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(ω, c)-periodic solutions for a class of fractional integrodifferential equations

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Abstract

In this paper we investigate the following fractional order in time integrodifferential problem

$$\mathbb{D}_t^\alpha u(t) + Au(t) = f(t, u(t)) + \int_{-\infty}^t k(t-s)g(s, u(s))\, ds, \quad t \in \mathbb{R}.$$

Here, $\mathbb{D}_t^\alpha$ is the Caputo derivative. We obtain results on the existence and uniqueness of (ω, c)-periodic mild solutions assuming that $-A$ generates an analytic semigroup on a Banach space X and f, g, and k satisfy suitable conditions. Finally, an interesting example that fits our framework is given.

MSC: 35R11; 45K05; 34G20; 47D06

Keywords: (ω, c)-periodic mild solutions; Fractional integrodifferential equations; Nonlocal Cauchy problem; Fractional powers

1 Introduction

The aim of this paper is to investigate the existence of (ω, c)-periodic mild solutions for a class of fractional integrodifferential equations in Banach spaces. More precisely, let X be a Banach space. Our objective is to study the following problem

$$\mathbb{D}_t^\alpha u(t) + Au(t) = f(t, u(t)) + (Ku)(t), \quad t \in \mathbb{R}. \tag{1.1}$$

In (1.1), $0 < \alpha \le 1$, $\mathbb{D}_t^\alpha$ denotes the Caputo fractional derivative in the t variable that is defined by

$$\mathbb{D}_t^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u'(\tau)\, d\tau,$$

where $-A$ generates an analytic semigroup $S(t)$ in X, and f, g are continuous functions from $\mathbb{R} \times X$ to X, and

$$(Ku)(t) := \int_{-\infty}^t k(t-s)g(s, u(s))\, ds,$$

where k is a continuous function from $\mathbb{R}^+$ to $\mathbb{R}$.

In many areas of science and technology, the theory of fractional differential equations and their applications is of significant importance because certain situations do not fit into classical models, see [18, 25, 26] and the references therein.

Recently, Alvarez et al. presented the concept of vector-valued (ω, c)-periodic solutions and its properties in [6]. Moreover, they proved the existence and uniqueness of (ω, c)-periodic mild solutions to the problem (1.1) with $K = 0$. Then, several authors have studied related problems, see, for example, [1, 4, 5, 7, 10–15, 17, 22, 23, 27]. Also, there exist various generalizations of this kind of functions and applications to real-life problems [2, 3, 20, 21].

The problem of the existence and uniqueness of a pseudoalmost-periodic PC-mild solution for

$$\mathbb{D}_t^{\alpha} u(t) + Au(t) = f\big(t, u(t)\big) + \int_{-\infty}^{t} k(t-s) g\big(s, u(s)\big)\, ds + \sum_{j=-\infty}^{\infty} G_j\big(u(t)\big)\delta(t - t_j), \quad t \in \mathbb{R},$$

where G_j are continuous impulsive operators, $\delta(\cdot)$ is the Dirac delta function, and τ_j are a sequence in $\mathbb{Z}$ was investigated by Xia in [29] for $0 < \alpha < 1$, and by Gu and Li in [19] for $1 < \alpha < 2$. The existence of almost-periodic mild solutions for the case without impulsive effects was studied in [8].

It is worth mentioning that not much seems to be known about (ω, c)-periodic mild solutions for the integrodifferential equation (1.1). This is precisely our aim in this article.

We succeed in solving this open problem using Banach fixed-point arguments and the fractional powers of operators to derive some sufficient conditions guaranteeing the existence and uniqueness of (ω, c)-periodic mild solutions to (1.1).

The paper is structured as follows. In Sect. 2, we recall the definition of (ω, c)-periodic functions, the fractional power of an operator, and the definition of Mittag–Leffler functions and their properties that will be used throughout the manuscript. In Sect. 3, we investigate the main problem where we obtain a novel regularity result related to (ω, c)-periodic mild solutions of (1.1). Finally, an interesting example is given in Sect. 4.

2 Preliminaries

Throughout this paper, $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$, X will denote a Banach space with norm $\|\cdot\|_X$ and we will denote the set of continuous functions on $\mathbb{R}$ by

$$C(\mathbb{R}, X) := \{f : \mathbb{R} \to X : f \text{ is continuous}\},$$

and the set of continuous functions on $\mathbb{R} \times X$ by

$$C(\mathbb{R} \times X, X) := \{f : \mathbb{R} \times X \to X : f \text{ is continuous}\}.$$

We recall that a function $f \in C(\mathbb{R}, X)$ is said to be (ω, c)-periodic if $f(t + \omega) = cf(t)$ for all $t \in \mathbb{R}$, see [6]. The collection of those functions with the same c-period ω will be denoted by $P_{\omega c}(\mathbb{R}, X)$. Also, in the same article, it was proved that $P_{\omega c}(\mathbb{R}, X)$ is a Banach space with the norm

$$\|f\|_{\omega c} := \sup_{t \in [0,\omega]} \big\| |c|^{\wedge}(-t) f(t) \big\|.$$

Definition 2.1 ([28, Sect. 2.6]) Assume that $-A$ generates an analytic semigroup $\{S(t)\}_{t\geq 0}$ in a Banach space X and $0 \in \rho(A)$. For any $\beta > 0$, we define the fractional power $A^{-\beta}$ of the operator A by

$$A^{-\beta} := \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} S(t)\, dt.$$

We further define $A^{-0} := I$.

Lemma 2.2 ([28, Lemma 6.3]) *Let the operator $-A$ be an infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ in the Banach space X and $0 \in \rho(A)$. There exists a constant C_β such that*

$$\left\|A^{-\beta} x\right\|_X \leq C_\beta \|x\|_X, \quad \textit{for all } x \in X,$$

where $0 \leq \beta \leq 1$.

Theorem 2.3 ([28, Theorem 6.13]) *Let $-A$ be an infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$. If $0 \in \rho(A)$, then*

1. *$S(t) : X \to D(A^\beta)$ for all $t > 0$ and $\beta \geq 0$;*
2. *For all $x \in D(A^\beta)$, it follows that $S(t)A^\beta x = A^\beta S(t)x$;*
3. *For all $t > 0$, the operator $A^\beta S(t)$ is bounded and*

$$\left\|A^\beta S(t)\right\|_{\mathcal{L}(X)} \leq M_\beta t^{-\beta} e^{-\lambda t}, \quad M_\beta > 0, \lambda > 0,$$

where M_β is a positive constant and $\lambda > 0$ satisfies that $-A + \lambda I$ remains the infinitesimal generator of the analytic semigroup $S(t)$.

4. *For $0 < \beta \leq 1$ and $x \in D(A^\beta)$, there exists $C_\beta > 0$ such that*

$$\left\|S(t)x - x\right\|_X \leq C_\beta t^\beta \left\|A^\beta x\right\|_X.$$

Theorem 2.4 ([28]) *The space $X_\beta := D(A^\beta) \subset X$ with norm $\|x\|_\beta := \|A^\beta x\|_X$ is a Banach space.*

We recall that the Mittag–Leffler-type function (or the two-parameter Mittag–Leffler function) is given by

$$E_{\alpha,\beta}(t) = \sum_{n=0}^\infty \frac{t^n}{\Gamma(\alpha n + \beta)}, \quad (\alpha > 0, \beta \in \mathbb{C}).$$

When $\beta = 1$, we write simply $E_\alpha(t)$ instead of $E_{\alpha,1}(t)$. For more details about the Mittag–Leffler function, the reader may want to consult [18].

Proposition 2.5 ([25]) *Let $0 < \alpha < 1$. If $\theta \geq 0$, the following properties are satisfied:*

(a)

$$M_\alpha(\theta) \geq 0.$$

(b)

$$\int_0^\infty \theta^n M_\alpha(\theta)\,d\theta = \frac{\Gamma(n+1)}{\Gamma(\alpha n+1)}, \quad n \geq -1.$$

(c)

$$\int_0^\infty M_\alpha(\theta)e^{-t\theta}\,d\theta = E_\alpha(-t).$$

Lemma 2.6 ([25]) *Let $0 < \alpha < 1$. If $-A$ is an infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ on X, $0 \in \rho(A)$ and $x \in X$, then*

$$E_\alpha\bigl(-t^\alpha A\bigr)x = \int_0^\infty M_\alpha(\theta)S\bigl(\theta t^\alpha\bigr)x\,d\theta, \quad t \geq 0$$

and

$$E_{\alpha,\alpha}\bigl(-t^\alpha A\bigr)x = \int_0^\infty \alpha\theta M_\alpha(\theta)S\bigl(\theta t^\alpha\bigr)x\,d\theta, \quad t \geq 0.$$

Theorem 2.7 ([9]) *Let $\alpha, \beta \in (0,1)$. If $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ and $0 \in \rho(A)$, there exists a constant M_E such that*

$$\bigl\|E_\alpha\bigl(-t^\alpha A\bigr)x\bigr\|_\beta \leq M_E t^{-\alpha\beta}\|x\|_X \quad \textit{and} \quad \bigl\|E_{\alpha,\alpha}\bigl(-t^\alpha A\bigr)x\bigr\|_\beta \leq M_E t^{-\alpha\beta}\|x\|_X$$

for all $t > 0$.

Lemma 2.8 ([25]) *The operators $E_{\alpha,\alpha}(-t^\alpha A)$ and $E_\alpha(-t^\alpha A)$ are strongly continuous, which means that for all $x \in X$ and $s, t > 0$, we have that*

$$\bigl\|E_{\alpha,\alpha}\bigl(-t^\alpha A\bigr)x - E_{\alpha,\alpha}\bigl(-s^\alpha A\bigr)x\bigr\|_X \to 0 \quad \textit{and} \quad \bigl\|E_\alpha\bigl(-t^\alpha A\bigr)x - E_\alpha\bigl(-s^\alpha A\bigr)x\bigr\|_X \to 0$$

when $s \to t$.

Proposition 2.9 ([26]) *Let $0 < \alpha < 1$, $t > 0$. There are two asymptotic representations set up for $E_\alpha(-t^\alpha)$:*

$$E_\alpha\bigl(-t^\alpha\bigr) \sim \begin{cases} E_\alpha^0(-t^\alpha) := \exp(-\frac{t^\alpha}{\Gamma(1+\alpha)}), & t \to 0; \\ E_\alpha^\infty(-t^\alpha) := \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = \frac{\sin(\alpha\pi)}{\pi}\frac{\Gamma(\alpha)}{t^\alpha}, & t \to \infty. \end{cases}$$

3 (ω, c)-periodic mild solutions

In this section we prove the main result of this article. Under suitable conditions, we show the existence and uniqueness of (ω, c)-periodic mild solutions for (1.1).

Let us consider the following Cauchy problem

$$\begin{cases} \mathbb{D}_t^\alpha u(t) + Au(t) = f(t, u(t)) + (Ku)(t), & t > t_0, \\ u(t_0) = u_0, & t_0 \in \mathbb{R}, u_0 \in X, \end{cases} \tag{3.1}$$

where the $\mathbb{D}_t^\alpha$ denotes the fractional Caputo derivative, $0 < \alpha < 1$, $-A : D(-A) \subset X \to X$ generates an analytic semigroup $S(t)$ in a Banach space X, and f, g are continuous functions from $\mathbb{R} \times X$ to X and $(Ku)(t) := \int_{-\infty}^{t} k(t-s)g(s,u(s))\,ds$. Here, k is a continuous function from $\mathbb{R}^+$ to $\mathbb{R}$.

We assume the following:

(H1) $-A$ is an infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ such that $0 \in \rho(A)$ and

$$\left\|S(t)\right\|_X \leq Ce^{-\sigma t} \quad \text{for } t \geq 0,$$

where σ and C are positive constants.

(H2) $|k(t)| \leq C_k e^{-\eta t}$ for some positive constants C_k, η.

(H3) $f \in C(\mathbb{R} \times X_\beta, X_\beta)$ and there exists $(\omega, c) \in \mathbb{R}^+ \times (\mathbb{C} \setminus \{0\})$ such that $f(t+\omega, cx) = cf(t,x)$ for all $t \in \mathbb{R}$ and all $x \in X_\beta$. Also, there exists a positive constant L_f such that

$$\left\|f(t,u) - f(t,v)\right\|_X \leq L_f \|u - v\|_\beta, \quad t \in \mathbb{R}, u, v \in X_\beta.$$

(H4) $g \in C(\mathbb{R} \times X_\beta, X_\beta)$ and $g(t+\omega, cx) = cg(t,x)$ (where ω and c are the same as given in (H3)) for all $t \in \mathbb{R}$ and all $x \in X_\beta$. Also, there exists a positive constant L_g such that

$$\left\|g(t,u) - g(t,v)\right\|_X \leq L_g \|u - v\|_\beta, \quad t \in \mathbb{R}, u, v \in X_\beta.$$

The next definition is similar to [16, Definition 3.1] and [29, Definition 3.1].

Definition 3.1 A mild solution of (3.1) is a continuous function u from $\mathbb{R}$ to X that satisfies the following integral equation:

$$u(t) = E_\alpha\big(-(t-t_0)^\alpha A\big)u_0 + \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)\big(f\big(s,u(s)\big) + (Ku)(s)\big)\,ds. \quad (3.2)$$

Proposition 3.2 *Suppose that* (H1) *holds. If u is a mild solution of* (3.1), *then*

$$\lim_{t_0 \to -\infty} u(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)\big(f\big(s,u(s)\big) + (Ku)(s)\big)\,ds. \quad (3.3)$$

Proof According to the definition of an improper integral, we have

$$\begin{aligned}
&\lim_{t_0 \to -\infty}\left(\int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)\big(f\big(s,u(s)\big) + (Ku)(s)\big)\,ds\right) \\
&\quad = \int_{-\infty}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)\big(f\big(s,u(s)\big) + (Ku)(s)\big)\,ds.
\end{aligned} \quad (3.4)$$

On the other hand, we will prove that $\lim_{t_0\to-\infty} E_\alpha(-(t-t_0)^\alpha A)u_0 = 0$. In fact, by Proposition 2.5 and (H1), we obtain

$$\begin{aligned}\left\| E_\alpha\big(-(t-t_0)^\alpha A\big)u_0\right\|_X &= \left\| \int_0^\infty M_\alpha(\theta) S\big((t-t_0)^\alpha\theta\big)u_0\,d\theta\right\|_X\\ &\le \int_0^\infty M_\alpha(\theta) Ce^{-\sigma(t-t_0)^\alpha\theta}\|u_0\|_X\,d\theta\\ &\le C\|u_0\|_X E_\alpha\big(-\big(\sigma^{1/\alpha}(t-t_0)\big)^\alpha\big).\end{aligned}$$

Now, by Proposition 2.9, we obtain

$$\left\| E_\alpha\big(-(t-t_0)^\alpha A\big)u_0\right\|_X \le C\|u_0\|_X\left(\frac{\sin(\alpha\pi)}{\pi}\cdot\frac{\Gamma(\alpha)}{\sigma(t-t_0)^\alpha}\right)\xrightarrow[t_0\to-\infty]{} 0,$$

which shows that $\lim_{t_0\to-\infty} E_\alpha(-(t-t_0)^\alpha A)u_0 = 0$. Using this fact, together with (3.2) and (3.4), we obtain the desired result. □

The previous proposition motivates the following definition.

Definition 3.3 A mild solution of (1.1) is a continuous function u from $\mathbb{R}$ to X that satisfies the following integral equation:

$$u(t) = \int_{-\infty}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)\big(f\big(s,u(s)\big) + (Ku)(s)\big)\,ds, \tag{3.5}$$

provided that (H1) holds.

The next results are crucial for the proof of our main result.

Lemma 3.4 *If* (H3) *and* (H4) *are satisfied and* $u \in P_{\omega c}(\mathbb{R}, X_\beta)$, *then* $f_u = f(\cdot, u(\cdot))$, $g_u = g(\cdot, u(\cdot))$ *lies in* $P_{\omega c}(\mathbb{R}, X_\beta)$.

Proof Let $t \in \mathbb{R}$. Then,

$$f_u(t+\omega) = f\big(t+\omega, u(t+\omega)\big) = f\big(t+\omega, cu(t)\big) = cf\big(t, u(t)\big) = cf_u(t).$$

By [6, Theorem 2.11] we have that $f_u \in P_{\omega c}(\mathbb{R}, X_\beta)$. Analogously, we can prove the claim for g_u. □

Lemma 3.5 *Suppose that* (H2)–(H4) *are satisfied. If* $u \in P_{\omega c}(\mathbb{R}, X_\beta)$, *then*

$$h(\cdot) := f\big(\cdot, u(\cdot)\big) + (Ku)(\cdot) \in P_{\omega c}(\mathbb{R}, X_\beta). \tag{3.6}$$

Proof First, we will show that $h \in C(\mathbb{R}, X_\beta)$. In order to prove that h is continuous for each $t \in \mathbb{R}$, we claim that $\lim_{\rho \to 0^+} \|h(t+\rho) - h(t)\|_\beta = 0$. Indeed, let $\rho > 0$. Then,

$$\begin{aligned} \bigl\|h(t+\rho) - h(t)\bigr\|_\beta &= \bigl\|f\bigl(t+\rho, u(t+\rho)\bigr) + (Ku)(t+\rho) - f\bigl(t, u(t)\bigr) - (Ku)(t)\bigr\|_\beta \\ &\le \bigl\|f\bigl(t+\rho, u(t+\rho)\bigr) - f\bigl(t, u(t)\bigr)\bigr\|_\beta \\ &\quad + \underbrace{\int_t^{t+\rho} \bigl\|k(t+\rho-s) g\bigl(s, u(s)\bigr)\bigr\|_\beta \, ds}_{I} \\ &\quad + \underbrace{\int_{-\infty}^t \bigl\|\bigl(k(t+\rho-s) - k(t-s)\bigr) g\bigl(s, u(s)\bigr)\bigr)\bigr\|_\beta \, ds}_{II}. \end{aligned}$$

Note that by (H3), we have $\|f(t+\rho, u(t+\rho)) - f(t, u(t))\|_\beta \xrightarrow[\rho \to 0^+]{} 0$. Now, we estimate I and II separately. By (H2), (H4), and Lemma 3.4, we have

$$\begin{aligned} I &= \int_t^{t+\rho} \bigl\|k(t+\rho-s) g\bigl(s, u(s)\bigr)\bigr\|_\beta \, ds \\ &\le C_k \|g_u\|_{\omega c} e^{-\eta(t+\rho)} \int_t^{t+\rho} e^{s(\frac{\ln|c| + \eta\omega}{\omega})} \, ds \\ &\le C_k \|g_u\|_{\omega c} \left(\frac{\omega}{\ln|c| + \eta\omega}\right) \bigl(e^{(t+\rho)\frac{\ln|c|}{\omega}} - e^{t\frac{\ln|c|}{\omega} - \rho\eta}\bigr) \xrightarrow[\rho \to 0^+]{} 0. \end{aligned}$$

On the other hand, by (H4) and Lemma 3.4, we obtain

$$\begin{aligned} II &= \int_{-\infty}^t \bigl\|\bigl(k(t+\rho-s) - k(t-s)\bigr) g\bigl(s, u(s)\bigr)\bigr)\bigr\|_\beta \, ds \\ &\le \|g_u\|_{\omega c} \int_{-\infty}^t \bigl|k(t+\rho-s) e^{s\frac{\ln|c|}{\omega}} - k(t-s) e^{s\frac{\ln|c|}{\omega}}\bigr| \, ds. \end{aligned}$$

Since $k \in C(\mathbb{R}^+, \mathbb{R})$ and $s < t + \rho$ for $\rho > 0$, we have that

$$s \mapsto k(t+\rho-s) e^{s\frac{\ln|c|}{\omega}} : (-\infty, t+\rho) \to \mathbb{R} \tag{3.7}$$

is continuous. In particular,

$$\bigl|k(t+\rho-s) e^{s\frac{\ln|c|}{\omega}} - k(t-s) e^{s\frac{\ln|c|}{\omega}}\bigr| \xrightarrow[\rho \to 0^+]{} 0.$$

Moreover, by (H2)

$$\bigl|k(t+\rho-s) e^{s\frac{\ln|c|}{\omega}} - k(t-s) e^{s\frac{\ln|c|}{\omega}}\bigr| \le C_k \bigl(e^{-\eta(t+\rho)} + e^{-\eta t}\bigr) e^{s(\frac{\ln|c|}{\omega} + \eta)}.$$

Due to the facts that $\rho > 0$ and $\eta > 0$, we have

$$e^{-\eta(t+\rho)} < e^{-\eta t}.$$

The above implies that

$$C_k\left(e^{-\eta(t+\rho)}+e^{-\eta t}\right)e^{s(\frac{\ln|c|}{\omega}+\eta)}<2C_k e^{-\eta t}e^{s(\frac{\ln|c|}{\omega}+\eta)},$$

and therefore,

$$\left|k(t+\rho-s)e^{s\frac{\ln|c|}{\omega}}-k(t-s)e^{s\frac{\ln|c|}{\omega}}\right|\le 2C_k\left(e^{-\eta t}\right)e^{s(\frac{\ln|c|}{\omega}+\eta)}.$$

Also, the function $s\mapsto 2C_k e^{-\eta t}e^{s(\frac{\ln|c|+\eta\omega}{\omega})}$ is integrable in $(-\infty,t)$, since

$$\int_{-\infty}^{t}2C_k e^{-\eta t}e^{s(\frac{\ln|c|+\eta\omega}{\omega})}\,ds=2C_k\left(\frac{\omega}{\ln|c|+\eta\omega}\right)e^{\frac{t\ln|c|}{\omega}}<\infty.$$

Hence, the criterion of comparison of improper integrals guarantees that

$$s\mapsto\left|k(t+\rho-s)e^{s\frac{\ln|c|}{\omega}}-k(t-s)e^{s\frac{\ln|c|}{\omega}}\right|$$

is integrable in $(-\infty,t)$. By virtue of the Dominated Convergence Theorem, it follows that

$$II\le\|g_u\|_{\omega c}\int_{-\infty}^{t}\left|k(t+\rho-s)e^{s\frac{\ln|c|}{\omega}}-k(t-s)e^{s\frac{\ln|c|}{\omega}}\right|\,ds\xrightarrow[\rho\to0^+]{}0,$$

obtaining the claim.

Analogously, we can show that $\lim_{\rho\to0^-}\|h(t+\rho)-h(t)\|_\beta=0$.

Now, we will prove that $h(t+\omega)=ch(t)$ for all $t\in\mathbb{R}$. In fact, since $u\in P_{\omega c}(\mathbb{R},X)$, by the definition of (ω,c)-periodicity, (H3), and (H4), we obtain

$$\begin{aligned}h(t+\omega)&=f\big(t+\omega,u(t+\omega)\big)+(Ku)(t+\omega)\\&=f\big(t+\omega,cu(t)\big)+\int_{-\infty}^{t}k(t-r)g\big(r+\omega,cu(r)\big)\,dr\\&=cf\big(t,u(t)\big)+\int_{-\infty}^{t}k(t-r)cg\big(r,u(r)\big)\,dr=ch(t).\end{aligned}$$

Consequently, $h\in P_{\omega c}(\mathbb{R},X_\beta)$. □

Lemma 3.6 *Suppose that* (H1)–(H4) *are satisfied. If* $u\in P_{\omega c}(\mathbb{R},X_\beta)$, *then*

$$(\Theta u)(t)=\int_{-\infty}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-(t-s)^\alpha A\right)\left(f\big(s,u(s)\big)+(Ku)(s)\right)ds \tag{3.8}$$

lies in $P_{\omega c}(\mathbb{R},X_\beta)$.

Proof Define $h(s):=f(s,u(s))+(Ku)(s)$ for all $s\in\mathbb{R}$. According to Lemma 3.5, we have $h\in P_{\omega c}(\mathbb{R},X_\beta)$.

First, we will show that $(\Theta u) \in C(\mathbb{R}, X_\beta)$. For this, we claim that $\lim_{\xi\to 0^+} \|(\Theta u)(t+\xi) - (\Theta u)(t)\|_\beta = 0$. Indeed, let $\xi > 0$. Then,

$$\begin{aligned}
&\left\|(\Theta u)(t+\xi) - (\Theta u)(t)\right\|_\beta \\
&= \left\| \int_{-\infty}^{t+\xi} (t+\xi-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t+\xi-s)^\alpha A\big)h(s)\,ds \right. \\
&\quad \left. - \int_{-\infty}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)h(s)\,ds \right\|_\beta \\
&\le \underbrace{\int_{-\infty}^{t} \left\|(t+\xi-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t+\xi-s)^\alpha A\big)h(s) - (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)h(s)\right\|_\beta ds}_{I} \\
&\quad + \underbrace{\int_{t}^{t+\xi} (t+\xi-s)^{\alpha-1} \left\|E_{\alpha,\alpha}\big(-(t+\xi-s)^\alpha A\big)h(s)\right\|_\beta ds}_{II}.
\end{aligned}$$

We will estimate I and II. Indeed, for $s \in (-\infty, t)$, by Theorem 2.7 and Lemma 3.5, we have

$$\begin{aligned}
&\left\|(t+\xi-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t+\xi-s)^\alpha A\big)h(s) - (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)h(s)\right\|_\beta \\
&\quad \le \left\|(t+\xi-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t+\xi-s)^\alpha A\big)h(s) - (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t+\xi-s)^\alpha A\big)h(s)\right\|_\beta \\
&\qquad + \left\|(t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t+\xi-s)^\alpha A\big)h(s) - (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)h(s)\right\|_\beta \\
&\quad \le \left(\frac{M_E C_\beta \|h\|_{\omega c} e^{s\frac{\ln|c|}{\omega}}}{(t+\xi-s)^{\alpha\beta}}\right) \left| \left(\frac{1}{t+\xi-s}\right)^{1-\alpha} - \left(\frac{1}{t-s}\right)^{1-\alpha} \right| \\
&\qquad + (t-s)^{\alpha-1} \left\|E_{\alpha,\alpha}\big(-(t+\xi-s)^\alpha A\big)h(s) - E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)h(s)\right\|_\beta \xrightarrow[\xi\to 0^+]{} 0.
\end{aligned}$$

Therefore,

$$\left\|(t+\xi-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t+\xi-s)^\alpha A\big)h(s) - (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)h(s)\right\|_\beta \xrightarrow[\xi\to 0^+]{} 0.$$

Again, by Theorem 2.7 and Lemma 3.5, we obtain

$$\begin{aligned}
&\left\|(t+\xi-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t+\xi-s)^\alpha A\big)h(s) - (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)h(s)\right\|_\beta \\
&\quad \le \left\|(t+\xi-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t+\xi-s)^\alpha A\big)h(s)\right\|_\beta + \left\|(t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)h(s)\right\|_\beta \\
&\quad \le M_E C_\beta \|h\|_{\omega c} \left(\frac{e^{s\frac{\ln|c|}{\omega}}}{(t+\xi-s)^{1-\alpha+\alpha\beta}} + \frac{e^{s\frac{\ln|c|}{\omega}}}{(t-s)^{1-\alpha+\alpha\beta}} \right), \quad s \in (-\infty, t).
\end{aligned}$$

Due to $\xi > 0$ and $0 < 1-\alpha+\alpha\beta < 1$, we have

$$M_E C_\beta \|h\|_{\omega c} \left(\frac{e^{s\frac{\ln|c|}{\omega}}}{(t+\xi-s)^{1-\alpha+\alpha\beta}} + \frac{e^{s\frac{\ln|c|}{\omega}}}{(t-s)^{1-\alpha+\alpha\beta}} \right) \le M_E C_\beta \|h\|_{\omega c} \left(\frac{2e^{s\frac{\ln|c|}{\omega}}}{(t-s)^{1-\alpha+\alpha\beta}} \right),$$

and therefore,

$$\begin{aligned}&\big\|(t+\xi-s)^{\alpha-1}E_{\alpha,\alpha}\big(-(t+\xi-s)^{\alpha}A\big)h(s)-(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-(t-s)^{\alpha}A\big)h(s)\big\|_{\beta}\\&\quad\le 2M_E C_\beta\|h\|_{\omega c}\left(\frac{e^{s\frac{\ln|c|}{\omega}}}{(t-s)^{1-\alpha+\alpha\beta}}\right),\quad s\in(-\infty,t).\end{aligned}$$

In addition, the function $s\mapsto 2M_E C_\beta\|h\|_{\omega c}(\frac{e^{s\frac{\ln|c|}{\omega}}}{(t-s)^{1-\alpha+\alpha\beta}})$ is integrable in $(-\infty,t)$, since

$$\int_{-\infty}^{t}2M_E C_\beta\|h\|_{\omega c}\left(\frac{e^{s\frac{\ln|c|}{\omega}}}{(t-s)^{1-\alpha+\alpha\beta}}\right)ds=2M_E C_\beta\|h\|_{\omega c}e^{t\frac{\ln|c|}{\omega}}\left(\frac{\omega}{\ln|c|}\right)^{\alpha(1-\beta)}\Gamma\big(\alpha(1-\beta)\big).$$

Hence, the criterion of comparison of improper integrals guarantees that

$$s\mapsto\big\|(t+\xi-s)^{\alpha-1}E_{\alpha,\alpha}\big(-(t+\xi-s)^{\alpha}A\big)h(s)-(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-(t-s)^{\alpha}A\big)h(s)\big\|_{\beta}$$

is integrable in $(-\infty,t)$. Thus, by the Dominated Convergence Theorem, it follows that

$$\begin{aligned}I&=\int_{-\infty}^{t}\big\|(t+\xi-s)^{\alpha-1}E_{\alpha,\alpha}\big(-(t+\xi-s)^{\alpha}A\big)h(s)\\&\qquad-(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-(t-s)^{\alpha}A\big)h(s)\big\|_{\beta}\,ds\\&\xrightarrow[\xi\to0^+]{}0.\end{aligned}$$

On the other hand, using similar arguments to those in the estimates of I, we obtain

$$\begin{aligned}II&=\int_{t}^{t+\xi}(t+\xi-s)^{\alpha-1}\big\|E_{\alpha,\alpha}\big(-(t+\xi-s)^{\alpha}A\big)h(s)\big\|_{\beta}\,ds\\&\le M_E C_\beta\|h\|_{\omega c}\int_{t}^{t+\xi}(t+\xi-s)^{\alpha-\alpha\beta-1}e^{s\frac{\ln|c|}{\omega}}\,ds\\&\le M_E C_\beta\|h\|_{\omega c}e^{(t+\xi)\frac{\ln|c|}{\omega}}\int_{0}^{\xi}r^{\alpha-\alpha\beta-1}e^{-r\frac{\ln|c|}{\omega}}\,dr.\end{aligned}$$

Note that, using a change of variable and the definition of the incomplete Gamma function γ, we have

$$\begin{aligned}\int_{0}^{\xi}r^{\alpha-\alpha\beta-1}e^{-r\frac{\ln|c|}{\omega}}\,dr&=\int_{0}^{\frac{\ln|c|}{\omega}\xi}\left(s\frac{\omega}{\ln|c|}\right)^{\alpha(1-\beta)-1}e^{-s}\frac{\omega}{\ln|c|}\,ds\\&=\left(\frac{\ln|c|}{\omega}\right)^{\alpha(\beta-1)}\gamma\left(\alpha(1-\beta),\frac{\ln|c|}{\omega}\xi\right).\end{aligned}$$

Thus,

$$II\le M_E C_\beta\|h\|_{\omega c}e^{(t+\xi)\frac{\ln|c|}{\omega}}\left(\frac{\ln|c|}{\omega}\right)^{\alpha(\beta-1)}\gamma\left(\alpha(1-\beta),\frac{\ln|c|}{\omega}\xi\right)\xrightarrow[\xi\to0^+]{}0.$$

Therefore, $\|(\Theta u)(t+\xi)-(\Theta u)(t)\|_\beta\to0$ when $\xi\to0^+$, proving the claim. In a similar way, we can show that $\lim_{\xi\to0^-}\|(\Theta u)(t+\xi)-(\Theta u)(t)\|_\beta=0$.

Now, we will show that $(\Theta u)(t+\omega) = c(\Theta u)(t)$ for all $t \in \mathbb{R}$. Indeed, since $h \in P_{\omega c}(\mathbb{R}, X_\beta)$, by the definition of (ω, c)-periodicity, we have

$$\begin{aligned}(\Theta u)(t+\omega) &= \int_{-\infty}^{t} (t-r)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-r)^\alpha A\big) h(r+\omega)\,dr \\ &= c\int_{-\infty}^{t} (t-r)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-r)^\alpha A\big) h(r)\,dr = c(\Theta u)(t).\end{aligned}$$

Hence, we deduce that $(\Theta u) \in P_{\omega c}(\mathbb{R}, X_\beta)$. □

Theorem 3.7 *Suppose that* (H1)–(H4) *are satisfied and* $1 < |c| < e^{\eta\omega}$. *If* $\delta < 1$ *where*

$$\delta := M_E \left(\frac{\omega}{\ln|c|}\right)^{\alpha(1-\beta)} \Gamma\big(\alpha(1-\beta)\big)\left(L_f + \frac{C_k L_g \omega}{\eta\omega + \ln|c|}\right), \tag{3.9}$$

then (1.1) *has a unique mild solution* $u \in P_{\omega c}(\mathbb{R}, X_\beta)$.

Proof Let us define the operator $\Theta : P_{\omega c}(\mathbb{R}, X_\beta) \to P_{\omega c}(\mathbb{R}, X_\beta)$ given by

$$(\Theta u)(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)\big(f\big(s,u(s)\big) + (Ku)(s)\big)\,ds.$$

According to Lemma 3.6, we have $\Theta u \in P_{\omega c}(\mathbb{R}, X_\beta)$ for all $u \in P_{\omega c}(\mathbb{R}, X_\beta)$.

Let us see that Θ is a contraction. In fact, let $u, v \in P_{\omega c}(\mathbb{R}, X_\beta)$. By (H3) and Theorem 2.7, we have

$$\begin{aligned}&\left\| |c|^\wedge(-t)\int_{-\infty}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)\big(f\big(s,u(s)\big) - f\big(s,v(s)\big)\big)\,ds\right\|_\beta \\ &\quad\le M_E L_f \|u-v\|_{\omega c} \int_{-\infty}^{t} (t-s)^{\alpha(1-\beta)-1} |c|^\wedge\big(-(t-s)\big)\,ds \\ &\quad\le M_E L_f \left(\frac{\omega}{\ln|c|}\right)^{\alpha(1-\beta)} \Gamma\big(\alpha(1-\beta)\big)\|u-v\|_{\omega c}.\end{aligned} \tag{3.10}$$

On the other hand, by (H2) and (H4), we obtain

$$\begin{aligned}\big\|(Ku)(s) - (Kv)(s)\big\|_X &\le \int_{-\infty}^{s} \big|k(s-r)\big|\,\big\|g\big(r,u(r)\big) - g\big(r,v(r)\big)\big\|_X\,dr \\ &\le C_k L_g \|u-v\|_{\omega c} e^{-\eta s}\left(\int_{-\infty}^{s} e^{(\eta + \frac{\ln|c|}{\omega})r}\,dr\right) \\ &\le C_k L_g \left(\frac{\omega}{\eta\omega + \ln|c|}\right)\|u-v\|_{\omega c}|c|^\wedge(s).\end{aligned}$$

Using this fact together with Theorem 2.7, we obtain

$$\begin{aligned}&\left\| |c|^\wedge(-t)\int_{-\infty}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)\big((Ku)(s) - (Kv)(s)\big)\,ds\right\|_\beta \\ &\quad\le M_E \int_{-\infty}^{t} |c|^\wedge(-t)(t-s)^{\alpha-1-\alpha\beta}\big\|(Ku)(s) - (Kv)(s)\big\|_X\,ds\end{aligned}$$

$$
\begin{aligned}
&\leq M_E \int_{-\infty}^{t} |c|^\wedge(-t)(t-s)^{\alpha-1-\alpha\beta}\left(C_k L_g\left(\frac{\omega}{\eta\omega+\ln|c|}\right)\|u-v\|_{\omega c}|c|^\wedge(s)\right) ds \\
&\leq M_E C_k L_g\left(\frac{\omega}{\eta\omega+\ln|c|}\right)\left(\frac{\omega}{\ln|c|}\right)^{\alpha(1-\beta)}\Gamma\big(\alpha(1-\beta)\big)\|u-v\|_{\omega c}.
\end{aligned}
\tag{3.11}
$$

Now, by (3.10) and (3.11), we have

$$
\begin{aligned}
&\big\|(\Theta u)(t)-(\Theta v)(t)\big\|_{\omega c} \\
&\leq \sup_{t\in[0,\omega]}\left(\left\| |c|^\wedge(-t)\int_{-\infty}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)\big(f\big(s,u(s)\big)-f\big(s,v(s)\big)\big)\,ds\right\|_\beta\right. \\
&\quad\left.+\left\| |c|^\wedge(-t)\int_{-\infty}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)\big((Ku)(s)-(Kv)(s)\big)\,ds\right\|_\beta\right) \\
&\leq \sup_{t\in[0,\omega]}\left(M_E L_f\left(\frac{\omega}{\ln|c|}\right)^{\alpha(1-\beta)}\Gamma\big(\alpha(1-\beta)\big)\|u-v\|_{\omega c}\right. \\
&\quad\left.+M_E C_k L_g\left(\frac{\omega}{\eta\omega+\ln|c|}\right)\left(\frac{\omega}{\ln|c|}\right)^{\alpha(1-\beta)}\Gamma\big(\alpha(1-\beta)\big)\|u-v\|_{\omega c}\right) \\
&\leq \left(M_E\left(\frac{\omega}{\ln|c|}\right)^{\alpha(1-\beta)}\Gamma\big(\alpha(1-\beta)\big)\left(L_f+\frac{C_k L_g\omega}{\eta\omega+\ln|c|}\right)\right)\|u-v\|_{\omega c} \\
&\leq \delta\|u-v\|_{\omega c}.
\end{aligned}
$$

Since $\delta < 1$, Θ is a contraction. Therefore, Banach's Fixed-Point Theorem guarantees the existence of a unique fixed point $u \in P_{\omega c}(\mathbb{R}, X_\beta)$ of the operator Θ, which satisfies

$$
(\Theta u)(t) = u(t) = \int_{-\infty}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-(t-s)^\alpha A\big)\big(f\big(s,u(s)\big)+(Ku)(s)\big)\,ds.
$$

This completes the proof of the theorem. □

4 An application

In this section we present an example that fits our framework.

Let $X = (L^2[0,1], \|\cdot\|_{L^2})$. Consider the following problem:

$$
\begin{cases}
\partial_t^\alpha w(t,x)+\partial_x^2 w(t,x) = a(t)\cos(b(t)w(t,x))+Kw(t,x), & t\in\mathbb{R}, x\in(0,1),\\
w(t,0)=w(t,1)=0, & t\in\mathbb{R},
\end{cases}
\tag{4.1}
$$

where $0 < \alpha < 1$, ∂_t^α denotes the Caputo fractional derivative with respect to t and

$$
Kw(t,x) = \int_{-\infty}^{t} k(t-s)\big(a(s)\sin\big(b(s)w(s,x)\big)\big)\,ds.
$$

The functions k and a, b will be specified later.

We define the linear operator $-A$ on X by

$$
\begin{aligned}
&D(-A) = \big\{u\in X : u, u' \in X \text{ are absolutely continuous}, u''\in X \text{ and } u(0)=u(1)=0\big\},\\
&-Au(x) = u''(x), \quad \forall x\in(0,1), u\in D(-A).
\end{aligned}
$$

It is well known that $-A$ is an infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ on X (see, for example, [24, Example 4.1.7] with a little modification). In addition, $-A$ has a discrete spectrum, namely, the eigenvalues $-\lambda_n = -n^2$, $n \in \mathbb{N}$. The associated normalized eigenfunctions are given by $e_n(x) = \sqrt{2}\sin(n\pi x)$, $n \in \mathbb{N}$. Moreover, the semigroup is

$$S(t)u(x) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle u, e_n\rangle_{L^2} e_n(x).$$

Also, $\|S(t)\|_{L^2} \leq e^{-\pi^2 t}$ for $t \geq 0$. This shows that (H1) holds. This in turn implies that the fractional powers of A can be defined as in Sect. 2. More precisely, since A has a compact resolvent, we have that

$$A^\beta u = \sum_{n=1}^{\infty}\langle u, e_n\rangle_{L^2} e_n \lambda_n^\beta = \sum_{n=1}^{\infty}\langle u, e_n\rangle_{L^2} e_n n^{2\beta},$$

with domain

$$\left\{u \in X : \sum_{n=1}^{\infty}\left|\langle u, e_n\rangle_{L^2}\right|^2 n^{4\beta} < \infty\right\}.$$

Now, let $k(t) = e^{-\pi^2 t}$. Then, $|e^{-\pi^2 t}| \leq (2/3)e^{-9t}$. Thus, (H2) holds with $C_k = \frac{2}{3}$ and $\eta = 9$.

Let $a \in P_{\omega,c}(\mathbb{R}, X_\beta)$ and $b \in P_{\omega,\frac{1}{c}}(\mathbb{R}, X_\beta)$ with $1 < |c| < e^{9\omega}$.

Let us define $f(t,x) = a(t)\cos(b(t)x)$ and $g(t,x) = a(t)\sin(b(t)x)$. Then, the problem (4.1) can be reformulated as (1.1) with A, k, f, and g defined as above.

Next, we will show that (H3) and (H4) hold. Indeed,

$$\begin{aligned} f(t+\omega, cx) &= a(t+\omega)\cos\bigl(b(t+\omega)cx\bigr) \\ &= ca(t)\cos\left(\frac{1}{c}b(t)cx\right) \\ &= ca(t)\cos\bigl(b(t)x\bigr) = cf(t,x). \end{aligned}$$

Since $a \in P_{\omega c}(\mathbb{R}, X_\beta)$, $b \in P_{\omega\frac{1}{c}}(\mathbb{R}, X_\beta)$, we have $f \in C(\mathbb{R} \times X_\beta, X_\beta)$. Also, for $x, y \in X_\beta$, we obtain

$$\begin{aligned} \bigl\|f(t,x) - f(t,y)\bigr\|_{L^2} &\leq \|a\|_{L^2}\bigl\|\cos\bigl(b(t)x\bigr) - \cos\bigl(b(t)y\bigr)\bigr\|_{L^2} \\ &\leq \|a\|_{L^2} 2\left\|\sin\left(\frac{b(t)x - b(t)y}{2}\right)\right\|_{L^2}\left\|\sin\left(\frac{b(t)x + b(t)y}{2}\right)\right\|_{L^2} \\ &\leq \|a\|_{L^2} 2\left\|\frac{b(t)x - b(t)y}{2}\right\|_{L^2}\cdot 1 \\ &\leq \|a\|_{L^2}\|b\|_{L^2}\|x-y\|_{L^2} \\ &\leq \|a\|_{L^2}\|b\|_{L^2} C_\beta\|x-y\|_\beta, \quad \forall t \in \mathbb{R}, x, y \in X_\beta, \end{aligned}$$

obtaining $(H3)$. The proof for (H4) is analogous. More precisely,

$$\bigl\|g(t,x) - g(t,y)\bigr\|_{L^2} \leq \|a\|_{L^2}\|b\|_{L^2} C_\beta\|x-y\|_\beta, \quad \forall t \in \mathbb{R}, x, y \in X_\beta.$$

From the estimated

$$\begin{aligned}\left\|A^{-\beta}x\right\|_{L^2} &\le \frac{\|x\|_{L^2}}{\Gamma(\beta)}\int_0^\infty\left(\frac{s}{\pi^2}\right)^{\beta-1}e^{-s}\frac{ds}{\pi^2}\\ &\le \frac{1}{\pi^{2\beta}}\|x\|_{L^2}, \quad x\in X_\beta,\end{aligned}$$

we see that the constant C_β can be chosen as $\frac{1}{\pi^{2\beta}}$ (see Lemma 2.2).

The constant M_β of Theorem 2.3 can be taken as $\frac{1}{\pi^{2(1-\beta)}}$. In fact, note that $\|S(t)x(\cdot)\|_{L^2}\le e^{-\pi^2 t}\|x\|_{L^2}$ for $t\ge 0$ and $\|AS(t)x(\cdot)\|_{L^2}\le t^{-1}e^{-\pi^2 t}\|x\|_{L^2}$ for $t>0$. Moreover, for $x\in X_\beta$, we have

$$\begin{aligned}\left\|A^{\beta}S(t)x\right\|_{L^2} &= \left\|A^{-(1-\beta)}AS(t)x\right\|_{L^2}\\ &\le \frac{1}{\Gamma(1-\beta)}\int_0^\infty s^{-\beta}(t+s)^{-1}e^{-\pi^2(t+s)}\|x\|_{L^2}\,ds\\ &\le \frac{1}{\pi^{2(1-\beta)}}t^{-\beta}e^{-\pi^2 t}\|x\|_{L^2}.\end{aligned}$$

Finally, the constant M_E of Theorem 2.7 can be taken as $M_E=\frac{1}{\pi^{2(1-\beta)}}(\frac{\Gamma(1-\beta)}{\Gamma(\alpha(1-\beta))})$. Indeed,

$$\begin{aligned}\left\|E_{\alpha,\alpha}\left(-t^\alpha A\right)x\right\|_\beta &= \left\|A^\beta\int_0^\infty \alpha\theta M_\alpha(\theta)S\left(\theta t^\alpha\right)x\,d\theta\right\|_{L^2}\\ &\le \int_0^\infty \alpha\theta M_\alpha(\theta)\left\|A^\beta S\left(\theta t^\alpha\right)x\right\|_{L^2}d\theta\\ &\le \int_0^\infty \alpha\theta M_\alpha(\theta)\left(\frac{1}{\pi^{2(1-\beta)}}\left(\theta t^\alpha\right)^{-\beta}e^{-\pi^2\theta t^\alpha}\|x\|_{L^2}\right)d\theta\\ &\le \frac{\alpha}{\pi^{2(1-\beta)}}\left(\int_0^\infty M_\alpha(\theta)\theta^{1-\beta}\,d\theta\right)t^{-\alpha\beta}\|x\|_{L^2}.\end{aligned}$$

Due to the Proposition 2.5, it is fulfilled that $\int_0^\infty\theta^n M_\alpha(\theta)\,d\theta=\frac{\Gamma(n+1)}{\Gamma(\alpha n+1)}$, for $n\ge -1$ and by the definition of the Gamma function one has that $\Gamma(\theta+1)=\theta\Gamma(\theta)$.

Then,

$$\begin{aligned}\left\|E_{\alpha,\alpha}\left(-t^\alpha A\right)x\right\|_\beta &\le \frac{\alpha}{\pi^{2(1-\beta)}}\left(\frac{\Gamma(1-\beta+1)}{\Gamma(\alpha(1-\beta)+1)}\right)t^{-\alpha\beta}\|x\|_{L^2}\\ &\le \frac{1}{\pi^{2(1-\beta)}}\left(\frac{\Gamma(1-\beta)}{\Gamma(\alpha(1-\beta))}\right)t^{-\alpha\beta}\|x\|_{L^2}.\end{aligned}$$

Consequently $M_E=\frac{1}{\pi^{2(1-\beta)}}(\frac{\Gamma(1-\beta)}{\Gamma(\alpha(1-\beta))})$.

Now, by (3.9), we have

$$\delta = M_E\left(\frac{\omega}{\ln|c|}\right)^{\alpha(1-\beta)}\Gamma\left(\alpha(1-\beta)\right)\left(L_f+\frac{C_k L_g\omega}{\eta\omega+\ln|c|}\right),$$

and therefore,

$$\delta = \left(\frac{\Gamma(1-\beta)}{\pi^{2(1-\beta)}\Gamma(\alpha(1-\beta))}\right)$$

$$
\begin{aligned}
&\times\left(\frac{\omega}{\ln|c|}\right)^{\alpha(1-\beta)}\Gamma\bigl(\alpha(1-\beta)\bigr)\cdot\|a\|_{L^2}\|b\|_{L^2}\cdot\frac{1}{\pi^{2\beta}}\left(1+\frac{(2/3)\omega}{9\omega+\ln|c|}\right)\\
&=\frac{\Gamma(1-\beta)}{\pi^2}\cdot\|a\|_{L^2}\|b\|_{L^2}\left(\frac{\omega^{\alpha(1-\beta)}}{(\ln|c|)^{\alpha(1-\beta)}}\right)\left(1+\frac{(2/3)\omega}{9\omega+\ln|c|}\right).
\end{aligned}
$$

According to Theorem 3.7, the fractional problem (4.1) has a unique (ω,c)-periodic mild solution whenever $\delta<1$. Moreover, the solution is given by

$$
\begin{aligned}
u(t)={}&\int_{-\infty}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-(t-s)^{\alpha}A\bigr)\\
&\times\left(a(s)\cos\bigl(b(s)u(s)\bigr)+\int_{-\infty}^{s}e^{\pi^2(s-r)}a(s)\sin\bigl(b(s)u(s)\bigr)\,dr\right)ds.
\end{aligned}
$$

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
E.A. and R.G. had the main idea, R.M. worked in all computations and E.A., R:G. and R.M. wrote the main manuscript.

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