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Blow-up criteria of the simplified Ericksen–Leslie system

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Abstract

In this paper, we establish scaling invariant blow-up criteria for a classical solution to the simplified Ericksen–Leslie system in terms of the positive part of the second eigenvalue of the strain matrix and orientation field in mixed-norm Lebesgue spaces. Our result may be also regarded as an extension or improvement of the corresponding results obatined by Neustupa and Penel (Trends in Partial Differential Equations of Mathematical Physics, pp. 197–212, 2005), Miller (Arch. Ration. Mech. Anal. 235(1):99–139, 2020) and Huang and Wang (Commun. Partial Differ. Equ. 37(5):875–884, 2012).

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1 Introduction

In this paper, we consider the following Cauchy problem for the three-dimensional simplified Ericksen–Leslie system:

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = -\nabla\cdot(\nabla d\odot\nabla d) & \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ \partial_t d + (u\cdot\nabla)d = \Delta d + |\nabla d|^2 d & \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ \nabla\cdot u = 0, \qquad |d| = 1 & \text{in } \mathbb{R}^3\times\mathbb{R}_+, \end{cases} \tag{1.1}$$

where $u:\mathbb{R}^3\times\mathbb{R}_+\to\mathbb{R}^3$ is the unknown velocity field of the flow, $p:\mathbb{R}^3\times\mathbb{R}_+\to\mathbb{R}$ is the scalar pressure and $d:\mathbb{R}^3\times\mathbb{R}_+\to\mathbb{S}^2$, is the macroscopic average of the nematic liquid-crystal orientation field in $\mathbb{R}^3$, $\nabla\cdot u = 0$ represents the incompressible condition, and the notation $\nabla d\odot\nabla d$ denotes the 3×3 matrix whose (i,j)th component is given by $\partial_i d\cdot\partial_j d$ $(1\le i,j\le 3)$. We will consider the Cauchy problem (1.1) with the initial conditions

$$u|_{t=0} = u_0(x), \quad \text{and} \quad d|_{t=0} = d_0(x), \qquad \left|d_0(x)\right| = 1 \quad \text{in } \mathbb{R}^3$$

and far-field behaviors

$$u\to 0, \qquad d\to \bar{d}_0 \quad \text{as } |x|\to\infty,$$

where u_0 is a given initial velocity with $\nabla \cdot u_0 = 0$ in the distribution sense, $d_0 : \mathbb{R}^3 \to \mathbb{S}^2$ is a given initial liquid-crystal orientation field and $\bar{d}_0$ is a constant vector with $|\bar{d}_0| = 1$.

The parabolic system (1.1) was first proposed by Lin [25] as a simplification of the general Ericksen–Leslie system that models the hydrodynamic flow of nematic liquid-crystal material [11, 20]. The simplified Ericksen–Leslie system (1.1) can be viewed as the incompressible Navier–Stokes equations coupling with the heat flow of a harmonic map, which has been successful in modeling various dynamical behaviors for nematic liquid crystals. When the velocity field u is identically vanishing, the system (1.1) becomes the heat flow of harmonic maps. Chang, Ding, and Ye [6] proved that strong solution blowup in finite time to the harmonic heat-flow equation. Wang [35] established a Serrin-type regularity criteria, which implies that if the solution d blowup at time T_*, then

$$\sup_{0\le t<T_*} \left\| \nabla d(\cdot,t) \right\|_{L^n} = \infty.$$

For a more detailed physical background, please refer to [22, 24] and the references therein.

From the mathematical point of view, the simplified Ericksen–Leslie system (1.1) has recently acquired much interest in the research community. Recently, Lin, Lin, and Wang [23] and Hong [17] independently proved the global existence of Leray–Hopf-type weak solutions to the system (1.1) for any smooth bounded domain in $\mathbb{R}^2$ and the whole space $\mathbb{R}^2$, respectively. For the case of three dimensions, Li and Wang [21] established the existence of a local strong solution with large initial value and the global strong solution with a small initial value for the initial boundary value problem of system (1.1). To characterize the first singular time, Huang and Wang [18] considered the so-called Beale–Kato–Majda-type blow-up criterion, more precisely, they proved $0 < T_* < +\infty$ is the maximal time interval if and only if

$$n = 3, \qquad |\omega| + |\nabla d|^2 \notin L_t^1 L_x^\infty\big(\mathbb{R}^3 \times [0, T_*]\big); \tag{1.2}$$

and

$$n = 2, \qquad |\nabla d|^2 \notin L_t^1 L_x^\infty\big(\mathbb{R}^2 \times [0, T_*]\big), \tag{1.3}$$

where $\omega = \nabla \times u$. Liu and Zhao [26] showed that the smooth solution (u, d) of system (1.1) blows up at the time T_* if and only if

$$\int_0^{T_*} \frac{\|\omega(\cdot,t)\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d(\cdot,t)\|^2_{\dot{B}^2_{\infty,\infty}}}{\sqrt{1 + \ln(e + \|\omega(\cdot,t)\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d(\cdot,t)\|_{\dot{B}^0_{\infty,\infty}})}}\, dt = \infty. \tag{1.4}$$

Chen, Tan, and Wu [9] obtained the Serrin-type regularity criterion [34], which states that

$$u \in L^q\big(0, T; L^p\big), \qquad \nabla d \in L^r\big(0, T; L^s\big), \tag{1.5}$$

with $\frac{2}{q} + \frac{3}{p} = 1$, $\frac{2}{r} + \frac{3}{s} = 1$, $3 < p \le \infty$, $3 < s \le \infty$. Lee [19] obtained the Beirão da Veiga-type blow-up criterion [10], which states that

$$\operatorname{curl} u \in L^q\big(0, T; L^p\big), \qquad \nabla d \in L^r\big(0, T; L^s\big), \tag{1.6}$$

with $\frac{2}{q}+\frac{3}{p}=2$, $\frac{2}{r}+\frac{3}{s}=1$, $\frac{3}{2}<p\leq\infty$, $3<s\leq\infty$. For system (1.1), some refined blow-up criteria of (1.5) and (1.6) are later proven in [13, 14, 41].

When the macroscopic average of the nematic liquid-crystal orientation field is neglected, i.e., d = constant vector, system (1.1) reduces to the incompressible Navier–Stokes equations (in short NSE). Many classical Serrin-type criteria and Beirão da Veiga-type criteria for the regularity of weak solutions have been proved, please refer to [2–4, 7, 8, 12, 16, 32, 33, 37–40].

There is numerical evidence for the Navier–Stokes or Euler equation in [15] regarding the tendency of the vorticity to align with the eigenvector of the strain tensor corresponding to the intermediate eigenvalue λ_2 and later Neustupa and Penel, Chae, and Miller independently gave the analytical evidence of this fact in [5, 27, 29–31]. Specifically, Neustupa and Penel [29–31] and Miller [27] proved that

$$\lambda_2^+\in L^q\bigl(0,T;L^p(\Omega)\bigr),\qquad \frac{2}{q}+\frac{3}{p}=2,\qquad \frac{3}{2}<p\leq\infty \tag{1.7}$$

implies the smoothness of the solution to the Navier–Stokes equations, where $\lambda_2^+(x)=\max\{\lambda_2(x),0\}$, Ω be a bounded domain or $\Omega=\mathbb{R}^3$. Chae [5] proved that the dynamical behaviors of the L^2 norm of vorticity is controlled completely by the second largest eigenvalue λ_2^+ of the deformation tensor for the 3D incompressible Euler equations. Recently, the second named author [36] extended the above regularity criteria to the Multiplier space and Besov space. More recently, Miller [28] extended the Serrin-type and the Beirão da Veiga-type criteria to the Lebeguse sum spaces for singularities of a local smooth solution.

From the physical point of view, the fluid behavior can be different in different directions. Therefore, understanding the asymptotic behaviors of solutions to the simplified Ericksen–Leslie system in anisotropic functional spaces seems to be an interesting topic. This leads us to focus on the blow-up criteria for the 3D simplified Ericksen–Leslie system (1.1) on the framework of the mixed-norm Lebesgue space.

It is well known that if the initial velocity $u_0\in H^s(\mathbb{R}^n,\mathbb{R}^n)$ with $\nabla\cdot u_0=0$ and $d_0\in H^{s+1}(\mathbb{R}^n,\mathbb{S}^2)$ for $s\geq n$, then there is $T_0>0$ depending only on $\|u_0\|_{H^s}$ and $\|d_0\|_{H^{s+1}}$ such that (1.1) has a unique, classical solution (u,d) in $\mathbb{R}^n\times[0,T_0)$ satisfying

$$\begin{aligned}
&u\in C\bigl([0,T],H^s\bigl(\mathbb{R}^n\bigr)\bigr)\cap C^1\bigl([0,T],H^{s-1}\bigl(\mathbb{R}^n\bigr)\bigr),\\
&d\in C\bigl([0,T],H^{s+1}\bigl(\mathbb{R}^n,\mathbb{S}^2\bigr)\bigr)\cap C^1\bigl([0,T],H^{s}\bigl(\mathbb{R}^n,\mathbb{S}^2\bigr)\bigr)
\end{aligned}\tag{1.8}$$

for any $0<T<T_0$. Assume $T_*>0$ is the maximum value such that (1.8) holds with $T_0=T_*$. We would like to characterize such a T_*. To facilitate the presentation of the result, let us first recall the definition of the mixed-norm Lebesgue space.

Definition 1.1 For a given $\vec{p}=(p_1,p_2,p_3)\in[1,\infty)^3$, the mixed norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^d)$ is defined to be the space consisting of all measurable functions $f:\mathbb{R}^3\to\mathbb{R}$ such that the norm

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^3)}=\biggl(\int_{-\infty}^{\infty}\biggl(\int_{-\infty}^{\infty}\biggl(\int_{-\infty}^{\infty}\bigl|f(x)\bigr|^{p_1}\,dx_1\biggr)^{\frac{p_2}{p_1}}\,dx_2\biggr)^{\frac{p_3}{p_2}}\,dx_3\biggr)^{\frac{1}{p_3}}<\infty.$$

Similar definitions can be formulated if any of $\{p_1,p_2,p_3\}$ is the same as ∞.

Now, we state our main result as follows:

Theorem 1.1 *For $s \geq 3$, $u_0 \in H^s(\mathbb{R}^3, \mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ and $d_0 \in H^{s+1}(\mathbb{R}^3, \mathbb{S}^2)$, let $T_* > 0$ be the maximum value such that system (1.1) has a unique solution (u, d) satisfying (1.8) with T_0 replaced by T_*, and let $\lambda_1(x) \leq \lambda_2(x) \leq \lambda_3(x)$ be the eigenvalues of the strain tensor $S = \nabla_{\mathrm{sym}} u = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. Let $\lambda_2^+(x) = \max\{\lambda_2(x), 0\}$. If $T_* < +\infty$, then*

$$\int_0^{T_*} \frac{\|\lambda_2^+(t)\|_{L^{\vec{p}}}^q + \|\nabla d(t)\|_{L^{2\vec{p}}}^{2q}}{1 + \ln(e + \|u\|_{L^{\vec{s}}} + \|\nabla d\|_{L^{\vec{s}}})}\, dt = \infty, \tag{1.9}$$

with $2 < p_i \leq \infty$, $\frac{2}{q} + \sum_{i=1}^3 \frac{1}{p_i} = 2$, $1 - \sum_{i=1}^3 \frac{1}{p_i} \geq 0$ and $2 < s_3 < \infty$, $\sum_{i=1}^3 \frac{1}{s_i} = \frac{1}{2}$.

Remark 1.1 We note that when $p_1 = p_2 = p_3 = p$, the mixed-norm Lebesgue space $L^{\vec{p}}$ is reduced to the usual Lebesgue space L^p. Theorem 1.1 naturally extends and improves the blow-up criteria as stated in [18, 19, 27]. In addition, we show the logarithmic blow-up criterion. To the authors' knowledge, Theorem 1.1 is the improvement result on blow-up criteria via the mixed Lebesgue norm in the denominator.

Remark 1.2 From Theorem 1.1, it is easy to see that if there exists a constant $M > 0$ such that

$$\int_0^{T_*} \frac{\|\lambda_2^+(t)\|_{L^{\vec{p}}}^q + \|\nabla d(t)\|_{L^{2\vec{p}}}^{2q}}{1 + \ln(e + \|u\|_{L^{\vec{s}}} + \|\nabla d\|_{L^{\vec{s}}})}\, dt \leq M,$$

with $2 < p_i \leq \infty$, $\frac{2}{q} + \sum_{i=1}^3 \frac{1}{p_i} = 2$, $1 - \sum_{i=1}^3 \frac{1}{p_i} \geq 0$, and $2 < s_3 < \infty$, $\sum_{i=1}^3 \frac{1}{s_i} = \frac{1}{2}$, then the local smooth solution (u, d) can be extended beyond the time T_*.

The proof of Theorem 1.1 will be given in Sect. 2. Before concluding this section, we list the following lemmas that are needed in Sect. 2.

Lemma 1.1 ([27]) *For all $-\frac{3}{2} < \alpha < \frac{3}{2}$ and for all u divergence free in the sense that $\xi \cdot \hat{u}(\xi) = 0$ almost everywhere,*

$$\|S\|_{\dot{H}^\alpha}^2 = \|A\|_{\dot{H}^\alpha}^2 = \frac{1}{2}\|\omega\|_{\dot{H}^\alpha}^2 = \frac{1}{2}\|\nabla \otimes u\|_{\dot{H}^\alpha}^2, \tag{1.10}$$

where the symmetric part $S = S_{ij} = \frac{1}{2}(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j})$, which we refer to as the strain tensor, and the antisymmetric part $A = A_{ij} = \frac{1}{2}(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j})$, $\omega = \nabla \times u$.

Lemma 1.2 ([27]) *Suppose $S \in L^\infty([0,T]; L^2(\mathbb{R}^3)) \cap L^2([0,T] : \dot{H}^1(\mathbb{R}^3))$ is a local strong solution to the Navier–Stokes strain equation and $S(x)$ has eigenvalues $\lambda_1(x) \leq \lambda_2(x) \leq \lambda_3(x)$. Define*

$$\lambda_2^+(x) = \max\{\lambda_2(x), 0\},$$

then

$$-\det(S) \leq \frac{1}{2}|S|^2 \lambda_2^+.$$

Lemma 1.3 ([42]) *For $p_1, p_2, p_3 \in [2,\infty)$ and $0 \le \sum_{i=1}^{3}\frac{1}{p_i} - \frac{1}{2} \le 1$, there exists a positive constant C such that*

$$\begin{aligned}\|f\|_{L^{\vec{p}}(\mathbb{R}^3)} &\le C\|\partial_1 f\|_{L^2(\mathbb{R}^3)}^{\frac{p_1-2}{2p_1}}\|\partial_2 f\|_{L^2(\mathbb{R}^3)}^{\frac{p_2-2}{2p_2}}\|\partial_3 f\|_{L^2(\mathbb{R}^3)}^{\frac{p_3-2}{2p_3}}\|f\|_{L^2(\mathbb{R}^3)}^{\sum_{i=1}^{3}\frac{1}{p_i-\frac{1}{2}}} \\ &\le C\|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}-\sum_i\frac{1}{p_i}}\|f\|_{L^2(\mathbb{R}^3)}^{\sum_i\frac{1}{p_i}-\frac{1}{2}}.\end{aligned}$$

Lemma 1.4 ([1]) *Let $\vec{s} = (s_1, s_2, s_3) \in [2,\infty]^3$ satisfy*

$$\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} = \frac{1}{2} \quad \textit{and} \quad s_3 \in (2,\infty).$$

Then, there exist constants $C = C(\vec{s})$ such that

$$\|u\|_{L^{\vec{s}}(\mathbb{R}^3)} \le C\big[\|Du\|_{L^2(\mathbb{R}^3)} + \|u\|_{L^2(\mathbb{R}^3)}\big], \quad \forall u \in W^{1,2}(\mathbb{R}^3).$$

2 The proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We assume that (1.9) was not true, then there exists a positive constant K such that

$$\int_0^{T_*}\frac{\|\lambda_2^+(t)\|_{L^{\vec{p}}}^q + \|\nabla d(t)\|_{L^{2\vec{p}}}^{2q}}{1 + \ln(e + \|u\|_{L^{\vec{s}}} + \|\nabla d\|_{L^{\vec{s}}})}\,dt \le K. \tag{2.1}$$

Due to the Beale–Kato–Majda-type blowup criterion (1.2) in [18], it suffices to present the bound

$$\int_0^{T_*}\left(\|\omega(t)\|_{L^\infty} + \|\nabla d(t)\|_{L^\infty}^2\right)dt \le C$$

under condition (2.1), which is enough to guarantee the extension of a local smooth solution (u,d) beyond the time T_*. That is to say, $[0,T_*)$ is not a maximal existence interval, and we obtain the desired contradiction.

Proposition 2.1 (Strain reformulation of the dynamics) *Suppose (u,d) is a classical solution to the system (1.1). Then, strain tensor $S = \nabla_{\mathrm{sym}}(u)$ sastifies*

$$\partial_t S + (u\cdot\nabla)S - \Delta S + S^2 + \frac{1}{4}\omega\otimes\omega - \frac{1}{4}|\omega|^2 I_3 + \mathrm{Hess}(P) = -\nabla_{\mathrm{sym}}(\Delta d\cdot\nabla d),$$

where $P = p + \frac{|\nabla d|^2}{2}$.

Proof By Proposition 2.1 in [27] and noting that

$$\nabla\cdot(\nabla d\otimes\nabla d) = \nabla\left(\frac{|\nabla d|^2}{2}\right) + \Delta d\cdot\nabla d,$$

this implies Proposition 2.1 immediately. □

Proof of Theorem 1.1 First, we give the basic energy estimate of system (1.1). Taking the inner product of $(1.1)_1$ with u and $(1.1)_2$ with $-\Delta d$ in $L^2(\mathbb{R}^3)$, respectively, and adding together, one has

$$\frac{1}{2}\frac{d}{dt}\big\|(u,\nabla d)\big\|_{L^2}^2+\|\nabla u\|_{L^2}^2=-\int_{\mathbb{R}^3}\Delta d\cdot\big(\Delta d+|\nabla d|^2 d\big)\,dx$$

and then, we discover that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big\|(u,\nabla d)\big\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\big\|\Delta d+|\nabla d|^2 d\big\|_{L^2}^2\\&\quad=\int_{\mathbb{R}^3}|\nabla d|^2 d\cdot\big(\triangle d+|\nabla d|^2 d\big)\,dx=0,\end{aligned}$$

where we have used the facts that

$$|d|^2=1\quad\Rightarrow\quad 0=\frac{1}{2}\Delta|d|^2=d\cdot\Delta d+|\nabla d|^2.$$

Next, we derive the H^1 estimate for $(u,\nabla d)$. Taking $\nabla\times$ on the first equation of (1.1), we obtain

$$\partial_t\omega+(u\cdot\nabla)\omega-\Delta\omega=S\omega-\nabla\times(\Delta d\cdot\nabla d)\tag{2.2}$$

and then taking the operator ∇_{sym} (*i.e.*, $S=\nabla_{\mathrm{sym}}(u)_{ij}=\frac{1}{2}(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j})$) to the $(1.1)_1$, one obtains

$$\partial_t S+(u\cdot\nabla)S-\Delta S+S^2+\frac{1}{4}\omega\otimes\omega-\frac{1}{4}|\omega|^2 I_3+\mathrm{Hess}(P)=-\nabla_{\mathrm{sym}}(\Delta d\cdot\nabla d),\tag{2.3}$$

where $P=p+\frac{|\nabla d|^2}{2}$. Multiplying (2.2) by ω and integrating by parts over $\mathbb{R}^3$, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2&=\int_{\mathbb{R}^3}S\omega\cdot\omega\,dx-\int_{\mathbb{R}^3}\nabla\times(\Delta d\cdot\nabla d)\cdot\omega\,dx\\&=\int_{\mathbb{R}^3}S\omega\cdot\omega\,dx+\int_{\mathbb{R}^3}(\Delta d\cdot\nabla d)\cdot\Delta u\,dx.\end{aligned}\tag{2.4}$$

Multiplying (2.3) by S, we deduce that

$$\begin{aligned}&\frac{d}{dt}\|S\|_{L^2}^2+2\|\nabla S\|_{L^2}^2\\&\quad=-2\int_{\mathbb{R}^3}\mathrm{tr}\big(S^3\big)\,dx-\frac{1}{2}\int_{\mathbb{R}^3}\omega\otimes\omega\cdot S\,dx+\int_{\mathbb{R}^3}(\Delta d\cdot\nabla d)\cdot\Delta u\,dx,\end{aligned}\tag{2.5}$$

where we used the following facts that were proved by Proposition 2.4 and Theorem 4.5 in [27]

$$\big\langle|\omega|^2 I_3,S\big\rangle_{L^2}=0,$$

$$\big\langle\mathrm{Hess}(P),S\big\rangle_{L^2}=0,$$

$$\left\langle S^2, S\right\rangle_{L^2} = \int_{\mathbb{R}^3} \operatorname{tr}\left(S^3\right),$$
$$2\left\langle \nabla_{\mathrm{sym}}(\Delta d \cdot \nabla d), S\right\rangle_{L^2} = \left\langle (\Delta d \cdot \nabla d), -\Delta u\right\rangle_{L^2}.$$

From Lemma 1.1, for the identity (2.4), it follows that

$$\frac{d}{dt}\|S\|_{L^2}^2 + 2\|\nabla S\|_{L^2}^2 = \int_{\mathbb{R}^3} S\omega \otimes \omega \, dx + \int_{\mathbb{R}^3} (\Delta d \cdot \nabla d) \cdot \Delta u \, dx. \tag{2.6}$$

Adding $\frac{2}{3}$ (2.5) and $\frac{1}{3}$ (2.6), we conclude that

$$\begin{aligned} \frac{d}{dt}\|S\|_{L^2}^2 + 2\|\nabla S\|_{L^2}^2 &= -\frac{4}{3} \int_{\mathbb{R}^3} \operatorname{tr}\left(S^3\right) + \int_{\mathbb{R}^3} (\Delta d \cdot \nabla d) \cdot \Delta u \, dx \\ &\stackrel{\text{def}}{=} I_1 + I_2, \end{aligned} \tag{2.7}$$

where

$$I_1 = -\frac{4}{3} \int_{\mathbb{R}^3} \operatorname{tr}\left(S^3\right), \qquad I_2 = \int_{\mathbb{R}^3} (\Delta d \cdot \nabla d) \cdot \Delta u \, dx. \tag{2.8}$$

We first estimate the term I_1, since $\operatorname{tr}(S) = \nabla \cdot u = 0$, it follows from Lemma 1.2 and Lemma 1.3 that

$$\begin{aligned} I_1 &= -\frac{4}{3} \int_{\mathbb{R}^3} \operatorname{tr}\left(S^3\right) dx \\ &= -\frac{4}{3} \int_{\mathbb{R}^3} \lambda_1^3 + \lambda_2^3 + \lambda_3^3 \, dx \\ &= -\frac{4}{3} \int_{\mathbb{R}^3} \lambda_1^3 + \lambda_2^3 + (-\lambda_1 - \lambda_2)^3 \, dx \\ &= -4 \int_{\mathbb{R}^3} (-\lambda_1 - \lambda_2)\lambda_1\lambda_2 \, dx = -4 \int_{\mathbb{R}^3} \lambda_1\lambda_2\lambda_3 \, dx \\ &= -4 \int_{\mathbb{R}^3} \det(S) \, dx \le 2 \int_{\mathbb{R}^3} |S|^2 \lambda_2^+ \, dx \\ &\le C\left\|\lambda_2^+\right\|_{L^{\vec{p}}} \|S\|_{L_1^{\frac{2p_1}{p_1-2}} L_2^{\frac{2p_2}{p_2-2}} L_3^{\frac{2p_3}{p_3-2}}} \|S\|_{L^2} \\ &\le C\left\|\lambda_2^+\right\|_{L^{\vec{p}}} \|\nabla S\|_{L^2}^{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}} \|S\|_{L^2}^{2-(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3})} \\ &\le C\left\|\lambda_2^+\right\|_{L^{\vec{p}}}^q \|S\|_{L^2}^2 + \frac{1}{8}\|\nabla S\|_{L^2}^2, \end{aligned} \tag{2.9}$$

where $q = \frac{2}{2-(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3})}$, $2 < p_i \le \infty$.

For the term I_2, it follows from Lemma 1.3, Hölder's inequality, and Young's inequality that

$$\begin{aligned} I_2 &\le C\|\nabla d\|_{L^{2\vec{p}}} \|\Delta d\|_{L_1^{\frac{2p_1}{p_1-1}} L_2^{\frac{2p_2}{p_2-1}} L_3^{\frac{2p_3}{p_3-1}}} \|\Delta u\|_{L^2} \\ &\le C\|\nabla d\|_{L^{2\vec{p}}} \|\nabla \Delta d\|_{L^2}^{\frac{1}{2p_1} + \frac{1}{2p_2} + \frac{1}{2p_3}} \|\Delta d\|_{L^2}^{1-(\frac{1}{2p_1} + \frac{1}{2p_2} + \frac{1}{2p_3})} \|\Delta u\|_{L^2} \end{aligned} \tag{2.10}$$

$$
\begin{aligned}
&\le C\|\nabla d\|_{L^{2\vec{p}}}^2 \|\nabla\Delta d\|_{L^2}^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}} \|\Delta d\|_{L^2}^{2-(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})} + \frac{1}{8}\|\nabla S\|_{L^2}^2 \\
&\le C\|\Delta d\|_{L^2}^2 \|\nabla d\|_{L^{2\vec{p}}}^{2q} + \frac{1}{8}\|\nabla\Delta d\|_{L^2}^2 + \frac{1}{8}\|\nabla S\|_{L^2}^2,
\end{aligned}
$$

where $q = \frac{2}{2-(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})}$, $2 < p_i \le \infty$.

Inserting (2.9) and (2.10) into (2.7), it follows that

$$
\frac{d}{dt}\|S\|_{L^2}^2 + \|\nabla S\|_{L^2}^2 \le C\|\lambda_2^+\|_{L^{\vec{p}}}^q \|S\|_{L^2}^2 + C\|\Delta d\|_{L^2}^2 \|\nabla d\|_{L^{2\vec{p}}}^{2q} + \frac{1}{8}\|\nabla\Delta d\|_{L^2}^2. \tag{2.11}
$$

For the estimate of Δd, taking Δ on the second equation of (1.1), multiplying by Δd, and integrating over $\mathbb{R}^3$, one obtains

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta d\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2 &= -\int_{\mathbb{R}^3} \Delta\big((u\cdot\nabla)d\big)\cdot\Delta d\,dx + \int_{\mathbb{R}^3} \Delta\big(|\nabla d|^2 d\big)\cdot\Delta d\,dx \\
&\overset{\text{def}}{=} J_1 + J_2,
\end{aligned} \tag{2.12}
$$

where

$$
J_1 = -\int_{\mathbb{R}^3} \partial_{ll}(u_i\partial_i d_k)\partial_{jj} d_k\,dx, \qquad J_2 = \int_{\mathbb{R}^3} \Delta\big(|\nabla d|^2 d\big)\cdot\Delta d\,dx. \tag{2.13}
$$

We now obtain the estimate of the term J_1. Since

$$
\int_{\mathbb{R}^3} (u\cdot\nabla\Delta d)\cdot\Delta d\,dx = \frac{1}{2}\int_{\mathbb{R}^3} (u\cdot\nabla)\big(|\Delta d|^2\big)\,dx = 0,
$$

it follows from Lemma 1.3 and Young's inequality that

$$
\begin{aligned}
J_1 &= -\int_{\mathbb{R}^3} \partial_{ll}(u_i\partial_i d_k)\partial_{jj} d_k\,dx \\
&= \int_{\mathbb{R}^3} \partial_l(u_i\partial_i d_k)\partial_l\partial_{jj} d_k\,dx \\
&= \int_{\mathbb{R}^3} \partial_l u_i\partial_i d_k\partial_l\partial_{jj} d_k\,dx + \int_{\mathbb{R}^3} u_i\partial_i\partial_l d_k\partial_l\partial_{jj} d_k\,dx \\
&= \int_{\mathbb{R}^3} \partial_l u_i\partial_i d_k\partial_l\partial_{jj} d_k\,dx - \int_{\mathbb{R}^3} \partial_l u_i\partial_i\partial_l d_k\partial_{jj} d_k\,dx \\
&\le C\int_{\mathbb{R}^3} |\nabla u||\nabla d||\nabla\Delta d|\,dx \\
&\le C\|\nabla d\|_{L^{2\vec{p}}}\|\nabla u\|_{L_1^{\frac{2p_1}{p_1-1}}L_2^{\frac{2p_2}{p_2-1}}L_3^{\frac{2p_3}{p_3-1}}}\|\nabla\Delta d\|_{L^2} \\
&\le C\|\nabla d\|_{L^{2\vec{p}}}\big\|\nabla^2 u\big\|_{L^2}^{\frac{1}{2p_1}+\frac{1}{2p_2}+\frac{1}{2p_3}}\|\nabla u\|_{L^2}^{1-(\frac{1}{2p_1}+\frac{1}{2p_2}+\frac{1}{2p_3})}\|\nabla\Delta d\|_{L^2} \\
&\le C\|S\|_{L^2}^2\|\nabla d\|_{L^{2\vec{p}}}^{2q} + \frac{1}{8}\|\nabla\Delta d\|_{L^2}^2 + \frac{1}{4}\|\nabla S\|_{L^2}^2,
\end{aligned} \tag{2.14}
$$

where $q = \frac{2}{2-(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})}$, $2 < p_i \le \infty$.

For the term J_2, after integration by parts, by using the Young inequality and Lemma 1.3, and the fact $|d| = 1$, one has

$$\begin{aligned}
J_2 &\le \left| \int_{\mathbb{R}^3} \Delta\left(|\nabla d|^2 d\right) \cdot \Delta d \, dx \right| = \left| \int_{\mathbb{R}^3} \nabla\left(|\nabla d|^2 d\right) \cdot \nabla \Delta d \, dx \right| \\
&\le \left| \int_{\mathbb{R}^3} \nabla d |\nabla d|^2 \cdot \nabla \Delta d \, dx \right| + \left| \int_{\mathbb{R}^3} d \nabla\left(|\nabla d|^2\right) \cdot \nabla \Delta d \, dx \right| \\
&\le \left| \int_{\mathbb{R}^3} \nabla\left(\nabla d |\nabla d|^2\right) \cdot \Delta d \, dx \right| + \left| \int_{\mathbb{R}^3} d \nabla\left(|\nabla d|^2\right) \cdot \nabla \Delta d \, dx \right| \\
&\le C \left| \int_{\mathbb{R}^3} |\nabla d|^2 \left|\nabla^2 d\right|^2 dx \right| + C \left| \int_{\mathbb{R}^3} |\nabla d| \left|\nabla^2 d\right| |\nabla \Delta d| \, dx \right| \\
&\le C \|\nabla d\|_{L^{2\vec{p}}}^2 \left\|\nabla^2 d\right\|^2_{L_1^{\frac{2p_1}{p_1-1}} L_2^{\frac{2p_2}{p_2-1}} L_3^{\frac{2p_3}{p_3-1}}} + \frac{1}{8} \|\nabla \Delta d\|_{L^2}^2 \\
&\le C \|\nabla d\|_{L^{2\vec{p}}}^2 \|\nabla \Delta d\|_{L^2}^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}} \|\Delta d\|_{L^2}^{2-(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})} + \frac{1}{8} \|\nabla \Delta d\|_{L^2}^2 \\
&\le C \|\Delta d\|_{L^2}^2 \|\nabla d\|_{L^{2\vec{p}}}^{2q} + \frac{1}{4} \|\nabla \Delta d\|_{L^2}^2.
\end{aligned} \tag{2.15}$$

Combining (2.12) and (2.14) with (2.15), we obtain

$$\frac{d}{dt} \|\Delta d\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2 \le C\left(\|S\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\right) \|\nabla d\|_{L^{2\vec{p}}}^{2q} + \frac{1}{4} \|\nabla S\|_{L^2}^2. \tag{2.16}$$

Adding (2.11) and (2.16) together and using Lemma 1.4, we arrive at

$$\begin{aligned}
&\frac{d}{dt}\left(\|S\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\right) + \frac{1}{2}\|\nabla S\|_{L^2}^2 + \frac{1}{2}\|\nabla \Delta d\|_{L^2}^2 \\
&\quad \le C \left\|\lambda_2^+\right\|_{L^{\vec{p}}}^q \|S\|_{L^2}^2 + C\left(\|S\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\right) \|\nabla d\|_{L^{2\vec{p}}}^{2q} \\
&\quad \le C \frac{\|\lambda_2^+\|_{L^{\vec{p}}}^q + \|\nabla d\|_{L^{2\vec{p}}}^{2q}}{1 + \ln(e + \|u\|_{L^{\vec{s}}} + \|\nabla d\|_{L^{\vec{s}}})} \left(1 + \ln\left(e + \|u\|_{L^{\vec{s}}} + \|\nabla d\|_{L^{\vec{s}}}\right)\right) \\
&\qquad \cdot \left(\|S\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\right) \\
&\quad \le C \frac{\|\lambda_2^+\|_{L^{\vec{p}}}^q + \|\nabla d\|_{L^{2\vec{p}}}^{2q}}{1 + \ln(e + \|u\|_{L^{\vec{s}}} + \|\nabla d\|_{L^{\vec{s}}})} \\
&\qquad \cdot \left(1 + \ln\left(e + \|u\|_{L^2}^2 + \|S\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + C\right)\right) \\
&\qquad \cdot \left(\|S\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\right).
\end{aligned} \tag{2.17}$$

Combining the basic energy estimates and (2.17), we see that

$$\begin{aligned}
&\frac{d}{dt}\left(1 + \ln\left(e + \|u\|_{L^2}^2 + \|S\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + C\right)\right) \\
&\quad \le C \frac{\|\lambda_2^+\|_{L^{\vec{p}}}^q + \|\nabla d\|_{L^{2\vec{p}}}^{2q}}{1 + \ln(e + \|u\|_{L^{\vec{s}}} + \|\nabla d\|_{L^{\vec{s}}})} \\
&\qquad \cdot \left(1 + \ln\left(e + \|u\|_{L^2}^2 + \|S\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + C\right)\right),
\end{aligned} \tag{2.18}$$

which together with the Grönwall inequality leads to

$$\begin{aligned}
&\sup_{0\le t\le T_*}\ln\bigl(e+\|u\|_{L^2}^2+\|S\|_{L^2}^2+\|\nabla d\|_{L^2}^2+\|\Delta d\|_{L^2}^2+C\bigr)\\
&\quad\le\bigl(1+\ln\bigl(e+\|u_0\|_{L^2}^2+\|S_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2+\|\Delta d_0\|_{L^2}^2+C\bigr)\bigr)\\
&\qquad\cdot\exp C\int_0^{T_*}\frac{\|\lambda_2^+\|_{L^{\bar p}}^q+\|\nabla d\|_{L^{2\bar p}}^{2q}}{1+\ln(e+\|u\|_{L^{\bar s}}+\|\nabla d\|_{L^{\bar s}})}\,dt,
\end{aligned}\tag{2.19}$$

which implies that

$$u,\nabla d\in L^\infty\bigl(0,T_*;H^1\bigl(\mathbb{R}^3\bigr)\bigr)\cap L^2\bigl(0,T_*;H^2\bigl(\mathbb{R}^3\bigr)\bigr).\tag{2.20}$$

To estimate

$$\int_0^{T_*}\bigl(\bigl\|\omega(t)\bigr\|_{L^\infty}+\bigl\|\nabla d(t)\bigr\|_{L^\infty}^2\bigr)\,dt\le C,$$

let us establish a higher-order estimate for $(u,\nabla d)$. Applying Δ and $\nabla\Delta$ to the first equation of (1.1) and the second equation of (1.2), respectively, and then multiplying the resulting equations by Δu and $\nabla\Delta d$, we have

$$\begin{aligned}
&\frac12\frac{d}{dt}\bigl(\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2\bigr)+\|\nabla\Delta u\|_{L^2}^2+\bigl\|\Delta^2 d\bigr\|_{L^2}^2\\
&\quad=-\int_{\mathbb{R}^3}\Delta(u\cdot\nabla u)\cdot\Delta u\,dx-\int_{\mathbb{R}^3}\Delta(\nabla d\cdot\Delta d)\cdot\Delta u\,dx\\
&\qquad-\int_{\mathbb{R}^3}\nabla\Delta(u\cdot\nabla d)\cdot\nabla\Delta d\,dx+\int_{\mathbb{R}^3}\nabla\Delta\bigl(|\nabla d|^2d\bigr)\cdot\nabla\Delta d\,dx\\
&\quad\stackrel{\text{def}}{=}K_1+K_2+K_3+K_4,
\end{aligned}\tag{2.21}$$

where

$$K_1=-\int_{\mathbb{R}^3}\Delta(u\cdot\nabla u)\cdot\Delta u\,dx,\qquad K_2=-\int_{\mathbb{R}^3}\Delta(\nabla d\cdot\Delta d)\cdot\Delta u\,dx\tag{2.22}$$

and

$$K_3=-\int_{\mathbb{R}^3}\nabla\Delta(u\cdot\nabla d)\cdot\nabla\Delta d\,dx,\qquad K_4=\int_{\mathbb{R}^3}\nabla\Delta\bigl(|\nabla d|^2d\bigr)\cdot\nabla\Delta d\,dx.\tag{2.23}$$

Noting the fact that

$$u,\nabla d\in L^\infty\bigl(0,T_*;H^1\bigl(\mathbb{R}^3\bigr)\bigr)\cap L^2\bigl(0,T_*;H^2\bigl(\mathbb{R}^3\bigr)\bigr),$$

we shall establish the bounds of K_1, K_2, K_3, and K_4. By integrating by parts, Hölder's inequality, and Young's inequality, we can estimate K_1 as

$$\begin{aligned}
K_1 &\le \epsilon\|\nabla\Delta u\|_{L^2}^2 + C\big(\big\|(\nabla u\cdot\nabla)u\big\|_{L^2}^2 + \big\|(u\cdot\nabla)\nabla u\big\|_{L^2}^2\big) \\
&\le \epsilon\|\nabla\Delta u\|_{L^2}^2 + C\big(\|\nabla u\|_{L^4}^4 + \|u\|_{L^6}^2\|\Delta u\|_{L^3}^2\big) \\
&\le \epsilon\|\nabla\Delta u\|_{L^2}^2 + C\big(\|\nabla u\|_{L^2}^{\frac{5}{2}}\|\nabla\Delta u\|_{L^2}^{\frac{3}{2}} + \|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}\|\nabla\Delta u\|_{L^2}\big) \\
&\le \epsilon\|\nabla\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{10} + \epsilon\|\nabla\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\Delta u\|_{L^2}^2 + \epsilon\|\nabla\Delta u\|_{L^2}^2 \\
&\le 3\epsilon\|\nabla\Delta u\|_{L^2}^2 + C\big(\|\Delta u\|_{L^2}^2 + 1\big).
\end{aligned} \tag{2.24}$$

For the term K_2, by applying the Hölder inequality, the interpolation inequality, and the Young inequality, we obtain

$$\begin{aligned}
K_2 &= \int_{\mathbb{R}^3} \nabla(\nabla d\cdot\Delta d)\nabla\Delta u\,dx \\
&\le \big(\|\nabla d\|_{L^4}\|\nabla\Delta d\|_{L^4} + \|\Delta d\|_{L^4}\big\|\nabla^2 d\big\|_{L^4}\big)\|\nabla\Delta u\|_{L^2} \\
&\le C\big(\|\Delta d\|_{L^2}^{\frac{3}{4}}\|\nabla\Delta d\|_{L^4} + \|\Delta d\|_{L^4}\big\|\nabla^2 d\big\|_{L^4}\big)\|\nabla\Delta u\|_{L^2} \\
&\le C\|\nabla\Delta d\|_{L^4}^2 + C\big\|\nabla^2 d\big\|_{L^4}^4 + \epsilon\|\nabla\Delta u\|_{L^2}^2 \\
&\le C\|\nabla\Delta d\|_{L^2}^{\frac{1}{2}}\big\|\Delta^2 d\big\|_{L^2}^{\frac{3}{2}} + C\|\Delta d\|_{L^2}^{\frac{5}{2}}\big\|\Delta^2 d\big\|_{L^2}^{\frac{3}{2}} + \epsilon\|\nabla\Delta u\|_{L^2}^2 \\
&\le C\big(\|\nabla\Delta d\|_{L^2}^2 + 1\big) + \epsilon\big\|\Delta^2 d\big\|_{L^2}^2 + \epsilon\|\nabla\Delta u\|_{L^2}^2.
\end{aligned} \tag{2.25}$$

Similarly, for the terms K_3 and K_4, we discover that

$$\begin{aligned}
K_3 &= \int_{\mathbb{R}^3} \Delta\big((u\cdot\nabla)d\big)\cdot\Delta^2 d\,dx \\
&\le \epsilon\big\|\Delta^2 d\big\|_{L^2}^2 + C\big(\big\|(\Delta u\cdot\nabla)d\big\|_{L^2}^2 + \big\|(\nabla u\cdot\nabla)\nabla d\big\|_{L^2}^2 + \big\|(u\cdot\nabla)\Delta d\big\|_{L^2}^2\big) \\
&\le \epsilon\big\|\Delta^2 d\big\|_{L^2}^2 + C\big(\|\Delta u\|_{L^3}^2\|\nabla d\|_{L^6}^2 + \|\nabla u\|_{L^4}^2\|\Delta d\|_{L^4}^2 + \|u\|_{L^6}^2\|\nabla\Delta d\|_{L^3}^2\big) \\
&\le \epsilon\big\|\Delta^2 d\big\|_{L^2}^2 + C\big(\|\Delta u\|_{L^2}\|\nabla\Delta u\|_{L^2} + \|\nabla u\|_{L^2}^{\frac{5}{4}}\|\nabla\Delta u\|_{L^2}^{\frac{3}{4}}\|\Delta d\|_{L^2}^{\frac{5}{4}}\big\|\Delta^2 d\big\|_{L^2}^{\frac{3}{4}} \\
&\quad + \|\nabla\Delta d\|_{L^2}\big\|\Delta^2 d\big\|_{L^2}\big) \\
&\le 2\epsilon\big\|\Delta^2 d\big\|_{L^2}^2 + \epsilon\|\nabla\Delta u\|_{L^2}^2 + C\big(\|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2 + 1\big)
\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}
K_4 &= -\int_{\mathbb{R}^3} \Delta\big(|\nabla d|^2 d\big)\Delta^2 d\,dx \\
&= \int_{\mathbb{R}^3} \big[\Delta\big(|\nabla d|^2\big)d + 2\nabla|\nabla d|^2\nabla d + |\nabla d|^2\Delta d\big]\Delta^2 d \\
&\le C\big(\|\Delta d\Delta d\|_{L^2} + \|\nabla d\nabla\Delta d\|_{L^2} + \|\nabla d\nabla d\Delta d\|_{L^2} + \|d\Delta d\Delta d\|_{L^2}\big)\big\|\Delta^2 d\big\|_{L^2}
\end{aligned} \tag{2.27}$$

$$
\begin{aligned}
&\leq C\big(\|\Delta d\|_{L^4}^2 + \|\nabla d\|_{L^4}\|\nabla\Delta d\|_{L^4}\big)\big\|\Delta^2 d\big\|_{L^2}\\
&\leq C\|\Delta d\|_{L^2}^{\frac{5}{4}}\big\|\Delta^2 d\big\|_{L^2}^{\frac{3}{4}}\big\|\Delta^2 d\big\|_{L^2} + C\|\nabla\Delta d\|_{L^2}^{\frac{1}{4}}\big\|\Delta^2 d\big\|_{L^2}^{\frac{3}{4}}\big\|\Delta^2 d\big\|_{L^2}\\
&\leq C\big(\|\nabla\Delta d\|_{L^2}^2 + 1\big) + \epsilon\big\|\Delta^2 d\big\|_{L^2}^2.
\end{aligned}
$$

Inserting the above estimates (2.24)–(2.27) into (2.21), we conclude that

$$
\begin{aligned}
&\frac{d}{dt}\big(\|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2\big) + \|\nabla\Delta u\|_{L^2}^2 + \big\|\Delta^2 d\big\|_{L^2}^2\\
&\quad\leq C\big(\|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2 + 1\big),
\end{aligned}
\tag{2.28}
$$

which leads to

$$
u, \nabla d \in L^\infty\big(0, T_*; H^2\big(\mathbb{R}^3\big)\big) \cap L^2\big(0, T_*; H^3\big(\mathbb{R}^3\big)\big). \tag{2.29}
$$

Due to Sobolev embedding $H^2(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$, we have

$$
\int_0^{T_*}\big(\big\|\omega(t)\big\|_{L^\infty} + \big\|\nabla d(t)\big\|_{L^\infty}^2\big)\,dt \leq C.
$$

This completes the proof of Theorem 1.1. □

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There was no mention of data availability in this paper.

Declarations

Competing interests

The authors declare no competing interests.

Author contributions

F. Wu posed the problem and wrote the Sect. 1. Z. Chen wrote the Sect. 2. Both authors discussed and checked all the details. Both authors contributed equally to this work.

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