$$\begin{aligned}&-\varepsilon\Big(\alpha-\frac{1}{4}\varepsilon\gamma\Big)(u,u_t))\\&\leq\varepsilon\Big(\alpha-\frac{1}{4}\varepsilon\gamma\Big)\|u\|\|u_t\|\\&\leq\frac{1}{2}\varepsilon^2\Big(\alpha-\frac{1}{4}\varepsilon\gamma\Big)\|u\|^2+\frac{1}{2}\Big(\alpha-\frac{1}{4}\varepsilon\gamma\Big)\|u_t\|^2.\end{aligned}\tag{4.13}$$

By (4.12) and (4.13) we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\|u_t\|^2+\nu\|u\|^2+\|\Delta u\|^2+2\int_{\mathbb{R}^n}F\big(x,u(t,x)\big)\,dx+\varepsilon(u,u_t)\Big)\\&\quad+\frac{1}{4}\varepsilon\gamma\Big(\|u_t\|^2+\nu\|u\|^2+\|\Delta u\|^2+2\int_{\mathbb{R}^n}F\big(x,u(t,x)\big)\,dx+\varepsilon(u,u_t)\Big)\\&\quad+\Big(\frac{1}{2}\alpha-\varepsilon-\frac{1}{8}\varepsilon\gamma\Big)\|u_t\|^2+\varepsilon\Big(1-\frac{1}{4}\gamma\Big)\|\Delta u\|^2\\&\quad+\frac{1}{2}\varepsilon\Big(\nu-\frac{1}{2}\nu\gamma-\varepsilon\alpha+\frac{1}{4}\varepsilon^2\gamma\Big)\|u\|^2\\&\leq c_5\big(1+\big\|g(t)\big\|^2+\big|\zeta_\delta(\theta_t\omega)\big|^{2+\frac{4}{p}}\big).\end{aligned}\tag{4.14}$$

Multiplying (4.14) by $e^{\frac{1}{4}\varepsilon\gamma t}$, and then integrating the inequality $[\tau-t,\tau]$, after replacing ω by $\theta_{-\tau}\omega$, we obtain

$$\begin{aligned}&\big\|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\big\|^2+\nu\big\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big\|^2+\big\|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big\|^2\\&\quad+2\int_{\mathbb{R}^n}F\big(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big)\,dx\\&\quad+\varepsilon\big(u(\tau,\tau-t,\theta_{-\tau}\omega,u_0),u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\big)\\&\quad+\Big(\frac{1}{2}\alpha-\varepsilon-\frac{1}{8}\varepsilon\gamma\Big)\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\big\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_{1,0})\big\|^2\,ds\end{aligned}$$

$$
\begin{aligned}
&+ \varepsilon\left(1-\frac{1}{4}\gamma\right)\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\left\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2 ds \\
&+ \frac{1}{2}\varepsilon\left(\nu-\frac{1}{2}\nu\gamma-\varepsilon\alpha+\frac{1}{4}\varepsilon^2\gamma\right)\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\left\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2 ds \\
\le{}& e^{-\frac{1}{4}\varepsilon\gamma t}\left(\|u_{1,0}\|^2+\nu\|u_0\|^2+\|\Delta u_0\|^2+2\int_{\mathbb{R}^n}F(x,u_0)\,dx+\varepsilon(u_0,u_{1,0})\right) \\
&+ c_5\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\left(1+\left\|g(s)\right\|^2+\left|\zeta_\delta(\theta_{s-\tau}\omega)\right|^{2+\frac{4}{p}}\right) ds. \qquad (4.15)
\end{aligned}
$$

For the first term on the right-hand side of (4.15), by (4.4) we obtain

$$
\begin{aligned}
& e^{-\frac{1}{4}\varepsilon\gamma t}\left(\|u_{1,0}\|^2+\nu\|u_0\|^2+\|\Delta u_0\|^2+2\int_{\mathbb{R}^n}F(x,u_0)\,dx+\varepsilon(u_0,u_{1,0})\right) \\
&\le c_6 e^{-\frac{1}{4}\varepsilon\gamma t}\left(1+\|u_{1,0}\|^2+\|u_0\|^2_{H^2(\mathbb{R}^n}+\|u_0\|^{p+2}_{H^2(\mathbb{R}^n)}\right) \\
&\le c_7 e^{-\frac{1}{4}\varepsilon\gamma t}\left(1+\left\|D(\tau-t,\theta_{-t}\omega)\right\|^{p+2}\right)\to 0, \quad \text{as } t\to\infty. \qquad (4.16)
\end{aligned}
$$

By (4.15) and (4.16) we find that there exists $T=T(\tau,\omega,D)>0$ such that for all $t\ge T$,

$$
\begin{aligned}
&\left\|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right\|^2+\nu\left\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2+\left\|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2 \\
&\quad+2\int_{\mathbb{R}^n}F\big(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big)\,dx \\
&\quad+\varepsilon\big(u(\tau,\tau-t,\theta_{-\tau}\omega,u_0),u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\big) \\
&\quad+\left(\frac{1}{2}\alpha-\varepsilon-\frac{1}{8}\varepsilon\gamma\right)\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\left\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right\|^2 ds \\
&\quad+\varepsilon\left(1-\frac{1}{4}\gamma\right)\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\left\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2 ds \\
&\quad+\frac{1}{2}\varepsilon\left(\nu-\frac{1}{2}\nu\gamma-\varepsilon\alpha+\frac{1}{4}\varepsilon^2\gamma\right)\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\left\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2 ds \\
&\le 1+c_5\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\left(1+\left\|g(s)\right\|^2+\left|\zeta_\delta(\theta_{s-\tau}\omega)\right|^{2+\frac{4}{p}}\right) ds. \qquad (4.17)
\end{aligned}
$$

By (4.7) we obtain

$$
\begin{aligned}
&\varepsilon\big(u(\tau,\tau-t,\theta_{-\tau}\omega,u_0),u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\big) \\
&\le \frac{1}{2}\varepsilon\left\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2+\frac{1}{2}\varepsilon\left\|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right\|^2 \\
&\le \frac{1}{2}\nu\varepsilon\left\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2+\frac{1}{2}\varepsilon\left\|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right\|^2. \qquad (4.18)
\end{aligned}
$$

It follows from (4.2), (4.17), and (4.18) that for all $t\ge T$,

$$
\begin{aligned}
&\frac{1}{2}\left\|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right\|^2+\frac{1}{2}\nu\left\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2+\left\|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2 \\
&\quad+\left(\frac{1}{2}\alpha-\varepsilon-\frac{1}{8}\varepsilon\gamma\right)\int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\left\|u_t(s,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right\|^2 ds
\end{aligned}
$$

$$+\varepsilon\left(1-\frac{1}{4}\gamma\right)\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\left\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2 ds$$
$$+\frac{1}{2}\varepsilon\left(\nu-\frac{1}{2}\nu\gamma-\varepsilon\alpha+\frac{1}{4}\varepsilon^2\gamma\right)\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\left\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2 ds$$
$$\le 1+2\|\varphi_1\|_{L^1(\mathbb{R}^n)}+c_5\int_{-\infty}^{0}e^{\frac{1}{4}\varepsilon\gamma s}\left(1+\left\|g(s+\tau)\right\|^2+\left|\zeta_\delta(\theta_s\omega)\right|^{2+\frac{4}{p}}\right)ds.$$

Then, the proof is completed. □

Based on Lemma 4.1, we can easily obtain the following Lemma that implies the existence of tempered random absorbing sets of Φ.

Lemma 4.2 *If* (3.2), (3.3), (3.6), (4.1), (4.2), *and* (4.5)–(4.9) *hold, then the cocycle* Φ *possesses a closed measurable* $\mathcal{D}$*-pullback absorbing set* $B=\{B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, *which is given by*

$$B(\tau,\omega)=\left\{(u_0,u_{1,0})\in H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n):\|u_0\|_{H^2(\mathbb{R}^n)}^2+\|u_{1,0}\|^2\le L(\tau,\omega)\right\},\tag{4.19}$$

where

$$L(\tau,\omega)=M_1+M_1\int_{-\infty}^{0}e^{\frac{1}{4}\varepsilon\gamma s}\left(1+\left\|g(s+\tau)\right\|^2+\left|\zeta_\delta(\theta_s\omega)\right|^{2+\frac{4}{p}}\right)ds.$$

In order to derive the uniform tail-estimates of the solutions of (1.1) for large space variables when time is long enough, we need to derive the regularity of the solutions in a space higher than $H^2(\mathbb{R}^n)$.

Lemma 4.3 *Let* (3.2), (3.3), (3.6), (4.1), (4.2), *and* (4.5)–(4.8) *hold. Then, for any* $\tau\in\mathbb{R}$, $\omega\in\Omega$ *and* $D\in\mathcal{D}$, *there exists* $T=T(\tau,\omega,D)>0$ *such that for all* $t\ge T$, *the solution of* (1.1) *satisfies*

$$\left\|A^{\frac{1}{4}}u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right\|^2+\left\|A^{\frac{3}{4}}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2$$
$$+\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\left(\left\|A^{\frac{1}{4}}u_t(s,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right\|^2+\left\|A^{\frac{3}{4}}u(s,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2\right)ds$$
$$\le M_2+M_2\int_{-\infty}^{0}e^{\frac{1}{4}\varepsilon\gamma s}\left(1+\left\|g(s+\tau)\right\|_1^2+\left|\zeta_\delta(\theta_s\omega)\right|^2\right)ds,$$

where $(u_0,u_{1,0})\in D(\tau-t,\theta_{-\tau}\omega)$ *and* M_2 *is a positive number independent of* τ, ω, *and* D.

Proof Taking the inner product of $(1.1)_1$ with $A^{\frac{1}{2}}u$ in $L^2(\mathbb{R}^n)$, we have

$$\frac{d}{dt}\left(A^{\frac{1}{4}}u_t,A^{\frac{1}{4}}u\right)+\alpha\left(A^{\frac{1}{4}}u_t,A^{\frac{1}{4}}u\right)+\left\|A^{\frac{3}{4}}u\right\|^2+\nu\left\|A^{\frac{1}{4}}u\right\|^2+\left(f(x,u),A^{\frac{1}{2}}u\right)$$
$$=\left\|A^{\frac{1}{4}}u_t\right\|^2+\left(g(t)+h(t,\cdot,u)\zeta_\delta(\theta_t\omega),A^{\frac{1}{2}}u\right).\tag{4.20}$$

Taking the inner product of $(1.1)_1$ with $A^{\frac{1}{2}}u_t$ in $L^2(\mathbb{R}^n)$, we find that

$$\frac{d}{dt}\left(\left\|A^{\frac{1}{4}}u_t\right\|^2+\nu\left\|A^{\frac{1}{4}}u\right\|^2+\left\|A^{\frac{3}{4}}u\right\|^2\right)$$

$$= -2\alpha\|A^{\frac{1}{4}}u_t\|^2 - 2\big(f(x,u),A^{\frac{1}{2}}u_t\big) + 2\big(g(t)+h(t,\cdot,u)\zeta_\delta(\theta_t\omega),A^{\frac{1}{2}}u_t\big). \tag{4.21}$$

By (4.20) and (4.21), we obtain

$$\begin{aligned}
&\frac{d}{dt}\big(\|A^{\frac{1}{4}}u_t\|^2 + \nu\|A^{\frac{1}{4}}u\|^2 + \|A^{\frac{3}{4}}u\|^2 + \varepsilon\big(A^{\frac{1}{4}}u_t,A^{\frac{1}{4}}u\big)\big) \\
&\quad + (2\alpha-\varepsilon)\|A^{\frac{1}{4}}u_t\|^2 + \varepsilon\alpha\big(A^{\frac{1}{4}}u_t,A^{\frac{1}{4}}u\big) \\
&\quad + \varepsilon\|A^{\frac{3}{4}}u\|^2 + \varepsilon\nu\|A^{\frac{1}{4}}u\|^2 + \varepsilon\big(f(x,u),A^{\frac{1}{2}}u\big) + 2\big(f(x,u),A^{\frac{1}{2}}u_t\big) \\
&= \big(g(t)+h(t,\cdot,u)\zeta_\delta(\theta_t\omega),\varepsilon A^{\frac{1}{2}}u + 2A^{\frac{1}{2}}u_t\big).
\end{aligned} \tag{4.22}$$

For the right-hand side of (4.22), using (4.5), (4.6), and Lemma 4.1, we have

$$\begin{aligned}
&\big(g(t)+h\big(t,\cdot,u(t)\big)\zeta_\delta(\theta_t\omega),\varepsilon A^{\frac{1}{2}}u + 2A^{\frac{1}{2}}u_t\big) \\
&\quad\le \big(\|g(t)\|_1 + \|h\big(t,\cdot,u(t)\big)\zeta_\delta(\theta_t\omega)\|_1\big)\big(\varepsilon\|A^{\frac{1}{4}}u\| + 2\|A^{\frac{1}{2}}u_t\|\big) \\
&\quad\le \frac{1}{2}\varepsilon\nu\|A^{\frac{1}{4}}u\|^2 + \alpha\|A^{\frac{1}{2}}u_t\|^2 + \left(\alpha^{-1} + \frac{1}{2}\varepsilon\nu^{-1}\right)\big(\|g(t)\|_1 + \|h\big(t,\cdot,u(t)\big)\zeta_\delta(\theta_t\omega)\|_1\big)^2 \\
&\quad\le \frac{1}{2}\varepsilon\nu\|A^{\frac{1}{4}}u\|^2 + \alpha\|A^{\frac{1}{4}}u_t\|^2 + \big(2\alpha^{-1}+\varepsilon\nu^{-1}\big)\|g(t)\|_1^2 \\
&\qquad + \big(2\alpha^{-1}+\varepsilon\nu^{-1}\big)\|h\big(t,\cdot,u(t)\big)\zeta_\delta(\theta_t\omega)\|_1^2 \\
&\quad\le \frac{1}{2}\varepsilon\nu\|A^{\frac{1}{4}}u\|^2 + \alpha\|A^{\frac{1}{4}}u_t\|^2 + \big(2\alpha^{-1}+\varepsilon\nu^{-1}\big)\|g(t)\|_1^2 + c_8|\zeta_\delta(\theta_t\omega)|^2.
\end{aligned} \tag{4.23}$$

From (4.3) and Lemma 4.1 we find

$$\begin{aligned}
&\big|\varepsilon\big(f(x,u),A^{\frac{1}{2}}u\big) + 2\big(f(x,u),A^{\frac{1}{2}}u_t\big)\big| \\
&\quad\le 2\int_{\mathbb{R}^n}\left|\frac{\partial f}{\partial u}(x,u)\cdot A^{\frac{1}{4}}u\cdot A^{\frac{1}{4}}u_t + \frac{\partial f}{\partial x}(x,u)\cdot A^{\frac{1}{4}}u_t\right|dx \\
&\qquad + \varepsilon\int_{\mathbb{R}^n}\left|\frac{\partial f}{\partial u}(x,u)\cdot A^{\frac{1}{4}}u\cdot A^{\frac{1}{4}}u + \frac{\partial f}{\partial x}(x,u)\cdot A^{\frac{1}{4}}u\right|dx \\
&\quad\le 2\iota\int_{\mathbb{R}^n}|u|^p\cdot|A^{\frac{1}{4}}u|\cdot|A^{\frac{1}{4}}u_t|\,dx + 2\varsigma\int_{\mathbb{R}^n}|A^{\frac{1}{4}}u|\cdot|A^{\frac{1}{4}}u_t|\,dx \\
&\qquad + 2\int_{\mathbb{R}^n}|\varphi_4|\cdot|A^{\frac{1}{4}}u_t|\,dx \\
&\qquad + \varepsilon\iota\int_{\mathbb{R}^n}|u|^p\cdot|A^{\frac{1}{4}}u|\cdot|A^{\frac{1}{4}}u|\,dx + \varepsilon\varsigma\int_{\mathbb{R}^n}|A^{\frac{1}{4}}u|\cdot|A^{\frac{1}{4}}u|\,dx \\
&\qquad + \varepsilon\int_{\mathbb{R}^n}|\varphi_4|\cdot|A^{\frac{1}{4}}u|\,dx \\
&\quad\le 2\iota\|u\|^p_{L^{\frac{10p}{4}}}\cdot\|A^{\frac{1}{4}}u\|_{L^{10}}\cdot\|A^{\frac{1}{4}}u_t\| + 2\varsigma\|A^{\frac{1}{4}}u\|\cdot\|A^{\frac{1}{4}}u_t\| + \frac{\varepsilon}{4}\|A^{\frac{1}{4}}u_t\|^2 + \frac{4}{\varepsilon}\|\varphi_4\|^2 \\
&\qquad + \varepsilon\iota\|u\|^p\cdot\|A^{\frac{1}{4}}u\|^2 + \varepsilon\varsigma\|A^{\frac{1}{4}}u\|^2 + \frac{\varepsilon}{2}\|A^{\frac{1}{4}}u\|^2 + \frac{\varepsilon}{2}\|\varphi_4\|^2 \\
&\quad\le \varepsilon\|A^{\frac{1}{4}}u_t\|^2 + \frac{2C^{p+1}\iota^2}{\varepsilon}L^p\|A^{\frac{3}{4}}u\|^2 + c_9,
\end{aligned}$$

where the definition of L see Lemma 4.2, and C is the positive constant satisfying

$$C\|\Delta u\|^2 \geq \left(\int_{\mathbb{R}^n} |u|^{10}\,dx\right)^{\frac{1}{5}}, \qquad C\|u\|_2^2 \geq \left(\int_{\mathbb{R}^n} |u|^{\frac{10p}{4}}\,dx\right)^{\frac{2}{10p}}.$$

Choosing

$$0 < \iota^2 \leq \frac{\varepsilon^2}{4L^p C^{p+1}},$$

we obtain

$$\left|\varepsilon\big(f(x,u),A^{\frac{1}{2}}u\big) + 2\big(f(x,u),A^{\frac{1}{2}}u_t\big)\right| \leq \varepsilon\left\|A^{\frac{1}{4}}u_t\right\|^2 + \frac{\varepsilon}{2}\left\|A^{\frac{3}{4}}u\right\|^2 + c_9. \tag{4.24}$$

By (4.22)–(4.24), we obtain

$$\begin{aligned}
&\frac{d}{dt}\big(\left\|A^{\frac{1}{4}}u_t\right\|^2 + \nu\left\|A^{\frac{1}{4}}u\right\|^2 + \left\|A^{\frac{3}{4}}u\right\|^2 + \varepsilon\big(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u\big)\big) + (\alpha - 2\varepsilon)\left\|A^{\frac{1}{4}}u_t\right\|^2 \\
&\quad + \varepsilon\alpha\big(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u\big) + \frac{\varepsilon}{2}\left\|A^{\frac{3}{4}}u\right\|^2 + \frac{\varepsilon}{2}\nu\left\|A^{\frac{1}{4}}u\right\|^2 \\
&\leq c_{10}\big(1 + \left\|g(t)\right\|_1^2 + \left|\zeta_\delta(\theta_t\omega)\right|^2\big),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\frac{d}{dt}\big(\left\|A^{\frac{1}{4}}u_t\right\|^2 + \nu\left\|A^{\frac{1}{4}}u\right\|^2 + \left\|A^{\frac{3}{4}}u\right\|^2 + \varepsilon\big(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u\big)\big) \\
&\quad + \frac{1}{4}\varepsilon\gamma\big(\left\|A^{\frac{1}{4}}u_t\right\|^2 + \nu\left\|A^{\frac{1}{4}}u\right\|^2 + \left\|A^{\frac{3}{4}}u\right\|^2 + \varepsilon\big(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u\big)\big) \\
&\quad + \left(\alpha - 2\varepsilon - \frac{1}{4}\varepsilon\gamma\right)\left\|A^{\frac{1}{4}}u_t\right\|^2 + \frac{\varepsilon}{2}\left(1 - \frac{\gamma}{2}\right)\left\|A^{\frac{3}{4}}u\right\|^2 + \frac{\varepsilon}{2}\nu\left(1 - \frac{\gamma}{2}\right)\left\|A^{\frac{1}{4}}u\right\|^2 \\
&\leq c_{10}\big(1 + \left\|g(t)\right\|_1^2 + \left|\zeta_\delta(\theta_t\omega)\right|^2\big) - \varepsilon\left(\alpha - \frac{1}{4}\varepsilon\gamma\right)\big(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u\big).
\end{aligned} \tag{4.25}$$

For the last term on the right-hand side of (4.25) we have

$$\begin{aligned}
&-\varepsilon\left(\alpha - \frac{1}{4}\varepsilon\gamma\right)\big(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u\big) \\
&\leq \varepsilon\left(\alpha - \frac{1}{4}\varepsilon\gamma\right)\left\|A^{\frac{1}{4}}u\right\|\left\|A^{\frac{1}{4}}u_t\right\| \\
&\leq \frac{1}{2}\varepsilon^2\left(\alpha - \frac{1}{4}\varepsilon\gamma\right)\left\|A^{\frac{1}{4}}u\right\|^2 + \frac{1}{2}\left(\alpha - \frac{1}{4}\varepsilon\gamma\right)\left\|A^{\frac{1}{4}}u_t\right\|^2,
\end{aligned}$$

from which together with (4.25), we obtain

$$\begin{aligned}
&\frac{d}{dt}\big(\left\|A^{\frac{1}{4}}u_t\right\|^2 + \nu\left\|A^{\frac{1}{4}}u\right\|^2 + \left\|A^{\frac{3}{4}}u\right\|^2 + \varepsilon\big(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u\big)\big) \\
&\quad + \frac{1}{4}\varepsilon\gamma\big(\left\|A^{\frac{1}{4}}u_t\right\|^2 + \nu\left\|A^{\frac{1}{4}}u\right\|^2 + \left\|A^{\frac{3}{4}}u\right\|^2 + \varepsilon\big(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u\big)\big) \\
&\quad + \left(\frac{\alpha}{2} - 2\varepsilon - \frac{1}{8}\varepsilon\gamma\right)\left\|A^{\frac{1}{4}}u_t\right\|^2 + \frac{\varepsilon}{2}\left(1 - \frac{\gamma}{2}\right)\left\|A^{\frac{3}{4}}u\right\|^2
\end{aligned}$$

$$+\frac{\varepsilon}{2}\left(\nu-\frac{\nu}{2}\gamma-\frac{\varepsilon}{2}\alpha+\frac{1}{8}\varepsilon^2\gamma\right)\|A^{\frac{1}{4}}u\|^2$$
$$\le c_{10}\left(1+\|g(t)\|_1^2+|\zeta_\delta(\theta_t\omega)|^2\right). \tag{4.26}$$

Similar to the remainder of Lemma 4.1, we can obtain the desired result. □

Lemma 4.4 *Let* (3.2), (3.3), (3.6), (4.1), (4.2), *and* (4.5)–(4.8) *hold. Then, for every* $\eta>0$, $\tau\in\mathbb{R}$, $\omega\in\Omega$ *and* $D\in\mathcal{D}$, *there exists* $T_0=T_0(\eta,\tau,\omega,D)>0$ *and* $m_0=m_0(\eta,\tau,\omega)\ge 1$ *such that for all* $t\ge T_0$, $m\ge m_0$ *and* $(u_0,u_{1,0})\in D(\tau-t,\theta_{-\tau}\omega)$, *the solution of* (1.1) *satisfies*

$$\int_{|x|\ge m}\left(|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})|^2+|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2\right.$$
$$\left.+|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2\right)dx<\eta.$$

Proof Let $\rho:\mathbb{R}^n\to\mathbb{R}$ be a smooth function such that $0\le\rho(x)\le 1$ for all $x\in\mathbb{R}^n$, and

$$\rho(x)=0\quad\text{for }|x|\le\frac{1}{2};\quad\text{and}\quad\rho(x)=1\quad\text{for }|x|\ge 1.$$

For every $m\in\mathbb{N}$, let

$$\rho_m(x)=\rho(x/m),\quad x\in\mathbb{R}^n.$$

Then, there exist positive constants c_{11} and c_{12} independent of m such that $|\nabla\rho_m(x)|\le\frac{1}{m}c_{11}$, $|\Delta\rho_m(x)|\le\frac{1}{m}c_{12}$ for all $x\in\mathbb{R}^n$ and $m\in\mathbb{N}$.

Similar to the energy equation (3.11), we have

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)\left(|u_t(t,x)|^2+\nu|u(t,x)|^2+|\Delta u(t,x)|^2+2F(x,u(t,x))\right)dx$$
$$+2\alpha\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2\,dx$$
$$=-4\int_{\mathbb{R}^n}\nabla\rho_m(x)\cdot\Delta u(t,x)\cdot\nabla u_t(t,x)\,dx-2\int_{\mathbb{R}^n}\Delta\rho_m(x)\cdot\Delta u(t,x)\cdot u_t(t,x)\,dx$$
$$+2\int_{\mathbb{R}^n}\rho_m(x)g(t,x)u_t(t,x)\,dx$$
$$+2\zeta_\delta(\theta_t\omega)\int_{\mathbb{R}^n}\rho_m(x)h(t,x,u(t,x))u_t(t,x)\,dx. \tag{4.27}$$

Taking the inner product of $(1.1)_1$ with $\rho_m(x)u$ in $L^2(\mathbb{R}^n)$, we have

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)u(t,x)u_t(t,x)\,dx+\alpha\int_{\mathbb{R}^n}\rho_m(x)u(t,x)u_t(t,x)\,dx$$
$$+\int_{\mathbb{R}^n}\rho_m(x)|\Delta u(t,x)|^2\,dx+\nu\int_{\mathbb{R}^n}\rho_m(x)|u(t,x)|^2\,dx$$
$$+\int_{\mathbb{R}^n}\rho_m(x)f(x,u(t,x))u(t,x)\,dx$$
$$=\int_{\mathbb{R}^n}\rho_m(x)|u_t(t,x)|^2\,dx-2\int_{\mathbb{R}^n}\nabla\rho_m(x)\cdot\Delta u(t,x)\cdot\nabla u(t,x)\,dx$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^n} \Delta\rho_m(x)\cdot\Delta u(t,x)\cdot u(t,x)\,dx \\
&+ \int_{\mathbb{R}^n} \rho_m(x)\big(g(t,x)+h\big(t,x,u(t,x)\big)\zeta_\delta(\theta_t\omega)\big)u(t,x)\,dx. \qquad (4.28)
\end{aligned}
$$

By (4.27) and (4.28), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} \rho_m(x)\big(\big|u_t(t,x)\big|^2+\nu\big|u(t,x)\big|^2+\big|\Delta u(t,x)\big|^2 \\
&\quad + 2F\big(x,u(t,x)\big)+\varepsilon u(t,x)u_t(t,x)\big)\,dx \\
&\quad + (2\alpha-\varepsilon)\int_{\mathbb{R}^n}\rho_m(x)\big|u_t(t,x)\big|^2\,dx+\varepsilon\alpha\int_{\mathbb{R}^n}\rho_m(x)u(t,x)u_t(t,x)\,dx \\
&\quad + \varepsilon\int_{\mathbb{R}^n}\rho_m(x)\big|\Delta u(t,x)\big|^2\,dx \\
&\quad + \varepsilon\nu\int_{\mathbb{R}^n}\rho_m(x)\big|u(t,x)\big|^2\,dx+\varepsilon\int_{\mathbb{R}^n}\rho_m(x)f\big(x,u(t,x)\big)u(t,x)\,dx \\
&= \int_{\mathbb{R}^n}\rho_m(x)\big(g(t,x)+h\big(t,x,u(t,x)\big)\zeta_\delta(\theta_t\omega)\big)\big(\varepsilon u(t,x)+2u_t(t,x)\big)\,dx \\
&\quad - 2\varepsilon\int_{\mathbb{R}^n}\nabla\rho_m(x)\cdot\Delta u(t,x)\cdot\nabla u(t,x)\,dx-\varepsilon\int_{\mathbb{R}^n}\Delta\rho_m(x)\cdot\Delta u(t,x)\cdot u(t,x)\,dx \\
&\quad - 4\int_{\mathbb{R}^n}\nabla\rho_m(x)\cdot\Delta u(t,x)\cdot\nabla u_t(t,x)\,dx \\
&\quad - 2\int_{\mathbb{R}^n}\Delta\rho_m(x)\cdot\Delta u(t,x)\cdot u_t(t,x)\,dx. \qquad (4.29)
\end{aligned}
$$

By (4.1) and (4.29) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} \rho_m(x)\big(\big|u_t(t,x)\big|^2+\nu\big|u(t,x)\big|^2+\big|\Delta u(t,x)\big|^2 \\
&\quad + 2F\big(x,u(t,x)\big)+\varepsilon u(t,x)u_t(t,x)\big)\,dx \\
&\quad + (2\alpha-\varepsilon)\int_{\mathbb{R}^n}\rho_m(x)\big|u_t(t,x)\big|^2\,dx+\varepsilon\alpha\int_{\mathbb{R}^n}\rho_m(x)u(t,x)u_t(t,x)\,dx \\
&\quad + \varepsilon\int_{\mathbb{R}^n}\rho_m(x)\big|\Delta u(t,x)\big|^2\,dx \\
&\quad + \varepsilon\nu\int_{\mathbb{R}^n}\rho_m(x)\big|u(t,x)\big|^2\,dx+\varepsilon\gamma\int_{\mathbb{R}^n}\rho_m(x)F\big(x,u(t,x)\big)\,dx \\
&\le \int_{\mathbb{R}^n}\rho_m(x)\big(g(t,x)+h\big(t,x,u(t,x)\big)\zeta_\delta(\theta_t\omega)\big)\big(\varepsilon u(t,x)+2u_t(t,x)\big)\,dx \\
&\quad - \varepsilon\int_{\mathbb{R}^n}\rho_m(x)\varphi_3(x)\,dx \\
&\quad - 2\varepsilon\int_{\mathbb{R}^n}\nabla\rho_m(x)\cdot\Delta u(t,x)\cdot\nabla u(t,x)\,dx-\varepsilon\int_{\mathbb{R}^n}\Delta\rho_m(x)\cdot\Delta u(t,x)\cdot u(t,x)\,dx \\
&\quad - 4\int_{\mathbb{R}^n}\nabla\rho_m(x)\cdot\Delta u(t,x)\cdot\nabla u_t(t,x)\,dx \\
&\quad - 2\int_{\mathbb{R}^n}\Delta\rho_m(x)\cdot\Delta u(t,x)\cdot u_t(t,x)\,dx. \qquad (4.30)
\end{aligned}
$$

Similar to the arguments of (4.11), we know that the first term on the right-hand side of (4.30) is bounded by

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \rho_m(x)\big(g(t,x)+h\big(t,x,u(t,x)\big)\zeta_\delta(\theta_t\omega)\big)\big(\varepsilon u(t,x)+2u_t(t,x)\big)\,dx\right| \\
&\quad\le \frac{1}{2}\varepsilon\nu \int_{\mathbb{R}^n} \rho_m(x)\big|u(t,x)\big|^2\,dx + \alpha\int_{\mathbb{R}^n} \rho_m(x)\big|u_t(t,x)\big|^2\,dx \\
&\qquad + \frac{1}{2}\varepsilon\gamma \int_{\mathbb{R}^n} \rho_m(x)F\big(x,u(t,x)\big)\,dx \\
&\qquad + c_{13}\int_{\mathbb{R}^n} \rho_m(x)\big(\big|g(t,x)\big|^2 + \big|\varphi_1(x)\big| + \big|\zeta_\delta(\theta_t\omega)\varphi_6(x)\big|^2 \\
&\qquad + \big|\zeta_\delta(\theta_t\omega)\varphi_5(x)\big|^{2+\frac{4}{p}}\big)\,dx,
\end{aligned} \tag{4.31}$$

where c_{13} depends only on α, ν, γ, and ε.

By (4.30) and (4.31) we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} \rho_m(x)\big(\big|u_t(t,x)\big|^2 + \nu\big|u(t,x)\big|^2 + \big|\Delta u(t,x)\big|^2 \\
&\qquad + 2F\big(x,u(t,x)\big) + \varepsilon u(t,x)u_t(t,x)\big)\,dx \\
&\qquad + (\alpha-\varepsilon)\int_{\mathbb{R}^n} \rho_m(x)\big|u_t(t,x)\big|^2\,dx + \varepsilon\alpha\int_{\mathbb{R}^n} \rho_m(x)u(t,x)u_t(t,x)\,dx \\
&\qquad + \varepsilon\int_{\mathbb{R}^n} \rho_m(x)\big|\Delta u(t,x)\big|^2\,dx \\
&\qquad + \frac{1}{2}\varepsilon\nu\int_{\mathbb{R}^n} \rho_m(x)\big|u(t,x)\big|^2\,dx + \frac{1}{2}\varepsilon\gamma\int_{\mathbb{R}^n} \rho_m(x)F\big(x,u(t,x)\big)\,dx \\
&\quad\le c_{14}\int_{\mathbb{R}^n} \rho_m(x)\big(\big|g(t,x)\big|^2 + \big|\varphi_1(x)\big| + \big|\varphi_3(x)\big| + \big|\zeta_\delta(\theta_t\omega)\varphi_6(x)\big|^2 \\
&\qquad + \big|\zeta_\delta(\theta_t\omega)\varphi_5(x)\big|^{2+\frac{4}{p}}\big)\,dx \\
&\qquad + \frac{c_{14}}{m}\big(\|u\| + \|\nabla u\| + \|u_t\| + \|\nabla u_t\|\big)\|\Delta u\|,
\end{aligned} \tag{4.32}$$

where $c_{14} > 0$ depends only on α, ν, γ, and ε, but not on m.

By (4.32) we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} \rho_m(x)\big(\big|u_t(t,x)\big|^2 + \nu\big|u(t,x)\big|^2 + \big|\Delta u(t,x)\big|^2 \\
&\qquad + 2F\big(x,u(t,x)\big) + \varepsilon u(t,x)u_t(t,x)\big)\,dx \\
&\qquad + \frac{1}{4}\varepsilon\gamma\int_{\mathbb{R}^n} \rho_m(x)\big(\big|u_t(t,x)\big|^2 + \nu\big|u(t,x)\big|^2 + \big|\Delta u(t,x)\big|^2 \\
&\qquad + 2F\big(x,u(t,x)\big) + \varepsilon u(t,x)u_t(t,x)\big)\,dx \\
&\qquad + \left(\alpha - \varepsilon - \frac{1}{4}\varepsilon\gamma\right)\int_{\mathbb{R}^n} \rho_m(x)\big|u_t(t,x)\big|^2\,dx \\
&\qquad + \varepsilon\left(\alpha - \frac{1}{4}\gamma\right)\int_{\mathbb{R}^n} \rho_m(x)u(t,x)u_t(t,x)\,dx
\end{aligned}$$

$$
\begin{aligned}
&+ \varepsilon\left(1-\frac{1}{4}\gamma\right)\int_{\mathbb{R}^n}\rho_m(x)\left|\Delta u(t,x)\right|^2\,dx \\
&+ \frac{1}{2}\varepsilon\nu\left(1-\frac{1}{2}\gamma\right)\int_{\mathbb{R}^n}\rho_m(x)\left|u(t,x)\right|^2\,dx + \frac{1}{2}\varepsilon\gamma\int_{\mathbb{R}^n}\rho_m(x)F\big(x,u(t,x)\big)\,dx \\
\le{}& c_{14}\int_{\mathbb{R}^n}\rho_m(x)\big(\left|g(t,x)\right|^2 + \left|\varphi_1(x)\right| + \left|\varphi_3(x)\right| + \left|\zeta_\delta(\theta_t\omega)\varphi_6(x)\right|^2 \\
&+ \left|\zeta_\delta(\theta_t\omega)\varphi_5(x)\right|^{2+\frac{4}{p}}\big)\,dx \\
&+ \frac{c_{14}}{m}\big(\|u\| + \|\nabla u\| + \|u_t\| + \|\nabla u_t\|\big)\|\Delta u\|.
\end{aligned}
\tag{4.33}
$$

By Young's inequality we obtain

$$
\begin{aligned}
&\left|\varepsilon\left(\alpha-\frac{1}{4}\gamma\right)\int_{\mathbb{R}^n}\rho_m(x)u(t,x)u_t(t,x)\,dx\right| \\
&\quad\le \frac{1}{2}\varepsilon^2\left(\alpha-\frac{1}{4}\gamma\right)\int_{\mathbb{R}^n}\rho_m(x)\left|u(t,x)\right|^2\,dx \\
&\qquad+ \frac{1}{2}\left(\alpha-\frac{1}{4}\gamma\right)\int_{\mathbb{R}^n}\rho_m(x)\left|u_t(t,x)\right|^2\,dx.
\end{aligned}
\tag{4.34}
$$

By (4.33) and (4.34) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)\big(\left|u_t(t,x)\right|^2 + \nu\left|u(t,x)\right|^2 + \left|\Delta u(t,x)\right|^2 \\
&\quad+ 2F\big(x,u(t,x)\big) + \varepsilon u(t,x)u_t(t,x)\big)\,dx \\
&\quad+ \frac{1}{4}\varepsilon\gamma\int_{\mathbb{R}^n}\rho_m(x)\big(\left|u_t(t,x)\right|^2 + \nu\left|u(t,x)\right|^2 + \left|\Delta u(t,x)\right|^2 \\
&\quad+ 2F\big(x,u(t,x)\big) + \varepsilon u(t,x)u_t(t,x)\big)\,dx \\
&\quad+ \left(\frac{1}{2}\alpha - \varepsilon - \frac{1}{8}\varepsilon\gamma\right)\int_{\mathbb{R}^n}\rho_m(x)\left|u_t(t,x)\right|^2\,dx \\
&\quad+ \varepsilon\left(1-\frac{1}{4}\gamma\right)\int_{\mathbb{R}^n}\rho_m(x)\left|\Delta u(t,x)\right|^2\,dx \\
&\quad+ \frac{1}{2}\varepsilon\left(\nu - \frac{1}{2}\nu\gamma - \varepsilon\alpha + \frac{1}{4}\varepsilon^2\gamma\right)\int_{\mathbb{R}^n}\rho_m(x)\left|u(t,x)\right|^2\,dx \\
&\quad+ \frac{1}{2}\varepsilon\gamma\int_{\mathbb{R}^n}\rho_m(x)F\big(x,u(t,x)\big)\,dx \\
&\le c_{14}\int_{\mathbb{R}^n}\rho_m(x)\big(\left|g(t,x)\right|^2 + \left|\varphi_1(x)\right| + \left|\varphi_3(x)\right| + \left|\zeta_\delta(\theta_t\omega)\varphi_6(x)\right|^2 \\
&\quad+ \left|\zeta_\delta(\theta_t\omega)\varphi_5(x)\right|^{2+\frac{4}{p}}\big)\,dx \\
&\quad+ \frac{c_{14}}{m}\big(\|u\| + \|\nabla u\| + \|u_t\| + \|\nabla u_t\|\big)\|\Delta u\|.
\end{aligned}
\tag{4.35}
$$

By (4.7) and (4.35) we have

$$
\frac{d}{dt}\int_{\mathbb{R}^n}\rho_m(x)\big(\left|u_t(t,x)\right|^2 + \nu\left|u(t,x)\right|^2 + \left|\Delta u(t,x)\right|^2 + 2F\big(x,u(t,x)\big) + \varepsilon u(t,x)u_t(t,x)\big)\,dx
$$

$$
\begin{aligned}
&+\frac{1}{4}\varepsilon\gamma\int_{\mathbb{R}^n}\rho_m(x)\big(\big|u_t(t,x)\big|^2+\nu\big|u(t,x)\big|^2+\big|\Delta u(t,x)\big|^2 \\
&+2F\big(x,u(t,x)\big)+\varepsilon u(t,x)u_t(t,x)\big)\,dx \\
\leq{}& c_{14}\int_{\mathbb{R}^n}\rho_m(x)\big(\big|g(t,x)\big|^2+\big|\varphi_1(x)\big|+\big|\varphi_3(x)\big|+\big|\zeta_\delta(\theta_t\omega)\varphi_6(x)\big|^2 \\
&+\big|\zeta_\delta(\theta_t\omega)\varphi_5(x)\big|^{2+\frac{4}{p}}\big)\,dx \\
&+\frac{c_{14}}{m}\big(\|u\|+\|\nabla u\|+\|u_t\|+\|\nabla u_t\|\big)\|\Delta u\|.
\end{aligned}
\tag{4.36}
$$

Multiplying (4.36) by $e^{\frac{1}{4}\varepsilon\gamma t}$, and then integrating the inequality $[\tau-t,\tau]$, after replacing ω by $\theta_{-\tau}\omega$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\rho_m(x)\big(\big|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\big|^2+\nu\big|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big|^2 \\
&+\big|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big|^2 \\
&+2F\big(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big)+\varepsilon u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\big)\,dx \\
\leq{}& e^{-\frac{1}{4}\varepsilon\gamma t}\int_{\mathbb{R}^n}\rho_m(x)\big(|u_{1,0}|^2+\nu|u_0|^2+|\Delta u_0|^2+2F\big(x,u_0(x)\big)+\varepsilon u_0(x)u_{1,0}(x)\big)\,dx \\
&+c_{14}\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{\mathbb{R}^n}\rho_m(x)\big(\big|g(s,x)\big|^2+\big|\varphi_1(x)\big|+\big|\varphi_3(x)\big|\big)\,dx\,ds \\
&+c_{14}\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{\mathbb{R}^n}\rho_m(x)\big(\big|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_6(x)\big|^2+\big|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_5(x)\big|^{2+\frac{4}{p}}\big)\,dx\,ds \\
&+\frac{2c_{14}}{m}\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\big(\big\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big\|^2_{H^2(\mathbb{R}^n)} \\
&+\big\|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\big\|^2_{H^1(\mathbb{R}^n)}\big)\,ds.
\end{aligned}
\tag{4.37}
$$

Next, we estimate the right-hand side of (4.37). By (4.16), we know that there exists $T_1(\eta,\tau,\omega,D)>0$ such that for all $t\geq T_1$,

$$
\begin{aligned}
&e^{-\frac{1}{4}\varepsilon\gamma t}\int_{\mathbb{R}^n}\rho_m(x)\big(|u_{1,0}|^2+\nu|u_0|^2+|\Delta u_0|^2 \\
&+2F\big(x,u_0(x)\big)+\varepsilon u_0(x)u_{1,0}(x)\big)\,dx<\eta.
\end{aligned}
\tag{4.38}
$$

For the second and the third terms on the right-hand side of (4.37) we obtain

$$
\begin{aligned}
&c_{14}\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{\mathbb{R}^n}\rho_m(x)\big(\big|g(s,x)\big|^2+\big|\varphi_1(x)\big|+\big|\varphi_3(x)\big|\big)\,dx\,ds \\
&+c_{14}\int_{\tau-t}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{\mathbb{R}^n}\big(\rho_m(x)\big|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_6(x)\big|^2+\big|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_5(x)\big|^{2+\frac{4}{p}}\big)\,dx\,ds \\
\leq{}& c_{14}\int_{-\infty}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{|x|\geq\frac{1}{2}m}\big(\big|g(s,x)\big|^2+\big|\varphi_1(x)\big|+\big|\varphi_3(x)\big|\big)\,dx\,ds \\
&+c_{14}\int_{-\infty}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{|x|\geq\frac{1}{2}m}\big(\big|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_6(x)\big|^2+\big|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_5(x)\big|^{2+\frac{4}{p}}\big)\,dx\,ds \\
\leq{}& c_{14}\int_{-\infty}^{\tau}e^{\frac{1}{4}\varepsilon\gamma(s-\tau)}\int_{|x|\geq\frac{1}{2}m}\big(\big|g(s,x)\big|^2+\big|\varphi_1(x)\big|+\big|\varphi_3(x)\big|\big)\,dx\,ds
\end{aligned}
$$

$$+ c_{14} \int_{-\infty}^{0} e^{\frac{1}{4}\varepsilon\gamma s} \left|\zeta_\delta(\theta_s\omega)\right|^2 ds \int_{|x|\geq \frac{1}{2}m} \left|\varphi_6(x)\right|^2 dx$$
$$+ c_{14} \int_{-\infty}^{0} e^{\frac{1}{4}\varepsilon\gamma s} \left|\zeta_\delta(\theta_s\omega)\right|^{2+\frac{4}{p}} ds \int_{|x|\geq \frac{1}{2}m} \left|\varphi_5(x)\right|^{2+\frac{4}{p}} dx. \tag{4.39}$$

By (4.8) and with the conditions of $\varphi_i(x)$ $(i = 1,3,5,6)$ satisfied, we know that there exists $m_1 = m_1(\eta,\tau,\omega) \geq 1$ such that for all $m \geq m_1$, the right-hand of side of (4.39) is bounded by η, i.e.,

$$\begin{aligned} &c_{14} \int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)} \int_{\mathbb{R}^n} \rho_m(x)\big(\left|g(s,x)\right|^2 + \left|\varphi_1(x)\right| + \left|\varphi_3(x)\right|\big)\,dx\,ds \\ &\quad + c_{14} \int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)} \int_{\mathbb{R}^n} \big(\rho_m(x)\left|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_6(x)\right|^2 + \left|\zeta_\delta(\theta_{s-\tau}\omega)\varphi_5(x)\right|^{2+\frac{4}{p}}\big)\,dx\,ds \\ &< \eta. \end{aligned} \tag{4.40}$$

For the last term in (4.37), by Lemma 4.1 and Lemma 4.3, we know that there exists $T_2(\eta,\tau,\omega,D) \geq T_1$ such that for all $t \geq T_2$,

$$\begin{aligned} &\frac{2c_{14}}{m} \int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)} \big(\left\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right\|_{H^2(\mathbb{R}^n)}^2 \\ &\quad + \left\|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right\|_{H^1(\mathbb{R}^n)}^2\big)\,ds \leq \frac{c_{15}}{m}, \end{aligned}$$

where $c_{15} > 0$ depends only on α, ν, γ, ε, τ, and ω, but not on m. Thus, there exists $m_2 = m_2(\eta,\tau,\omega) \geq m_1$ such that for all $m \geq m_2$ and $t \geq T_2$,

$$\begin{aligned} &\frac{2c_{14}}{m} \int_{\tau-t}^{\tau} e^{\frac{1}{4}\varepsilon\gamma(s-\tau)} \big(\left\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right\|_{H^2(\mathbb{R}^n)}^2 \\ &\quad + \left\|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right\|_{H^1(\mathbb{R}^n)}^2\big)\,ds \leq \eta. \end{aligned} \tag{4.41}$$

By (4.37), (4.38), (4.40), and (4.41) we see that for all $m \geq m_2$ and $t \geq T_2$,

$$\begin{aligned} &\int_{\mathbb{R}^n} \rho_m(x)\big(\left|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right|^2 + \nu\left|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right|^2 \\ &\quad + \left|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right|^2 \\ &\quad + 2F\big(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big) + \varepsilon u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\big)\,dx \\ &< 3\eta. \end{aligned} \tag{4.42}$$

By (4.7) we have

$$\begin{aligned} &\varepsilon \int_{\mathbb{R}^n} \rho_m(x)u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\,dx \\ &\leq \frac{1}{2}\nu \int_{\mathbb{R}^n} \rho_m(x)\left|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right|^2 dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^n} \rho_m(x)\left|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right|^2 dx, \end{aligned}$$

which together with (4.2) and (4.42) yields that for all $m \geq m_2$ and $t \geq T_2$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho_m(x)\Big(\frac{1}{2}\big|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\big|^2 + \frac{1}{2}\nu\big|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big|^2 \\
&\qquad + \big|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big|^2\,dx \\
&\quad \leq 3\eta + 2\int_{\mathbb{R}^n} \rho_m(x)\varphi_1(x)\,dx.
\end{aligned} \tag{4.43}$$

Since $\varphi_1 \in L^1(\mathbb{R}^n)$, there exists $m_3 = m_3(\eta,\tau,\omega) \geq m_2$ such that for all $m \geq m_3$,

$$2\int_{\mathbb{R}^n} \rho_m(x)\varphi_1(x)\,dx = 2\int_{|x|\geq \frac{1}{2}m} \rho_m(x)\varphi_1(x)\,dx \leq 2\int_{|x|\geq \frac{1}{2}m} \big|\varphi_1(x)\big|\,dx < \eta. \tag{4.44}$$

From (4.43) and (4.44) we obtain, for all $m \geq m_3$ and $t \geq T_2$,

$$\begin{aligned}
&\int_{|x|\geq m} \rho_m(x)\Big(\frac{1}{2}\big|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\big|^2 + \frac{1}{2}\nu\big|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big|^2 \\
&\qquad + \big|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big|^2\,dx \\
&\quad \leq \int_{\mathbb{R}^n} \rho_m(x)\Big(\frac{1}{2}\big|u_t(\tau,\tau-t,\theta_{-\tau}\omega,u_{1,0})\big|^2 + \frac{1}{2}\nu\big|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big|^2 \\
&\qquad + \big|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big|^2\,dx \\
&\quad < 4\eta.
\end{aligned}$$

□

5 Existence of random attractors

In this section, we present the existence and uniqueness of $\mathcal{D}$-pullback random attractors of (1.1).

Let u be the solution of (1.1). Denote $u = \tilde{v} + v$, where $\tilde{v}$ and v are the solutions of the following equations, respectively,

$$\begin{cases} \tilde{v}_{tt} + \alpha\tilde{v}_t + \Delta^2\tilde{v} + \nu\tilde{v} = g(t), & t > \tau, \\ \tilde{v}(\tau) = u_0, \quad \tilde{v}_t(\tau) = u_{1,0}, \end{cases} \tag{5.1}$$

and

$$\begin{cases} v_{tt} + \alpha v_t + \Delta^2 v + \nu v = -f(x,u) + h(t,x,u)\zeta_\delta(\theta_t\omega), & t > \tau, \\ v(\tau) = 0, \quad v_t(\tau) = 0. \end{cases} \tag{5.2}$$

Lemma 5.1 *Suppose (4.7) and (4.8) hold. Then, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D \in \mathcal{D}$, there exists $T = T(\tau,\omega,D) > 0$ such that for all $t \geq T$ and $r \in [-t,0]$, the solution $\tilde{v}$ of (5.1) satisfies*

$$\begin{aligned}
&\big\|\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\big\|^2_{H^2(\mathbb{R}^n)} + \big\|\tilde{v}_r(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\big\|^2 \\
&\quad \leq e^{-\frac{1}{2}\varepsilon r} M_2\bigg(1 + \int_{\infty}^{0} e^{\frac{1}{2}\varepsilon s}\big\|g(s+\tau)\big\|^2\,ds\bigg),
\end{aligned}$$

where $(u_0,u_{1,0}) \in D(\tau-t,\theta_{-t}\omega)$ and M_2 is a positive number independent of τ, ω, and D.

Proof From (3.8), (3.9), and (5.1) we see that

$$\begin{aligned}
&\frac{d}{dt}\big(\|\tilde{v}_t\|^2 + \|\Delta\tilde{v}\|^2 + \nu\|\tilde{v}\|^2 + \varepsilon\big(\tilde{v}(t),\tilde{v}_t(t)\big)\big) + (2\alpha - \varepsilon)\|\tilde{v}_t\|^2 \\
&\quad + \varepsilon\|\Delta\tilde{v}\|^2 + \varepsilon\nu\|\tilde{v}\|^2 + \varepsilon\alpha\big(\tilde{v}(t),\tilde{v}_t(t)\big) \\
&= \big(g(t),\varepsilon\tilde{v}(t) + 2\tilde{v}_t(t)\big) \\
&\le \varepsilon\big\|g(t)\big\|\big\|\tilde{v}(t)\big\| + 2\big\|g(t)\big\|\big\|\tilde{v}_t(t)\big\| \\
&\le \frac{1}{2}\varepsilon^2\big\|\tilde{v}(t)\big\|^2 + \alpha\big\|\tilde{v}_t(t)\big\|^2 + \left(\frac{1}{2} + \alpha^{-1}\right)\big\|g(t)\big\|^2.
\end{aligned} \tag{5.3}$$

In addition, we obtain

$$\left|\left(\alpha - \frac{1}{2}\varepsilon\right)\varepsilon\big(\tilde{v}(t),\tilde{v}_t(t)\big)\right| \le \frac{1}{2}\left(\alpha - \frac{1}{2}\varepsilon\right)\big(\varepsilon^2\big\|\tilde{v}(t)\big\|^2 + \big\|\tilde{v}_t(t)\big\|^2\big). \tag{5.4}$$

By (5.3) and (5.4) we have

$$\begin{aligned}
&\frac{d}{dt}\big(\|\tilde{v}_t\|^2 + \|\Delta\tilde{v}\|^2 + \nu\|\tilde{v}\|^2 + \varepsilon\big(\tilde{v}(t),\tilde{v}_t(t)\big)\big) + \left(\frac{1}{2}\alpha - \frac{3}{4}\varepsilon\right)\|\tilde{v}_t\|^2 \\
&\quad + \varepsilon\|\Delta\tilde{v}\|^2 + \varepsilon\left(\nu - \frac{1}{2}\varepsilon - \frac{1}{2}\varepsilon\alpha + \frac{1}{4}\varepsilon^2\right)\|\tilde{v}\|^2 + \frac{1}{2}\varepsilon^2\big(\tilde{v}(t),\tilde{v}_t(t)\big) \\
&\le \left(\frac{1}{2} + \alpha^{-1}\right)\big\|g(t)\big\|^2,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\frac{d}{dt}\big(\|\tilde{v}_t\|^2 + \|\Delta\tilde{v}\|^2 + \nu\|\tilde{v}\|^2 + \varepsilon\big(\tilde{v}(t),\tilde{v}_t(t)\big)\big) \\
&\quad + \frac{1}{2}\varepsilon\big(\|\tilde{v}_t\|^2 + \|\Delta\tilde{v}\|^2 + \nu\|\tilde{v}\|^2 + \varepsilon\big(\tilde{v}(t),\tilde{v}_t(t)\big)\big) \\
&\quad + \left(\frac{1}{2}\alpha - \frac{5}{4}\varepsilon\right)\|\tilde{v}_t\|^2 + \frac{1}{2}\varepsilon\|\Delta\tilde{v}\|^2 + \frac{1}{2}\varepsilon\left(\nu - \varepsilon - \varepsilon\alpha + \frac{1}{2}\varepsilon^2\right)\|\tilde{v}\|^2 \\
&\le \left(\frac{1}{2} + \alpha^{-1}\right)\big\|g(t)\big\|^2.
\end{aligned} \tag{5.5}$$

It follows from (4.7) and (5.5) that

$$\begin{aligned}
&\frac{d}{dt}\big(\|\tilde{v}_t\|^2 + \|\Delta\tilde{v}\|^2 + \nu\|\tilde{v}\|^2 + \varepsilon\big(\tilde{v}(t),\tilde{v}_t(t)\big)\big) \\
&\quad + \frac{1}{2}\varepsilon\big(\|\tilde{v}_t\|^2 + \|\Delta\tilde{v}\|^2 + \nu\|\tilde{v}\|^2 + \varepsilon\big(\tilde{v}(t),\tilde{v}_t(t)\big)\big) \\
&\le \left(\frac{1}{2} + \alpha^{-1}\right)\big\|g(t)\big\|^2.
\end{aligned} \tag{5.6}$$

Applying Gronwall's lemma to (5.6), we obtain for all $\tau \in \mathbb{R}$, $t \ge 0$, $r \in [-t, 0]$ and $\omega \in \Omega$,

$$\big\|\tilde{v}_r(\tau + r, \tau - t, \theta_{-\tau}\omega, u_{1,0})\big\|^2 + \big\|\Delta\tilde{v}(\tau + r, \tau - t, \theta_{-\tau}\omega, u_0)\big\|^2$$

$$
\begin{aligned}
&+\nu\left\|\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2\\
&+\varepsilon\left(\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0),\tilde{v}_r(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right)\\
\le{}& e^{-\frac{1}{2}\varepsilon r}e^{-\frac{1}{2}\varepsilon t}\left(\|u_{1,0}\|^2+\nu\|u_0\|^2+\|\Delta u_0\|^2+\varepsilon(u_0,u_{1,0})\right)\\
&+\left(\frac{1}{2}+\alpha^{-1}\right)e^{-\frac{1}{2}\varepsilon r}\int_{\tau-t}^{\tau+r}e^{\frac{1}{2}\varepsilon(s-\tau)}\left\|g(s)\right\|^2\,ds. \qquad (5.7)
\end{aligned}
$$

By (4.7) we have

$$
\begin{aligned}
&\varepsilon\left(\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0),\tilde{v}_r(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right)\\
&\le\frac{1}{2}\varepsilon\left\|\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2+\frac{1}{2}\varepsilon\left\|\tilde{v}_r(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right\|^2\\
&\le\frac{1}{2}\nu\left\|\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2+\frac{1}{2}\left\|\tilde{v}_r(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right\|^2. \qquad (5.8)
\end{aligned}
$$

By (5.7) and (5.8) we see that for all $\tau\in\mathbb{R}$, $t\ge 0$, $r\in[-t,0]$ and $\omega\in\Omega$,

$$
\begin{aligned}
&\frac{1}{2}\left\|\tilde{v}_r(\tau+r,\tau-t,\theta_{-\tau}\omega,u_{1,0})\right\|^2+\left\|\Delta\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2\\
&\quad+\frac{1}{2}\nu\left\|\tilde{v}(\tau+r,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2\\
&\le e^{-\frac{1}{2}\varepsilon r}e^{-\frac{1}{2}\varepsilon t}\left(\|u_{1,0}\|^2+\nu\|u_0\|^2+\|\Delta u_0\|^2+\varepsilon(u_0,u_{1,0})\right)\\
&\quad+\left(\frac{1}{2}+\alpha^{-1}\right)e^{-\frac{1}{2}\varepsilon r}\int_{\tau-t}^{\tau+r}e^{\frac{1}{2}\varepsilon(s-\tau)}\left\|g(s)\right\|^2\,ds. \qquad (5.9)
\end{aligned}
$$

Similar to (4.16), one can verify that

$$
e^{-\frac{1}{2}\varepsilon t}\left(\|u_{1,0}\|^2+\nu\|u_0\|^2+\|\Delta u_0\|^2+\varepsilon(u_0,u_{1,0})\right)\to 0,\quad \text{as } t\to\infty,
$$

which along with (5.9) yields the desired result. □

Based on Lemma 5.1, we infer that system (5.1) has a tempered pullback random absorbing set.

Lemma 5.2 *Suppose* (4.8) *and* (4.9) *hold, then* (5.1) *possesses a closed measurable $\mathcal{D}$-pullback absorbing set $B_1=\{B_1(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, which is given by*

$$
B_1(\tau,\omega)=\left\{(u_0,u_{1,0})\in H^2\left(\mathbb{R}^n\right)\times L^2\left(\mathbb{R}^n\right):\|u_0\|^2_{H^2(\mathbb{R}^n)}+\|u_{1,0}\|^2\le L_1(\tau,\omega)\right\}, \qquad (5.10)
$$

where

$$
L_1(\tau,\omega)=M_2+M_2\int_{-\infty}^{0}e^{\frac{1}{2}\varepsilon s}\left\|g(s+\tau)\right\|^2\,ds.
$$

Lemma 5.3 *Suppose* (4.8) *and* (4.9) *hold, then the sequence of the solutions to* (5.1)

$$
\left\{\tilde{v}\left(\tau,\tau-t_n,\theta_{-\tau}\omega,u_0^{(n)}\right),\tilde{v}_t\left(\tau,\tau-t_n,\theta_{-\tau}\omega,u_{1,0}^{(n)}\right)\right\}_{n=1}^{\infty}
$$

converges in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *for any* $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D \in \mathcal{D}$, $t_n \to \infty$ *monotonically, and* $(u_0^{(n)}, u_{1,0}^{(n)}) \in D(\tau - t_n, \theta_{-t_n}\omega)$.

Proof Let $m > n$ and

$$\begin{aligned} & v_{n,m}(t,\tau - t_n, \theta_{-\tau}\omega) \\ &\quad = \tilde{v}\big(t,\tau - t_n, \theta_{-\tau}\omega, u_0^{(n)}\big) - \tilde{v}\big(t,\tau - t_m, \theta_{-\tau}\omega, u_0^{(m)}\big) \\ &\quad = \tilde{v}\big(t,\tau - t_n, \theta_{-\tau}\omega, u_0^{(n)}\big) - \tilde{v}\big(t,\tau - t_n, \theta_{-\tau}\omega, \tilde{v}\big(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, u_0^{(m)}\big) \end{aligned} \tag{5.11}$$

for $t \geq \tau - t_n$.

By (5.1) we obtain

$$\begin{cases} \partial_{tt}^2 v_{n,m}(t) + \alpha \partial_t v_{n,m}(t) + \Delta^2 v_{n,m}(t) + \nu v_{n,m}(t) = 0, \quad t > \tau - t_n, \\ v_{n,m}(\tau - t_n) = u_0^{(n)} - \tilde{v}(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, u_0^{(m)}), \\ \partial_t v_{n,m}(\tau - t_n) = u_{1,0}^{(n)} - \tilde{v}_t. \end{cases} \tag{5.12}$$

Similar to (5.9) with $r = 0$, $t = t_n$, and $g = 0$, we obtain

$$\begin{aligned} & \frac{1}{2}\big\|\partial_t v_{n,m}(\tau,\tau - t_n, \theta_{-\tau}\omega)\big\|^2 + \big\|\Delta v_{n,m}(\tau,\tau - t_n, \theta_{-\tau}\omega)\big\|^2 + \frac{1}{2}\nu v_{n,m}(\tau,\tau - t_n, \theta_{-\tau}\omega)\|^2 \\ &\quad \leq e^{-\frac{1}{2}\varepsilon t_n}\big(\big\|\partial_t v_{n,m}(\tau - t_n)\big\|^2 + \big\|v_{n,m}(\tau - t_n)\big\|^2 + \big\|\Delta v_{n,m}(\tau - t_n)\big\|^2\big), \end{aligned} \tag{5.13}$$

which together with $(5.12)_2$, gives

$$\begin{aligned} & \big\|\partial_t v_{n,m}(\tau,\tau - t_n, \theta_{-\tau}\omega)\big\|^2 + 2\big\|\Delta v_{n,m}(\tau,\tau - t_n, \theta_{-\tau}\omega)\big\|^2 + \nu v_{n,m}(\tau,\tau - t_n, \theta_{-\tau}\omega)\|^2 \\ &\quad \leq 2e^{-\frac{1}{2}\varepsilon t_n}(\|\tilde{v}_t(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, u_{1,0}^{(m)}\|^2 + \|\tilde{v}\big(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, u_0^{(m)}\|_{H^2}^2) \\ &\qquad + 2e^{-\frac{1}{2}\varepsilon t_n}\big(\big\|u_{1,0}^{(n)}\big\|^2 + \big\|u_0^{(n)}\big\|^2 + \big\|\Delta u_0^{(n)}\big\|^2\big). \end{aligned} \tag{5.14}$$

By (5.9) with $r = -t_n$, and $t = t_m$, we obtain

$$\begin{aligned} & \big\|\tilde{v}_t\big(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, u_{1,0}^{(m)}\big)\big\|^2 + 2\big\|\Delta\tilde{v}\big(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, u_0^{(m)}\big)\big\|^2 \\ &\qquad + \nu\big\|\tilde{v}\big(\tau - t_n, \tau - t_m, \theta_{-\tau}\omega, u_0^{(m)}\big)\big\|^2 \\ &\quad \leq 2e^{\frac{1}{2}\varepsilon t_n}e^{-\frac{1}{2}\varepsilon t_m}\big(\big\|u_{1,0}^{(n)}\big\|^2 + \nu\big\|u_0^{(n)}\big\|^2 + \big\|\Delta u_0^{(n)}\big\|^2 + \varepsilon\big(u_0^{(n)}, u_{1,0}^{(n)}\big)\big) \\ &\qquad + \big(1 + 2\alpha^{-1}\big)e^{\frac{1}{2}\varepsilon t_n}\int_{\tau - t_m}^{\tau - t_n} e^{\frac{1}{2}\varepsilon(s-\tau)}\big\|g(s)\big\|^2\,ds. \end{aligned} \tag{5.15}$$

It follows from (5.14) and (5.15) that for $m > n \to \infty$,

$$\big\|\partial_t v_{n,m}(\tau,\tau - t_n, \theta_{-\tau}\omega)\big\|^2 + \big\|v_{n,m}(\tau,\tau - t_n, \theta_{-\tau}\omega)\big\|_{H^2(\mathbb{R}^n)}^2 \to 0,$$

which together with (5.11) implies $\{\tilde{v}(\tau,\tau - t_n, \theta_{-\tau}\omega, u_0^{(n)}), \tilde{v}_t(\tau,\tau - t_n, \theta_{-\tau}\omega, u_{1,0}^{(n)})\}_{n=1}^{\infty}$ is a Cauchy sequence in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. This complete the proof. □

Lemma 5.4 *Suppose* (4.8) *and* (4.9) *hold, then* (5.1) *has a unique* $\mathcal{D}$*-pullback random attractor* $\mathcal{A}_1 = \{\mathcal{A}_1(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ *in* $H^2(\mathbb{R}^a n) \times L^2(\mathbb{R}^n)$, *which is actually a singleton; that is,* $\mathcal{A}_1(\tau,\omega)$ *consisting of a single point for all* $\tau \in \mathbb{R}$, $\omega \in \Omega$.

Proof From Lemmas 5.2 and 5.3 by applying the abstract results in [19], we can obtain the existence and uniqueness of the $\mathcal{D}$-pullback random attractor $\mathcal{A}_1 \in \mathcal{D}$ of (5.1) in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ immediately.

Next, we prove $\mathcal{A}_1$ is a singleton. Suppose $\{t_n\}_{n=1}^{\infty}$ 1 is a sequence of numbers such that $t_n \to \infty$ as $n \to \infty$. Given $\tau \in \mathbb{R}$, $\omega \in \Omega$, let $(z_0^{(n)}, z_{1,0}^{(n)}), (y_0^{(n)}, y_{1,0}^{(n)}) \in \mathcal{A}_1(\tau - t_n, \theta_{-t_n}\omega)$.

Similar to (5.13) we have

$$\begin{aligned}
&\big\|\tilde{v}_t\big(\tau,\tau-t_n,\theta_{-\tau}\omega,z_{1,0}^{(n)}\big)-\tilde{v}_t\big(\tau,\tau-t_n,\theta_{-\tau}\omega,y_{1,0}^{(n)}\big)\big\|^2\\
&\quad+2\big\|\Delta\tilde{v}\big(\tau,\tau-t_n,\theta_{-\tau}\omega,z_0^{(n)}\big)-\Delta\tilde{v}\big(\tau,\tau-t_n,\theta_{-\tau}\omega,y_0^{(n)}\big)\big\|^2\\
&\quad+\nu\big\|\tilde{v}\big(\tau,\tau-t_n,\theta_{-\tau}\omega,z_0^{(n)}\big)-\tilde{v}\big(\tau,\tau-t_n,\theta_{-\tau}\omega,y_0^{(n)}\big)\big\|^2\\
&\le e^{-\frac{1}{2}\varepsilon t_n}\big(\big\|z_{1,0}^{(n)}-y_{1,0}^{(n)}\big\|^2+\big\|z_0^{(n)}-y_0^{(n)}\big\|^2+\big\|\Delta z_0^{(n)}-\Delta z_0^{(n)}\big\|^2\big)\\
&\le 2e^{-\frac{1}{2}\varepsilon t_n}\big(\big\|z_{1,0}^{(n)}\big\|^2+\big\|z_0^{(n)}\big\|^2_{H^2(\mathbb{R}^n)}+\big\|y_{1,0}^{(n)}\big\|^2+\big\|y_{1,0}^{(n)}\big\|^2_{H^2(\mathbb{R}^n)}\big)\\
&\le 4e^{-\frac{1}{2}\varepsilon t_n}\big\|\mathcal{A}_1(\tau-t_n,\theta_{-t_n}\omega)\big\|^2_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}.
\end{aligned}\tag{5.16}$$

Due to $\mathcal{A}_1 \in \mathcal{D}$, we see that the right-hand side of (5.16) tends to zero as $n \to \infty$, and thus we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\big(\tilde{v}_t\big(\tau,\tau-t_n,\theta_{-\tau}\omega,z_{1,0}^{(n)}\big)-\tilde{v}_t\big(\tau,\tau-t_n,\theta_{-\tau}\omega,y_{1,0}^{(n)}\big)\big)=0\quad \text{in } L^2\big(\mathbb{R}^n\big),\\
&\lim_{n\to\infty}\big(\tilde{v}\big(\tau,\tau-t_n,\theta_{-\tau}\omega,z_0^{(n)}\big)-\tilde{v}_t\big(\tau,\tau-t_n,\theta_{-\tau}\omega,y_0^{(n)}\big)\big)=0\quad \text{in } H^2\big(\mathbb{R}^n\big),
\end{aligned}$$

which, together with the invariance of $\mathcal{A}_1$, shows that the $\mathcal{D}$-pullback random attractor $\mathcal{A}_1$ is a singleton. This complete the proof. □

To obtain the asymptotic compactness of the solutions of (5.2), we need the following Lemma.

Lemma 5.5 *Let* $u_0 \in H^2(\mathbb{R}^n)$, $u_{1,0} \in L^2(\mathbb{R}^n)$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $T > 0$. *If* (3.2), (3.3), (3.6), (4.1), (4.2), *and* (4.5)–(4.8) *hold, then the solution of* (5.2) *satisfies, for all* $t \in [\tau, \tau + T]$,

$$\big\|A^{\frac{3}{4}}v(t,\tau,\omega)\big\| + \big\|A^{\frac{1}{4}}v_t(t,\tau,\omega)\big\| \le C,$$

where C *is a positive number depending on* τ, ω, T *and* R *when* $\|(u_0, u_{1,0})\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \le R$.

Proof This is an immediate consequence of Lemma 4.3. □

Lemma 5.6 *Let* (3.2), (3.3), (3.6), (4.1), (4.3), *and* (4.5)–(4.9) *hold. Then, the cocycle* Φ *is* $\mathcal{D}$*-pullback asymptotically compact in* $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$; *that is, the sequence* $\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, (u_0^{(n)}, u_{1,0}^{(n)})\}_{n=1}^{\infty}$ *has a convergent subsequence in* $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *for any* $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D \in \mathcal{D}$, $t_n \to \infty$, *and* $(u_0^{(n)}, u_{1,0}^{(n)}) \in D(\tau - t_n, \theta_{-t_n}\omega)$.

Proof Given $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $(u_0, u_{1,0} \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, define

$$\Phi_1\big(t,\tau,\omega,(u_0,u_{1,0})\big) = \big(\tilde{v}(t+\tau,\tau,\theta_{-\tau}\omega,u_0),\tilde{v}_t(t+\tau,\tau,\theta_{-\tau}\omega,u_{1,0})\big),$$
$$\Phi_2\big(t,\tau,\omega,(u_0,u_{1,0})\big) = \big(v(t+\tau,\tau,\theta_{-\tau}\omega,u_0),v_t(t+\tau,\tau,\theta_{-\tau}\omega,u_{1,0})\big),$$

where $\tilde{v}$ and v are the solutions of (5.1) and (5.2), respectively.

By (3.72) we have

$$\Phi\big(t,\tau,\omega,(u_0,u_{1,0})\big) = \Phi_1\big(t,\tau,\omega,(u_0,u_{1,0})\big) + \Phi_2\big(t,\tau,\omega,(u_0,u_{1,0})\big). \tag{5.17}$$

Let $B \in \mathcal{D}$ be the $\in \mathcal{D}$-pullback absorbing set of Φ given by (4.19). From Lemmas 4.2, 4.4, and 5.4 we see that for every $\delta > 0$ there exists $t_0 = t_0(\delta,\tau,\omega,B) > 0$ and $k_0 = k_0(\delta,\tau,\omega) \geq 1$ such that for all $(u_0,u_{1,0}) \in B(\tau - t_0, \theta_{-t_0}\omega)$,

$$\big\|\Phi\big(t_0,\tau-t_0,\theta_{-t_0}\omega,(u_0,u_{1,0})\big)|_{\tilde{\mathcal{O}}_{k_0}}\big\|_{H^2(\tilde{\mathcal{O}}_{k_0})\times L^2(\tilde{\mathcal{O}}_{k_0})} < \delta, \tag{5.18}$$

with $\tilde{\mathcal{O}}_{k_0} = \{x \in \mathbb{R}^n : |x| > k_0\}$, and

$$\Phi_1\big(t_0,\tau-t_0,\theta_{-t_0}\omega,B(\tau-t_0,\theta_{-t_0}\omega)\big) \quad \text{is covered by a ball of radius } \delta \tag{5.19}$$

in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

In addition, by Lemma 5.5 we know that for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $k \in \mathbb{N}$,

$$\Phi_2\big(t,\tau-t,\theta_{-t}\omega,B(\tau-t,\theta_{-t}\omega)\big) \quad \text{is bounded in } H^3\big(\mathbb{R}^n\big) \times H^1\big(\mathbb{R}^n\big),$$

and thus for each $k \in \mathbb{N}$,

$$\Phi_2\big(t,\tau-t,\theta_{-t}\omega,B(\tau-t,\theta_{-t}\omega)\big)|_{\mathcal{O}_k} \quad \text{is precompact } H^2(\mathcal{O}_k) \times L^2(\mathcal{O}_k), \tag{5.20}$$

with $\mathcal{O}_k = \{x \in \mathbb{R}^n : |x| < k\}$.

It follows from (5.17)–(5.20) that all conditions of Theorem 2.1 are satisfied, hence, Φ is $\mathcal{D}$-pullback asymptotically compact in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. □

Since Lemma 4.2 implies a closed measurable $\mathcal{D}$-pullback absorbing set for Φ, and Φ is $\mathcal{D}$-pullback asymptotically compact in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ from Lemma 5.6, we immediately obtain the following existence theorem by Theorem 2.2.

Theorem 5.1 *Let* (3.2), (3.3), (3.6), (4.1), (4.3), *and* (4.5)–(4.9) *hold. Then, the cocycle* Φ *has a unique* $\mathcal{D}$*-pullback random attractor in* $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

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References

1. Arnold, L.: Random Dynamical Systems. Springer Monographs in Mathematics. Springer, Berlin (1998)
2. Barbosaa, A.R.A., Ma, T.F.: Long-time dynamics of an extensible plate equation with thermal memory. J. Math. Anal. Appl. **416**, 143–165 (2014)
3. Gu, A., Guo, B., Wang, B.: Long term behavior of random Navier-Stokes equations driven by colored noise. Discrete Contin. Dyn. Syst., Ser. B **25**, 2495–2532 (2020)
4. Gu, A., Wang, B.: Asymptotic behavior of random FitzHugh-Nagumo systems driven by colored noise. Discrete Contin. Dyn. Syst., Ser. B **23**, 1689–1720 (2018)
5. Khanmamedov, A.K.: Existence of global attractor for the plate equation with the critical exponent in an unbounded domain. Appl. Math. Lett. **18**, 827–832 (2005)
6. Khanmamedov, A.K.: Global attractors for the plate equation with a localized damping and a critical exponent in an unbounded domain. J. Differ. Equ. **225**, 528–548 (2006)
7. Khanmamedov, A.K.: A global attractor for the plate equation with displacement-dependent damping. Nonlinear Anal. **74**, 1607–1615 (2011)
8. Lions, J.L.: Quelques Methodes de Resolution des Problemes aux Limites Non Lineaires. Dunod, Paris (1969)
9. Liu, T.T., Ma, Q.Z.: The existence of time-dependent strong pullback attractors for non-autonomous plate equations. Chin. J. Contemp. Math. **2**, 101–118 (2017)
10. Liu, T.T., Ma, Q.Z.: Time-dependent asymptotic behavior of the solution for plate equations with linear memory. Discrete Contin. Dyn. Syst., Ser. B **23**, 4595–4616 (2018)
11. Liu, T.T., Ma, Q.Z.: Time-dependent attractor for plate equations on $\mathbb{R}^n$. J. Math. Anal. Appl. **479**, 315–332 (2019)
12. Ma, Q.Z., Ma, W.J.: Asymptotic behavior of solutions for stochastic plate equations with strongly damped and white noise. J. Northwest Norm. Univ. Nat. Sci. **50**, 6–17 (2014)
13. Ma, W.J., Ma, Q.Z.: Attractors for the stochastic strongly damped plate equations with additive noise. Electron. J. Differ. Equ. **111**, 1 (2013)
14. Ridolfi, L., D'Odorico, P., Laio, F.: Noise-Induced Phenomena in the Environmental Sciences. Cambridge University Press, Cambridge (2011)
15. Shen, X.Y., Ma, Q.Z.: The existence of random attractors for plate equations with memory and additive white noise. Korean J. Math. **24**, 447–467 (2016)
16. Shen, X.Y., Ma, Q.Z.: Existence of random attractors for weakly dissipative plate equations with memory and additive white noise. Comput. Math. Appl. **73**, 2258–2271 (2017)
17. Uhlenbeck, G., Ornstein, L.: On the theory of Brownian motion. Phys. Rev. **36**, 823–841 (1930)
18. Wang, B.: Attractors for reaction-diffusion equations in unbounded domains. Physica D **128**, 41–52 (1999)
19. Wang, B.: Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems. J. Differ. Equ. **253**, 1544–1583 (2012)
20. Wang, B.: Asymptotic behavior of supercritical wave equations driven by colored noise on unbounded domains. Discrete Contin. Dyn. Syst., Ser. B (2022). https://doi.org/10.3934/dcdsb.2021223
21. Wang, M., Uhlenbeck, G.: On the theory of Brownian motion. II. Rev. Mod. Phys. **17**, 323–342 (1945)
22. Wang, R., Shi, L., Wang, B.: Asymptotic behavior of fractional nonclassical diffusion equations driven by nonlinear colored noise on $\mathbb{R}^n$. Nonlinearity **32**, 4524–4556 (2019)
23. Wang, X., Lu, K., Wang, B.: Wong-Zakai approximations and attractors for stochastic reaction-diffusion equations on unbounded domains. J. Differ. Equ. **246**, 378–424 (2018)
24. Wu, H.: Long-time behavior for a nonlinear plate equation with thermal memory. J. Math. Anal. Appl. **348**, 650–670 (2008)
25. Xiao, H.B.: Asymptotic dynamics of plate equations with a critical exponent on unbounded domain. Nonlinear Anal. **70**, 1288–1301 (2009)
26. Yang, L., Zhong, C.K.: Uniform attractor for non-autonomous plate equations with a localized damping and a critical nonlinearity. J. Math. Anal. Appl. **338**, 1243–1254 (2008)
27. Yang, L., Zhong, C.K.: Global attractor for plate equation with nonlinear damping. Nonlinear Anal. **69**, 3802–3810 (2008)
28. Yao, B.X., Ma, Q.Z.: Global attractors for a Kirchhoff type plate equation with memory. Kodai Math. J. **40**, 63–78 (2017)
29. Yao, B.X., Ma, Q.Z.: Global attractors of the extensible plate equations with nonlinear damping and memory. J. Funct. Spaces **2017**, Article ID 4896161 (2017)
30. Yao, X.B.: Existence of a random attractor for non-autonomous stoc- hastic plate equations with additive noise and nonlinear damping on $\mathbb{R}^n$. Bound. Value Probl. **49** (2020). https://doi.org/10.1186/s13661-020-01346-z

31. Yao, X.B.: Random attractors for non-autonomous stochastic plate equations with multiplicative noise and nonlinear damping. AIMS Math. **5**, 2577–2607 (2020)
32. Yao, X.B.: Random attractors for stochastic plate equations with memory in unbounded domains. Open Math. **19**, 1435–1460 (2021)
33. Yao, X.B.: Asymptotic behavior for stochastic plate equations with memory and additive noise on unbounded domains. Discrete Contin. Dyn. Syst., Ser. B **27**, 443–468 (2022)
34. Yao, X.B., Liu, X.: Asymptotic behavior for non-autonomous stochastic plate equation on unbounded domains. Open Math. **17**, 1281–1302 (2019)
35. Yao, X.B., Ma, Q.Z., Liu, T.T.: Asymptotic behavior for stochastic plate equations with rotational inertia and Kelvin-Voigt dissipative term on unbounded domains. Discrete Contin. Dyn. Syst., Ser. B **24**, 1889–1917 (2019)
36. Yue, G.C., Zhong, C.K.: Global attractors for plate equations with critical exponent in locally uniform spaces. Nonlinear Anal. **71**, 4105–4114 (2009)
37. Zhou, J.: Global existence and blow-up of solutions for a Kirchhoff type plate equation with damping. Appl. Math. Comput. **265**, 807–818 (2015)