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On qualitative analysis of boundary value problem of variable order fractional delay differential equations

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Abstract

Variable order differential equations are the natural extension of the said area. In many situations, such problems have the ability to describe real-world problems more concisely. Therefore, keeping this validity in mind, we have considered a class of boundary value problems (BVPs) under the variable order differentiation. For the suggested problems, we have developed the existence and uniqueness (EU) by using some fixed point results due to Banach and Schauder. Sufficient adequate results have been established for the required need. Some stability results have also been elaborated based on the concepts of Ulam, Hyers, and Rassias. Proper examples have also been provided with detailed analysis to verify our results.

Keywords: Variable order derivative; Delay differential equation; Existence results; Stability results

1 Introduction

Calculus devoted to non-integer order derivatives and integrals has gotten significant interest in the last few decades from researchers in different fields of science and technology because the derivative of the non-integer order derivative of a function produces its complete spectrum, which involves the corresponding integer-order counterpart as a special case. For some significant applications in various disciplines like rheology, fluid dynamics, viscoelasticity, financial mathematics, distribution theory, etc. For instance, author in [1] applied concepts of fractional calculus in bioengineering. For fractional dynamics and physics, see [2], a chaos neuron model using fractional calculus in [3]. Similarly, see the delay fractional order model for the malarial disease and HIV/AIDS in [4]. Basic concepts, theory, and applications of fractional calculus, we refer to [5] and [6], respectively. Further, the area devoted to deal differential equations of non-integer order has been considered by many researchers. Because using such equations in mathematical models of real-world processes has a greater degree of freedom and produces comprehensive dynamics of the phenomenon. Some reputed results in this regard can be seen about applications viscoelasticity, physics, and dynamics in [7–9], and [10], respectively. Therefore, researchers have established various aspects like qualitative theory, stability analysis, and numerical interpretation. For instance, some basic theories of the said can be read in [11] and fundamental concepts we refer to [12].

It is remarkable that boundary value problems (BVPs) have numerous applications in engineering disciplines. Therefore, qualitative aspects of the said area have been well investigated for arbitrary order differential equations. The required qualitative results for a variety of BVPs under fractional calculus have been studied using a fixed-point approach. Some significant results in this respect can be referred to as: for existence and uniqueness of different boundary and initial value problems using fixed point theory we refer to [13–16], and [17], respectively. Also, some authors have extended the fixed point approach to study nonlocal, multi-point, and initial value problems for existence theory in [18–21], and [22], respectively. It is important that connecting a present phenomenon with its past can be investigated by delay-type equations of an integer as well as factional order. There are three kinds of delay problems, including discreet delay, proportional, and continuous delay equations. Therefore, in the last few years, the area of fractional order delay differential equations (FODDEs) has been considered very well. FODDEs play important roles in modeling various physical and biological processes and phenomena. FODDEs have various applications in different fields, including electrodynamics, probability theory of structures, growth cells, quantum mechanics, dynamics of both linear and non-linear systems, and astrophysics. In this regard, various classes of fractional pantograph differential equations for numerical analysis have been investigated by wavelet method, polynomials, and other tools in [23, 24], and [25], respectively. So far, the area devoted to fixed fractional order derivatives has been explored very well. In this regard, various differential operators have been introduced whose detail can be seen as for computational algorithm and existence uniqueness results of delay problems, we refer to [26] and [27]. Treating FODDEs by decomposition method, see [28], numerical analysis of aforesaid problems by various numerical methods, including wavelet, operational matrices, etc. we refer to [29, 30], and [31], respectively. For qualitative theory by fixed point theory and degree method, we refer to [32] and [33], respectively. In addition, here we cite some works where authors have investigated different problems for qualitative analysis as [34–37].

Recently, another form of differential operator where the order is taken as a continuous function has attracted attention. The said idea was given by Samko and his co-author in 1993 [38]. Selecting order as a variable makes the operator more flexible with more degree of freedom. Also, a variety of problems whose dynamics cannot be well studied using traditional type fractional order operators still exist. Therefore, in the last two decades, researchers have increasingly used variable-order differential operators to derive EU, stability, and numerical results. Authors [39] have studied some problems under variable order for theoretical analysis. Results about the existence theory, stability analysis, and investigation of extremal solutions have been developed in [40, 41], and [42]. Also, some useful applications in real-world problems of the said area have been given in [43] and [44].

For dynamical problems, stability theory is usually demanded. For usual traditional problems of fractional calculus, Lyapunov and Mittag-Leffler as well as exponential kinds stabilities have been developed very well. Proper attention in recent times has been given to H-U stability. For instance, the said stability result has been established for a class of Hilfer FODEs in [45]. Also, the mentioned stability for the FODDEs system of tumor-immune has been derived in [46]. The mentioned stability has also been derived for a coupled system of FODEs in [47]. Some existence results by fixed point approach and stability results have been studied in [48]. Existence theory and stability analysis of some FODDEs have been investigated in [49] and [50], respectively. For a class of linear FODDEs, the mentioned stability has also derived in [51].

Here we claim that FODDEs involving mixed-type delays have not been properly investigated. Therefore, by overcoming this gap, here we consider the following BVPs of mixed-type delays FODEs as

$$\begin{cases} {}^{C}\mathbf{D}_{+0}^{\ell(t)}\mathbf{z}(t) = f(t,\mathbf{z}(\rho t),\mathbf{z}(t-\tau)), \quad 0<\rho<1, 0<\ell(t)\leq 1, \\ \mathbf{z}(0) = \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)} g(\mathbf{z}(\varsigma))\,d\varsigma + \mathbf{z}_0, \quad 0<\delta\leq 1, \end{cases} \tag{1}$$

where ${}^{C}\mathbf{D}_{+0}^{\ell(t)}$ represents a variable order derivative using Caputo sense, $\mathbf{J} = [0,T]$, and $f : \mathbf{J}\times\mathscr{R}^2 \to \mathscr{R}$ and $g : [0,T] \to \mathscr{R}$ are continuous function. Here, we state that recently some improved results on variable order problems have been published in [52, 53]. We will follow the same procedures as mentioned in these articles. In addition, the traditional fixed point approach and nonlinear functional analysis are used to develop EU and stability results to the above variable FODDEs involving mixed-type delay terms. We investigate various kinds of UH stability, including generalized Hyers Ulam, Rassias and generalized Hyers–Ulam Rassia abbreviated as gUH, UHR, and gUHR. Also, by proper test examples, we justify our analysis. The concerned stability has been investigated for fixed fractional order problems under boundary conditions [54–56]. We will use fixed point theory [57] to develop our study.

2 Preliminaries results

Some axillary results we need as follow.

Definition 2.1 ([38]) The variable order fractional integral of $\mathbf{z}\in L(\mathbf{J})$ in the Riemann–Liouville sense is recollected as

$$\mathbf{I}_{+0}^{\ell(t)}\mathbf{z}(t) = \frac{1}{\Gamma(\ell(t))}\int_0^t (t-s)^{\ell(\varsigma)-1}\mathbf{z}(\varsigma)\,d\varsigma, \tag{2}$$

where $\ell : \mathbf{J}\to(0,1]$ is continuous function.

Definition 2.2 ([38]) The variable order fractional derivative in the sense of Caputo is recollected as

$${}^{c}\mathrm{D}_0^{\ell(t)}\mathbf{z}(t) = \frac{1}{\Gamma(n-\ell(t))}\int_0^t (t-\varsigma)^{\ell(\varsigma)-1}\mathbf{z}^{(n)}(\varsigma)\,d\varsigma. \tag{3}$$

Lemma 2.1 ([6]) *For fractional order $\ell\in(0,1]$ of function $\mathbf{z}$, the relation holds*

$$\mathbf{I}_{+0}^{\ell}\,{}^{C}\mathbf{D}_{+0}^{\ell}\mathbf{z}(t) = \mathbf{z}(t) - \mathbf{z}(0).$$

Theorem 2.3 ([57]) *Let $\mathbb{B}\neq\emptyset$ be a closed, convex subset of a Banach space $\mathbb{X}$, and if $\mathscr{P} : \mathbb{B}\to\mathbb{B}$ is a continuous function such that $\mathscr{P}(\mathbb{B})$ is a relatively compact subset of $\mathbb{X}$, then $\mathscr{P}$ has at least one fixed point in $\mathbb{B}$.*

3 Main results

Here, we derive our main results about EU for the proposed problem (1). Let $n \in \{1, 2, 3, \ldots.\}$, then consider a partition of $\mathbf{J}$ as

$$\big\{\mathbf{J}_1 = [0, t_1], \mathbf{J}_2 = (t_1, t_2], \mathbf{J}_3 = (t_2, t_3], \ldots, \mathbf{J}_n = (t_{n-1}, T]\big\},$$

and let $\ell : \mathbf{J} \to (0, 1]$ be a piecewise function such that

$$\ell(t) = \sum_{i=1}^{n} \ell_i(t) I_i(t) = \begin{cases} \ell_1 & \text{if } t \in \mathbf{J}_1, \\ \ell_2 & \text{if } t \in \mathbf{J}_2, \\ \vdots & \\ \ell_n & \text{if } t \in \mathbf{J}_n, \end{cases}$$

where $0 < \ell_i \leq 1$ represent constants, and I_i is the indicator function of $\mathbf{J}_i = (t_{i-1}, t_n]$ with $i = 1, 2, \ldots, n$, such that $t_0 = 0, t_n = T$ and

$$I_i(t) = \begin{cases} 1 & \text{for } t \in \mathbf{J}_i, \\ 0 & \text{elsewhere.} \end{cases}$$

Consider the Banach space $\mathrm{X}_i = C[\mathbf{J}_i, R]$, with $i = 1, 2, \ldots, n$, under the norm define as $\|\mathbf{z}\| = \max_{t \in J} |\mathbf{z}(t)|$. Therefore, we can write the left side of the considered problem as

$$ {}^{C}\mathbf{D}_{+0}^{\ell(t)} \mathbf{z}(t) = \int_0^{t_1} \frac{(t-\varsigma)^{-\ell_1}}{\Gamma(1-\ell_1)} \mathbf{z}^{(1)}(\varsigma)\, d\varsigma + \cdots + \int_{t_{n-1}}^{t} \frac{(t-\varsigma)^{-\ell_i}}{\Gamma(1-\ell_i)} \mathbf{z}^{(1)}(\varsigma)\, d\varsigma. \tag{4}$$

In view of (4), we can write our considered problem as

$$\int_0^{t_1} \frac{(t-\varsigma)^{-\ell_1}}{\Gamma(1-\ell_1)} \mathbf{z}^{(1)}(\varsigma)\, d\varsigma + \cdots + \int_{t_{n-1}}^{t} \frac{(t-\varsigma)^{-\ell_i}}{\Gamma(1-\ell_i)} \mathbf{z}^{(1)}(\varsigma)\, d\varsigma = f\big(t, \mathbf{z}(\rho t), \mathbf{z}(t-\tau)\big). \tag{5}$$

Therefore, let $\mathbf{z} \in C([0, T], R)$, such that we need to deal

$$\begin{cases} {}^{C}\mathbf{D}_{t_{i-1}}^{\ell_i} \mathbf{z}(t) = f(t, \mathbf{z}(\rho t), \mathbf{z}(t-\tau)), & t \in \mathbf{J}_i, \\ \mathbf{z}(t_{i-1}) = \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)} g(\mathbf{z}(\varsigma))\, d\varsigma + \mathbf{z}_0. \end{cases} \tag{6}$$

Lemma 3.1 *If $h \in L(\mathbf{J}_i)$, then the problem*

$$\begin{cases} {}^{C}\mathbf{D}_{+0}^{\ell_i} \mathbf{z}(t) = h(t), & t \in \mathbf{J}_i, \\ \mathbf{z}(t_{i-1}) = \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)} g(\varsigma, \mathbf{z}(\varsigma))\, d\varsigma + \mathbf{z}_0 \end{cases} \tag{7}$$

has solution as

$$\mathbf{z}(t) = \mathbf{z}_0 + \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)} g\big(\varsigma, \mathbf{z}(\varsigma)\big)\, d\varsigma + \frac{1}{\Gamma(\ell_i)} \int_{t_{i-1}}^{t} (t-\varsigma)^{\ell_i - 1} h(\varsigma)\, d\varsigma.$$

Proof By applying the integral $\mathbf{I}_{t_{i-1}}^{\ell_i}$ to Problem (7) and then using Lemma 2.1, one has

$$\mathbf{z}(t) = c + \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^{t}(t-\varsigma)^{\ell_i-1}h(\varsigma)\,d\varsigma \tag{8}$$

Putting $t \to 0$ in (8) and using the initial condition, we get

$$c = \mathbf{z}_0 + \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)}g\bigl(\varsigma,\mathbf{z}(\varsigma)\bigr)\,d\varsigma.$$

And hence we get

$$\mathbf{z}(t) = \mathbf{z}_0 + \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)}g\bigl(\varsigma,\mathbf{z}(\varsigma)\bigr)\,d\varsigma + \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^{t}(t-\varsigma)^{\ell_i-1}h(\varsigma)\,d\varsigma. \qquad \square$$

Corollary 1 *Thank to Lemma 3.1, the proposed problem (1) has a solution given as*

$$\begin{aligned}\mathbf{z}(t) &= \mathbf{z}_0 + \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)}g\bigl(\varsigma,\mathbf{z}(\varsigma)\bigr)\,d\varsigma \\ &\quad + \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^{t}(t-\varsigma)^{\ell_i-1}f\bigl(\varsigma,\mathbf{z}(\rho\varsigma),\mathbf{z}(\varsigma-\tau)\bigr)\,d\varsigma\end{aligned}$$

Next, for the EU of the problem (1), we define the given hypothesis:

(A_1) If $\mathbf{z},\bar{\mathbf{z}},\mathbf{y},\bar{\mathbf{y}} \in \mathscr{R}$, then for $L_f > 0$, one has

$$\bigl|f(t,\mathbf{z},\mathbf{y}) - f(t,\bar{\mathbf{z}},\bar{\mathbf{z}})\bigr| \le L_f\bigl(|\mathbf{z}-\bar{\mathbf{z}}| + |\mathbf{y}-\bar{\mathbf{y}}|\bigr).$$

(A_2) For $\mathbf{L}_g > 0$, with $\mathbf{z},\bar{\mathbf{z}} \in \mathscr{R}$, we have

$$\bigl|g(\mathbf{z}) - g(\bar{\mathbf{z}})\bigr| \le \mathbf{L}_g|\mathbf{z}-\bar{\mathbf{z}}|.$$

Theorem 3.1 *If hypothesis* (A_1, A_2) *holds, and* $\frac{T^\delta \mathbf{L}_g}{\Gamma(\delta+1)} + \frac{2T_i^\ell L_f}{\Gamma(\ell_i+1)} < 1$, *then the considered problem (1) preserves a unique solution.*

Proof Define the operator $\mathscr{L} : \mathbb{X}_i \to \mathbb{X}_i$ by

$$\mathscr{L}\mathbf{z}(t) = \mathbf{z}_0 + \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)}g\bigl(\mathbf{z}(\varsigma)\bigr)\,d\varsigma + \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^{t}(t-\varsigma)^{\ell_i-1}f\bigl(\varsigma,\mathbf{z}(\rho\varsigma),\mathbf{z}(\varsigma-\tau)\bigr)\,d\varsigma.$$

Now to show that $\mathscr{L}$ is a condensing operator, for this let $\mathbf{z},\bar{\mathbf{z}} \in \mathrm{X}_i$, using $(T-t_{i-1})^{\ell_i} \le T^\ell$, and consider

$$\begin{aligned}&\|\mathscr{L}\mathbf{z} - \mathscr{L}\bar{\mathbf{z}}\| \\ &\quad = \max_{t\in\mathbf{J}}\Bigl|\Bigl\{\mathbf{z}_0 + \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)}g\bigl(\mathbf{z}(\varsigma)\bigr)\,d\varsigma \\ &\qquad + \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^{t}(t-\varsigma)^{\ell_i-1}f\bigl(\varsigma,\mathbf{z}(\rho\varsigma),\mathbf{z}(\varsigma-\tau)\bigr)\,d\varsigma\Bigr\}\end{aligned}$$

$$
\begin{aligned}
&-\bigg\{\mathbf{z}_0+\int_0^T\frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)}g\big(\bar{\mathbf{z}}(\varsigma)\big)\,d\varsigma\\
&+\frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^t(t-\varsigma)^{\ell_i-1}f\big(\varsigma,\bar{\mathbf{z}}(\rho\varsigma),\bar{\mathbf{z}}(\varsigma-\tau)\big)\,d\varsigma\bigg\}\bigg|\\
\le{}&\max_{t\in\mathbf{J}}\bigg\{\int_0^T\frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)}\big|g\big(\mathbf{z}(\varsigma)\big)-g\big(\bar{\mathbf{z}}(\varsigma)\big)\big|\\
&+\frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^t(t-\varsigma)^{\ell_i-1}\big|f\big(\varsigma,\mathbf{z}(\rho\varsigma),\mathbf{z}(\varsigma-\tau)\big)-f\big(\varsigma,\bar{\mathbf{z}}(\rho\varsigma),\bar{\mathbf{z}}(\varsigma-\tau)\big)\big|\,d\varsigma\bigg\}\\
\le{}&\frac{\mathbf{L}_g\|\mathbf{z}-\bar{\mathbf{z}}\|}{\Gamma(\delta)}\int_0^T(T-\varsigma)^{\delta-1}\,d\varsigma+\frac{2L_f\|\mathbf{z}-\bar{\mathbf{z}}\|}{\Gamma(\ell_i)}\int_{t_{i-1}}^T(T-\varsigma)^{\ell_i-1}\,d\varsigma\\
\le{}&\left\{\frac{\mathbf{L}_gT^\delta}{\Gamma(\delta+1)}+\frac{2L_fT^{\ell_i}}{\Gamma(\ell_i+1)}\right\}\|\mathbf{z}-\bar{\mathbf{z}}\|.
\end{aligned}
$$

Hence proved. □

(A_3) for $\mathscr{K}_f>0$, one has

$$\big|f(t,\mathbf{z},\bar{\mathbf{z}})\big|\le\mathscr{K}_f\big\{|\mathbf{z}|+|\bar{\mathbf{z}}|\big\},\quad\text{for }\mathbf{z},\bar{\mathbf{z}}\in\mathscr{R};$$

(A_4) If $\mathbf{K}_g>0$, then

$$\big|g(\mathbf{z})\big|\le\mathbf{K}_g|\mathbf{z}|,\quad\text{for }\mathbf{z}\in\mathscr{R};$$

Theorem 3.2 *In view of assumptions* (A_1-A_4), *the suggested problem* (1) *has at least one solution in bounded set* $\mathbb{B}=\{\mathbf{z}\in\mathbb{X}_i:\|\mathbf{z}\|\le\gamma\}$, *with* $\Delta=\frac{\mathbf{L}_gT^\delta}{\Gamma(\delta+1)}+\frac{2L_fT^{\ell_i}}{\Gamma(\ell_i+1)}$.

Proof Let us perform the given steps to establish the required result:
Step 1: We need to prove that $\mathscr{P}:\mathbb{B}\to\mathbb{B}$, is bounded. Let $\mathbf{z}\in\mathbb{B}$, one has

$$
\begin{aligned}
\big\|\mathscr{P}(\mathbf{z})\big|={}&\max_{t\in\mathbf{J}}\bigg|\mathbf{z}_0+\int_0^T\frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)}g\big(\mathbf{z}(\varsigma)\big)\,d\varsigma\\
&+\frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^t(t-\varsigma)^{\ell_i-1}f\big(\varsigma,\mathbf{z}(\rho\varsigma),\mathbf{z}(\varsigma-\tau)\big)\,d\varsigma\bigg|,\\
\le{}&\mathbf{z}_0+\int_0^T\frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)}\big|g\big(\mathbf{z}(\varsigma)\big)\big|\,d\varsigma\\
&+\frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^T(T-\varsigma)^{\ell_i-1}\big|f\big(\varsigma,\mathbf{z}(\rho\varsigma),\mathbf{z}(\varsigma-\tau)\big)\big|\,d\varsigma,\\
\le{}&\mathbf{z}_0+\left\{\frac{\mathbf{K}_gT^\delta}{\Gamma(\delta+1)}+\frac{2\mathscr{K}_fT^{\ell_i}}{\Gamma(\ell_i+1)}\right\}\gamma\le\gamma.
\end{aligned}
$$

Hence $\mathscr{P}(\mathbf{z})\in\mathbb{B}$, therefore $\mathscr{P}$ maps bounded set into bounded in $\mathbb{X}_i$.
Step 2: For the continuity of $\mathscr{P}$, let a sequence $\mathbf{z}_n$ converge to $\mathbf{z}$ in $\mathbb{B}$, and for every $t\in\mathbf{J}_i$, we have

$$\big\|\mathscr{P}(\mathbf{z}_n)-\mathscr{P}(\mathbf{z})\big\|$$

$$\begin{aligned}
&= \max_{t\in\mathbf{J}_i}\bigg|\bigg\{\mathbf{z}_0 + \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)} g\big(\mathbf{z}_n(\varsigma)\big)\,d\varsigma \\
&\quad + \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^{t} (t-\varsigma)^{\ell_i-1} f\big(\varsigma,\mathbf{z}_n(\rho\varsigma),\mathbf{z}_n(\varsigma-\tau)\big)\,d\varsigma\bigg\} \\
&\quad - \bigg\{\mathbf{z}_0 + \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)} g\big(\mathbf{z}(\varsigma)\big)\,d\varsigma \\
&\quad + \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^{t} (t-\varsigma)^{\ell_i-1} f\big(\varsigma,\mathbf{z}(\rho\varsigma),\mathbf{z}(\varsigma-\tau)\big)\,d\varsigma\bigg\}\bigg| \\
&\le \max_{t\in\mathbf{J}_i}\bigg\{\int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)} \big|g\big(\mathbf{z}_n(\varsigma)\big) - g\big(\mathbf{z}(\varsigma)\big)\big|\,d\varsigma \\
&\quad + \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^{t} (t-\varsigma)^{\ell_i-1} \big|f\big(\varsigma,\mathbf{z}_n(\rho\varsigma),\mathbf{z}_n(\varsigma-\tau)\big) - f\big(\varsigma,\mathbf{z}(\rho\varsigma),\mathbf{z}(\varsigma-\tau)\big)\big|\,d\varsigma\bigg\} \\
&\le \max_{t\in\mathbf{J}_i}\bigg\{\mathbf{K}_g\int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)}\,d\varsigma + \frac{2\mathscr{K}_f}{\Gamma(\ell_i)}\int_{t_{i-1}}^{t} (t-\varsigma)^{\ell_i-1}\,d\varsigma\bigg\}\|\mathbf{z}_n-\mathbf{z}\|.
\end{aligned}$$

As

$$\|\mathbf{z}_n-\mathbf{z}\| \to 0 \quad \text{by } n\to 0.$$

Also, $\mathscr{P}$ is bounded. So, $\|\mathscr{P}(\mathbf{z}_n)-\mathscr{P}(\mathbf{z})\| \to 0$ as $n \to 0$. Therefore, $\mathscr{P}$ is continuous.

Step 3: $\mathscr{P}$ map bounded set into equi-continuous set in $\mathbb{X}_i$. If $t_1, t_2 \in \mathbf{J}, t_1 < t_2$, we have

$$\begin{aligned}
&\big|\mathscr{P}\big(\mathbf{z}(t_1)\big) - \mathscr{P}\big(\mathbf{z}(t_2)\big)\big| \\
&\quad = \bigg|\bigg\{\mathbf{z}_0 + \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)} g\big(\mathbf{z}(\varsigma)\big)\,d\varsigma \\
&\qquad + \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^{t_1} (t_1-\varsigma)^{\ell_i-1} f\big(\varsigma,\mathbf{z}(\rho\varsigma),\mathbf{z}(\varsigma-\tau)\big)\,d\varsigma\bigg\} \\
&\qquad - \bigg\{\mathbf{z}_0 + \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)} g\big(\mathbf{z}(\varsigma)\big)\,d\varsigma \\
&\qquad + \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^{t_2} (t_2-\varsigma)^{\ell_i-1} f\big(\varsigma,\mathbf{z}(\rho\varsigma),\mathbf{z}(\varsigma-\tau)\big)\,d\varsigma\bigg\}\bigg| \\
&\quad \le \bigg\{\frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^{t_1} (t_1-\varsigma)^{\ell_i-1}\,d\varsigma - \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^{t_2} (t_2-\varsigma)^{\ell_i-1}\,d\varsigma\bigg\} 2\mathscr{K}_f\gamma.
\end{aligned}$$

As $t_1 \to t_2$, then $|\mathscr{P}\mathbf{z}(t_1) - \mathscr{P}\mathbf{z}(t_2)| \to 0$ and from *Step 1–2* $\mathscr{P}$ is bounded and continuous. Hence, $\|\mathscr{P}\mathbf{z}(t_1) - \mathscr{P}\mathbf{z}(t_2)\| \to 0$. Hence, $\mathscr{P}$ is completely continuous.

Step 4: For a prior bounds, we need to show that the set $\mathbb{Q} = \{\mathbf{z} \in \mathbb{X}_i : \mathbf{z} = \eta\mathscr{P}\mathbf{z}\}$ for some $0 \le \eta \le 1$, is bounded. Let for any $\mathbf{z} \in \mathbb{Q}$, we have

$$\begin{aligned}
\|\mathbf{z}\| &= \max_{t_i\in\mathbf{J}_i}\big|\eta\mathscr{P}\mathbf{z}(t)\big|, \\
&\le \max_{t\in\mathbf{J}_i}\bigg|\mathbf{z}_0 + \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)} g\big(\mathbf{z}(\varsigma)\big)\,d\varsigma \\
&\quad + \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^{t} (t-\varsigma)^{\ell_i-1} f\big(\varsigma,\mathbf{z}(\rho\varsigma),\mathbf{z}(\varsigma-\tau)\big)\,d\varsigma\bigg|,
\end{aligned}$$

$$\leq \mathbf{z}_0 + \left\{ \frac{\mathbf{L}_g T^{\delta}}{\Gamma(\delta+1)} + \frac{2L_f T^{\ell_i}}{\Gamma(\ell_i+1)} \right\} \gamma \leq \gamma.$$

We see that $\gamma \geq \frac{\mathbf{z}_0}{1-\Delta}$, which implies that

$$\|\mathbf{z}\| \leq \gamma.$$

Hence by Theorem 2.3, the proposed problem (1) has at least one solution. □

4 Stability analysis

In this section, we discuss some basic results corresponding to stability analysis for our proposed problem (1). To achieve the required result, we have the following results.

Definition 4.1 The solution $\mathbf{z}$ of the considered problem (1) is UH stable. If we can take a constant $\mathbf{N}_f > 0$, such that for every $\widehat{\varrho} > 0$, and each solution $\mathbf{z} \in \mathbb{X}_i$ of the inequality

$$\left| {}^{C}\mathbf{D}^{\ell_i}_{t_{i-1}} \mathbf{z}(t) - f\big(t, \mathbf{z}(\rho t), \mathbf{z}(t-\tau)\big) \right| \leq \widehat{\varrho} \quad \text{for all } t \in \mathbf{J}_i, \tag{9}$$

for the unique solution $\mathbf{z}^*$ of problem (1) in $\mathbb{X}_i$, such that

$$\left\| \mathbf{z} - \mathbf{z}^* \right\| \leq \mathbf{N}_f \widehat{\varrho}.$$

Definition 4.2 The solution of the proposed problem (1) is gUH stable, if one can find $\widehat{\phi} \in C[\mathscr{R}^+, \mathscr{R}^+]$ with $\widehat{\phi}(0) = 0$, and for any solution of the inequality (9), one has

$$\left\| \mathbf{z} - \mathbf{z}^* \right\| \leq \mathbf{N}_f \widehat{\phi}(\widehat{\varrho}).$$

Remark 1 Let $\mathbf{z}$ be the solution in $\mathbb{X}_i$ for the inequality (9), if there exists $\beta \in C(\mathbf{J}_i)$, for every $t \in \mathbf{J}_i$, such that

(i) $|\beta(t)| \leq \widehat{\varrho}$;
(ii) ${}^{C}\mathbf{D}^{\ell_i}_{t_{i-1}} \mathbf{z}(t) = f(t, \mathbf{z}(\rho t), \mathbf{z}(t-\tau)) + \beta(t)$.

Definition 4.3 Problem (1) for $\mathbf{z} \in \mathbb{X}_i$ is UHR stable under $\psi \in \mathbb{X}_i$, if $\mathbf{N}_f > 0$, such that

$$\left| {}^{C}\mathbf{D}^{\ell_i}_{t_{i-1}} \mathbf{z}(t) - f\big(t, \mathbf{z}(\rho t), \mathbf{z}(t-\tau)\big) \right| \leq \psi(t)\widehat{\varrho}, \quad \text{for all } t \in \mathbf{J}_i, \tag{10}$$

for unique $\mathbf{z}^* \in \mathbb{X}_i$ of (1), one has

$$\left\| \mathbf{z} - \mathbf{z}^* \right\| \leq \mathbf{N}_f \psi(t) \widehat{\varrho}.$$

Definition 4.4 The solution of the considered problem (1) will be gUHR stable if

$$\left\| \mathbf{z} - \mathbf{z}^* \right\| \leq \mathbf{N}_f \psi(t) \widehat{\phi}(\widehat{\varrho}), \quad \text{where } \widehat{\phi}(0) = 0.$$

Remark 2 For mapping $\beta \in L(\mathbf{J}_i)$, one has

(i) $|\beta(t)| \leq \widehat{\varrho}\psi(t)$;
(ii) ${}^{C}\mathbf{D}^{\ell_i}_{t_{i-1}} \mathbf{z}(t) = f(t, \mathbf{z}(\rho t), \mathbf{z}(t-\tau)) + \beta(t)$.

Lemma 4.1 *Thank to Remarks 1 and Lemma 3.1, the solution of perturb problem*

$$\begin{cases} {}^{C}\mathbf{D}_{t_{i-1}}^{\ell_i}\mathbf{z}(t) = f(t,\mathbf{z}(\rho t),\mathbf{z}(t-\tau)) + \beta(t), & 0 < \ell_i \leq 1, \\ \mathbf{z}(0) = \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)} g(\mathbf{z}(\varsigma))\,d\varsigma + \mathbf{z}_0, \end{cases} \tag{11}$$

satisfies the following

$$\left|\mathbf{z}(t) - \mathscr{L}\left(\mathbf{z}(t)\right)\right| \leq \frac{T^{\ell_i}}{\Gamma(\ell_i+1)}\widehat{\varrho}, \quad \text{for all } t \in \mathbf{J}_i, \tag{12}$$

with

$$\mathscr{L}\left(\mathbf{z}(t)\right) = \mathbf{z}_0 + \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)} g\left(\varsigma,\mathbf{z}(\varsigma)\right)d\varsigma + \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^t (t-\varsigma)^{\ell_i-1}y(\varsigma)\,d\varsigma.$$

Proof Using Lemma 3.1, problem (11) yields

$$\begin{aligned} \mathbf{z}(t) = \mathbf{z}_0 &+ \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)} g\left(\varsigma,\mathbf{z}(\varsigma)\right)d\varsigma + \frac{1}{\Gamma(\ell_i)}\int_0^t (t-\varsigma)^{\ell_i-1}y(\varsigma)\,d\varsigma \\ &+ \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^t (t-\varsigma)^{\ell_i-1}\beta(\varsigma)\,d\varsigma, \end{aligned}$$

which implies that

$$\left|\mathbf{z}(t) - \mathscr{L}\left(\mathbf{z}(t)\right)\right| \leq \frac{T^{\ell_i}}{\Gamma(\ell_i+1)}\widehat{\varrho}.$$

□

Theorem 4.5 *Under the hypothesis $A_1 - A_4$, the desired solution of concerned problem (1) is* UH *and* gUH *stable, if* $1 \neq \frac{T^{\ell_i}}{\Gamma(\ell_i+1)}$.

Proof Thank to Lemma 4.1, if $\mathbf{z}$ and $\mathbf{z}^*$ are solutions of (1), then

$$\begin{aligned} \left|\mathbf{z}(t) - \mathbf{z}^*(t)\right| &= \left|\mathbf{z}(t) - \mathscr{L}\left(\mathbf{z}(t)\right)\right| \\ &= \left|\mathbf{z}(t) - \mathscr{L}\left(\mathbf{z}(t)\right) + \mathscr{L}\left(\mathbf{z}(t)\right) - \mathscr{L}\left(\mathbf{z}*(t)\right)\right| \\ &\leq \left|\mathbf{z}(t) - \mathscr{L}\left(\mathbf{z}(t)\right)\right| + \left|\mathscr{L}\left(\mathbf{z}(t)\right) - \mathscr{L}\left(\mathbf{z}^*(t)\right)\right| \\ &\leq \frac{T^{\ell_i}}{\Gamma(\ell_i+1)}\widehat{\varrho} + \left\{\frac{\mathbf{L}_g T^{\delta}}{\Gamma(\delta+1)} + \frac{2\mathbf{L}_f T^{\ell_i}}{\Gamma(\ell_i+1)}\right\}\left|\mathbf{z}(t) - \mathbf{z}^*(t)\right|, \end{aligned}$$

which further yields that

$$\left\|\mathbf{z} - \mathbf{z}^*\right\| \leq \frac{T^{\ell_i}}{(1-\Delta)\Gamma(\ell_i+1)}\widehat{\varrho}.$$

Expressing by $\Delta = \{\frac{\mathbf{L}_g T^{\delta}}{\Gamma(\delta+1)} + \frac{2\mathbf{L}_f T^{\ell_i}}{\Gamma(\ell_i+1)}\}$, then the required results hold. □

Lemma 4.2 *If Remark 2 holds, then from the solution of (11),*

$$\left|\mathbf{z}(t) - \mathscr{P}\mathbf{z}(t)\right| \leq \frac{T^{\ell_i}}{\Gamma(\ell_i + 1)}\widehat{\varrho}, \quad \textit{for all } t \in \mathbf{J}_i,$$

holds.

Proof Applying Lemma 3.1, problem (11) yields

$$\begin{aligned}
\mathbf{z}(t) = \mathbf{z}_0 &+ \int_0^T \frac{(T-\varsigma)^{\delta-1}}{\Gamma(\delta)} g\big(\varsigma, \mathbf{z}(\varsigma)\big)\, d\varsigma + \frac{1}{\Gamma(\ell(t))}\int_0^t (t-\varsigma)^{\ell(\varsigma)-1} y(\varsigma)\, d\varsigma \\
&+ \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^t (t-\varsigma)^{\ell_i-1}\beta(\varsigma)\, d\varsigma,
\end{aligned}$$

which implies that

$$\begin{aligned}
\left|\mathbf{z}(t) - \mathscr{L}\big(\mathbf{z}(t)\big)\right| &\leq \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^t (t-\varsigma)^{\ell_i-1}\left|\beta(\varsigma)\right| d\varsigma, \\
&\leq \frac{1}{\Gamma(\ell_i)}\int_{t_{i-1}}^t (t-\varsigma)^{\ell_i-1}\widehat{\varrho}\psi(t)\, d\varsigma, \\
&\leq \frac{\widehat{\varrho}}{\Gamma(\ell(t))}\int_{t_{i-1}}^t (t-\varsigma)^{\ell(\varsigma)-1}\psi(t)\, d\varsigma, \\
&\leq \frac{1}{\Gamma(\ell_i)}\widehat{\varrho}\Psi(t),
\end{aligned}$$

□

where $\Psi(t) = \int_{t_{i-1}}^t (t-\varsigma)^{\ell_i-1}\psi(\varsigma)\, d\varsigma$.

Theorem 4.6 *In view of hypothesis* (A_1, A_2), *if* $\Gamma(\ell_i) \neq 1$, *then the problem* (1) *is* UHR *and* gUHR *stable.*

5 Examples

Here to demonstrate our results, some problems are treated as:

Example 1 Consider the delay fractional order problem as

$$\begin{cases}
{}^C\mathbf{D}_{t_{i-1}}^{\ell_i}\mathbf{z}(t) = t^2\left(\frac{\mathbf{z}(\frac{1}{4}t)}{13+|\mathbf{z}(\frac{1}{4}t)|} + \frac{\mathbf{z}(t-0.35)}{13+|\mathbf{z}(t-0.35)|}\right), \quad t \in [0,2], \\
\mathbf{z}(0) = \frac{1}{\Gamma(0.7)}\int_0^1 (1-\varsigma)^{-0.3}\frac{\mathbf{z}(\varsigma)}{12+|\mathbf{z}(\varsigma)|}\, d\varsigma + 0.037.
\end{cases} \tag{13}$$

Clearly $T = 2, \delta = 0.7$, then

$$\ell(t) = \begin{cases} 0.75, & t \in [0,1], \\ 0.5, & t \in]1,2], \end{cases}$$

clearly $i = 1,2$, and $f(t, \mathbf{z}(\rho t), \mathbf{z}(t-\tau) = t^2\left(\frac{\mathbf{z}(\frac{1}{4}t)}{13+|\mathbf{z}(\frac{1}{4}t)|} + \frac{\mathbf{z}(t-0.35)}{13+|\mathbf{z}(t-0.35)|}\right)$ and $g(\mathbf{z}(t)) = \frac{\mathbf{z}(t)}{12+|\mathbf{z}(t)|}$. So, let $\mathbf{z}, \bar{\mathbf{z}} \in \mathbf{X}_i, i = 1,2$, one has

$$\left|f(t, \mathbf{z}(\rho t), \mathbf{z}(t-\tau) - f(t, \bar{\mathbf{z}}(\rho t), \bar{\mathbf{z}}(t-\tau)\right|$$

$$\leq |t^2| \left| \frac{\mathbf{z}(\frac{1}{4}t)}{13 + |\mathbf{z}(\frac{1}{4}t)|} - \frac{\bar{\mathbf{z}}(\frac{1}{4}t)}{13 + |\bar{\mathbf{z}}(\frac{1}{4}t)|} + \frac{\mathbf{z}(t-0.35)}{13 + |\mathbf{z}(t-0.35)|} - \frac{\bar{\mathbf{z}}(t-0.35)}{13 + |\bar{\mathbf{z}}(t-0.35)|} \right|$$
$$\leq \frac{1}{13} \left\{ \left| \mathbf{z}\left(\frac{1}{4}t\right) - \bar{\mathbf{z}}\left(\frac{1}{4}t\right) \right| + \left| \mathbf{z}(t-0.35) - \bar{\mathbf{z}}(t-0.35) \right| \right\}$$

and

$$\left| g(\mathbf{z}(t)) - g(\bar{\mathbf{z}}(t)) \right| \leq \frac{1}{10} \left| \mathbf{z}(t) - \bar{\mathbf{z}}(t) \right|.$$

Then, in the first case, we have

$$\begin{cases} {}^C\mathbf{D}_0^{\ell_1}\mathbf{z}(t) = t^2(\frac{\mathbf{z}(\frac{1}{4}t)}{13+|\mathbf{z}(\frac{1}{4}t)|} + \frac{\mathbf{z}(t-0.35)}{13+|\mathbf{z}(t-0.35)|}), \quad t \in [0,1], \\ \mathbf{z}(0) = \frac{1}{\Gamma(0.7)} \int_0^1 (1-\varsigma)^{-0.3} \frac{\mathbf{z}(\varsigma)}{12+|\mathbf{z}(\varsigma)|} \, d\varsigma + 0.037. \end{cases} \tag{14}$$

Here, $\mathbf{L}_f = \frac{1}{13}$, $\mathbf{L}_g = \frac{1}{10}$. Hence clearly the assumptions A_1 and A_2 hold. We also examine that

$$\frac{T^\delta \mathbf{L}_g}{\Gamma(\delta+1)} + \frac{2T^{\ell_1}\mathbf{L}_f}{\Gamma(\ell_1+1)} \approx 0.277449 < 1.$$

Thus, problem (14) has a unique solution via Theorem 3.1. Moreover, one has $\frac{T^{\ell_1}}{\Gamma(\ell_1+1)} = 1.08807 \neq 1$, thus the solution of the stated problem (13) is UH and gUH stable. By the same line if $\Psi(t) = \frac{t}{2}, t \in [0,1]$ is a mapping, then we deduce that the said problem is UHR and gUHR stable. In addition, if $i = 2$, we have

$$\begin{cases} {}^C\mathbf{D}_0^{\ell_2}\mathbf{z}(t) = t^2(\frac{\mathbf{z}(\frac{1}{4}t)}{13+|\mathbf{z}(\frac{1}{4}t)|} + \frac{\mathbf{z}(t-0.35)}{13+|\mathbf{z}(t-0.35)|}), \quad t \in (1,2], \\ \mathbf{z}(0) = \frac{1}{\Gamma(0.7)} \int_0^1 (1-\varsigma)^{-0.3} \frac{\mathbf{z}(\varsigma)}{12+|\mathbf{z}(\varsigma)|} \, d\varsigma + 0.037. \end{cases} \tag{15}$$

Following the same procedure as in (14), we can prove for $T = 2$ that

$$\frac{T^\delta \mathbf{L}_g}{\Gamma(\delta+1)} + \frac{2T^{\ell_2}\mathbf{L}_f}{\Gamma(\ell_2+1)} \approx 0.89765 < 1.$$

The problem (15) has a unique solution using Theorem 3.1. Additionally, $\frac{T^{\ell_2}}{\Gamma(\ell_2+1)} = 1.595769 \neq 1$, hence (15) is UH and gUH stable using Theorem 4.5. Moreover, let $\Psi(t) = \frac{t}{2}, t \in (1,2]$ be a function, then one can deduce that the said problem is UHR and gUHR stable via Theorem 4.6.

Example 2 For further analysis, we also give the following example.

$$\begin{cases} {}^C\mathbf{D}_{t_{i-1}}^{\ell_i}\mathbf{z}(t) = \exp(-t^2)(\frac{\mathbf{z}(\frac{1}{2}t)}{130+|\mathbf{z}(\frac{1}{2}t)|} + \frac{\mathbf{z}(t-0.25)}{130+|\mathbf{z}(t-0.25)|}), \quad t \in [0,3], \\ \mathbf{z}(0) = \frac{1}{\Gamma(0.5)} \int_0^1 (1-\varsigma)^{-0.5} \frac{\mathbf{z}(\varsigma)}{40+|\mathbf{z}(\varsigma)|} \, d\varsigma + 0.01. \end{cases} \tag{16}$$

Clearly $T = 3, \delta = 0.5$, then

$$\ell(t) = \begin{cases} 0.8, \quad t \in [0,1], \\ 0.875, \quad t \in (1,3], \end{cases}$$

clearly $i = 1,2$, and $f(t,\mathbf{z}(\rho t),\mathbf{z}(t-\tau) = \exp(-t^2)(\frac{\mathbf{z}(\frac{1}{2}t)}{130+|\mathbf{z}(\frac{1}{2}t)|} + \frac{\mathbf{z}(t-0.25)}{130+|\mathbf{z}(t-0.25)|})$ and $g(\mathbf{z}(t)) = \frac{\mathbf{z}(t)}{40+|\mathbf{z}(t)|}$. So, let $\mathbf{z},\bar{\mathbf{z}} \in \mathbf{X}_i, i = 1,2$, one has

$$\begin{aligned}
&\left|f(t,\mathbf{z}(\rho t),\mathbf{z}(t-\tau)-f(t,\bar{\mathbf{z}}(\rho t),\bar{\mathbf{z}}(t-\tau)\right| \\
&\quad \le \left|\frac{\mathbf{z}(\frac{1}{2}t)}{130+|\mathbf{z}(\frac{1}{2}t)|} - \frac{\bar{\mathbf{z}}(\frac{1}{2}t)}{130+|\bar{\mathbf{z}}(\frac{1}{2}t)|} + \frac{\mathbf{z}(t-0.25)}{130+|\mathbf{z}(t-0.25)|} - \frac{\bar{\mathbf{z}}(t-0.25)}{130+|\bar{\mathbf{z}}(t-0.25)|}\right| \\
&\quad \le \frac{1}{130}\left\{\left|\mathbf{z}\left(\frac{1}{2}t\right) - \bar{\mathbf{z}}\left(\frac{1}{2}t\right)\right| + \left|\mathbf{z}(t-0.25) - \bar{\mathbf{z}}(t-0.25)\right|\right\}
\end{aligned}$$

and

$$\left|g(\mathbf{z}(t)) - g(\bar{\mathbf{z}}(t))\right| \le \frac{1}{40}\left|\mathbf{z}(t) - \bar{\mathbf{z}}(t)\right|.$$

Let in the first case, one has

$$\begin{cases} {}^{C}\mathbf{D}_0^{\ell_1}\mathbf{z}(t) = \exp(-t^2)(\frac{\mathbf{z}(\frac{1}{2}t)}{130+|\mathbf{z}(\frac{1}{2}t)|} + \frac{\mathbf{z}(t-0.25)}{130+|\mathbf{z}(t-0.25)|}), & t \in [0,3], \\ \mathbf{z}(0) = \frac{1}{\Gamma(0.5)}\int_0^1 (1-\varsigma)^{-0.5}\frac{\mathbf{z}(\varsigma)}{40+|\mathbf{z}(\varsigma)|}\,d\varsigma + 0.01. \end{cases} \tag{17}$$

Here, $\mathbf{L}_f = \frac{1}{130}$, $\mathbf{L}_g = \frac{1}{40}$. Hence clearly the assumptions A_1 and A_2 hold. We also examine that

$$\frac{T^{\delta}\mathbf{L}_g}{\Gamma(\delta+1)} + \frac{2T^{\ell_1}\mathbf{L}_f}{\Gamma(\ell_1+1)} < 1.$$

The problem 17 has a unique solution using Theorem 3.1. Moreover, one can easily prove the conditions of Theorem 3.2. $\frac{T^{\ell_1}}{\Gamma(\ell_1+1)} = 1.08807 \neq 1$, thus (16) is UH and gUH stable on using Theorem 4.5. Obviously, one can prove the results for UHR and gUHR stabling using Theorem 4.6. In addition, if $i = 2$, one has

$$\begin{cases} {}^{C}\mathbf{D}_0^{\ell_2}\mathbf{z}(t) = \exp(-t^2)(\frac{\mathbf{z}(\frac{1}{2}t)}{130+|\mathbf{z}(\frac{1}{2}t)|} + \frac{\mathbf{z}(t-0.25)}{130+|\mathbf{z}(t-0.25)|}), & t \in (1,3], \\ \mathbf{z}(0) = \frac{1}{\Gamma(0.5)}\int_0^1 (1-\varsigma)^{-0.5}\frac{\mathbf{z}(\varsigma)}{40+|\mathbf{z}(\varsigma)|}\,d\varsigma + 0.01. \end{cases} \tag{18}$$

Following the same procedure as in (17), we can prove for $T = 3$ that

$$\frac{T^{\delta}\mathbf{L}_g}{\Gamma(\delta+1)} + \frac{2T^{\ell_2}\mathbf{L}_f}{\Gamma(\ell_2+1)} < 1.$$

Hence, we can deduce the conditions of Theorem 3.1 that problem (18) has a unique result. Furthermore, it is easy to show the said problem using Theorem 3.2 has at least one result. Moreover, one has $\frac{T^{\ell_2}}{\Gamma(\ell_2+1)} \neq 1$, hence we can deduce that problem (18) is UH and gUH stable on using Theorem 4.5. Obviously, the result of UHR, and gUHR stabling also holds using Theorem 4.6.

6 Conclusion

Keeping the useful applications in the mind of variable order problems, we have studied a class of BVPs with integral boundary condition. Further, on the applications of Schauder and Banach theorems, we have established sufficient results for the EU of solution to the proposed problem. In addition, using different concepts of UH, adequate results have been deduced to discuss the stability analysis. Various results in this regard have been derived. Moreover, considering pertinent test problems, all the derived results have been justified. In the future, such kind of analysis can be performed for those problems involving non-singular type kernels and fractal type variable order derivatives.

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