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Nonnegative nontrivial solutions for a class of $p(x)$-Kirchhoff equation involving concave-convex nonlinearities

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Abstract
In this paper, we study the existence of a class of $p(x)$-Kirchhoff equation involving concave-convex nonlinearities. The main tools used are the perturbation technique, variational method, and a priori estimation.

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1 Introduction and main result

Let $0 \in \Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper, we consider the following $p(x)$-Kirchhoff problem:

$$\begin{cases} -(a + b\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx)\Delta_{p(x)}u = |u|^{2p_+-2}u + \lambda|u|^{p_--2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $a \geq 0$, $b > 0$, $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, and $\lambda > 0$ is a parameter, $p(x)$ satisfies the following assumptions:

(P_1) $p \in C(\overline{\Omega})$, $p_- = \min\{p(x)|x \in \overline{\Omega}\}$, $p_+ = \max\{p(x)|x \in \overline{\Omega}\}$;

(P_2) $1 < p_- < N$ and $p_- < 2p_+ < p_-^*$, where $p_-^* = \frac{Np_-}{N-p_-}$;

(P_3) $p(0) = p_+$, $p(x) \leq p_+ - c|x|^\alpha$ for all $x \in \overline{\Omega}$, where $c > 0$, $\alpha = 1 - \frac{N(2p_+-p_-)}{2p_+p_-} > 0$.

The study on Kirchhoff-type equations and variational problems with $p(x)$-growth condition has attracted more and more interest in the recent years, see [7–9, 17, 29] and the references therein. It was proposed by Kirchhoff in 1883 as a generalization of the well-known D'Alembert wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

where ρ, P_0, h, L, and E are constants, by considering the changes in the length of the string during the vibrations, see [16]. This type of operators arises in a natural way in many different applications such as image processing, quantum mechanics, elastic mechanics, electrorheological fluids, see [5, 23] and the references therein. Set $M(t) = a + bt$, problem (1) is called nondegenerate if $a > 0$ and $b \geq 0$, while it is named degenerate if $a = 0$ and $b > 0$. In the large literature of degenerate Kirchhoff problems, the transverse oscillations of a stretched string with nonlocal flexural rigidity depends continuously on the Sobolev deflection norm of u via $M(\|u\|^2)$. From a physical point of view, the fact that $M(0) = 0$ means that the base tension of the string is zero, a very realistic model. More specifically, M measures the change of the tension on the string caused by the change of its length during the vibration. The presence of the nonlinear coefficient M is crucial to be considered when the changes in tension during the motion cannot be neglected. For more information, the reader can refer to [1, 28].

In 1994, Ambrosetti, Brezis, and Cerami in [2] considered the following problem:

$$\begin{cases} -\Delta u = \lambda u^q + u^r, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$, $0 < q < 1 < r < 2^* - 1$, and they established multiple results.

At the same time, many authors researched $p(x)$-Laplacian equations containing concave-convex nonlinearities. In particular, Mihăilescu in [19] studied the following $p(x)$-Laplacian equation involving concave-convex nonlinearities:

$$\begin{cases} -\Delta_{p(x)} u = \lambda |u|^{q(x)-2} u + |u|^{r(x)-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $1 < q(x) < p_- < p_+ < r(x) < p_-^*$, λ is a positive constant. Using Ekeland's variational principle and the mountain pass lemma, he proved that problem (4) has two positive solutions for $\lambda > 0$ small enough. Subsequently, the more general case was considered in [20]. In 2009, Dai and Hao in [9] studied the following $p(x)$-Kirchhoff-type equation:

$$\begin{cases} -(a + b \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx) \Delta_{p(x)} u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p(x) \in C(\overline{\Omega})$, $a, b > 0$, and $f(x, u) : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy certain condition. They established the existence and multiplicity of solutions by the variational method. Especially, the standard arguments given in [9] show that the verification of the Palais–Smale condition at the mountain pass level relies on the well-known Ambrosetti–Rabinowitz condition((AR) condition, for short):

(AR) There exist $T > 0$ and $\theta > 2p^+$ such that

$$0 < \theta F(x, t) = \theta \int_0^t f(x, \tau)\, d\tau \leq tf(x, t), \quad |t| \geq T, \text{ a.e. } x \in \Omega.$$

Actually, the (AR) condition is quite natural and important not only to ensure that the Euler–Lagrange functional has a mountain pass geometry, but also to guarantee the boundedness of Palais–Smale sequences. However, this condition is somewhat restrictive, not being satisfied by many nonlinearities. In fact, from the (AR) condition it follows that for some $C_1, C_2 > 0$

$$F(x,t) \geq C_1|t|^\theta - C_2, \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Thus, for example, the function $f(x,t) = |t|^{p^+-2}t\ln(1+|t|)$ does not satisfy the (AR) condition. In fact, many papers still required nonlinearity to satisfy the superlinear growth condition

$$f(x,t)t > 2p(x)F(x,t) \quad \text{for all } x \in \Omega \text{ and } |t| \text{ is large enough.}$$

However, it is easy to see that condition (P_3) in problem (1) violates this condition. It allows

$$f(x,t)t \leq 2p(x)F(x,t) \quad \text{for some } x \in \Omega \text{ and any } t > 0,$$

where $f(x,t) = t^{2p_+-1} + \lambda t^{p_--1}$. As described in [13], we need to overcome some difficulties to show the existence of nonnegative nontrivial solutions. Similar problems with concave-convex nonlinearities have been discussed by many authors (see [12, 15, 22, 25–27, 30]).

The main result of this paper reads as follows.

Theorem 1.1 *Suppose that $a \geq 0$, $b > 0$, conditions $(P_1)-(P_3)$ hold. Then there exists $\lambda_* > 0$ such that problem (1) has at least two nonnegative nontrivial solutions for any $\lambda \in (0, \lambda_*)$.*

Remark 1.2 When $a = 0$, we use the perturbation method and Moser iteration mainly to deal with degenerate cases. Most of the literature considers only one of the degenerate and nondegenerate scenarios. However, we discuss the above two cases at same time in Theorem 1.1.

To discuss problem (1), we need the functional space $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \left\{ u : u : \Omega \to \mathbb{R} \text{ is measurable}, \int_\Omega |u|^{p(x)}\,dx < \infty \right\}$$

with the norm

$$|u|_{p(x)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \leq 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces if $1 < p^- \le p^+ < \infty$ (see [11]). Moreover, we know that $\|u\| = |\nabla u|_{p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

Lemma 1.3 (see [11]) *If $q \in C(\overline{\Omega})$ satisfies $1 \le q(x) < p^*(x)$ ($p^*(x) = \frac{Np(x)}{N-p(x)}$, if $N > p(x)$; $p^*(x) = +\infty$, if $N \le p(x)$) for $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.*

Lemma 1.4 (see [11]) *Set $\rho(u) = \int_\Omega |u|^{p(x)}\,dx$ for $u \in L^{p(x)}(\Omega)$. If $u \in L^{p(x)}(\Omega)$ and $\{u_k\}_{k\in\mathbb{N}} \subset L^{p(x)}(\Omega)$, then we have*

(i) $|u|_{p(x)} < 1\ (=1; >1) \Leftrightarrow \rho(u) < 1\ (=1; >1)$;
(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p_-} \le \rho(u) \le |u|_{p(x)}^{p_+}$;
(iii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p_+} \le \rho(u) \le |u|_{p(x)}^{p_-}$;
(iv) $\lim_{k\to\infty} |u_k - u|_{p(x)} = 0 \Leftrightarrow \lim_{k\to\infty} \rho(u_k - u) = 0 \Leftrightarrow u_k \to u$ *in measure in* Ω *and* $\lim_{k\to\infty} \rho(u_k) = \rho(u)$.

Similar to Lemma 1.4, it is easy to obtain the following lemma.

Lemma 1.5 *Set $L(u) = \int_\Omega |\nabla u|^{p(x)}\,dx$ for $u \in W_0^{1,p(x)}(\Omega)$. If $u \in W_0^{1,p(x)}(\Omega)$ and $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$, we have*

(i) $\|u\| < 1\ (=1; >1) \Leftrightarrow L(u) < 1\ (=1; >1)$;
(ii) $\|u\| > 1 \Rightarrow \|u\|^{p_-} \le L(u) \le \|u\|^{p_+}$;
(iii) $\|u\| < 1 \Rightarrow \|u\|^{p_+} \le L(u) \le \|u\|^{p_-}$;
(iv) $\|u_k\| \to 0 \Leftrightarrow L(u_k) \to 0$; $\|u_k\| \to \infty \Leftrightarrow L(u_k) \to \infty$.

Lemma 1.6 (see [9]) *Set $\phi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx$ for $u \in W_0^{1,p(x)}(\Omega)$. The functional $\phi : X \to \mathbb{R}$ is convex. The mapping $\phi' : X \to X^*$ is a strictly monotone, bounded homeomorphism and is of (S_+) type, namely*

$$u_n \rightharpoonup u \quad \textit{and} \quad \overline{\lim}_{n\to\infty} \phi'(u_n)(u_n - u) \le 0 \quad \textit{implies } u_n \to u,$$

where $X = W_0^{1,p(x)}(\Omega)$.

Lemma 1.7 (see [24]) *In the Euclidean space $\mathbb{R}^N$, an optimal Gagliardo–Nirenberg inequality has the form*

$$\left(\int_{\mathbb{R}^N} |u|^r\,dx\right)^{\frac{p}{r\theta}} \le A(p,q,r)\left(\int_{\mathbb{R}^N} |\nabla u|^p\,dx\right)\left(\int_{\mathbb{R}^N} |u|^q\,dx\right)^{\frac{p(1-\theta)}{\theta q}}$$

with $1 < p < N$, $1 \le q < r \le p^*$, and $\theta = \theta(p,q,r) = \frac{Np(r-q)}{r(q(p-N)+Np)} \in (0,1]$, $A(p,q,r)$ the best constant.

Lemma 1.8 (see [3]) *Let X be a real Banach space, let $I : X \to \mathbb{R}$ be a functional of class $C^1(X,\mathbb{R})$ that satisfies the Palais–Smale condition (i.e. any sequence $\{u_n\} \subset X$ such that $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ has a convergent subsequence), $I(0) = 0$, and the following conditions hold:*

(i) *There exist positive constants ρ and α such that $I(u) \ge \alpha$ for any $u \in X$ with $\|u\| = \rho$;*

(ii) *There exists a function* $e \in X$ *such that* $\|e\| > \rho$ *and* $I(e) \leq 0$.

Then the functional I *has a critical value* $c \geq \alpha$, *that is, there exists* $u \in X$ *such that* $I(u) = c$ *and* $I'(u) = 0$ *in* X^*.

Lemma 1.9 (see [10]) *Let* X *be a complete metric space with metric* d, *and let* $I : X \mapsto (-\infty, +\infty]$ *be a low semicontinuous function bounded from below and not identical to* $+\infty$. *Let* ε *be given and* $U \in X$ *be such that*

$$I(U) \leq \inf_X I + \varepsilon.$$

Then there exists $V \in X$ *such that*

$$I(U) \leq I(V), \qquad d(U, V) \leq 1,$$

and for each $W \in X$, *one has*

$$I(U) \leq I(W) + \varepsilon\, d(V, W).$$

To end this section, we describe the basic ideas in the proof of Theorem 1.1. If $a = 0$ and $p(0) = p_+$, it is not easy to verify the boundedness of Palais–Smale sequence for the functional corresponding to problem (1). Inspired by [6], we first modify the nonlinear term to obtain a perturbation equation of problem (1). Then, using Ekeland's variational principle and the mountain pass lemma, we prove that the perturbation equation has at least two nonnegative nontrivial solutions for $\lambda > 0$ sufficiently small. Finally, we use the Moser iteration to prove that the solutions to the perturbation equation are uniformly bounded. Therefore, we show that two nonnegative nontrivial solutions of the perturbation equation are also the solutions of the original problem (1).

Throughout this paper, let $B_\delta = \{x : |x| < \delta\} \subset \Omega$ and $\Omega_\delta = \Omega \setminus B_\delta$. We use $\| \cdot \|$ to denote the usual norms of $W_0^{1,p(x)}(\Omega)$, the letters C and C_μ stand for positive constants which may take different values at different places.

2 Solutions of the perturbation equation

Since $p(x)$ is a continuous function, from (P_2) and (P_3), we see that there exists $r > 0$ such that

$$1 < p_- - r < 2p_+ + r < p_-^* \tag{6}$$

and

$$r < \frac{2[Np_- - 2p_+(N - p_-)]}{N}. \tag{7}$$

Let $\psi(t) \in C_0^\infty(\mathbb{R}, [0, 1])$ be a smooth even function with the following properties: $\psi(t) = 1$ for $|t| \leq 1$, $\psi(t) = 0$ for $|t| \geq 2$ and $\psi(t)$ is monotonically decreasing on the interval $(0, +\infty)$. Define

$$b_\mu(t) = \psi(\mu t), \qquad m_\mu(t) = \int_0^t b_\mu(\tau)\, d\tau$$

for $\mu \in (0,1]$. We will deal with the perturbation equation

$$\begin{cases} -(a + b\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx)\Delta_{p(x)}u = (\frac{u}{m_\mu(u)})^r u^{2p_+-1} + \lambda u^{p_--1} & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases} \tag{8}$$

Define

$$g_\mu(t) = \left(\frac{t}{m_\mu(t)}\right)^r t_+^{2p_+-1}, \qquad G_\mu(t) = \int_0^t g_\mu(\tau)\,d\tau.$$

Then the formal energy functional J_μ associated with equation (8) is defined by

$$J_\mu(u) = a\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx + \frac{b}{2}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx\right)^2 - \int_\Omega G_\mu(u)\,dx - \frac{\lambda}{p_-}\int_\Omega u_+^{p_-}\,dx.$$

Lemma 2.1 *The function $G_\mu(t)$ defined above satisfies the following inequality*:

$$G_\mu(t) \le \frac{1}{2p_+} tg_\mu(t), \qquad G_\mu(t) \le \frac{1}{2p_+ + r} tg_\mu(t) + C_\mu,$$

where $C_\mu > 0$ is a positive constant.

Proof By the definition of function g_μ, the conclusion is clear for $t \le 0$. Since $b_\mu(t)$ is monotonically decreasing on the interval $(0,+\infty)$, we have

$$\frac{d}{dt}\left(\frac{t}{m_\mu(t)}\right) = \frac{m_\mu(t) - tb_\mu(t)}{m_\mu^2(t)} = \frac{t(b_\mu(\xi) - b_\mu(t))}{m_\mu^2(t)} \ge 0$$

for $t > 0$, where $\xi \in (0,t)$. Therefore, $\frac{t}{m_\mu(t)}$ is monotonically increasing on the interval $(0,+\infty)$. Hence, $\frac{g_\mu(t)}{t^{2p_+-1}} = (\frac{t}{m_\mu(t)})^r$ is also monotonically increasing on the interval $(0,+\infty)$. It follows that

$$G_\mu(t) = \int_0^t g_\mu(\tau)\,d\tau \le \int_0^t \frac{g_\mu(t)}{t^{2p_+-1}}\tau^{2p_+-1}\,d\tau = \frac{1}{2p_+} tg_\mu(t) \quad \text{for } t > 0. \tag{9}$$

By the definition of function m_μ, we have $m_\mu(t) = \frac{A}{\mu}$ for $t \ge \frac{2}{\mu}$, where $A = 1 + \int_1^2 \psi(\tau)\,d\tau$. For $t > \frac{2}{\mu}$, one has

$$\begin{aligned} G_\mu(t) &= \int_0^{\frac{2}{\mu}} g_\mu(\tau)\,d\tau + \int_{\frac{2}{\mu}}^t \left(\frac{\mu}{A}\right)^r \tau^{2p_++r-1}\,d\tau \\ &= \int_0^{\frac{2}{\mu}} \left(g_\mu(\tau) - \left(\frac{\mu}{A}\right)^r \tau^{2p_++r-1}\right) d\tau + \int_0^t \left(\frac{\mu}{A}\right)^r \tau^{2p_++r-1}\,d\tau \\ &\le \frac{g_\mu(t)t}{2p_+ + r} + C_\mu. \end{aligned}$$

The proof is complete. □

Lemma 2.2 *Suppose that $a \ge 0$, $b > 0$, conditions (P_1) and (P_2) hold. Then, for any $\mu \in (0,1]$, there exists $\lambda_1 > 0$ such that J_μ satisfies the (PS) condition for $\lambda \in (0,\lambda_1)$.*

Proof Let $\{u_n\}$ be a (PS) sequence of J_μ in $W_0^{1,p(x)}(\Omega)$. This means that there exists $C > 0$ such that

$$\left|J_\mu(u_n)\right| \leq C, \qquad J'_\mu(u_n) \to 0 \quad \text{as } n \to \infty. \tag{10}$$

Now we show that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. If $\|u_n\| \leq 1$, we are done. Otherwise, by Lemma 1.5, we have

$$\|u\|^{p_-} \leq \int_\Omega |\nabla u|^{p(x)}\,dx. \tag{11}$$

It follows from the Sobolev embedding theorem that

$$\int_\Omega u_+^{p_-}\,dx \leq C\int_\Omega |\nabla u|^{p_-}\,dx \leq C\left(1 + \left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^2\right). \tag{12}$$

From (6), (11), (12), and Lemma 2.1, we derive that there exists $\lambda_1 > 0$ such that

$$\begin{aligned}
&J_\mu(u_n) - \frac{1}{2p_+ + r}\langle J'_\mu(u_n), u_n\rangle \\
&\quad = a\int_\Omega \left(\frac{1}{p(x)} - \frac{1}{2p_+ + r}\right)|\nabla u_n|^{p(x)}\,dx + \frac{b}{2}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)^2 \\
&\qquad - \frac{b}{2p_+ + r}\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx \int_\Omega |\nabla u_n|^{p(x)}\,dx + \int_\Omega \left(\frac{g_\mu(u_n)u_n}{2p_+ + r} - G_\mu(u_n)\right)dx \\
&\qquad + \left(\frac{1}{2p_+ + r} - \frac{1}{p_-}\right)\lambda \int_\Omega (u_n)_+^{p_-}\,dx \\
&\quad \geq \left(\frac{1}{2p_+} - \frac{1}{2p_+ + r}\right)\frac{b}{p_+}\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right)^2 \\
&\qquad + \left(\frac{1}{2p_+ + r} - \frac{1}{p_-}\right)\lambda \int_\Omega (u_n)_+^{p_-}\,dx - C_\mu|\Omega| \\
&\quad \geq \left(\frac{br}{2p_+^2(2p_+ + r)} - \frac{2p_+ + r - p_-}{p_-(2p_+ + r)}C\lambda\right)\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right)^2 - C\lambda - C_\mu|\Omega| \\
&\quad \geq C_1\|u_n\|^{2p_-} - C_2
\end{aligned}$$

for $\lambda \in (0, \lambda_1)$. It implies from (10) that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

With the loss of generality, up to a subsequence, we may assume that

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,p(x)}(\Omega), \\ u_n \to u & \text{in } L^s(\Omega), 1 \leq s < p_-^*. \end{cases}$$

Thus, we have

$$\begin{aligned}
\langle J'(u_n), u_n - u\rangle &= \left(a + b\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)\,dx \\
&\quad - \int_\Omega g_\mu(u_n)(u_n - u)\,dx - \lambda\int_\Omega (u_n)_+^{p_- - 1}(u_n - u)\,dx \to 0.
\end{aligned}$$

It is easy to see that

$$\left|g_\mu(t)\right| \le C\left(|t|^{2p_+-1} + |t|^{2p_++r-1}\right).$$

Using the Sobolev inequality and the Hölder inequality yields

$$\begin{aligned}\left|\int_\Omega \left(g_\mu(u_n)\right)(u_n-u)\,dx\right| &\le C\int_\Omega |u_n|^{2p_+-1}|u_n-u|\,dx + C\int_\Omega |u_n|^{2p_++r-1}|u_n-u|\,dx \\ &\le C\|u_n\|_{2p_+}^{2p_+-1}\|u_n-u\|_{2p_+} + C\|u_n\|_{2p_++r}^{2p_++r-1}\|u_n-u\|_{2p_++r} \\ &\le C\|u_n-u\|_{2p_+} + C\|u_n-u\|_{2p_++r} \to 0 \end{aligned} \tag{13}$$

and

$$\begin{aligned}\left|\int_\Omega (u_n)_+^{p_--1}(u_n-u)\,dx\right| &\le \int_\Omega |u_n|^{p_--1}|u_n-u|\,dx \\ &\le \|u_n\|_{p_-}^{p_--1}\|u_n-u\|_{p_-} \\ &\le C\|u_n-u\|_{p_-} \to 0 \end{aligned} \tag{14}$$

as $n \to +\infty$. From (13) and (14), one has

$$\left(a + b\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)\,dx \to 0 \quad \text{as } n \to +\infty.$$

Notice that $a \ge 0$ and $b > 0$, we have

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)\,dx \to 0 \quad \text{as } n \to +\infty.$$

It implies from Lemma 1.6 that $\{u_n\}$ is strongly convergent to u. Hence J_μ satisfies the (PS) condition. □

In the following lemma, we will verify that J_μ possesses the mountain pass geometry.

Lemma 2.3 *Suppose that $a \ge 0$, $b > 0$, conditions (P_1)–(P_3) hold. Then there exists λ_2 such that the functional J_μ possesses the mountain pass geometry for any $\lambda \in (0, \lambda_2)$, namely*
(i) *there exist $m, \rho > 0$ such that $J_\mu(u) > m$ for any $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| = \rho$;*
(ii) *there exists $w \in W_0^{1,p(x)}(\Omega)$ such that $\|w\| > \rho$ and $J_\mu(w) < 0$.*

Proof By the definition of function G_μ, we have

$$\int_\Omega G_\mu(u)\,dx \le C_\mu \int_\Omega \left(|u|^{2p_+} + |u|^{2p_++r}\right)dx.$$

By the Sobolev embedding theorem and Lemma 1.5, we obtain

$$\int_\Omega |u|^{2p_++r}\,dx \le C\|u\|^{2p_++r} \le C\left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^{\frac{2p_++r}{p_+}}$$

for any $u \in W_0^{1,p(x)}(\Omega)$ with $\int_\Omega |\nabla u|^{p(x)}\,dx < 1$. Let $B_{\delta_0} \subset \Omega$ satisfy that there exists $\varepsilon_0 > 0$ such that $p(x) \le p_+ - \varepsilon_0$ for any $x \in \Omega_{\delta_0}$. By Lemma 1.5, the Hölder inequality, and the Sobolev embedding theorem, we have

$$\begin{aligned}
\int_\Omega |u|^{2p_+}\,dx &= \int_{B_{\delta_0}} |u|^{2p_+}\,dx + \int_{\Omega_{\delta_0}} |u|^{2p_+}\,dx \\
&\le |B_{\delta_0}|^{\frac{r}{2p_++r}} \left(\int_\Omega |u|^{2p_++r}\,dx\right)^{\frac{2p_+}{2p_++r}} + C\|u\|^{2p_+} \\
&\le C|B_{\delta_0}|^{\frac{r}{2p_++r}} \left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^2 + C\left(\int_{\Omega_{\delta_0}} |\nabla u|^{p(x)}\,dx\right)^{\frac{2p_+}{p_+-\varepsilon_0}} \\
&\le C|B_{\delta_0}|^{\frac{r}{2p_++r}} \left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^2 + C\left(\int_{\Omega} |\nabla u|^{p(x)}\,dx\right)^{\frac{2p_+}{p_+-\varepsilon_0}}
\end{aligned}$$

for any $u \in W_0^{1,p(x)}(\Omega)$ with $\int_\Omega |\nabla u|^{p(x)}\,dx < 1$. Therefore,

$$\begin{aligned}
\int_\Omega G_\mu(u)\,dx \le{}& C_\mu \left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^{\frac{2p_++r}{p_+}} + C_\mu |B_{\delta_0}|^{\frac{r}{2p_++r}} \left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^2 \\
&+ C_\mu \left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^{\frac{2p_+}{p_+-\varepsilon_0}} \qquad (15)
\end{aligned}$$

for any $u \in W_0^{1,p(x)}(\Omega)$ with $\int_\Omega |\nabla u|^{p(x)}\,dx < 1$. Set $\rho_0 = \int_\Omega |\nabla u|^{p(x)}\,dx$. Fix $\mu \in (0,1]$, it implies from (12) and (15) that

$$\begin{aligned}
J_\mu(u) &\ge \frac{a}{p_+}\rho_0 + \frac{b}{2p_+^2}\rho_0^2 - C_\mu \rho_0^{\frac{2p_++r}{p_+}} - C_\mu |B_{\delta_0}|^{\frac{r}{2p_++r}} \rho_0^2 - C_\mu \rho_0^{\frac{2p_+}{p_+-\varepsilon_0}} - C\lambda\left(1+\rho_0^2\right) \\
&\ge \frac{b}{4p_+^2}\rho_0^2 - C\lambda\left(1+\rho_0^2\right)
\end{aligned}$$

for $\delta_0,\ \rho_0 > 0$ small enough. Let $\lambda_2 = \frac{b\rho_0^2}{8Cp_+^2(1+\rho_0^2)}$. We have $J_\mu(u) > \frac{b}{8p_+^2}\rho_0^2$ for any $\lambda \in (0,\lambda_2)$. By Lemma 1.5, we know that there exist $m,\ \rho > 0$ such that $J_\mu(u) > m$ for any $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| = \rho$.

By the definition of function g_μ, we know $g_\mu(t) \ge t^{2p_+-1}$. Let $U_0 \subset \Omega_{\delta_0}$. Fix $v_0 \in W_0^{1,p(x)}(U_0)\backslash\{0\}$. Then, for $t > 0$ sufficiently large, we have

$$\begin{aligned}
J_\mu(tv_0) ={}& a\int_{U_0} \frac{1}{p(x)}|\nabla tv_0|^{p(x)}\,dx + \frac{b}{2}\left(\int_{U_0} \frac{1}{p(x)}|\nabla tv_0|^{p(x)}\,dx\right)^2 \\
&- \int_{U_0} G_\mu(tv_0)\,dx - \frac{\lambda}{p_-}\int_{U_0} |tv_0|_+^{p_-}\,dx \\
\le{}& at^{p_+-\varepsilon_0} \int_{U_0} \frac{1}{p(x)}|\nabla v_0|^{p(x)}\,dx + \frac{b}{2}t^{2(p_+-\varepsilon_0)}\left(\int_{U_0} \frac{1}{p(x)}|\nabla v_0|^{p(x)}\,dx\right)^2 \\
&- \frac{t^{2p_+}}{p_+}\left(\frac{\mu}{A}\right)^r \int_{U_0} |v_0|^{2p_+}\,dx < 0.
\end{aligned}$$

Choosing $w = tv_0$ with $t > 0$ sufficiently large, we have $\|w\| > \rho$ and $J_\mu(w) < 0$. The proof is complete. □

Proposition 2.4 *Suppose that $a \geq 0, b > 0$, conditions $(P_1)-(P_3)$ hold. Then there exist $\lambda_0 > 0$ and $L > 0$ independent of μ such that problem (8) has at least two nonnegative nontrivial solutions u'_μ and u''_μ satisfying*

$$J_\mu\big(u'_\mu\big) < 0 < J_\mu\big(u''_\mu\big) < L \quad \textit{for any } \lambda \in (0, \lambda_0).$$

Proof According to (P_1) and (P_2), we know that there exist $\varepsilon_1 > 0$ and $U_1 \subset \Omega$ such that $p(x) \geq p_+ - \varepsilon_1 > p_-$ for any $x \in U_1$. Fix $\varphi_0 \in W_0^{1,p(x)}(U_1)\backslash\{0\}$. Let $\lambda_0 = \min\{\lambda_1, \lambda_2\}$. For any $\lambda \in (0, \lambda_0)$ and $k > 0$ sufficiently small, we have

$$\begin{aligned}
J_\mu(k\varphi_0) &= a\int_{U_1}\frac{1}{p(x)}|\nabla k\varphi_0|^{p(x)}\,dx + \frac{b}{2}\left(\int_{U_1}\frac{1}{p(x)}|\nabla k\varphi_0|^{p(x)}\,dx\right)^2 \\
&\quad - \int_{U_1} G_\mu(k\varphi_0)\,dx - \frac{\lambda}{p_-}\int_{U_1}|k\varphi_0|_+^{p_-}\,dx \\
&\leq ak^{p_+-\varepsilon_1}\int_{U_1}\frac{|\nabla\varphi_0|^{p(x)}}{p(x)}\,dx + \frac{b}{2}k^{2(p_+-\varepsilon_1)}\left(\int_{U_1}\frac{|\nabla\varphi_0|^{p(x)}}{p(x)}\,dx\right)^2 \\
&\quad - \frac{k^{p_-}}{p_-}\lambda\int_{U_1}|\varphi_0|^{p_-}\,dx < 0.
\end{aligned}$$

Thus we deduce that

$$c_\mu = \inf_{u\in B_\rho(0)} J_\mu(u) < 0 < \inf_{u\in\partial B_\rho(0)} J_\mu(u).$$

By applying Ekeland's variational principle in $\overline{B_\rho(0)}$ (see [10]), we obtain that problem (8) has a solution u'_μ satisfying $J_\mu(u'_\mu) = c_\mu < 0$.

From Lemmas 2.1 and 2.2, we see that the functional J_μ satisfies the (PS) condition and has the mountain pass geometry. Define

$$\Gamma = \big\{\gamma \in C\big([0,1], W_0^{1,p(x)}(\Omega)\big) | \gamma(0) = 0, \gamma(1) = w\big\}, \quad \tilde{c}_\mu = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\mu\big(\gamma(t)\big).$$

By the mountain pass lemma (see [21]), we obtain that problem (8) has a solution u''_μ satisfying $J_\mu(u''_\mu) = \tilde{c}_\mu > 0$. Consider the functional

$$I(u) = a\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx + \frac{b}{2}\left(\int_\Omega\frac{|\nabla u|^{p(x)}}{p(x)}\,dx\right)^2 - \frac{1}{2p_+}\int_\Omega |u_+|^{2p_+}\,dx,$$

where $u_+ = \max\{\pm u, 0\}$. It is easy to see that $J_\mu(u) \leq I(u)$ for any $u \in W_0^{1,p(x)}(\Omega)$. We can choose $v_0 \in W_0^{1,p(x)}(\Omega)\backslash\{0\}$ such that $I(tv_0) \to -\infty$ as $t \to +\infty$. Then $J_\mu(u''_\mu) = \tilde{c}_\mu \leq \sup_{t>0} I(tv_0) = L$.

Since $J_\mu(u'_\mu) < J_\mu(0) < J_\mu(u''_\mu)$, we know that u'_μ and u''_μ are two nontrivial solutions of problem (8). Let u_μ be a nontrivial critical of J_μ and $u_\mu^\pm = \max\{\pm u_\mu, 0\}$. After a direct calculation, we derive that $(a + b\int_\Omega |\nabla u_\mu^-|^{p(x)}\,dx)\int_\Omega |\nabla u_\mu^-|^{p(x)}\,dx = \langle J'_\mu(u_\mu), u_\mu^-\rangle = 0$, which implies that $u_\mu^- = 0$. Hence, $u_\mu \geq 0$. Therefore, u'_μ and u''_μ are two nonnegative nontrivial solutions of problem (8). The proof is complete. □

3 L^∞-estimate of nontrivial solutions

In this section, we show that the solutions of perturbation equation (8) are indeed the solutions of the original problem (1). For this purpose, we need the following uniform L^∞-estimate for critical points of the functional J_μ.

Proposition 3.1 *Suppose that $a \geq 0$, $b > 0$, conditions $(P_1)-(P_3)$ hold. If v is a critical point of J_μ with $J_\mu(v) \leq L$, then there exist $\lambda_3 > 0$ and a positive constant $M = M(L)$ independent of μ such that $\|v\|_{L^\infty(\Omega)} \leq M$ for any $\lambda \in (0, \lambda_3)$.*

To prove Proposition 3.1, we need some preliminaries. Let $\beta = 0$ and $n = 0$ in Corollary 2 on page 139 of [18], we obtain the following lemma.

Lemma 3.2 *Let $1 \leq p < N$, $p \leq q \leq \frac{Np}{N-p}$, and $\alpha_1 = 1 - \frac{N(q-p)}{pq}$. Then*

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C\big\||x|^{\alpha_1}\nabla u\big\|_{L^p(\mathbb{R}^N)}$$

for all $u \in \mathfrak{D}(\mathbb{R}^N)$, where $\mathfrak{D}(\mathbb{R}^N)$ is the space of functions in $C^\infty(\mathbb{R}^N)$ with compact supports in $\mathbb{R}^N$.

Lemma 3.3 *Suppose that $(P_1)-(P_3)$ hold. Then there exists $C > 0$ such that*

$$\int_\Omega |u|^{p_-}\,dx \leq C\int_\Omega |x|^\alpha |\nabla u|^{p(x)}\,dx \tag{16}$$

for all $u \in W_0^{1,p(x)}(\Omega)$ with $\int_\Omega |x|^\alpha |\nabla u|^{p(x)}\,dx \geq 1$.

Proof If the conclusion does not hold, then there exists a sequence $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ such that

$$n\int_\Omega |x|^\alpha |\nabla u_n|^{p(x)}\,dx \leq \int_\Omega |u_n|^{p_-}\,dx \tag{17}$$

and

$$\int_\Omega |x|^\alpha |\nabla u_n|^{p(x)}\,dx \geq 1.$$

Therefore,

$$\eta_n^{p_-} = \int_\Omega |u_n|^{p_-}\,dx \geq n \to \infty \quad \text{as } n \to +\infty. \tag{18}$$

Set $u_n = \eta_n v_n$. Then

$$\int_\Omega |v_n|^{p_-}\,dx = \eta_n^{-p_-}\int_\Omega |u_n|^{p_-}\,dx = 1.$$

Combining (17) with (18), we have

$$\begin{aligned} n\int_\Omega |x|^\alpha |\nabla v_n|^{p(x)}\,dx &\leq n\eta_n^{-p_-}\int_\Omega |x|^\alpha |\nabla u_n|^{p(x)}\,dx \\ &\leq \eta_n^{-p_-}\int_\Omega |u_n|^{p_-}\,dx = 1, \end{aligned}$$

which implies that

$$\int_{\Omega} |x|^{\alpha} |\nabla v_n|^{p(x)}\, dx \to 0 \quad \text{as } n \to +\infty. \tag{19}$$

Therefore, for any $\delta > 0$, we obtain

$$\int_{\Omega_\delta} |\nabla v_n|^{p(x)}\, dx \le \frac{1}{\delta^{\alpha}} \int_{\Omega_\delta} |x|^{\alpha} |\nabla v_n|^{p(x)}\, dx \to 0 \quad \text{as } n \to +\infty.$$

By the Young inequality, for any $\varepsilon > 0$, one has

$$\int_{\Omega_\delta} |\nabla v_n|^{p_-}\, dx \le \int_{\Omega_\delta} \left(\varepsilon + C_\varepsilon |\nabla v_n|^{p(x)}\right) dx.$$

According to the arbitrariness of $\varepsilon > 0$, we have

$$\int_{\Omega_\delta} |\nabla v_n|^{p_-}\, dx \to 0 \quad \text{as } n \to +\infty. \tag{20}$$

Noticing that $\partial\Omega \subset \partial\Omega_\delta$ and $v_n = 0$ on $\partial\Omega$, by the Sobolev embedding theorem, we obtain

$$\int_{\Omega_\delta} |v_n|^{p_-^*}\, dx \to 0 \quad \text{as } n \to +\infty \tag{21}$$

for all $\delta > 0$. Set $p = p_-$ and $q = 2p_+$, it follows from Lemma 3.2 that

$$\|u\|_{L^{2p_+}(\mathbb{R}^N)} \le C\left(\int_{\mathbb{R}^N} |x|^{\alpha p_-} |\nabla u|^{p_-}\, dx\right)^{\frac{1}{p_-}} \tag{22}$$

for all $u \in \mathfrak{D}(\mathbb{R}^N)$. Let $\psi \in C_0^\infty(\mathbb{R}^N)$ satisfy $|\psi(x)| \le 1$, $\psi(x) = 1$ for $|x| \le \delta_1 < \frac{1}{2}$, $\psi(x) = 0$ for $|x| \ge 2\delta_1$, and $|\nabla\psi| \le C$ for $x \in \mathbb{R}^N$. Using the Hölder inequality, we deduce from (22) that

$$\begin{aligned}
\left(\int_{B_{\delta_1}} |v_n|^{2p_+}\, dx\right)^{\frac{p_-}{2p_+}} &\le \left(\int_{\mathbb{R}^N} |\psi v_n|^{2p_+}\, dx\right)^{\frac{p_-}{2p_+}} \le C\int_{\mathbb{R}^N} |x|^{\alpha p_-} |\nabla \psi v_n|^{p_-}\, dx \\
&\le C\int_{\mathbb{R}^N} |x|^{\alpha p_-} \left(|\psi|^{p_-} |\nabla v_n|^{p_-} + |v_n|^{p_-} |\nabla\psi|^{p_-}\right) dx \\
&\le C\int_{B_{2\delta_1}} |x|^{\alpha p_-} |\nabla v_n|^{p_-}\, dx + C\int_{\Omega_{\delta_1}} |v_n|^{p_-}\, dx \\
&\le C_\varepsilon \int_{B_{2\delta_1}} |x|^{\alpha} |\nabla v_n|^{p(x)}\, dx \\
&\quad + \varepsilon \int_{B_{2\delta_1}} |x|^{\alpha p_- \frac{p(x)-1}{p(x)-p_-}}\, dx + C\int_{\Omega_{\delta_1}} |v_n|^{p_-}\, dx \\
&\le C_\varepsilon \int_{B_{2\delta_1}} |x|^{\alpha} |\nabla v_n|^{p(x)}\, dx + \varepsilon \delta_1^N + C\left(\int_{\Omega_{\delta_1}} |v_n|^{p_-^*}\, dx\right)^{\frac{p_-}{p_-^*}}.
\end{aligned} \tag{23}$$

It implies from (19), (21), and (23) that

$$\int_{B_{\delta_1}} |v_n|^{p_-}\,dx \le C\left(\int_{B_{\delta_1}} |v_n|^{2p_+}\,dx\right)^{\frac{p_-}{2p_+}} \to 0 \quad \text{as } \delta_1 \to 0 \text{ and } n \to +\infty.$$

By the Hölder inequality, it follows from (21) that

$$\int_{\Omega_{\delta_1}} |v_n|^{p_-}\,dx \le C\left(\int_{\Omega_{\delta_1}} |v_n|^{p_-^*}\,dx\right)^{\frac{p_-}{p_-^*}} \to 0 \quad \text{as } n \to +\infty.$$

Therefore, we have

$$\int_\Omega |v_n|^{p_-}\,dx \to 0 \quad \text{as } n \to +\infty,$$

which contradicts the fact that

$$\int_\Omega |v_n|^{p_-}\,dx = 1.$$

The proof of Lemma 3.3 is completed. □

Lemma 3.4 *Suppose that $a \ge 0, b > 0$, conditions $(P_1)-(P_3)$ hold. If $J_\mu(u) \le L$ and $J_\mu'(u) = 0$, then there exist $\lambda_3 > 0$ and $C = C(L) > 0$ independent of μ such that*

$$\int_\Omega |x|^\alpha |\nabla u|^{p(x)}\,dx \le C \quad \textit{for any } \lambda \in (0, \lambda_3).$$

Proof If $\int_\Omega |x|^\alpha |\nabla u|^{p(x)}\,dx < 1$, we are done. Otherwise, by Lemma 2.1 and Lemma 3.3, we derive from (P_3) that

$$\begin{aligned}
L &\ge J_\mu(u) - \frac{1}{2p_+}\langle J_\mu'(u), u\rangle \\
&= a\int_\Omega \left(\frac{1}{p(x)} - \frac{1}{2p_+}\right)|\nabla u|^{p(x)}\,dx + \int_\Omega \left(\frac{g_\mu(u)u}{2p_+} - G_\mu(u)\right)dx \\
&\quad + \left(\frac{1}{2p_+} - \frac{1}{p_-}\right)\lambda\int_\Omega u_+^{p_-}\,dx + \frac{b}{2}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p_+}\right)|\nabla u|^{p(x)}\,dx \\
&\ge \frac{b}{2}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p_+}\right)|\nabla u|^{p(x)}\,dx - \left(\frac{1}{p_-} - \frac{1}{2p_+}\right)\lambda\int_\Omega u_+^{p_-}\,dx \\
&\ge \frac{b}{2}\int_\Omega \frac{c|x|^\alpha}{p_+(p_+ - c|x|^\alpha)}|\nabla u|^{p(x)}\,dx \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx - \left(\frac{1}{p_-} - \frac{1}{2p_+}\right)\lambda\int_\Omega u_+^{p_-}\,dx \\
&\ge C_3\int_\Omega |x|^\alpha|\nabla u|^{p(x)}\,dx \int_\Omega |\nabla u|^{p(x)}\,dx - C_4\lambda\int_\Omega |x|^\alpha|\nabla u|^{p(x)}\,dx \\
&\ge C_3\left(\int_\Omega |x|^\alpha|\nabla u|^{p(x)}\,dx\right)^2 - C_4\lambda\int_\Omega |x|^\alpha|\nabla u|^{p(x)}\,dx \\
&\ge (C_3 - C_4\lambda)\int_\Omega |x|^\alpha|\nabla u|^{p(x)}\,dx.
\end{aligned} \tag{24}$$

Set $C = 2L/C_5$ and $\lambda_3 = C_5/2C_4$. It implies from (24) that $\int_\Omega |x|^\alpha |\nabla u|^{p(x)}\,dx \le C$ for any $\lambda \in (0,\lambda_3)$. $\square$

Lemma 3.5 *Suppose that $a \ge 0$, $b > 0$, conditions $(P_1)-(P_3)$ hold. If $J_\mu(u) \le L$ and $J'_\mu(u) = 0$, then there exists $C = C(L) > 0$ independent of μ such that*

$$\int_\Omega |\nabla u|^{p(x)}\,dx \le C \quad \textit{for any } \lambda \in (0,\lambda_3).$$

Proof By the definition of function m_μ, we know that $m_\mu(t) = t$ for $t \le \frac{1}{\mu}$ and $m_\mu(t) \ge \frac{1}{\mu}$ for $t > \frac{1}{\mu}$. Therefore, we have

$$g_\mu(u) \le C\mu^r |u|^{2p_+ + r - 1} \le C\bigl(1 + |u|^{2p_+ + r - 1}\bigr) \tag{25}$$

for any $\mu \in (0,\ 1]$. It follows from $J'_\mu(u) = 0$ that u is a solution of problem (8). Multiply problem (8) by u and integrate to obtain

$$\begin{aligned} b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx \int_\Omega |\nabla u|^{p(x)}\,dx &\le \left(a + b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\int_\Omega |\nabla u|^{p(x)}\,dx \\ &= \int_\Omega \bigl(g_\mu(u) + \lambda u_+^{p_- - 1}\bigr)u\,dx \\ &\le C\left(1 + \int_\Omega |u|^{2p_+ + r}\,dx\right) \end{aligned} \tag{26}$$

for any $\lambda \in (0,\lambda_3)$. Choose $\theta = \frac{Nrp_-}{(2p_+ + r)[2p_+(p_- - N) + Np_-]}$, it is easy to verify $\theta \in (0,1]$. From Lemma 1.7 we have

$$\begin{aligned} \int_\Omega |u|^{2p_+ + r}\,dx &\le A\left(\int_\Omega |\nabla u|^{p_-}\,dx\right)^{\frac{(2p_+ + r)\theta}{p_-}} \left(\int_\Omega |u|^{2p_+}\,dx\right)^{\frac{(2p_+ + r)(1-\theta)}{2p_+}} \\ &\le A\left(\int_\Omega 1 + |\nabla u|^{p(x)}\,dx\right)^{\frac{(2p_+ + r)\theta}{p_-}} \left(\int_\Omega |u|^{2p_+}\,dx\right)^{\frac{(2p_+ + r)(1-\theta)}{2p_+}}. \end{aligned} \tag{27}$$

It follows from (26) and (27) that

$$\begin{aligned} \frac{b}{p_+}\left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^2 &\le b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx \int_\Omega |\nabla u|^{p(x)}\,dx \\ &\le C + C\int_\Omega |u|^{2p_+ + r}\,dx \\ &\le C + CA\left(\int_\Omega |\nabla u|^{p(x)}\,dx + |\Omega|\right)^{\frac{(2p_+ + r)\theta}{p_-}} \\ &\quad \times \left(\int_\Omega |u|^{2p_+}\,dx\right)^{\frac{(2p_+ + r)(1-\theta)}{2p_+}}. \end{aligned} \tag{28}$$

According to (7), we have

$$\frac{(2p_+ + r)\theta}{p_-} = \frac{Nr}{[2p_+(p_- - N) + Np_-]} < 2.$$

To prove that $\int_\Omega |\nabla u|^{p(x)}\,dx$ is bounded, we just prove that $\int_\Omega |u|^{2p_+}\,dx$ is bounded. Now we show that $\int_\Omega |u|^{2p_+}\,dx$ is uniformly bounded. By the Sobolev embedding theorem and Lemma 3.4, for any $\delta > 0$, we have

$$\begin{aligned}
\int_{\Omega_\delta} |u|^{p_-^*}\,dx &\le C\left(\int_{\Omega_\delta} |\nabla u|^{p_-}\,dx\right)^{\frac{p_-^*}{p_-}} \\
&\le C\left(1 + \int_{\Omega_\delta} |\nabla u|^{p(x)}\,dx\right)^{\frac{p_-^*}{p_-}} \\
&\le C + C_\delta\left(\int_{\Omega_\delta} |x|^\alpha |\nabla u|^{p(x)}\,dx\right)^{\frac{p_-^*}{p_-}} \\
&\le C + C_\delta\left(\int_{\Omega} |x|^\alpha |\nabla u|^{p(x)}\,dx\right)^{\frac{p_-^*}{p_-}} \\
&\le C_\delta.
\end{aligned} \tag{29}$$

Noticing that $1 < 2p_+ < p_-^*$, by the Hölder inequality and (29), we have

$$\int_{\Omega_\delta} |u|^{2p_+}\,dx \le C\left(\int_{\Omega_\delta} |u|^{p_-^*}\,dx\right)^{\frac{2p_+}{p_-^*}} \le C_\delta \tag{30}$$

for any $\delta > 0$. It implies from (23) that

$$\begin{aligned}
\left(\int_{B_\delta} |u|^{2p_+}\,dx\right)^{\frac{p_-}{2p_+}} &\le C_\varepsilon \int_{B_{2\delta}} |x|^\alpha |\nabla u|^{p(x)}\,dx + \varepsilon\delta^N + C\left(\int_{\Omega_\delta} |u|^{p_-^*}\,dx\right)^{\frac{p_-}{p_-^*}} \\
&\le C_\varepsilon \int_{\Omega} |x|^\alpha |\nabla u|^{p(x)}\,dx + \varepsilon\delta^N + C\left(\int_{\Omega_\delta} |u|^{p_-^*}\,dx\right)^{\frac{p_-}{p_-^*}}.
\end{aligned}$$

By Lemma 3.4, we obtain

$$\int_{B_\delta} |u|^{2p_+}\,dx \le C_\delta. \tag{31}$$

We deduce from (30) and (31) that

$$\int_{\Omega} |u|^{2p_+}\,dx \le C. \tag{32}$$

According to (28) and (32), we have $\int_\Omega |\nabla u|^{p(x)}\,dx$ is uniformly bounded. □

Proof of Proposition 3.1 Using the Sobolev embedding theorem and Lemma 3.5, we have

$$\int_{\Omega} |v|^{p_-^*}\,dx \le C\left(\int_{\Omega} |\nabla v|^{p_-}\,dx\right)^{\frac{p_-^*}{p_-}} \le C\left(\int_{\Omega} \left(1 + |\nabla v|^{p(x)}\right)dx\right)^{\frac{p_-^*}{p_-}} \le C. \tag{33}$$

Let $s > 0$ and $t = 2p_+ + r$. According to (25), multiply problem (8) by v^{sp_-+1} and integrate to obtain

$$\begin{aligned}
& b\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx \int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla v^{sp_-+1}\,dx \\
&\quad \le \left(a + b\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx\right)\int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla v^{sp_-+1}\,dx \\
&\quad = \int_\Omega \bigl(g_\mu(v) + \lambda v_+^{p_--1}\bigr)v^{sp_-+1}\,dx \\
&\quad \le C\left(1 + \int_\Omega |v|^{sp_-+t}\,dx\right)
\end{aligned}$$

for any $\lambda \in (0, \lambda_3)$. It implies that

$$\begin{aligned}
& b\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx \int_\Omega |\nabla v|^{p_-}v^{sp_-}\,dx \\
&\quad \le b\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx \int_\Omega \bigl(1 + |\nabla v|^{p(x)}\bigr)v^{sp_-}\,dx \\
&\quad = \frac{b}{p_-}\int_\Omega |\nabla v|^{p(x)}\,dx\left(\int_\Omega v^{sp_-}\,dx + \frac{1}{sp_- + 1}\int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla v^{sp_-+1}\,dx\right) \\
&\quad \le \frac{Cb}{p_-}\int_\Omega |\nabla v|^{p(x)}\,dx\left(1 + \int_\Omega |v|^{sp_-+t}\,dx\right).
\end{aligned} \tag{34}$$

On the one hand, by the Sobolev embedding theorem, we have

$$\begin{aligned}
\int_\Omega |\nabla v|^{p_-}v^{sp_-}\,dx &= \frac{1}{(1+s)^{p_-}}\int_\Omega \bigl|\nabla v^{1+s}\bigr|^{p_-}\,dx \\
&\ge \frac{C}{(1+s)^{p_-}}\left(\int_\Omega |v|^{(1+s)p_-^*}\,dx\right)^{\frac{p_-}{p_-^*}}.
\end{aligned} \tag{35}$$

On the other hand, by the Hölder inequality and (33), we have

$$\begin{aligned}
\int_\Omega |v|^{sp_-+t}\,dx &\le \left(\int_\Omega |v|^{p_-^*}\,dx\right)^{\frac{t-p_-}{p_-^*}}\left(\int_\Omega |v|^{p_-(1+s)\frac{p_-^*}{p_-^*-t+p_-}}\,dx\right)^{\frac{p_-^*-t+p_-}{p_-^*}} \\
&\le C\left(1 + \int_\Omega |\nabla v|^{p(x)}\,dx\right)^{\frac{t-p_-}{p_-}}\left(\int_\Omega |v|^{(1+s)\frac{p_-^*}{d}}\,dx\right)^{\frac{dp_-}{p_-^*}},
\end{aligned} \tag{36}$$

where $d = \frac{p_-^*-t+p_-}{p_-} > 1$. According to (34), (35), and (36), we obtain

$$\begin{aligned}
& \frac{b}{p_+}\int_\Omega |\nabla v|^{p(x)}\,dx\left(\int_\Omega |v|^{(1+s)p_-^*}\,dx\right)^{\frac{p_-}{p_-^*}} \\
&\quad \le b\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx\left(\int_\Omega |v|^{(1+s)p_-^*}\,dx\right)^{\frac{p_-}{p_-^*}}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{b}{p_-}\big(C(1+s)\big)^{p_-} \\
&\quad \times \int_\Omega |\nabla v|^{p(x)}\,dx\left(1+\left(1+\int_\Omega |\nabla v|^{p(x)}\,dx\right)^{\frac{t-p_-}{p_-}}\right)\left(\int_\Omega |v|^{(1+s)\frac{p_-^*}{d}}\,dx\right)^{\frac{dp_-}{p_-^*}}.
\end{aligned}
$$

Since $\frac{t-p_-}{p_-} > 1$ and $\int_\Omega |\nabla v|^{p(x)}\,dx < C$, we have

$$
\left(\int_\Omega |v|^{(1+s)p_-^*}\,dx\right)^{\frac{p_-}{p_-^*}} \le \big(C(1+s)\big)^{p_-}\max\left\{1,\left(\int_\Omega |v|^{(1+s)\frac{p_-^*}{d}}\,dx\right)^{\frac{dp_-}{p_-^*}}\right\},
$$

which implies that

$$
\begin{aligned}
&\max\left\{1,\left(\int_\Omega |v|^{(1+s)p_-^*}\,dx\right)^{\frac{1}{(1+s)p_-^*}}\right\} \\
&\quad \le \big(C(1+s)\big)^{\frac{1}{1+s}}\max\left\{1,\left(\int_\Omega |v|^{(1+s)\frac{p_-^*}{d}}\,dx\right)^{\frac{d}{(1+s)p_-^*}}\right\}.
\end{aligned}
\tag{37}
$$

Now we carry out an iteration process. Set $s_k = d^k - 1$ for $k = 1, 2, \ldots$. By (37), we have

$$
\begin{aligned}
&\max\left\{1,\left(\int_\Omega |v|^{d^k p_-^*}\,dx\right)^{\frac{1}{d^k p_-^*}}\right\} \\
&\quad \le \big(Cd^k\big)^{\frac{1}{d^k}}\max\left\{1,\left(\int_\Omega |v|^{d^{k-1} p_-^*}\,dx\right)^{\frac{1}{d^{k-1} p_-^*}}\right\} \\
&\quad \le \prod_{j=1}^{k}\big(Cd^j\big)^{\frac{1}{d^j}}\max\left\{1,\left(\int_\Omega |v|^{p_-^*}\,dx\right)^{\frac{1}{p_-^*}}\right\} \\
&\quad = C^{\sum_{j=1}^k d^{-j}}\cdot d^{\sum_{j=1}^k jd^{-j}}\max\left\{1,\left(\int_\Omega |v|^{p_-^*}\,dx\right)^{\frac{1}{p_-^*}}\right\}.
\end{aligned}
\tag{38}
$$

Since $d > 1$, the series $\sum_{j=1}^\infty d^{-j}$ and $\sum_{j=1}^\infty jd^{-j}$ are convergent. Letting $k \to \infty$, we conclude from (33) and (38) that $\|v\|_{L^\infty(\Omega)} \le M$. The proof is complete. □

Proof of Theorem 1.1 Let $\lambda_* = \min\{\lambda_0, \lambda_3\}$. By Proposition 2.4, we know that problem (8) has at least two nonnegative nontrivial solutions u'_μ and u''_μ satisfying

$$
J_\mu(u'_\mu) < 0 < J_\mu(u''_\mu) < L \quad \text{for all } \lambda \in (0, \lambda_*).
$$

By the definition of function m_μ, we have $m_\mu(t) = t$ for $t \le \frac{1}{\mu}$. Hence, problem (8) reduces to problem (1) for $|u| \le \frac{1}{\mu}$. Let $\mu < \frac{1}{2M}$. By Proposition 3.1, it is easy to see that u'_μ and u''_μ are indeed two nonnegative nontrivial solutions of problem (1). □

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References
1. Abita, R.: Existence and asymptotic behavior of solutions for degenerate nonlinear Kirchhoff strings with variable-exponent nonlinearities. Acta Math. Vietnam. **46**, 613–643 (2021)
2. Ambrosetti, A., Brezis, H., Cerami, G.: Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal. **122**, 519–543 (1994)
3. Ambrosetti, A., Rabinowitz, P.: Dual variational methods in critical point theory and applications. J. Funct. Anal. **4**, 349–381 (1973)
4. Aouaoui, S.: Multiple solutions to some degenerate quasilinear equation with variable exponents via perturbation method. J. Math. Anal. Appl. **458**, 1568–1596 (2018)
5. Chen, Y., Levine, S., Rao, M.: Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. **66**, 1383–1406 (2006)
6. Chu, C.M.: Positive solutions for a class of $p(x)$-Laplacian equation involving concave-convex nonlinearities. Adv. Differ. Equ. **26**, 341–362 (2021)
7. Chung, N.T.: Multiple solutions for a p(x)-Kirchhoff-type equation with sign-changing nonlinearities. Complex Var. Elliptic Equ. **58**, 1637–1646 (2013)
8. Chung, N.T.: Infinitely many solutions for a class of p(x)-Kirchhoff type problems with critical exponents. Ann. Pol. Math. **124**, 129–149 (2020)
9. Dai, G., Hao, R.: Existence of solutions for a $p(x)$-Kirchhoff-type equation. J. Math. Anal. Appl. **359**, 275–284 (2009)
10. Ekeland, I.: On the variational principle. J. Math. Anal. Appl. **47**, 324–353 (1974)
11. Fan, X.L., Zhang, Q.H.: Existence of solutions for $p(x)$-Laplacian Dirichlet problem. Nonlinear Anal. **52**, 1843–1852 (2003)
12. Garcia-Mellian, J., Rossi, J.D., De Lis, J.C.S.: A variable exponent diffusion problem of concave-convex nature. Topol. Methods Nonlinear Anal. **47**, 613–639 (2016)
13. Hashizume, M., Sano, M.: Strauss's radial compactness and nonlinear elliptic equation involving a variable critical exponent. J. Funct. Spaces (2018)
14. Ho, K., Sim, I.: Existence and multiplicity of solutions for degenerate $p(x)$-Laplace equations involving concave-convex type nonlinearities with two parameters. Taiwan. J. Math. **19**, 1469–1493 (2015)
15. Ji, C., Fang, F.: Infinitely many solutions for the $p(x)$-Laplacian equations without (AR)-type growth condition. Ann. Pol. Math. **105**, 87–99 (2010)
16. Kirchhoff, G.: Mechanik. Teubner, Leipzig (1883)
17. Li, Q., Teng, K., Wang, W.: Concentration phenomenon of solutions for a class of Kirchhoff-type equations with critical growth. J. Math. Anal. Appl. **491**, 124355 (2020)
18. Maz'ya, V.: Sobolev Space with Applications to Elliptic Partial Differential Equations, Second revised and augmented edn. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 342. Springer, Heidelberg (2011)
19. Mihăilescu, M.: Existence and multiple of solutions for an elliptic equation with $p(x)$ growth condition. Glasg. Math. J. **48**, 411–418 (2006)
20. Nguyen, T.C.: Multiple solutions for a class of $p(x)$-Laplacian problems involving concave-convex nonlinearities. Electron. J. Qual. Theory Differ. Equ. **2013**, 26 (2013)
21. Rabinowitz, P.H.: Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS Regional Conference Series in Mathematics, vol. 65. Am. Math. Soc., Providence (1986)
22. Rădulescu, V.: Nonlinear elliptic equations with variable exponent: old and new. Nonlinear Anal. **121**, 336–369 (2015)
23. Ruzicka, M.: Electrorheological Fluids: Modeling and Mathematical Theory. Springer, Berlin (2002)
24. Sun, X., Chen, W.: Optimal improved L^p-Riemannian Gagliardo-Nirenberg inequalities. Nonlinear Anal. **72**, 3159–3172 (2010)
25. Tan, Z., Fang, F.: On superlinear $p(x)$-Laplacian problems without Ambrosetti and Rabinowitz condition. Nonlinear Anal. **75**, 3902–3915 (2012)
26. Xie, W., Chen, H.: Infinitely many bound state solutions for Kirchhoff type problems. Appl. Math. Lett. **93**, 1–7 (2019)
27. Zang, A.B.: $p(x)$-Laplacian equations satisfying cerami condition. J. Math. Anal. Appl. **337**, 547–555 (2008)
28. Zhang, B.l., Alessio, F., Sihua, L.: Infinitely many solutions for critical degenerate Kirchhoff type equations involving the fractional p-Laplacian. Appl. Math. Optim. **80**, 63–80 (2019)
29. Zhang, B.L., Ge, B., Cao, X.F.: Multiple solutions for a class of new $p(x)$-Kirchhoff problem without the Ambrosetti-Rabinowitz conditions. Mathematics **8**, 2068 (2020)
30. Zhang, Q.H., Zhao, C.S.: Existence of strong solutions of a $p(x)$-Laplacian Dirichlet problem without the Ambrosetti-Rabinowitz condition. Comput. Math. Appl. **69**, 1–12 (2015)
31. Zhao, J., Liu, X., Liu, J.: p-Laplacian equations in $\mathbb{R}^N$ with finite potential via truncation method, the critical case. J. Math. Anal. Appl. **455**, 58–88 (2017)