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A class of Schrödinger elliptic equations involving supercritical exponential growth

Yony Raúl Santaria Leuyacc[1*]

*Correspondence:
ysantarial@unmsm.edu.pe
[1]Facultad de Ciencias Matemáticas, Universidad Nacional Mayor de San Marcos, Lima, Perú

Abstract

This paper studies the existence of nontrivial solutions to the following class of Schrödinger equations:

$$\begin{cases} -\operatorname{div}(w(x)\nabla u) = f(x,u), & x \in B_1(0), \\ u = 0, & x \in \partial B_1(0), \end{cases}$$

where $w(x) = (\ln(1/|x|))^\beta$ for some $\beta \in [0,1)$, the nonlinearity $f(x,s)$ behaves like $\exp(|s|^{\frac{2}{1-\beta}+h(|x|)})$, and h is a continuous radial function such that $h(r)$ can be unbounded as r tends to 1. Our approach is based on a new Trudinger–Moser-type inequality for weighted Sobolev spaces and variational methods.

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Keywords: Schrödinger equations; Trudinger–Moser inequality; Supercritical exponential growth; Variational methods

1 Introduction

Let consider the following Schrödinger equation:

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$. In the case $N \geq 3$, some pioneering works developed by Brézis [7], Brézis & Nirenberg [8], Bartsh & Willem [6], and Capozzi, Fortunato & Palmieri [14] considered the assumption $|f(x,u)| \leq c(1 + |u|^{q-1})$, with $1 < q \leq 2^* = 2N/(N-2)$. The above growth of the nonlinearity f is related to the Sobolev embedding $H_0^1(\Omega) \subset L^q(\Omega)$ for $1 \leq q \leq 2^*$. In the limiting case $N = 2$, one has $2^* = +\infty$, that is, $H_0^1(\Omega) \subset L^q(\Omega)$ for $q \geq 1$, in particular, the nonlinear function f in (1.1) may have arbitrary polynomial growth. Also, some examples show that $H_0^1(\Omega) \not\subset L^\infty(\Omega)$. An important result found independently by Yudovich [37], Pohozaev [28], and Trudinger [35] showed that the maximal growth of the nonlinearity in the bivariate case is of exponential type. More precisely, it was stated that

$$e^{\alpha u^2} \in L^1(\Omega), \quad \text{for all } u \in H_0^1(\Omega) \text{ and } \alpha > 0. \tag{1.2}$$

Furthermore, Moser [26] stated the existence of a positive constant $C = C(\alpha, \Omega)$ such that

$$\sup_{\substack{u\in H_0^1(\Omega),\\ \|\nabla u\|_2\le 1}} \int_\Omega e^{\alpha u^2}\,dx \begin{cases} \le C, & \alpha \le 4\pi, \\ = +\infty, & \alpha > 4\pi. \end{cases} \tag{1.3}$$

Estimates (1.2) and (1.3) from now on be referred to as Trudinger–Moser inequalities. The above results motivate us to say that the function f has subcritical exponential growth if

$$\lim_{s\to+\infty} \frac{f(x,s)}{e^{\alpha s^2}} = 0, \quad \text{for all } \alpha > 0,$$

and critical exponential growth if there exists $\alpha_0 > 0$ such that

$$\lim_{s\to+\infty} \frac{f(x,s)}{e^{\alpha s^2}} = \begin{cases} 0, & \alpha < \alpha_0, \\ +\infty, & \alpha > \alpha_0. \end{cases} \tag{1.4}$$

Equations of the type (1.1) considering nonlinearities involving subcritical and critical exponential growth were treated by Adimurthi [1], Adimurthi–Yadava [2], de Figueiredo, Miyagaki, and Ruf [18] (see also [1–4, 11, 13, 23, 27, 31]), and some results on Hamiltonian systems involving the above-mentioned growth can be found in [16, 17, 20, 24, 29, 33]. We shall write $g_1(s) \prec g_2(s)$ if there exist positive constants k and s_0 such that $g_1(s) \le g_2(ks)$ for $s \ge s_0$. Additionally, we shall say that g_1 and g_2 are equivalent and write $g_1(s) \sim g_2(s)$ if $g_1(s) \prec g_2(s)$ and $g_2(s) \prec g_1(s)$. Therefore, f possesses critical exponential growth if only if $f(x,s) = g(s)$ with $g(s) \sim e^{|s|^2}$.

Several extensions of the Trudinger–Moser inequalities were obtained considering weighted Sobolev spaces, weighted Lebesgue measures, or Lorentz–Sobolev spaces (see [3–5, 13, 15, 19, 24, 25, 34] among others). In the above-mentioned papers, the growth of the nonlinearity is of the type $f(x,s) = Q(x)g(s)$ where $g(s) \sim e^{|s|^p}$ with $p = 2$ on Sobolev spaces and $p > 1$ on Lorentz–Sobolev spaces and for some weight $Q(x)$. More precisely, on Lorentz–Sobolev spaces, Brezis and Wainger [9] have shown the following: Let Ω be a bounded domain in $\mathbb{R}^2$ and $s > 1$. Then, $e^{\alpha|u|^{\frac{s}{s-1}}}$ belongs to $L^1(\Omega)$ for all $u \in W_0^1L^{2,s}(\Omega)$ and $\alpha > 0$. Furthermore, Alvino [5] obtained the following refinement of (1.3): there exists a positive constant $C = C(\Omega, s, \alpha)$ such that

$$\sup_{\substack{u\in W_0^1L^{2,s}(\Omega),\\ \|\nabla u\|_{2,s}\le 1}} \int_\Omega e^{\alpha|u|^{\frac{s}{s-1}}}\,dx \begin{cases} \le C, & \alpha \le (4\pi)^{s/(s-1)}, \\ = +\infty, & \alpha > (4\pi)^{s/(s-1)}. \end{cases} \tag{1.5}$$

In order to extend equations (1.1), we will study Schrödinger equations involving a diffusion operator (see [10, 12, 32, 38, 39] among others). Let B_1 be the unit ball centered at the origin in $\mathbb{R}^2$ and $H^1_{0,\mathrm{rad}}(B_1, w)$ be the subspace of the radially symmetric functions in the closure of $\mathcal{C}_0^\infty(B_1)$ with respect to the norm

$$\|u\| := \|\nabla u\|_{L^2(B_1,w)} = \left(\int_{B_1} w(x)|\nabla u|^2\,dx\right)^{\frac{1}{2}}. \tag{1.6}$$

In particular, if $w \equiv 1$, we denote the above space by $H^1_{0,\mathrm{rad}}(B_1)$. Trudinger–Moser-type inequalities for radial Sobolev spaces with logarithmic weights were considered by Calanchi and Ruf in [11]. More precisely, the above-mentioned authors used the weight $w(x) = (\log 1/|x|)^\beta$ for some fixed $0 \le \beta < 1$, this logarithmic weight will be used in the rest of this article.

Proposition 1.1 (Calanchi–Ruf, [11]) *Suppose that $w(x) = (\log 1/|x|)^\beta$ and $0 \le \beta < 1$. Then,*

$$\int_{B_1} e^{\alpha|u|^{\frac{2}{1-\beta}}}\,dx < +\infty, \quad \text{for all } u \in H^1_{0,\mathrm{rad}}(B_1,w) \text{ and } \alpha > 0.$$

Furthermore, setting $\alpha^*_\beta = 2[2\pi(1-\beta)]^{\frac{1}{1-\beta}}$, there exists a positive constant $C = C(\alpha,\beta)$ such that

$$\sup_{\substack{u\in H^1_{0,\mathrm{rad}}(B_1,w),\\ \|u\|\le 1}} \int_{B_1} e^{\alpha|u|^{\frac{2}{1-\beta}}}\,dx \begin{cases} \le C, & \alpha \le \alpha^*_\beta, \\ = +\infty, & \alpha > \alpha^*_\beta. \end{cases}$$

In order to establish a Trudinger–Moser inequality proved by Ngô and Nguyen [27], we consider a continuous radial function $h : [0,1) \to \mathbb{R}$ such that

(h_1) $h(0) = 0$ and $h(r) > 0$ for $r \in (0,1)$;

(h_2) there exists $c > 0$ and $\gamma > 2$ such that

$$h(r) \le \frac{c}{(-\ln r)^\gamma} \quad \text{near } 0.$$

Proposition 1.2 (Ngô–Nguyen, [27]) *Suppose that h satisfies (h_1) and (h_2). Then, there exists a positive constant $C = C(\alpha,h)$ such that*

$$\sup_{\substack{u\in H^1_{0,\mathrm{rad}}(B_1),\\ \|\nabla u\|_2\le 1}} \int_{B_1} \exp\bigl(\alpha|u|^{2+h(|x|)}\bigr)\,dx \begin{cases} \le C, & \alpha \le 4\pi, \\ = +\infty, & \alpha > 4\pi. \end{cases}$$

Next we establish a new version of the Trudinger–Moser inequality which will be used throughout this paper.

Theorem 1.3 *Suppose h satisfies (h_1) and (h_2) and $w(x) = (\log 1/|x|)^\beta$ for some $\beta \in [0,1)$. Then, there exists a positive constant $C = C(\alpha,\beta,h)$ such that*

$$\sup_{\|u\|\le 1} \int_{B_1} \exp\bigl(\alpha|u|^{\frac{2}{1-\beta}+h(|x|)}\bigr)\,dx \le C. \tag{1.7}$$

If $\alpha > \alpha^*_\beta$, *then*

$$\sup_{\|u\|\le 1} \int_{B_1} \exp\bigl(\alpha |u|^{\frac{2}{1-\beta}+h(|x|)}\bigr)\,dx = +\infty. \tag{1.8}$$

The proof of Theorem 1.3 will be presented in the next section. In this work, we are interested in finding nontrivial weak solutions for the following class of Schrödinger equations:

$$\begin{cases} -\operatorname{div}(w(x)\nabla u) = f(x,u), & x \in B_1, \\ u = 0, & x \in \partial B_1, \end{cases} \tag{1.9}$$

where the growth of the nonlinearity of f is motivated by the Trudinger–Moser inequality given by Theorem 1.3. More precisely, we assume the following conditions on the nonlinearity f:

(H_1) $f: B_1 \times \mathbb{R} \to \mathbb{R}$ is a continuous and radially symmetric in the first variable function, that is, $f(x,s) = f(y,s)$ for $|x| = |y|$. Moreover, $f(x,s) = 0$ for all $x \in B_1$ and $s \le 0$.

(H_2) There exists a constant $\mu > 2$ such that

$$0 < \mu F(x,s) \le s f(x,s), \quad \text{for all } x \in B_1 \text{ and } s > 0,$$

where $F(x,s) = \int_0^s f(x,t)\,dt$.

(H_3) There exists a constant $M > 0$ such that

$$0 < F(x,s) \le M f(x,s), \quad \text{for all } s > 0.$$

(H_4) There holds

$$\limsup_{s\to 0} \frac{2F(x,s)}{s^2} < \lambda_1, \quad \text{uniformly in } x \in B_1,$$

where λ_1 is the first eigenvalue associated to $(-\operatorname{div}(w(x)\nabla u), H^1_{0,\mathrm{rad}}(B_1,w))$.

(H_5) There exists a constant $\alpha_0 > 0$ such that

$$\lim_{s\to\infty} \frac{f(x,s)}{\exp(\alpha |u|^{\frac{2}{1-\beta}+h(|x|)})} = \begin{cases} 0, & \alpha > \alpha_0, \\ +\infty, & \alpha < \alpha_0, \end{cases}$$

(H_6) There exist constants $p > 2$ and $C_p > 0$ such that

$$f(x,s) \ge C_p s^{p-1}, \quad \text{for all } s \ge 0,$$

where

$$C_p > \frac{(p-2)^{(p-2)/2} S_p^p}{p^{(p-2)/2}} \left(\frac{\alpha_0}{\alpha^*_\beta}\right)^{(1-\beta)(p-2)/2}$$

and

$$S_p := \sup_{0\ne u \in H^1_{0,\mathrm{rad}}(B_1,w)} \frac{(\int_{B_1} w(x)|\nabla u|^2\,dx)^{1/2}}{(\int_{B_1} |u|^p\,dx)^{1/p}}.$$

Throughout, we denote the space $E := H^1_{0,\mathrm{rad}}(B_1, w)$ endowed with the inner product

$$\langle u, v\rangle_E = \int_{B_1} w(x)\nabla u \nabla v\, dx, \quad \text{for all } u, v \in E,$$

to which corresponds the norm

$$\|u\| = \left(\int_{B_1} w(x)|\nabla u|^2\, dx\right)^{1/2}.$$

Also, we denote by E^* the dual space of E with its usual norm. We say that $u \in E$ is a weak solution of (1.9) if

$$\int_{B_1} w(x)\nabla u \nabla \phi\, dx = \int_{B_1} f(x,u)\phi\, dx, \quad \text{for all } \phi \in E. \tag{1.10}$$

Under the above assumptions on f, we consider the Euler–Lagrange functional $J : E \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\int_{B_1} w(x)|\nabla u|^2\, dx - \int_{B_1} F(x,u)\, dx, \quad \text{for all } u \in E.$$

Furthermore, using standard arguments (see [21]), J belongs to $\mathcal{C}^1(E, \mathbb{R})$ and

$$J'(u)\phi = \int_{B_1} w(x)\nabla u \nabla \phi\, dx - \int_{B_1} f(x,u)\phi\, dx, \quad \text{for all } u, \phi \in E.$$

Next, we present our existence result for the problem (1.9).

Theorem 1.4 *Suppose that f satisfies (H_1)–(H_6). Then, the problem (1.9) possesses a nontrivial weak solution.*

Notice that the class of Schrödinger equations (1.9) represents a natural extension of the equation (1.1). Under assumption (H_5), the nonlinearity f behaves like $\exp((\alpha + h(|x|))|s|^{\frac{2}{1-\beta}})$ as s tends to infinity. Moreover, if $\beta = 0$, we have that $w \equiv 1$ and the equation (1.9) is reduced to problem (1.1); the case with $\beta = 0$ and $h(x) = |x|^a$ for some $a > 0$ was studied in [27], and treated in many works considering $h = 0$ (see [1, 2, 18] among others). Additionally, we observe that (h_1) and (h_2) are conditions near the origin, in particular, h can tend to infinity for values of $|x|$ close to 1. Also, if β is close to 1, the power of $|s|^p$ where $p = 2/(1-\beta)$ can be sufficiently large. The above properties motivate us to say that f possesses supercritical exponential growth and represents an extension of other previously studied works. Finally, note that the class of functions which satisfies the conditions (H_1)–(H_6) is not empty, for instance, consider the following function $f : B_1 \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x,s) = \begin{cases} As^{p-1} + (p + |x|^\eta)s^{p-1+|x|^\eta} e^{s^{p+|x|^\eta}}, & s \geq 0, \\ 0, & s < 0. \end{cases}$$

for some positive constants η, $p = 2/(1-\beta)$, and A sufficiently large.

2 Preliminaries

The space $H^1_{0,\mathrm{rad}}(B_1,w)$ where $w(x)=(\log 1/|x|)^\beta$ for some $0\le\beta<1$, endowed with the norm given by (1.6), is a separable Banach space (see [22, Theorem 3.9]). Next, we present a compactness result.

Lemma 2.1 *The embedding $H^1_{0,\mathrm{rad}}(B_1,w)\hookrightarrow L^p(B_1)$ is continuous and compact for $1\le p<\infty$.*

Proof From the Cauchy–Schwarz inequality, we have

$$\int_{B_1}|\nabla u|\,dx\le\left(\int_{B_1}w(x)|\nabla u|^2\,dx\right)^{1/2}\cdot\left(\int_{B_1}w(x)^{-1}\,dx\right)^{1/2}.$$

Using the change of variable $|x|=e^{-s}$, we get

$$\frac{1}{2\pi}\int_{B_1}w(x)^{-1}\,dx=\int_0^{+\infty}e^{-2s}s^{-\gamma}\,ds=\int_0^1 e^{-2s}s^{-\gamma}\,ds+\int_1^{+\infty}e^{-2s}s^{-\gamma}\,ds.$$

Note that

$$\int_0^1 e^{-2s}s^{-\gamma}\,ds\le\int_0^1 s^{-\gamma}\,ds=\frac{1}{1-\gamma}$$

and

$$\int_1^{+\infty}e^{-2s}s^{-\gamma}\,ds\le\int_1^{+\infty}e^{-2s}\,ds=\frac{e^{-2}}{2}.$$

Therefore, we can find a positive constant C such that

$$\|\nabla u\|_1\le C\left(\int_{B_1}|\nabla u|^2 w(x)\,dx\right)^{1/2}.$$

Thus, $H^1_0(B_1,w)\hookrightarrow W^{1,1}_0(B_1)$ continuously, which implies the continuous and compact embedding

$$H^1_0(B_1,w)\hookrightarrow L^p(B_1),\quad\text{for all } p\ge 1.$$ □

Lemma 2.2 ([11]) *Let u be a function in $H^1_0(B_1,w)$. Then,*

$$\left|u(x)\right|\le\frac{(-\ln|x|)^{\frac{1-\beta}{2}}}{\sqrt{2\pi(1-\beta)}}\cdot\|u\|,\quad\textit{for all } x\in B_1.$$

2.1 Proof of Theorem 1.3

Proof To prove the first statement of the theorem, it is sufficient to consider $\alpha=\alpha^*_\beta$. From Lemma 2.2, for each $u\in E$ with $\|u\|\le 1$, we have

$$\alpha^*_\beta\left|u(r)\right|^{2/(1-\beta)}\le -2\ln r,\quad\text{for all } 0<r<1,\tag{2.1}$$

where $r = |x|$. Setting $r_1 := e^{-\alpha_\beta^*/2}$, we have

$$\left|u(r)\right| \le 1, \quad \text{for all } r \ge r_1. \tag{2.2}$$

Thus,

$$\int_{B_1\setminus B_{r_1}} \exp\left(\alpha_\beta^*|u|^{\frac{2}{1-\beta}+h(|x|)}\right) dx \le \int_{B_1\setminus B_{r_1}} \exp\left(\alpha_\beta^*\right) dx \le \exp\left(\alpha_\beta^*\right)|B_1|. \tag{2.3}$$

On the other hand, by (2.1), we can write

$$\begin{aligned}
&\int_{B_{r_1}} \exp\left(\alpha_\beta^*|u|^{\frac{2}{1-\beta}+h(|x|)}\right) dx \\
&\quad\le \int_{B_{r_1}} \exp\left(\alpha_\beta^*|u|^{\frac{2}{1-\beta}}|u|^{h(|x|)}\right) dx \\
&\quad\le \int_{B_{r_1}} \exp\left(\alpha_\beta^*|u|^{\frac{2}{1-\beta}}\left(\frac{-2\ln r}{\alpha_\beta^*}\right)^{\frac{(1-\beta)}{2}h(|x|)}\right) dx \\
&\quad\le \int_{B_{r_1}} \exp\left(\alpha_\beta^*|u|^{\frac{2}{1-\beta}}\right)(\exp\left(\alpha_\beta^*|u|^{\frac{2}{1-\beta}}\left(\left(\frac{-2\ln r}{\alpha_\beta^*}\right)^{\frac{(1-\beta)}{2}h(|x|)}-1\right)-1\right) dx \\
&\qquad+\int_{B_{r_1}} \exp\left(\alpha_\beta^*|u|^{\frac{2}{1-\beta}}\right) dx.
\end{aligned}$$

Note that $-2\ln r/\alpha_\beta^* \ge 1$ for $0 < r \le r_1$. By (h_2), there exist $c > 0$ and $0 < r_2 < r_1$ such that

$$h\left(|x|\right) \le \frac{c}{(-\ln r)^\gamma}, \quad \text{for all } 0 < r < r_2. \tag{2.4}$$

Using (2.1) and (2.4), we have

$$\begin{aligned}
&\exp(\alpha_\beta^*|u|^{\frac{2}{1-\beta}}\left(\left(\frac{-2\ln r}{\alpha_\beta^*}\right)^{\frac{(1-\beta)}{2}h(|x|)}-1\right)-1 \\
&\quad\le \exp(-2\ln r\left(\left(\frac{-2\ln r}{\alpha_\beta^*}\right)^{\frac{c(1-\beta)}{2(-\ln r)^\gamma}}-1\right)-1 := k(r).
\end{aligned}$$

Also, as $r \to 0^+$, one has

$$\begin{aligned}
\left(\frac{-2\ln r}{\alpha_\beta^*}\right)^{\frac{c(1-\beta)}{2(-\ln r)^\gamma}} &= \exp\left[\frac{c(1-\beta)}{2(-\ln r)^\gamma}\ln\left(\frac{-2\ln r}{\alpha_\beta^*}\right)\right] \\
&= 1 + \frac{c(1-\beta)}{2(-\ln r)^\gamma}\ln\left(\frac{-2\ln r}{\alpha_\beta^*}\right) + o\left(\frac{1}{(-\ln r)^\gamma}\ln\left(\frac{-2\ln r}{\alpha_\beta^*}\right)\right).
\end{aligned}$$

Therefore, as r is close to zero, we have

$$\begin{aligned}
-2\ln r\left(\left(\frac{-2\ln r}{\alpha_\beta^*}\right)^{\frac{c(1-\beta)}{2(-\ln r)^\gamma}}-1\right) &= \frac{c(1-\beta)}{(-\ln r)^{\gamma-1}}\ln\left(\frac{-2\ln r}{\alpha_\beta^*}\right) \\
&\quad+ o\left(\frac{1}{(-\ln r)^{\gamma-1}}\ln\left(\frac{-2\ln r}{\alpha_\beta^*}\right)\right).
\end{aligned}$$

Since $\gamma > 2$, we obtain

$$\frac{c(1-\beta)}{(-\ln r)^{\gamma-1}}\ln\left(\frac{-2\ln r}{\alpha_\beta^*}\right) \to 0, \quad \text{as } r \to 0^+. \tag{2.5}$$

Consequently,

$$\begin{aligned} k(r) &= \exp\left[\frac{c(1-\beta)}{(-\ln r)^{\gamma-1}}\ln\left(\frac{-2\ln r}{\alpha_\beta^*}\right) + o\left(\frac{1}{(-\ln r)^{\gamma-1}}\ln\left(\frac{-2\ln r}{\alpha_\beta^*}\right)\right)\right] - 1 \\ &= \frac{c(1-\beta)}{(-\ln r)^{\gamma-1}}\ln\left(\frac{-2\ln r}{\alpha_\beta^*}\right) + o\left(\frac{1}{(-\ln r)^{\gamma-1}}\ln\left(\frac{-2\ln r}{\alpha_\beta^*}\right)\right). \end{aligned}$$

Set

$$l(r) = \frac{c(1-\beta)}{(-\ln r)^{\gamma-1}}\ln\left(\frac{-2\ln r}{\alpha_\beta^*}\right).$$

In particular, k and l are continuous and positive in $(0, r_2)$. Moreover, there exist $C > 0$ and $0 < r_3 < r_2$ such that

$$k(r) \le Cl(r), \quad \text{for all } 0 < r \le r_3. \tag{2.6}$$

Therefore, by (2.1), (2.6), and the definition of $k(r)$, we have

$$\begin{aligned} &\int_{B_{r_3}} \exp\left(\alpha_\beta^*|u|^{\frac{2}{1-\beta}+h(|x|)}\right)dx \\ &\quad\le \int_{B_{r_3}} \exp\left(\alpha_\beta^*|u|^{\frac{2}{1-\beta}}\right)k\left(|x|\right)dx + \int_{B_{r_3}} \exp\left(\alpha_\beta^*|u|^{\frac{2}{1-\beta}}\right)dx \\ &\quad\le C_1\int_{B_{r_3}} \frac{1}{|x|^2}\ln\left(\frac{-2\ln|x|}{\alpha_\beta^*}\right)\frac{c(1-\beta)}{(-\ln|x|)^{\gamma-1}}\,dx + C_2 \\ &\quad= 2\pi C_1 c(1-\beta)\int_0^{\rho_3} \frac{1}{r}\ln\left(-\frac{2\ln r}{\alpha_\beta^*}\right)\frac{1}{(-\ln r)^{\gamma-1}}\,dr + C_2 \\ &\quad= 2\pi C_1 c(1-\beta)\int_{-\ln\rho_3}^{+\infty} \ln\left(\frac{2s}{\alpha_\beta^*}\right)\frac{1}{s^{\gamma-1}}\,ds + C_2, \end{aligned}$$

for some positive constants C_1 and C_2. Using the fact that $\gamma > 2$, we have

$$\int_{B_{r_3}} \exp\left(\alpha_\beta^*|u|^{\frac{2}{1-\beta}+h(|x|)}\right)dx \le C_2. \tag{2.7}$$

On the other hand, using (2.1), we have

$$1 \le \left|u(r)\right| \le \left(-\frac{2\ln r_3}{\alpha_\beta^*}\right)^{\frac{1-\beta}{2}}, \quad \text{for all } r_3 \le r \le r_1$$

Combining the above inequality with the boundedness of h in $B_{r_1}\backslash B_{r_3}$, we get

$$\int_{B_{r_1}\backslash B_{r_3}} \exp\left(\alpha_\beta^*|u|^{\frac{2}{1-\beta}+h(|x|)}\right)dx \le |B_{r_1}|M. \tag{2.8}$$

Consequently, from (2.3), (2.7), and (2.8), we obtain

$$\int_{B_1} \exp\bigl(\alpha^*_\beta |u|^{\frac{2}{1-\beta}+h(|x|)}\bigr)\,dx \le C,$$

which implies the first assertion of the theorem. In order to prove the sharpness, we consider the following sequence given in [15]:

$$\psi_k(x) = \left(\frac{1}{\alpha^*_\beta}\right)^{(1-\beta)/2} \begin{cases} k^{\frac{2}{1-\beta}} \ln(\frac{1}{|x|^2})^{1-\beta}, & 0 \le |x| \le e^{-k/2}, \\ k^{\frac{1-\beta}{2}}, & e^{-k/2} \le |x| \le 1. \end{cases}$$

Then, $\|\psi_k\| = 1$ for all $k \in \mathbb{N}$. Moreover, for $\alpha > \alpha^*_\beta$, we have

$$\int_{B_1} \exp\bigl(\alpha|\psi_k|^{\frac{2}{1-\beta}+h(|x|}\bigr)\,dx \ge \int_{B_1} \exp\bigl(\alpha|\psi_k|^{\frac{2}{1-\beta}}\bigr)\,dx \ge \int_{e^{-k/2}}^1 \exp\left(\frac{\alpha}{\alpha^*_\beta}k\right) r\,dr.$$

Then,

$$\int_{B_1} \exp\bigl(\bigl(\alpha + h\bigl(|x|\bigr)\bigr)|\psi_k|^{2/(1-\beta)}\bigr)\,dx \ge e^{k(\frac{\alpha}{\alpha^*_\beta}-1)}\bigl(e^k - 1\bigr) \to +\infty, \quad \text{as } k \to \infty,$$

and the proof is complete. □

Corollary 2.3 *Let $\eta > 0$. Then,*

$$\int_{B_1} \exp\bigl(\alpha|\psi_k|^{\frac{2}{1-\beta}+|x|^\eta}\bigr)\,dx < +\infty, \quad \text{for all } u \in H^1_{0,\mathrm{rad}}(B_1, w) \text{ and } \alpha > 0. \tag{2.9}$$

Furthermore, if $\alpha \le \alpha^*_\beta$, there exists a positive constant C such that

$$\int_{B_1} \exp\bigl(\alpha|\psi_k|^{\frac{2}{1-\beta}+|x|^\eta}\bigr)\,dx \le C. \tag{2.10}$$

If $\alpha > \alpha^*_\beta$, then

$$\sup_{\|u\|\le 1} \int_{B_1} \exp\bigl(\alpha|\psi_k|^{\frac{2}{1-\beta}+|x|^\eta}\bigr)\,dx = +\infty. \tag{2.11}$$

As it was observed in [27], the statements of Theorem 1.3 and its corollary are no longer true if one considers the space of nonradial functions $H^1_0(B_1, w)$. Additionally, using similar arguments as in Theorem 1.3, we can prove the natural extension of (1.2), that is, if $\alpha > 0$ and $u \in H^1_{0,\mathrm{rad}}(B_1, w)$, then

$$\int_{B_1} \exp\bigl(\alpha|u|^{\frac{2}{1-\beta}+h(|x|)}\bigr)\,dx < +\infty. \tag{2.12}$$

3 The geometry of the mountain pass theorem

This section is devoted to showing that the functional J satisfies the geometry of the mountain pass theorem.

Lemma 3.1 *Suppose that* (H_1), (H_4), *and* (H_5) *hold. Then, there exist* $\sigma, \rho > 0$ *such that*

$$J(u) \geq \sigma, \quad \text{for all } u \in E \text{ with } \|u\| = \rho.$$

Proof Consider $q > 2$ and $0 < \epsilon < \lambda_1/2$. From (H_1) and (H_4), we can find $c > 0$ such that

$$\left|F(x,s)\right| \leq \epsilon|s|^2 + c|s|^q \exp\bigl(2\alpha_0|u|^{\frac{2}{1-\beta}+h(|x|)}\bigr), \quad \text{for all } (x,s) \in B_1 \times \mathbb{R}.$$

Integrating on B_1 and applying the Cauchy–Schwarz inequality, we obtain

$$\int_{B_1} F(x,u)\,dx \leq \epsilon\|u\|_2^2 + c\|u\|_{2q}^q \left(\int_{B_1} \exp\bigl(4\alpha_0|u|^{\frac{2}{1-\beta}+h(|x|)}\bigr)\,dx\right)^{1/2}. \tag{3.1}$$

Let $h_0 = \max_{0\leq r\leq r_1} h(r)$ where r_1 is given by (2.2). By Theorem 1.3, we have

$$\begin{aligned} \int_{B_{r_1}} \exp\bigl(4\alpha_0|u|^{\frac{2}{1-\beta}+h(|x|)}\bigr)\,dr &\leq \int_{B_{r_1}} \exp\left[4\alpha_0\|u\|^{\frac{2}{1-\beta}+h(|x|)}\left(\frac{|u|}{\|u\|}\right)^{\frac{2}{1-\beta}+h(|x|)}\right]dx \\ &\leq \int_{B_{r_1}} \exp\left[4\alpha_0\|u\|^{\frac{2}{1-\beta}+h_0}\left(\frac{|u|}{\|u\|}\right)^{\frac{2}{1-\beta}+h(|x|)}\right]dx \\ &\leq C_1, \end{aligned} \tag{3.2}$$

provided that $\|u\| \leq \rho_0$ for some $0 < \rho_0 < 1$ such that $4\alpha_0\rho_0^{\frac{2}{1-\beta}+h_0} < \alpha_\beta^*$. Using (2.2), we have

$$\int_{B_1\setminus B_{r_1}} \exp\bigl(4\alpha_0|u|^{\frac{2}{1-\beta}+h(|x|)}\bigr)\,dx \leq \int_{B_1\setminus B_{r_1}} \exp(4\alpha_0)\,dx = C_2. \tag{3.3}$$

Replacing (3.2) and (3.3) in (3.1), we get some $c > 0$ such that

$$\int_{B_1} F(x,u)\,dx \leq \frac{\epsilon}{\lambda_1}\|u\|^2 + c\|u\|^q,$$

provided that $\|u\| \leq \rho_0$ for some $\rho_0 > 0$. Then,

$$J(u) \geq \frac{1}{2}\|u\|^2 - \int_{B_1} F(x,u)\,dx \geq \left(\frac{1}{2} - \frac{\epsilon}{\lambda_1}\right)\|u\|^2 - c\|u\|^q.$$

Therefore, we can find $\rho > 0$ and $\sigma > 0$ with $0 < \rho < \rho_0$ sufficiently small such that $J(u) \geq \sigma > 0$, for all $u \in E$ satisfying $\|u\| = \rho$. □

Lemma 3.2 *Suppose that* (H_1)–(H_2) *hold. Then, there exists* $e \in E$ *such that*

$$J(e) < \rho \quad \text{and} \quad \|e\| > \rho,$$

where $\rho > 0$ *is given by Lemma* 3.1.

Proof It follows from (H_2), that there exist $C > 0$ and $s_0 > 0$ such that

$$F(x,s) \geq Ce^{s/M}, \quad \text{for all } s \geq s_0.$$

Let $e_0 \geq 0$ and $e_0 \neq 0$ fixed. Then, there exists $\delta > 0$ such that $|\{x \in B_1 : e_0(x) \geq \delta\}| \geq \delta$. Thus, for $t \geq s_0/\delta$, we have

$$J(te_0) \geq \frac{t^2}{2}\|e_0\|^2 - \int_{\{x\in B_1:e_0\geq\delta\}} F(x,te_0)\,dx \geq \frac{t^2}{2}\|e_0\|^2 - C\delta e^{t\delta/M},$$

which implies that $J(te_0) \to -\infty$, as $t \to +\infty$. Therefore, we can take $e = t_0e_0$ with $t_0 > 0$ sufficiently large such that $J(e) < 0$ and $\|e\| > \rho$. □

4 Palais–Smale sequence

By Lemmas 3.1 and 3.2, in the mountain pass theorem (see [30, 36]), we can find a Palais–Smale sequence at level $d \geq \sigma$, where σ is given by Lemma 3.1, that is, there exists a sequence $(u_n) \subset E$ such that

$$J(u_n) \to d \quad \text{and} \quad \big\|J'(u_n)\big\|_{E^*} \to 0, \tag{4.1}$$

where $d > 0$ can be characterized as

$$d = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J\big(\gamma(t)\big), \tag{4.2}$$

and

$$\Gamma = \big\{\gamma \in \mathcal{C}\big([0,1],E\big) : \gamma(0) = 0, \gamma(1) = e\big\}.$$

Lemma 4.1 *Let $(u_n) \subset E$ be a Palais–Smale sequence for the functional J satisfying (4.1). Then, the sequence (u_n) is bounden in E.*

Proof From (H_2), we have

$$\begin{aligned} J(u_n) - \frac{1}{\mu}J'(u_n)u_n &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 - \frac{1}{\mu}\int_{B_1} \big(\mu F(x,u_n) - f(x,u_n)u_n\big)\,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2. \end{aligned}$$

Using (4.1), for n sufficiently large, we have

$$J(u_n) \leq d + 1 \quad \text{and} \quad \big\|J'(u_n)\big\|_{E^*} \leq \mu.$$

Therefore, for n sufficiently large, we obtain

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 \leq d + 1 + \|u_n\|,$$

which implies that the sequence (u_n) is bounded in E. □

Lemma 4.2 *Let (u_n) be a Palais–Smale sequence for the functional J satisfying (4.1) and suppose that $u_n \rightharpoonup u$ weakly in E. Then, there exists a subsequence of (u_n), still denoted by*

(u_n), *such that*

$$f(x,u_n) \to f(x,u) \quad \text{in } L^1(B_1) \tag{4.3}$$

and

$$F(x,u_n) \to F(x,u) \quad \text{in } L^1(B_1). \tag{4.4}$$

Proof From Lemma 2.1, we can suppose that (u_n) converges to u in $L^1(B_1)$. By Theorem 1.3, (H_1), and (H_4), we have that $f(x,u_n) \in L^1(B_1)$. Using Lemma 4.1, the sequence $(\|u_n\|)$ is bounded and the fact that $\|J'(u_n)\|_{E^*} \to 0$ allows us to obtain

$$\left|J'(u_n)u_n\right| \le \left\|J'(u_n)\right\|_{E^*}\|u_n\| \to 0, \quad \text{as } n \to +\infty.$$

Thus,

$$J'(u_n)u_n = \frac{\|u_n\|^2}{2} - \int_{B_1} f(x,u_n)u_n\,dx \to 0, \quad \text{as } n \to +\infty.$$

Therefore, the sequence $f(x,u_n)u_n$ is bounded in $L^1(B_1)$. Due to [18, Lemma 2.10], we conclude that $f(x,u_n) \to f(x,u)$ in $L^1(B_1)$. On the other hand, by the convergence (4.3), there exists $p \in L^1(B_1)$ such that

$$f(x,u_n) \le p(x), \quad \text{almost everywhere in } B_1 \text{ and for } n \text{ sufficiently large.}$$

From (H_3), we can write

$$F(x,u_n) \le Mp(x), \quad \text{almost everywhere in } B_1 \text{ and for } n \text{ sufficiently large.}$$

By Lebesgue's dominated convergence theorem, the convergence (4.4) follows. □

Lemma 4.3 *Let $(u_n) \subset E$ be a Palais–Smale sequence for the functional J satisfying* (4.1). *Then,*

$$d < \frac{1}{2}\left(\frac{\alpha_\beta^*}{\alpha_0}\right)^{1-\beta},$$

where d is the minimax level given by (4.2).

Proof Let $u_p \in E$ be a nonnegative function with $\|u_p\|_p = 1$ such that

$$S_p = \inf_{0\neq u\in H^1_{0,\mathrm{rad}}(B_1,w)} \frac{(\int_{B_1} w(x)|\nabla u|^2\,dx)^{1/2}}{(\int_{B_1}|u|^p\,dx)^{1/p}} = \|u_p\|.$$

From (H_6), we get

$$J(tu_p) = \frac{t^2}{2}\|u_p\|^2 - \int_{B_1} F(x,tu_p)\,dx \ge \frac{t^2}{2}\|u_p\|^2 - \frac{C_p t^p}{p}\int_{B_1}|u_p|^p\,dx.$$

Therefore, by the estimate on C_p, we have

$$\sup_{t\geq 0} J(tu_p) \leq \max_{t\geq 0}\left\{\frac{t^2S_p^2}{2} - \frac{C_pt^p}{p}\right\} = \frac{(p-2)S_p^{2p/(p-2)}}{2pC_p^{2/(p-2)}} < \frac{1}{2}\left(\frac{\alpha_\beta^*}{\alpha_0}\right)^{1-\beta}. \tag{4.5}$$

Take $e_0 = u_p$ in Lemma 3.2, that is, we consider $e = t_0u_p$ with $t_0 > 0$ given by Lemma 3.2. Setting $\gamma_0(t) = tt_0u_p$, in particular, we have $\gamma_0 \in \Gamma = \{\gamma \in \mathcal{C}([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$. Using (4.2) and (4.5), we obtain

$$d = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J\big(\gamma(t)\big) \leq \max_{t\in[0,1]} J\big(\gamma_0(t)\big) = \max_{t\in[0,1]} J(tt_0u_p) \leq \max_{t\geq 0} J(tu_p) < \frac{1}{2}\left(\frac{\alpha_\beta^*}{\alpha_0}\right)^{1-\beta}. \qquad \square$$

5 Proof of Theorem 1.4

Let $(u_n) \subset E$ be a Palais–Smale sequence of the functional J satisfying (4.1). Then,

$$J'(u_n)\phi = \int_{B_1} w(x)\nabla u_n\nabla\phi\,dx - \int_{B_1} f(x,u_n)\phi\,dx = o_n(1), \tag{5.1}$$

for all $\phi \in \mathcal{C}_{0,\mathrm{rad}}^\infty(B_1)$. By Lemma 4.1, the sequence (u_n) is bounded in E. Thus, up to a subsequence, we can assume that there exists $u \in E$ such that $u_n \rightharpoonup u$ weakly in E, and replacing the above convergence in (5.1) yields

$$\int_{B_1} w(x)\nabla u\nabla\phi\,dx - \int_{B_1} f(x,u)\phi\,dx = 0, \quad \text{for all } \phi \in \mathcal{C}_{0,\mathrm{rad}}^\infty(B_1).$$

Since $\mathcal{C}_{0,\mathrm{rad}}^\infty(B_1)$ is dense in E, we obtain

$$\int_{B_1} w(x)\nabla u\nabla\phi\,dx = \int_{B_1} f(x,u)\phi\,dx, \quad \text{for all } \phi \in E.$$

Therefore, $u \in E$ is a critical point of J. Now, we prove that u is nontrivial. Suppose, by contradiction, that $u \equiv 0$. From Lemma 2.1, we can assume that

$$u_n \to 0 \quad \text{in } L^p(B_1), \text{for all } p \geq 1. \tag{5.2}$$

Using the fact that $J(u_n) \to d$, we have

$$J(u_n) = \frac{\|u_n\|^2}{2} - \int_{B_1} F(x,u_n)\,dx = d + o_n(1). \tag{5.3}$$

Since, we suppose that $u_n \rightharpoonup 0$, by Lemma 4.2, we obtain

$$\int_{B_1} F(x,u_n)\,dx \to \int_{B_1} F(x,0)\,dx = 0.$$

Replacing the above limit in (5.3), one has

$$\frac{\|u_n\|^2}{2} = d + o_n(1). \tag{5.4}$$

By Lemma 4.3, we get

$$\|u_n\|^2 = 2d + o_n(1) < \left(\frac{\alpha_\beta^*}{\alpha_0}\right)^{1-\beta} + o_n(1).$$

Thus, we can assume that there exists $\delta > 0$ sufficiently small such that

$$\|u_n\|^{\frac{2}{1-\beta}} \leq \frac{\alpha_\beta^*}{\alpha_0} - 2\delta, \quad \text{for all } n \geq 1.$$

Now, we can find $\epsilon > 0$ sufficiently small and $m > 1$ sufficiently close to 1 such that

$$\|u_n\|^{\frac{2}{1-\beta}+\epsilon} \leq \frac{\alpha_\beta^*}{\alpha_0} - \delta, \quad \text{for all } n \geq 1, \tag{5.5}$$

and

$$m(\alpha_0 + \epsilon)\left(\frac{\alpha_\beta^*}{\alpha_0} - \delta\right) < \alpha_\beta^*. \tag{5.6}$$

From assumption (H_5) there exists a positive constant C such that

$$\left|f(x,s)\right| \leq C \exp\left((\alpha_0 + \epsilon)|s|^{\frac{2}{1-\beta}+h(|x|)}\right), \quad \text{for all } (x,s) \in B_1 \times \mathbb{R}.$$

By Hölder and the above inequalities, we have

$$\int_{B_1} f(x,u_n)u_n\,dx \leq C\|u_n\|_{m'}\left(\int_{B_1} \exp\left(m(\alpha_0+\epsilon)|u_n|^{\frac{2}{1-\beta}+h(|x|)}\right)dx\right)^{1/m}. \tag{5.7}$$

Since h is continuous and $h(0) = 0$, there exists $r_0 > 0$ such that

$$h\left(|x|\right) < \epsilon, \quad \text{for all } |x| \leq r_0.$$

Using (5.5), (5.6), and Theorem 1.3, we obtain $C_1 > 0$ such that

$$\begin{aligned}
&\int_{B_{r_0}} \exp\left(m(\alpha_0+\epsilon)|u_n|^{\frac{2}{1-\beta}+h(|x|)}\right)dx \\
&\quad\leq \int_{B_{r_0}} \exp\left[m(\alpha_0+\epsilon)\|u_n\|^{\frac{2}{1-\beta}+h(|x|)}\left(\frac{|u_n|}{\|u_n\|}\right)^{\frac{2}{1-\beta}+h(|x|)}\right]dx \\
&\quad\leq \int_{B_{r_0}} \exp(m(\alpha_0+\epsilon)\|u_n\|^{\frac{2}{1-\beta}+\epsilon}\left(\frac{|u_n|}{\|u_n\|}\right)^{\frac{2}{1-\beta}+h(|x|)}]\,dx \\
&\quad\leq \int_{B_{r_0}} \exp\left[\alpha_\beta^*\left(\frac{|u_n|}{\|u_n\|}\right)^{\frac{2}{1-\beta}+h(|x|)}\right]dx \leq C_1.
\end{aligned} \tag{5.8}$$

According to (2.2), we have $|u(x)| \leq 1$ for $r_1 \leq |x| < 1$. Thus, we can find $C_2 > 0$ such that

$$\int_{B_1\setminus B_{r_1}} \exp\left(m(\alpha_0+\epsilon)|u|^{\frac{2}{1-\beta}+h(|x|)}\right)dx \leq \int_{B_1\setminus B_{r_1}} \exp(m(\alpha_0+\epsilon)\,dx = C_2. \tag{5.9}$$

On the other hand, using the boundedness of $(\|u_n\|)$ and Lemma 2.2, we have

$$\left|u_n(x)\right| \le M_0, \quad \text{for all } r_0 \le |x| \le r_1 \text{ and } n \ge 1.$$

By the continuity of h, we can find $C_3 > 0$ such that

$$\int_{B_{r_1}\setminus B_{r_0}} \exp\bigl(m(\alpha_0 + \epsilon)|u_n|^{\frac{2}{1-\beta}+h(|x|)}\bigr)\,dx \le C_3. \tag{5.10}$$

Replacing (5.8), (5.9), and (5.10) in (5.7), we obtain

$$\int_{B_1} f(x,u_n)u_n\,dx \le C\|u_n\|_{m'}.$$

By (5.2), we get

$$\int_{B_R} f(x,u_n)u_n\,dx \to 0, \quad \text{as } n \to +\infty. \tag{5.11}$$

Using the fact that $(\|u_n\|)$ is bounded and $\|J'(u_n)\|_{E^*} \to 0$, we obtain $C > 0$ such that

$$\left|J'(u_n)u_n\right| \le \left\|J'(u_n)\right\|_{E^*}\|u_n\| \to 0, \quad \text{as } n \to +\infty. \tag{5.12}$$

Since,

$$J'(u_n)u_n = \|u_n\|^2 - \int_{B_1} f(x,u_n)u_n\,dx.$$

By (5.11) and (5.12), we have

$$\|u_n\|^2 = J'(u_n)u_n + \int_{B_1} f(x,u_n)u_n\,dx \to 0, \quad \text{as } n \to +\infty.$$

From (5.4), we have $\|u_n\|^2 \to 2d$. Hence, $d = 0$, which represents a contradiction with (4.2). Thus, u is a nontrivial critical point of J. Therefore, u is a nontrivial weak solution of the problem (1.9). This completes the proof.

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