Gao and Bai *Boundary Value Problems* (2023) 2023:43
https://doi.org/10.1186/s13661-023-01731-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

A Stackelberg reinsurance-investment game with derivatives trading

Rui Gao[1] and Yanfei Bai[2*]

*Correspondence:
baiyanfei0709@163.com
[2]School of Insurance, Shandong University of Finance and Economics, Jinan 250014, China
Full list of author information is available at the end of the article

Abstract

This paper studies a stochastic Stackelberg differential reinsurance-investment game with derivatives trading under a stochastic volatility model. The reinsurer who occupies a monopoly position can price a reinsurance premium and invest her wealth in the financial market consisting of a riskless asset and a stock and derivatives tied to the stock. The insurer, the follower of the Stackelberg game, purchases proportional reinsurance from the reinsurer and invests in the same financial market. The main target of the reinsurer and the insurer is to seek their own optimal strategy to maximize the CARA utility of the relative performance. An explicit equilibrium strategy with derivatives trading is deduced by solving Hamilton-Jacobi-Bellman (HJB) equations sequentially. The equilibrium investment strategy demonstrates that the insurer and the reinsurer imitate each other's investment strategies, showing a herd effect. In numerical experiments, the sensitivity of the equilibrium strategy to model parameters is analyzed. For the optimal investment strategy, we find that a short position in the derivative may switch to a long position with parameters changing, which provides investors with important decision-making information.

Keywords: Stochastic Stackelberg differential game; Reinsurance; Investment; Derivatives trading; Stochastic volatility

1 Introduction

Reinsurance and investment are important means for insurers to manage financial risks and increase returns. In recent years, the research on the optimization of reinsurance and investment under various objectives has received extensive attention. Among them, for example, maximizing the expected utility of terminal wealth (Li et al. [20], Huang et al. [17], Zhao and Rong [28], etc.), minimizing the probability of bankruptcy (Browne [9], Chen et al. [12], Li et al. [19], etc.), and the related research under mean-variance criterion (Bi et al. [8], Zhou et al. [29], Bai et al. [3], etc.).

Most of the above studies are from the perspective of insurer to investigate the optimal reinsurance-investment strategy. However, since the reinsurance contract is signed by both the reinsurer and the insurer, unilateral reinsurance optimization for the insurer may not be practical. In reality, the interests of both parties should be considered in the reinsurance contract. Under Stackelberg differential game framework, the reinsurer that occupies a monopoly position is usually regarded as the leader, and the insurer is the follower. In this paper, we mainly study the premium pricing, the optimal investment strategy of reinsurers, and the optimal reinsurance-investment strategy of insurers the under stochastic volatility model.

Nowadays, more and more attention has been focused on the game in the insurance market. Many experts have studied the zero-sum stochastic differential reinsurance-investment game (among them, Zeng [27], Taksar and Zeng [24], Li et al. [18]) and non-zero-sum stochastic differential reinsurance-investment game (among them, Bensoussan et al. [7], Meng et al. [23], Guan and Liang [15], Yan et al. [26], Deng et al. [14]). This kind of literature mostly studies the competitive game among insurers. In recent years, some experts have studied the stochastic Stackelberg differential game between the insurer and the reinsurer from the perspective of both parties. A stochastic Stackelberg differential reinsurance game is proposed by Chen and Shen [10] to describe the leader-follower relationship between the reinsurer and the insurer. Chen and Shen [11] further investigated a stochastic Stackelberg differential reinsurance game problem with time-inconsistent mean variance. The above literature on stochastic Stackelberg differential game only considers the reinsurance game, not the investment game. In view of the fact that investment can increase the company's earnings for stable operation, based on the CEV model, Bai et al. [4] and Bai et al. [5] studied the bilateral Stackelberg reinsurance-investment game and the multi-party hybrid reinsurance-investment game under different economic environment, respectively.

A basic assumption of the above research is that there is no opportunity to trade derivatives. Generally, the insurer and the reinsurer invest in stocks with stochastic volatility that is not completely related to the stock price, so the risk can not be hedged fully. Nowadays, derivatives trading is increasingly popular in the financial market (Ahn et al. [2], Bakshi and Madan [6], Liu and Pan [22], Liu et al. [21]). Many studies have shown that derivatives can complete the financial market and improve efficiency. Derivatives trading can provide differential exposure to the imperfect instantaneous correlation between stock returns and volatility. Recently, Xue et al. [25] studied the optimal strategy for the insurer with a constant absolute risk aversion (CARA) who manages its business risk using not only stock investment and proportional reinsurance but also trading derivatives.

Based on this intuition, this paper investigates a stochastic Stackelberg differential reinsurance-investment game problem with derivatives trading under stochastic volatility models. The insurer and the reinsurer manage the risk by means of investing in stocks and proportional reinsurance and trading options. The reinsurer who occupies a monopoly position can determine the reinsurance premium pricing and its asset allocation strategy invested in the stock and the option. The insurer, the follower of the Stackelberg game, can determine the proportion of reinsurance according to the price of reinsurance premium and its asset allocation strategy. The explicit Stackelberg equilibrium strategies are deduced by maximizing the CARA utility of relative performance. From the form of investment strategies of the insurer and the reinsurer, we find that they imitate each other's investment strategies, showing a herd effect. Furthermore, in numerical simulations, we analyze the sensitivity of the Stackelberg equilibrium strategy to model parameters.

This paper contributes to the existing research from at least the following two aspects. First, the price process of a risky asset described by Heston's stochastic volatility model is considered to study the equilibrium strategies in the stochastic Stackelberg differential reinsurance and investment game. Second, the derivatives trading is considered in the

Stackelberg game model, and we calculate the optimal asset allocation strategy of both parties of the game.

The remaining paper is constructed as follows. In Sect. 2, we describe the stochastic Stackelberg differential reinsurance and investment game model with derivatives trading and stochastic volatility. In Sect. 3, the equilibrium reinsurance and investment strategy is deduced. In Sect. 4, the numerical simulation is conducted. Section 5 is the conclusion.

2 Model setup

In this paper, we suppose the insurance market where there exists one reinsurer and one insurer. Denote the finite horizon by $[0,T]$ over which the investment and reinsurance behavior occurs. $(\Omega,\mathcal{F},P)$ is a complete probability space, where $\mathcal{F}=\{\mathcal{F}_t\}_{0\le t\le T}$ is a filtration.

2.1 Modeling the surplus processes

Referring to Browne [9], the insurer's risk claim process $\{R_F(t), 0\le t\le T\}$ satisfies

$$dR_F(t)=\lambda_F\mu_F\,dt-\sqrt{\lambda_F(\tilde{\sigma}_F)^2}\,dW_F(t), \tag{2.1}$$

where $\lambda_F>0$ depicts the claim intensity, $0<\mu_F<+\infty$ and $(\tilde{\sigma}_F)^2<+\infty$. $\{W_F(t), t\ge 0\}$ denotes the P-Brownian process.

The insurer's premium rate is denoted as c_F, and its calculation method adopts the expected value premium principle. Then, $c_F=(1+\theta_F)\lambda_F\mu_F$, where $\theta_F>0$ is the insurer's safety loading. The insurer's reinsurance strategy is described by $\{q(t), t\ge 0\}$ with $q(t)\in[0,1]$. Then, the reinsurer will cover $(1-q(t))100\%$ of the claims, and the insurer will cover the remaining. $p(t)\in[c_F,\bar{c}]$ denotes the price of the reinsurance premium at time t, where $\bar{c}=(1+\bar{\theta})\lambda_F\mu_F$, and $\bar{\theta}$ is an upper bound of the reinsurer's relative safety loading.

Then, the insurer's surplus process, $\{X_F(t), 0\le t\le T\}$ is as follows

$$\begin{aligned} dX_F(t)&=c_F\,dt-q(t)\,dR_F(t)-\big(1-q(t)\big)p(t)\,dt\\ &=\big[\theta_F a-\big(1-q(t)\big)\big(p(t)-a\big)\big]\,dt+q(t)\sigma_F\,dW_F(t), \qquad X_F(0)=x_F^0, \end{aligned} \tag{2.2}$$

where the insurer's initial surplus denoted by $x_F^0>0$, $a=\lambda_F\mu_F$, $\sigma_F=\sqrt{\lambda_F(\tilde{\sigma}_F)^2}$.

The reinsurer's surplus process, $\{X_L(t), 0\le t\le T\}$ is as follows

$$\begin{aligned} dX_L(t)&=\big(1-q(t)\big)p(t)\,dt-\big(1-q(t)\big)\,dR_F(t)\\ &=\big(1-q(t)\big)\big(p(t)-a\big)\,dt+\big(1-q(t)\big)\sigma_F\,dW_F(t), \qquad X_L(0)=x_L^0, \end{aligned} \tag{2.3}$$

where the reinsurer's initial surplus denoted by $x_L^0>0$.

2.2 Modeling the financial asset price

We consider a financial market consisting of one riskless asset, one risky asset, and the derivative security. r_0 is the riskless interest rate. The price model of the riskless asset, $\{S_0(t)\}_{t\ge 0}$, is

$$dS_0(t)=r_0S_0(t)\,dt, \qquad S_0(0)=1. \tag{2.4}$$

The risky asset represents the aggregate equity market. Assuming the stock price, $\{S_1(t)\}_{t\geq 0}$, is described by Heston's stochastic volatility model (refer to Heston [16]),

$$\begin{aligned} dS_1(t) &= S_1(t)\big[r_0 + \eta V(t)\big]\,dt + S_1(t)\sqrt{V(t)}\,dW_1(t), \\ dV(t) &= \kappa\big[\bar{v} - V(t)\big]\,dt + \sigma_1\sqrt{V(t)}\big[\rho\,dW_1(t) + \sqrt{1-\rho^2}\,dW_2(t)\big], \end{aligned} \tag{2.5}$$

where η, κ, $\bar{v}$, and σ_1 are all positive constants; $V(t)$ is the instantaneous variance; $\rho \in (-1,1)$ is the correlation between $S_1(t)$ and $V(t)$; $\{W_1(t), t \geq 0\}$ and $\{W_2(t), t \geq 0\}$ are independent P-Brownian processes, and both are independent of $\{W_F(t), t \geq 0\}$. Moreover, we assume $2\kappa\bar{v} > \sigma_1^2$ as mentioned in Cox et al. [13] to ensure that $V(t)$ is almost surely nonnegative.

Denote the derivative price by $S_2(t) = f(S_1(t), V(t))$ depending on the underlying stock price $S_1(t)$ and the volatility $V(t)$. Referring to Xue et al. [25], then, the derivative price satisfies

$$\begin{aligned} dS_2(t) &= r_0 S_2(t)\,dt + \big(f_{s_1} S_1(t) + \sigma_1\rho f_v\big)\big[\eta V(t)\,dt + \sqrt{V(t)}\,dW_1(t)\big] \\ &\quad + \sigma_1\sqrt{1-\rho^2}f_v\big[\xi V(t)\,dt + \sqrt{V(t)}\,dW_2(t)\big], \end{aligned} \tag{2.6}$$

where η and ξ are the risk premiums, $f_{s_1} \neq 0$ and $f_v \neq 0$ are the derivatives of f with respect to S_1 and V.

2.3 Modeling the wealth processes

Assuming no taxes and transaction fees, short-selling is allowed. $b_{F1}(t)$ and $b_{L1}(t)$ are the dollar amount invested in the stock for the insurer and the reinsurer, respectively. $b_{F2}(t)$ and $b_{L2}(t)$ are the dollar amount invested in the derivative. And, the remaining wealth $X_F^{\pi_F}(t) - b_{F1}(t) - b_{F2}(t)$ and $X_L^{\pi_L}(t) - b_{L1}(t) - b_{L2}(t)$ are invested in the riskless asset. We write $\pi_F(t) = (q(t), b_{F1}(t), b_{F2}(t))$ and $\pi_L(t) = (p(t), b_{L1}(t), b_{L2}(t))$.

Denote

$$B^{F_1}(t) = b_{F1}(t) + b_{F2}(t)\frac{f_{s_1}S_1(t) + \sigma_1\rho f_v}{S_2(t)}, \qquad B^{F_2}(t) = b_{F2}(t)\frac{\sigma_1\sqrt{1-\rho^2}f_v}{S_2(t)}, \tag{2.7}$$

$$B^{L_1}(t) = b_{L1}(t) + b_{L2}(t)\frac{f_{s_1}S_1(t) + \sigma_1\rho f_v}{S_2(t)}, \qquad B^{L_2}(t) = b_{L2}(t)\frac{\sigma_1\sqrt{1-\rho^2}f_v}{S_2(t)}. \tag{2.8}$$

Thus, we obtain the wealth processes for the insurer and the reinsurer, respectively, as follows.

$$\begin{aligned} dX_F^{\pi_F}(t) &= \frac{X_F^{\pi_F}(t) - b_{F1}(t) - b_{F2}(t)}{S_0(t)}\,dS_0(t) + \frac{b_{F1}(t)}{S_1(t)}\,dS_1(t) + \frac{b_{F2}(t)}{S_2(t)}\,dS_2(t) \\ &\quad + \big[\theta_F a - \big(1-q(t)\big)\big(p(t)-a\big)\big]\,dt + q(t)\sigma_F\,dW_F(t), \\ &= \big\{\big[r_0 X_F^{\pi_F}(t) + \eta V(t)B^{F_1}(t) + \xi V(t)B^{F_2}(t)\big] + \big[\theta_F a - \big(1-q(t)\big)\big(p(t)-a\big)\big]\big\}\,dt \\ &\quad + B^{F_1}(t)\sqrt{V(t)}\,dW_1(t) + B^{F_2}(t)\sqrt{V(t)}\,dW_2(t) + q(t)\sigma_F\,dW_F(t), \end{aligned} \tag{2.9}$$

$$\begin{aligned} dX_L^{\pi_L}(t) &= \frac{X_L^{\pi_L}(t) - b_{L1}(t) - b_{L2}(t)}{S_0(t)}\,dS_0(t) + \frac{b_{L1}(t)}{S_1(t)}\,dS_1(t) + \frac{b_{L2}(t)}{S_2(t)}\,dS_2(t) \\ &\quad + \big(1-q(t)\big)\big(p(t)-a\big)\,dt + \big(1-q(t)\big)\sigma_F\,dW_F(t), \end{aligned}$$

$$
\begin{aligned}
&= \Big\{ \big[r_0 X_L^{\pi_L}(t) + \eta V(t) B^{L_1}(t) + \xi V(t) B^{L_2}(t) \big] + \big(1 - q(t)\big)\big(p(t) - a\big) \Big\}\, dt \\
&\quad + B^{L_1}(t)\sqrt{V(t)}\, dW_1(t) + B^{L_2}(t)\sqrt{V(t)}\, dW_2(t) + \big(1 - q(t)\big)\sigma_F\, dW_F(t).
\end{aligned} \tag{2.10}
$$

Next, we proceed to deduce the optimal risk exposures $(B^{F_1}(t), B^{F_2}(t), B^{L_1}(t), B^{L_2}(t))$ and then deduce the optimal investment strategies for the insurer and the reinsurer by relations (2.7) and (2.8).

2.4 Modeling the stochastic Stackelberg differential game

This paper takes the derivatives trading into account in the stochastic Stackelberg differential reinsurance-investment game model. Referring to Chen and Shen [10], Chen and Shen [11], Bai et al. [4] and Bai et al. [5], the main target of the game is to find the Stackelberg equilibrium by solving the leader's (reinsurer) and follower's (insurer) optimization problems sequentially. The game problem can be solved by the following procedure:

- Step 1: The reinsurer moves first by announcing its any admissible strategy $(p(\cdot), b_{L_1}(\cdot), b_{L_2}(\cdot))$;
- Step 2: The insurer observes the reinsurer's strategy and decides on its optimal strategy $q^*(\cdot) = \alpha^*(\cdot, p(\cdot), b_{L_1}(\cdot), b_{L_2}(\cdot))$, $b^*_{F_1}(\cdot) = \beta_1^*(\cdot, p(\cdot), b_{L_1}(\cdot), b_{L_2}(\cdot))$, $b^*_{F_2}(\cdot) = \beta_2^*(\cdot, p(\cdot), b_{L_1}(\cdot), b_{L_2}(\cdot))$ by solving its own optimization problem;
- Step 3: Observing that the insurer would execute $\alpha^*(\cdot, p(\cdot), b_{L_1}(\cdot), b_{L_2}(\cdot))$, $\beta_1^*(\cdot, p(\cdot), b_{L_1}(\cdot), b_{L_2}(\cdot))$ and $\beta_2^*(\cdot, p(\cdot), b_{L_1}(\cdot), b_{L_2}(\cdot))$, the reinsurer then decides on its admissible strategy $(p^*(\cdot), b^*_{L_1}(\cdot), b^*_{L_2}(\cdot))$ by solving its own optimization problem.

Due to the psychology of comparison, the reinsurer and the insurer, as the two parties of the game, consider not only the expected utility of their own terminal wealth but also the expected utility of the wealth gap between themselves and the other party. That is to say, the target of the reinsurer is to seek the optimal reinsurance premium pricing strategy and investment strategy such that the expected utility of its relative performance is maximized. For insurer, it is similar. For simplicity, let $\hat{X}_F^{\pi_F}(t) = X_F^{\pi_F}(t) - k_1 X_L^{\pi_L}(t)$, $\hat{X}_L^{\pi_L}(t) = X_L^{\pi_L}(t) - k_2 X_F^{\pi_F}(t)$. Then, we have

$$
\begin{aligned}
d\hat{X}_F^{\pi_F}(t) &= \Big\{ r_0\big[X_F^{\pi_F}(t) - k_1 X_L^{\pi_L}(t)\big] + \big(B^{F_1}(t) - k_1 B^{L_1}(t)\big)\eta V(t) + \big(B^{F_2}(t) - k_1 B^{L_2}(t)\big)\xi V(t) \\
&\quad + \theta_F a - (1 + k_1)\big(1 - q(t)\big)\big(p(t) - a\big)\Big\}\, dt + \big(B^{F_1}(t) - k_1 B^{L_1}(t)\big)\sqrt{V(t)}\, dW_1(t) \\
&\quad + \big(B^{F_2}(t) - k_1 B^{L_2}(t)\big)\sqrt{V(t)}\, dW_2(t) + \big[(1 + k_1) q(t) - k_1\big]\sigma_F\, dW_F(t),
\end{aligned} \tag{2.11}
$$

$$
\begin{aligned}
d\hat{X}_L^{\pi_L}(t) &= \Big\{ r_0\big[X_L^{\pi_L}(t) - k_2 X_F^{\pi_F}(t)\big] + \big(B^{L_1}(t) - k_2 B^{F_1}(t)\big)\eta V(t) + \big(B^{L_2}(t) - k_2 B^{F_2}(t)\big)\xi V(t) \\
&\quad + (1 + k_2)\big(1 - q(t)\big)\big(p(t) - a\big) - k_2\theta_F a\Big\}\, dt + \big(B^{L_1}(t) - k_2 B^{F_1}(t)\big)\sqrt{V(t)}\, dW_1(t) \\
&\quad + \big(B^{L_2}(t) - k_2 B^{F_2}(t)\big)\sqrt{V(t)}\, dW_2(t) + \big[1 - (1 + k_2) q(t)\big]\sigma_F\, dW_F(t),
\end{aligned} \tag{2.12}
$$

where $k_1 \in [0,1]$ describes the sensitivity of the insurer to the reinsurer's performance, and $k_2 \in [0,1]$ describes the sensitivity of the reinsurer to the insurer's performance.

Let $X_L^{\pi_L}(t) = x_L$, $X_F^{\pi_F}(t) = x_F$, $\hat{X}_F^{\pi_F}(t) = X_F^{\pi_F}(t) - k_1 X_L^{\pi_L}(t) = x_F - k_1 x_L \doteq \hat{x}_F$, $\hat{X}_L^{\pi_L}(t) = X_L^{\pi_L}(t) - k_2 X_F^{\pi_F}(t) = x_L - k_2 x_F \doteq \hat{x}_L$, at time $t \in [0,T]$. Let $V(t) = v$, at time $t \in [0,T]$. Then, the admissible strategy is as follows.

Definition 1 (Admissible strategy) $\pi(\cdot) = \pi_L(\cdot) \times \pi_F(\cdot) = (p(\cdot), b_{L1}(\cdot), b_{L2}(\cdot)) \times (q(\cdot), b_{F1}(\cdot), b_{F2}(\cdot))$ is admissible if

(i) $\{\pi_L(t)\}_{t\in[0,T]}$ and $\{\pi_F(t)\}_{t\in[0,T]}$ are $\mathcal{F}$-progressively measurable processes, such that $p(t) \in [c_F, \bar{c}]$, $q(t) \in [0,1]$ for any $t \in [0,T]$;
(ii) $E\{\int_t^T [(B^{F_1}(t))^2 + (B^{F_2}(t))^2]V(t)\,d\ell\} < +\infty$, and $E\{\int_t^T [(B^{L_1}(t))^2 + (B^{L_2}(t))^2]V(t)\,d\ell\} < +\infty$, $\forall \ell \in [t,T]$;
(iii) the equation (2.11) associated with $\pi(\cdot)$ has a unique solution $\hat{X}_F^{\pi_F}(\cdot)$, which satisfies $\{E_{t,\hat{x}_F,v}[\sup|\hat{X}_F^{\pi_F}(\ell)|^2]\}^{\frac{1}{2}} < +\infty$, for $\forall (t,\hat{x}_F,v) \in [0,T]\times\mathbb{R}\times\mathbb{R}$, $\forall \ell \in [t,T]$.
(iv) the equation (2.12) associated with $\pi(\cdot)$ has a unique solution $\hat{X}_L^{\pi_L}(\cdot)$, which satisfies $\{E_{t,\hat{x}_L,v}[\sup|\hat{X}_L^{\pi_L}(\ell)|^2]\}^{\frac{1}{2}} < +\infty$, for $\forall (t,\hat{x}_L,v) \in [0,T]\times\mathbb{R}\times\mathbb{R}$, $\forall \ell \in [t,T]$.

Denote $\Pi = \Pi_L \times \Pi_F$ as the set of all admissible strategies, where Π_L is the set of all admissible strategies of the reinsurer, and Π_F is that of the insurer. Then, the Stackelberg game problem is described by Problem 1.

Problem 1 The insurer's problem can be described by the following optimization problem: for any $\pi_L(\cdot) = (p(\cdot), b_{L1}(\cdot), b_{L2}(\cdot)) \in \Pi_L(\cdot)$, find a map $\pi_F^*(\cdot) = (q^*(\cdot), b_{F_1}^*(\cdot), b_{F_2}^*(\cdot)) = (\alpha^*(\cdot,\pi_L(\cdot)), \beta_1^*(\cdot,\pi_L(\cdot)), \beta_2^*(\cdot,\pi_L(\cdot))) : [0,T]\times\Omega\times\Pi_L \to \Pi_F$ such that the following value function holds:

$$\begin{aligned} &J^F\big(t,\hat{x}_F,v;\pi_L(\cdot),\alpha^*\big(\cdot,\pi_L(\cdot)\big),\beta_1^*\big(\cdot,\pi_L(\cdot)\big),\beta_2^*\big(\cdot,\pi_L(\cdot)\big)\big) \\ &\quad= \sup_{\pi_F(\cdot)\in\Pi_F} J^F\big(t,\hat{x}_F,v;\pi_L(\cdot),\pi_F(\cdot)\big) \\ &\quad= \sup_{\pi_F(\cdot)\in\Pi_F} E_{t,\hat{x}_F,v}\big[U_F\big(\hat{X}_F^{\pi_F}(T)\big)\big], \end{aligned} \tag{2.13}$$

where U_F is the utility function of the insurer. The reinsurer's problem can be described by the following optimization problem: find a $\pi_L^*(\cdot) = (p^*(\cdot), b_{L1}^*(\cdot), b_{L2}^*(\cdot)) \in \Pi_L$ such that the following value function holds:

$$\begin{aligned} &J^L\big(t,\hat{x}_L,v;\pi_L^*(\cdot),\alpha^*\big(\cdot,\pi_L^*(\cdot)\big),\beta_1^*\big(\cdot,\pi_L^*(\cdot)\big),\beta_2^*\big(\cdot,\pi_L^*(\cdot)\big)\big) \\ &\quad= \sup_{\pi_L(\cdot)\in\Pi_L} J^L\big(t,\hat{x}_L,v;\pi_L(\cdot),\alpha^*\big(\cdot,\pi_L(\cdot)\big),\beta_1^*\big(\cdot,\pi_L(\cdot)\big),\beta_2^*\big(\cdot,\pi_L(\cdot)\big)\big) \\ &\quad= \sup_{\pi_L(\cdot)\in\Pi_L} E_{t,\hat{x}_L,v}\big[U_L\big(\hat{X}_L^{\pi_L}(T)\big)\big], \end{aligned} \tag{2.14}$$

where U_L represents the reinsurer's utility function.

Definition 2 The six-tuple $(p^*(\cdot), b_{L1}^*(\cdot), b_{L2}^*(\cdot), \alpha^*(\cdot,p^*(\cdot),b_{L_1}^*(\cdot),b_{L_2}^*(\cdot)), \beta_1^*(\cdot,p^*(\cdot),b_{L_1}^*(\cdot), b_{L_2}^*(\cdot)), \beta_2^*(\cdot,p^*(\cdot),b_{L_1}^*(\cdot),b_{L_2}^*(\cdot)))$ is an equilibrium solution to the Stackelberg game problem 1.

3 Equilibrium solution to the Stackelberg game for CARA preference

Compared with individual investors, the insurer and the reinsurer have considerable wealth, and their risk aversion coefficients are relatively stable and can be regarded as constants in the time interval $[0,T]$. The wealth of the insurer and the reinsurer may be negative due to the randomness of the number and size of future claims. Referring to Yan et al. [26] and so on, the positive condition of the wealth processes is very crucial for some common utility functions (for example, the CRRA utility function and the logarithmic utility function), which is hardly guaranteed, especially when considering relative performance under the game framework. In view of these facts, we assume that both the insurer and the reinsurer are constant absolute risk aversion (CARA) agents whose exponential utility functions are given by

$$U_F(\hat{x}_F) = -\frac{1}{\gamma_F}\exp(-\gamma_F\hat{x}_F), \tag{3.1}$$

$$U_L(\hat{x}_L) = -\frac{1}{\gamma_L}\exp(-\gamma_L\hat{x}_L), \tag{3.2}$$

where $\gamma_F > 0$ is the constant absolute risk aversion coefficients of the insurer, and $\gamma_L > 0$ is the constant absolute risk aversion coefficients of the reinsurer.

3.1 Optimal strategy and value function

In this section, we use the dynamic programming method combined with the procedure mentioned in Sect. 2.4 to solve the Stackelberg game problem.

Step 1. In the stochastic Stackelberg differential game, the reinsurer takes action first by announcing its any admissible strategy $(p(\cdot), b_{L1}(\cdot), b_{L2}(\cdot)) \in \Pi_L$.

Step 2. Based on the reinsurer's strategy $(p(\cdot), b_{L1}(\cdot), b_{L2}(\cdot)) \in \Pi_L$, we solve the insurer's optimization problem (2.13) under the CARA utility function.

Based on the value function of the insurer, we have

$$J^F(t,\hat{x}_F,\nu) = -\frac{1}{\gamma_F}\exp\bigl\{-\gamma_F\varphi^F(t)\hat{x}_F + g_1^F(t)\nu + g_2^F(t)\bigr\}, \tag{3.3}$$

where $\varphi^F(t)$, $g_1^F(t)$ and $g_2^F(t)$ are differentiable functions with $\varphi^F(T) = 1$, $g_1^F(T) = 0$ and $g_2^F(T) = 0$. Thus, for the insurer, the HJB equation is as follows

$$\begin{aligned}
0 = \sup_{\pi_F(\cdot)\in\Pi_F}\Bigl\{ & J_t^F + J_{\hat{x}_F}^F\bigl[r_0\hat{x}_F + \bigl(B^{F_1}(t) - k_1B^{L_1}(t)\bigr)\eta\nu + \bigl(B^{F_2}(t) - k_1B^{L_2}(t)\bigr)\xi\nu \\
& + \theta_F a - (1+k_1)\bigl(1-q(t)\bigr)\bigl(p(t)-a\bigr)\bigr] + \frac{1}{2}\bigl[\bigl(B^{F_1}(t) - k_1B^{L_1}(t)\bigr)^2\nu \\
& + \bigl(B^{F_2}(t) - k_1B^{L_2}(t)\bigr)^2\nu + \bigl[(1+k_1)q(t) - k_1\bigr]^2\sigma_F^2\bigr]J_{\hat{x}_F\hat{x}_F}^F + J_\nu^F\kappa(\bar{\nu}-\nu) \\
& + \frac{1}{2}J_{\nu\nu}^F\sigma_1^2\nu + J_{\hat{x}_F\nu}^F\bigl[\bigl(B^{F_1}(t) - k_1B^{L_1}(t)\bigr)\rho + \bigl(B^{F_2}(t) - k_1B^{L_2}(t)\bigr)\sqrt{1-\rho^2}\bigr]\sigma_1\nu\Bigr\},
\end{aligned} \tag{3.4}$$

where J_t^F, $J_{\hat{x}_F}^F$, and J_ν^F are the first derivatives of J^F with respect to t, $\hat{x}_F$ and ν, respectively. $J_{\hat{x}_F\hat{x}_F}^F$, $J_{\nu\nu}^F$ and $J_{\hat{x}_F\nu}^F$ are the second derivatives of J^F with respect to $\hat{x}_F$ and ν, respectively. Substituting these derivatives into the HJB equation (3.4) yields

$$\begin{aligned}
0 = \sup_{\pi_F(\cdot)\in\Pi_F} J^F(t,\hat{x}_F,\nu)\Bigl\{ & -\gamma_F\hat{x}_F\left(\frac{d\varphi^F(t)}{dt} + r_0\varphi^F(t)\right) + \nu\left[\frac{dg_1^F(t)}{dt} - g_1^F(t)\kappa + \frac{1}{2}\sigma_1^2\bigl(g_1^F(t)\bigr)^2\right. \\
& + \bigl(B^{F_1}(t) - k_1B^{L_1}(t)\bigr)\gamma_F\varphi^F(t)\bigl(-\eta - \rho\sigma_1 g_1^F(t)\bigr) + \bigl(B^{F_2}(t) - k_1B^{L_2}(t)\bigr)\gamma_F\varphi^F(t)\bigl(-\xi \\
& - \sqrt{1-\rho^2}\sigma_1 g_1^F(t)\bigr) + \bigl(B^{F_1}(t) - k_1B^{L_1}(t)\bigr)^2\frac{(\gamma_F\varphi^F(t))^2}{2} \\
& \left. + \bigl(B^{F_2}(t) - k_1B^{L_2}(t)\bigr)^2\frac{(\gamma_F\varphi^F(t))^2}{2}\right]
\end{aligned}$$

$$+\frac{dg_2^F(t)}{dt}+g_1^F(t)\kappa\bar{v}-\gamma_F\varphi^F(t)\theta_F a+\gamma_F\varphi^F(t)(1+k_1)\big(1-q(t)\big)\big(p(t)-a\big)$$
$$+\frac{(\gamma_F\sigma_F\varphi^F(t))^2}{2}\big[(1+k_1)q(t)-k_1\big]^2\bigg\}. \tag{3.5}$$

Based on the first order condition for maximizing the value in (3.5), we have

$$0=\nu\big[\gamma_F\varphi^F(t)\big(-\eta-\rho\sigma_1 g_1^F(t)\big)+\big(B^{F_1}(t)-k_1B^{L_1}(t)\big)\big(\gamma_F\varphi^F(t)\big)^2\big], \tag{3.6}$$
$$0=\nu\big[\gamma_F\varphi^F(t)\big(-\xi-\sqrt{1-\rho^2}\sigma_1 g_1^F(t)\big)+\big(B^{F_2}(t)-k_1B^{L_2}(t)\big)\big(\gamma_F\varphi^F(t)\big)^2\big], \tag{3.7}$$
$$0=-\gamma_F\varphi^F(t)(1+k_1)\big(p(t)-a\big)+\big(\gamma_F\sigma_F\varphi^F(t)\big)^2(1+k_1)\big[(1+k_1)q(t)-k_1\big]. \tag{3.8}$$

By solving equations (3.8), we have

$$B^{F_1}(t)=\frac{\eta+\rho\sigma_1 g_1^F(t)}{\gamma_F\varphi^F(t)}+k_1B^{L_1}(t), \tag{3.9}$$
$$B^{F_2}(t)=\frac{\xi+\sqrt{1-\rho^2}\sigma_1 g_1^F(t)}{\gamma_F\varphi^F(t)}+k_1B^{L_2}(t), \tag{3.10}$$
$$q(t)=\frac{(p(t)-a)}{\gamma_F\varphi^F(t)\sigma_F^2(1+k_1)}+\frac{k_1}{1+k_1}. \tag{3.11}$$

From the above equations, it is not difficult to find that the reinsurance strategy is independent of the investment strategy. Therefore,

$$\alpha^*\big(\cdot,p(\cdot),b_{L_1}(\cdot),b_{L_2}(\cdot)\big),\qquad \beta_1^*\big(\cdot,p(\cdot),b_{L_1}(\cdot),b_{L_2}(\cdot)\big),\qquad \beta_2^*\big(\cdot,p(\cdot),b_{L_1}(\cdot),b_{L_2}(\cdot)\big)$$

could be rewritten as $\alpha^*(\cdot,p(\cdot))$, $\beta_1^*(\cdot,b_{L_1}(\cdot),b_{L_2}(\cdot))$ and $\beta_2^*(\cdot,b_{L_1}(\cdot),b_{L_2}(\cdot))$. From the relations of $B^{F_1}(t)$ and $B^{F_2}(t)$ (i.e., (2.7)) and the scope of $q(t)$, we have

$$\begin{aligned}b_{F_1}^*(t)&=\beta_1^*\big(\cdot,b_{L_1}(\cdot),b_{L_2}(\cdot)\big)\\&=\frac{\sigma_1 f_\nu(\eta\sqrt{1-\rho^2}-\xi\rho)-f_{s_1}S_1(t)(\xi+\sqrt{1-\rho^2}\sigma_1 g_1^F(t))}{\gamma_F\varphi^F(t)\sigma_1\sqrt{1-\rho^2}f_\nu}\\&\quad+k_1B^{L_1}(t)-k_1B^{L_2}(t)\frac{f_{s_1}S_1(t)+\sigma_1\rho f_\nu}{\sigma_1\sqrt{1-\rho^2}f_\nu},\end{aligned} \tag{3.12}$$
$$b_{F_2}^*(t)=\beta_2^*\big(\cdot,b_{L_1}(\cdot),b_{L_2}(\cdot)\big)=\frac{S_2(t)}{\sigma_1\sqrt{1-\rho^2}f_\nu}\left[\frac{\xi+\sqrt{1-\rho^2}\sigma_1 g_1^F(t)}{\gamma_F\varphi^F(t)}+k_1B^{L_2}(t)\right], \tag{3.13}$$
$$q^*\big(t,p(t)\big)=\alpha^*\big(t,p(t)\big)=\left[\frac{(p(t)-a)}{\gamma_F\varphi^F(t)\sigma_F^2(1+k_1)}+\frac{k_1}{1+k_1}\right]\vee 0\wedge 1. \tag{3.14}$$

Substituting (3.9) and (3.10) into the HJB equation (3.5) gives that

$$0=-\gamma_F\hat{x}_F\left(\frac{d\varphi^F(t)}{dt}+r_0\varphi^F(t)\right)+\nu\left[\frac{dg_1^F(t)}{dt}-g_1^F(t)\kappa+\frac{1}{2}\sigma_1^2\big(g_1^F(t)\big)^2\right.$$
$$\left.-\frac{(\eta+\rho\sigma_1 g_1^F(t))^2}{2}-\frac{(\xi+\sqrt{1-\rho^2}\sigma_1 g_1^F(t))^2}{2}\right]+\frac{dg_2^F(t)}{dt}+g_1^F(t)\kappa\bar{v}$$

$$
\begin{aligned}
&- \gamma_F\varphi^F(t)\theta_F a + \gamma_F\varphi^F(t)(1+k_1)\bigl(1-q^*\bigl(t,p(t)\bigr)\bigr)\bigl(p(t)-a\bigr)\\
&+ \frac{(\gamma_F\sigma_F\varphi^F(t))^2}{2}\bigl[(1+k_1)q^*\bigl(t,p(t)\bigr)-k_1\bigr]^2.
\end{aligned}
\tag{3.15}
$$

It is not difficult to find that $q^*(t,p(t))$ is independent of variables $\hat{x}_F$ and v. Due to $\varphi^F(T)=1$, $g_1^F(T)=0$ and $g_2^F(T)=0$, we deduce that

$$
\varphi^F(t) = \exp\bigl\{r_0(T-t)\bigr\},
\tag{3.16}
$$

$$
g_1^F(t) = \begin{cases} \frac{1}{2}(\eta^2+\xi^2)(t-T), & \text{if } \kappa+\eta\sigma_1\rho+\xi\sigma_1\sqrt{1-\rho^2}=0;\\ \frac{(\eta^2+\xi^2)[e^{(\kappa+\eta\sigma_1\rho+\xi\sigma_1\sqrt{1-\rho^2})(t-T)}-1]}{2(\kappa+\eta\sigma_1\rho+\xi\sigma_1\sqrt{1-\rho^2})}, & \text{if } \kappa+\eta\sigma_1\rho+\xi\sigma_1\sqrt{1-\rho^2}\neq 0. \end{cases}
\tag{3.17}
$$

and

$$
\begin{aligned}
g_2^F(t) = {}& \gamma_F\theta_F a\int_T^t \varphi^F(s)\,ds - \kappa\bar{v}\int_T^t g_1^F(s)\,ds\\
&- \gamma_F(1+k_1)\int_T^t \varphi^F(s)\bigl(1-q^*\bigl(s,p(s)\bigr)\bigr)\bigl(p(s)-a\bigr)\,ds\\
&- \frac{(\gamma_F\sigma_F)^2}{2}\int_T^t \bigl(\varphi^F(s)\bigr)^2\bigl[(1+k_1)q^*\bigl(s,p(s)\bigr)-k_1\bigr]^2\,ds.
\end{aligned}
\tag{3.18}
$$

By equation (3.14) and $p(t)$, we obtain $\frac{(p(t)-a)}{\gamma_F\varphi^F(t)\sigma_F^2(1+k_1)}+\frac{k_1}{1+k_1}>0$. Next, these situations will be discussed:

- *Case (Fa)* If $\frac{(p(t)-a)}{\gamma_F\varphi^F(t)\sigma_F^2(1+k_1)}+\frac{k_1}{1+k_1}\geq 1$, then $q^*(t,p(t))=1$. Substituting $q^*(t,p(t))$ into equation (3.18), we get

$$
g_2^F(t) = g_2^{Fa}(t) \doteq \gamma_F\theta_F a\int_T^t \varphi^F(s)\,ds - \frac{(\gamma_F\sigma_F)^2}{2}\int_T^t \bigl(\varphi^F(s)\bigr)^2\,ds - \kappa\bar{v}\int_T^t g_1^F(s)\,ds.
\tag{3.19}
$$

- *Case (Fb)* If $\frac{(p(t)-a)}{\gamma_F\varphi^F(t)\sigma_F^2(1+k_1)}+\frac{k_1}{1+k_1}<1$, then $q^*(t,p(t))=\frac{(p(t)-a)}{\gamma_F\varphi^F(t)\sigma_F^2(1+k_1)}+\frac{k_1}{1+k_1}$. Substituting $q^*(t,p(t))$ into equation (3.18), we get

$$
\begin{aligned}
g_2^F(t) &= g_2^{Fb}(t)\\
&\doteq \gamma_F\theta_F a\int_T^t \varphi^F(s)\,ds - \kappa\bar{v}\int_T^t g_1^F(s)\,ds\\
&\quad - \gamma_F\int_T^t \varphi^F(s)\bigl(p(s)-a\bigr)\,ds + \frac{1}{2\sigma_F^2}\int_T^t \bigl(p(s)-a\bigr)^2\,ds.
\end{aligned}
\tag{3.20}
$$

Step 3. Observing the reinsurance and investment strategies of the insurer from equations (3.12), (3.13), and (3.14), the reinsurer decides on the strategy $\pi_L^*(\cdot)=(p^*(\cdot),b_{L1}^*(\cdot),b_{L2}^*(\cdot))\in\Pi_L$.

Based on the value function of the reinsurer, we have

$$
J^L(t,\hat{x}_L,v) = -\frac{1}{\gamma_L}\exp\bigl\{-\gamma_L\varphi^L(t)\hat{x}_L + g_1^L(t)v + g_2^L(t)\bigr\},
\tag{3.21}
$$

where $\varphi^L(t)$, $g_1^L(t)$, and $g_2^L(t)$ are differentiable functions with $\varphi^L(T) = 1$, $g_1^L(T) = 0$, and $g_2^L(T) = 0$. For the reinsurer, the HJB equation is as follows

$$\begin{aligned}
0 = \sup_{\pi_L(\cdot)\in\Pi_L} \Big\{ & J_t^L + J_{\hat{x}_L}^L \big[r_0\hat{x}_L + \big(B^{L_1}(t) - k_2B^{F_1}(t)\big)\eta\nu + \big(B^{L_2}(t) - k_2B^{F_2}(t)\big)\xi\nu - k_2\theta_F a \\
& + (1+k_2)\big(1 - q^*\big(t,p(t)\big)\big)\big(p(t) - a\big)\big] + \frac{1}{2}\big[\big(B^{L_1}(t) - k_2B^{F_1}(t)\big)^2\nu \\
& + \big(B^{L_2}(t) - k_2B^{F_2}(t)\big)^2\nu + \big[1 - (1+k_2)q^*\big(t,p(t)\big)\big]^2\sigma_F^2\big]J_{\hat{x}_L\hat{x}_L}^L + J_\nu^L\kappa(\bar{\nu} - \nu) + \frac{1}{2}J_{\nu\nu}^L\sigma_1^2\nu \\
& + J_{\hat{x}_L\nu}^L\big[\big(B^{L_1}(t) - k_2B^{F_1}(t)\big)\rho + \big(B^{L_2}(t) - k_2B^{F_2}(t)\big)\sqrt{1-\rho^2}\big]\sigma_1\nu\Big\}.
\end{aligned} \tag{3.22}$$

In the above, $J_t^L, J_{\hat{x}_L}^L, J_{\hat{x}_L\hat{x}_L}^L, J_\nu^L, J_{\nu\nu}^L$, and $J_{\hat{x}_L\nu}^L$ are partial derivatives of J^L. Substituting derivatives into the HJB equation (3.22) yields

$$\begin{aligned}
0 = \sup_{\pi_L(\cdot)\in\Pi_L} J^L(t,\hat{x}_L,\nu)\Big\{ & -\gamma_L\hat{x}_L\bigg(\frac{d\varphi^L(t)}{dt} + r_0\varphi^L(t)\bigg) + \nu\bigg[\frac{dg_1^L(t)}{dt} + \big(B^{L_1}(t) - k_2B^{F_1}(t)\big) \\
& \times \gamma_L\varphi^L(t)\big(-\eta - g_1^L(t)\sigma_1\rho\big) + \big(B^{L_2}(t) - k_2B^{F_2}(t)\big)\gamma_L\varphi^L(t)\big(-\xi - g_1^L(t)\sigma_1\sqrt{1-\rho^2}\big) \\
& + \frac{1}{2}\big(\gamma_L\varphi^L(t)\big)^2\big[\big(B^{L_1}(t) - k_2B^{F_1}(t)\big)^2 + \big(B^{L_2}(t) - k_2B^{F_2}(t)\big)^2\big] - g_1^L(t)\kappa + \frac{1}{2}\big(g_1^L(t)\sigma_1\big)^2\bigg] \\
& + \frac{dg_2^L(t)}{dt} + g_1^L(t)\kappa\bar{\nu} + \gamma_L\varphi^L(t)k_2\theta_F a - \gamma_L\varphi^L(t)(1+k_2)\big(1 - q^*\big(t,p(t)\big)\big)\big(p(t) - a\big) \\
& + \frac{1}{2}\big(\gamma_L\varphi^L(t)\big)^2\big[1 - (1+k_2)q^*\big(t,p(t)\big)\big]^2\sigma_F^2\Big\}.
\end{aligned} \tag{3.23}$$

Similarly, by equation (3.23), we have

$$\begin{aligned}
&0 = \nu\big[\gamma_L\varphi^L(t)\big(-\eta - \rho\sigma_1g_1^L(t)\big) + \big(B^{L_1}(t) - k_2B^{F_1}(t)\big)\big(\gamma_L\varphi^L(t)\big)^2\big], \\
&0 = \nu\big[\gamma_L\varphi^L(t)\big(-\xi - \sqrt{1-\rho^2}\sigma_1g_1^L(t)\big) + \big(B^{L_2}(t) - k_2B^{F_2}(t)\big)\big(\gamma_L\varphi^L(t)\big)^2\big].
\end{aligned}$$

Solving the above equation yields that

$$B^{L_1}(t) = \frac{\eta + \rho\sigma_1g_1^L(t)}{\gamma_L\varphi^L(t)} + k_2B^{F_1}(t), \tag{3.24}$$

$$B^{L_2}(t) = \frac{\xi + \sqrt{1-\rho^2}\sigma_1g_1^L(t)}{\gamma_L\varphi^L(t)} + k_2B^{F_2}(t). \tag{3.25}$$

From the ranges of k_1 and k_2, we know that $k_1k_2 \in [0,1]$. In order to solve the problem, we assume $k_1k_2 \neq 1$. Substituting the expressions of $B^{F_1}(t)$ and $B^{F_2}(t)$ (i.e., (3.9) and (3.10)) into the above formulas, we obtain

$$\begin{aligned}
&B^{L_1}(t) = \frac{1}{1-k_1k_2}\bigg[\frac{\eta + \rho\sigma_1g_1^L(t)}{\gamma_L\varphi^L(t)} + k_2\frac{\eta + \rho\sigma_1g_1^F(t)}{\gamma_F\varphi^F(t)}\bigg], \\
&B^{L_2}(t) = \frac{1}{1-k_1k_2}\bigg[\frac{\xi + \sqrt{1-\rho^2}\sigma_1g_1^L(t)}{\gamma_L\varphi^L(t)} + k_2\frac{\xi + \sqrt{1-\rho^2}\sigma_1g_1^F(t)}{\gamma_F\varphi^F(t)}\bigg].
\end{aligned}$$

From the relations of $B^{L_1}(t)$ and $B^{L_2}(t)$ (i.e., (2.8)), we can get the optimal investment strategy of the reinsurer:

$$\begin{aligned} b^*_{L_1}(t) &= \frac{1}{1-k_1k_2}\left[\frac{\eta+\rho\sigma_1 g_1^L(t)}{\gamma_L\varphi^L(t)} + k_2\frac{\eta+\rho\sigma_1 g_1^F(t)}{\gamma_F\varphi^F(t)}\right] \\ &\quad - \frac{1}{1-k_1k_2}\frac{f_{s_1}S_1(t)+\sigma_1\rho f_v}{\sigma_1\sqrt{1-\rho^2}f_v}\left[\frac{\xi+\sqrt{1-\rho^2}\sigma_1 g_1^L(t)}{\gamma_L\varphi^L(t)} + k_2\frac{\xi+\sqrt{1-\rho^2}\sigma_1 g_1^F(t)}{\gamma_F\varphi^F(t)}\right], \end{aligned} \tag{3.26}$$

$$b^*_{L_2}(t) = \frac{1}{1-k_1k_2}\frac{S_2(t)}{\sigma_1\sqrt{1-\rho^2}f_v}\left[\frac{\xi+\sqrt{1-\rho^2}\sigma_1 g_1^L(t)}{\gamma_L\varphi^L(t)} + k_2\frac{\xi+\sqrt{1-\rho^2}\sigma_1 g_1^F(t)}{\gamma_F\varphi^F(t)}\right], \tag{3.27}$$

For the insurer, the optimal investment strategy is as follows:

$$\begin{aligned} b^*_{F_1}(t) &= \beta_1^*\big(\cdot, b^*_{L_1}(\cdot)\big) \\ &= \frac{\sigma_1 f_v(\eta\sqrt{1-\rho^2}-\xi\rho) - f_{s_1}S_1(t)(\xi+\sqrt{1-\rho^2}\sigma_1 g_1^F(t))}{\gamma_F\varphi^F(t)\sigma_1\sqrt{1-\rho^2}f_v} + k_1 b^*_{L_1}(t), \end{aligned} \tag{3.28}$$

$$b^*_{F_2}(t) = \beta_2^*\big(\cdot, b^*_{L_2}(\cdot)\big) = \frac{S_2(t)}{\sigma_1\sqrt{1-\rho^2}f_v}\frac{\xi+\sqrt{1-\rho^2}\sigma_1 g_1^F(t)}{\gamma_F\varphi^F(t)} + k_1 b^*_{L_2}(t). \tag{3.29}$$

Substituting (3.24) and (3.25) into the reinsurer's HJB equation (3.23) yields that

$$\begin{aligned} 0 = \sup_{p(t)\in[c_F,\bar{c}]}\bigg\{ &-\gamma_L\hat{x}_L\left(\frac{d\varphi^L(t)}{dt} + r_0\varphi^L(t)\right) + v\left[\frac{dg_1^L(t)}{dt} - g_1^L(t)\big(\kappa+\eta\rho\sigma_1+\xi\sqrt{1-\rho^2}\sigma_1\big)\right. \\ &\left. - \frac{1}{2}\big(\eta^2+\xi^2\big)\right] + \frac{dg_2^L(t)}{dt} + g_1^L(t)\kappa\bar{v} - \gamma_L\varphi^L(t)(1+k_2)\big(1-q^*\big(t,p(t)\big)\big)\big(p(t)-a\big) \\ &+ \gamma_L\varphi^L(t)k_2\theta_F a + \frac{1}{2}\big(\gamma_L\varphi^L(t)\big)^2\big[1-(1+k_2)q^*\big(t,p(t)\big)\big]^2\sigma_F^2\bigg\}. \end{aligned} \tag{3.30}$$

It is not difficult to find that $p^*(t)$ is independent of variables $\hat{x}_L$ and v. Due to $\varphi^L(T) = 1$ and $g_1^L(T) = 0$, we deduce that

$$\varphi^L(t) = \exp\big\{r_0(T-t)\big\} = \varphi^F(t), \tag{3.31}$$

$$g_1^L(t) = g_1^F(t), \tag{3.32}$$

and

$$\begin{aligned} 0 = \sup_{p(t)\in[c_F,\bar{c}]}\bigg\{ &\frac{dg_2^L(t)}{dt} + g_1^L(t)\kappa\bar{v} - \gamma_L\varphi^L(t)(1+k_2)\big(1-q^*\big(t,p(t)\big)\big)\big(p(t)-a\big) \\ &+ \gamma_L\varphi^L(t)k_2\theta_F a + \frac{1}{2}\big(\gamma_L\varphi^L(t)\big)^2\big[1-(1+k_2)q^*\big(t,p(t)\big)\big]^2\sigma_F^2\bigg\}. \end{aligned} \tag{3.33}$$

For simplicity, we write

$$K = \frac{(1+k_1)\gamma_F + \gamma_L(1-k_1k_2)}{2(1+k_1)\gamma_F + \gamma_L(1+k_2)}, \qquad N^{\theta_F}(t) = \frac{\theta_F a}{\gamma_F \sigma_F^2 \varphi^F(t)},$$
$$N^{\bar{\theta}}(t) = \frac{\bar{\theta} a}{\gamma_F \sigma_F^2 \varphi^F(t)}. \tag{3.34}$$

The premium strategy $p(t)$ is discussed as follows:

- *Case (La)* When $\frac{(p(t)-a)}{\gamma_F\varphi^F(t)\sigma_F^2(1+k_1)} + \frac{k_1}{1+k_1} \geq 1$, then $q^*(t,p(t)) = 1$. Substituting $q^*(t,p(t))$ into equation (3.33), we get

$$0 = \sup_{p(t)\in[c_F,\bar{c}]} \left\{ \frac{dg_2^L(t)}{dt} + g_1^L(t)\kappa\bar{\nu} + \gamma_L\varphi^L(t)k_2\theta_F a + \frac{1}{2}\left(k_2\sigma_F\gamma_L\varphi^L(t)\right)^2 \right\}. \tag{3.35}$$

  It is easy to know $p^*(t) = p$, where $p \in [c_F,\bar{c}]$. Then, $q^*(t,p^*(t)) = 1$. $\frac{(p(t)-a)}{\gamma_F\varphi^F(t)\sigma_F^2(1+k_1)} + \frac{k_1}{1+k_1} \geq 1$ yields that $N^{\theta_F}(t) \geq 1$. From equation (3.35) and $g_2^L(T) = 0$, we have

$$\begin{aligned} g_2^L(t) &= g_2^{La}(t) \\ &\doteq -k_2\gamma_L\theta_F a \int_T^t \varphi^L(s)\,ds - \frac{1}{2}(k_2\sigma_F\gamma_L)^2 \int_T^t \left(\varphi^L(s)\right)^2 ds - \kappa\bar{\nu}\int_T^t g_1^L(s)\,ds. \end{aligned} \tag{3.36}$$

- *Case (Lb)* When $\frac{(p(t)-a)}{\gamma_F\varphi^F(t)\sigma_F^2(1+k_1)} + \frac{k_1}{1+k_1} < 1$, then $q^*(t,p(t)) = \frac{(p(t)-a)}{\gamma_F\varphi^F(t)\sigma_F^2(1+k_1)} + \frac{k_1}{1+k_1}$. Substituting $q^*(t,p(t))$ into equation (3.33), we have

$$\begin{aligned} 0 = \sup_{p(t)\in[c_F,\bar{c}]} \Bigg\{ & \frac{dg_2^L(t)}{dt} + g_1^L(t)\kappa\bar{\nu} + \gamma_L\varphi^L(t)k_2\theta_F a + \frac{1}{2}\left(\frac{1-k_1k_2}{1+k_1}\right)^2 \left(\sigma_F\gamma_L\varphi^L(t)\right)^2 \\ & - \left(p(t)-a\right)\frac{(1+k_2)\gamma_L\varphi^L(t)}{1+k_1}\left[1 + \frac{\gamma_L}{\gamma_F}\frac{(1-k_1k_2)}{(1+k_1)}\right] \\ & + \left(p(t)-a\right)^2 \frac{(1+k_2)\gamma_L}{(1+k_1)\gamma_F\sigma_F^2}\left[1 + \frac{\gamma_L}{2\gamma_F}\frac{(1+k_2)}{(1+k_1)}\right]\Bigg\}. \end{aligned} \tag{3.37}$$

  Similarly, we have

$$p^*(t) = \left[a + K\gamma_F\sigma_F^2\varphi^L(t)\right] \vee c_F \wedge \bar{c}. \tag{3.38}$$

  - *Subcase (Lb1)* When $a + K\gamma_F\sigma_F^2\varphi^L(t) \geq \bar{c}$, then $p^*(t) = \bar{c}$, $q^*(t,p^*(t)) = \frac{1}{1+k_1}(N^{\bar{\theta}}(t) + k_1)$. From $K < 1$ and $\varphi^L(t) = \varphi^F(t)$, then $\frac{(p(t)-a)}{\gamma_F\varphi^F(t)\sigma_F^2(1+k_1)} + \frac{k_1}{1+k_1} < 1$ yields that $\frac{1}{1+k_1}(N^{\bar{\theta}}(t) + k_1) < 1$. Then, by equation (3.37) and $g_2^L(T) = 0$, we have

$$\begin{aligned} g_2^L(t) &= g_2^{Lb1}(t) \\ &\doteq -k_2\gamma_L\theta_F a \int_T^t \varphi^L(s)\,ds - \kappa\bar{\nu}\int_T^t g_1^L(s)\,ds \end{aligned}$$

$$+\gamma_L\bar{\theta}a(1+k_2)\int_T^t\varphi^L(s)\left[1-\frac{1}{1+k_1}\left(N^{\bar{\theta}}(s)+k_1\right)\right]ds$$
$$-\frac{(\gamma_L\sigma_F)^2}{2}\int_T^t\left(\varphi^L(s)\right)^2\left[1-\frac{1+k_2}{1+k_1}\left(N^{\bar{\theta}}(s)+k_1\right)\right]^2ds. \tag{3.39}$$

And, equation (3.20) becomes

$$g_2^F(t)=g_2^{Fb1}(t)\doteq\gamma_F(\theta_F-\bar{\theta})a\int_T^t\varphi^F(s)\,ds-\kappa\bar{\nu}\int_T^t g_1^F(s)\,ds-\frac{(\bar{\theta}a)^2(T-t)}{2\sigma_F^2}. \tag{3.40}$$

– *Subcase (Lb2)* When $a+K\gamma_F\sigma_F^2\varphi^L(t)\le c_F$, then $p^*(t)=c_F$, $q^*(t,p^*(t))=\frac{1}{1+k_1}(N^{\theta_F}(t)+k_1)$. And $\frac{(p(t)-a)}{\gamma_F\varphi^F(t)\sigma_F^2(1+k_1)}+\frac{k_1}{1+k_1}<1$ becomes $\frac{1}{1+k_1}(N^{\theta_F}(t)+k_1)<1$. Then, by equation (3.37) and $g_2^L(T)=0$, we can get that

$$\begin{aligned}g_2^L(t)&=g_2^{Lb2}(t)\\&\doteq-k_2\gamma_L\theta_Fa\int_T^t\varphi^L(s)\,ds-\kappa\bar{\nu}\int_T^t g_1^L(s)\,ds\\&\quad+\gamma_L\theta_Fa(1+k_2)\int_T^t\varphi^L(s)\left[1-\frac{1}{1+k_1}\left(N^{\theta_F}(s)+k_1\right)\right]ds\\&\quad-\frac{(\gamma_L\sigma_F)^2}{2}\int_T^t\left(\varphi^L(s)\right)^2\left[1-\frac{1+k_2}{1+k_1}\left(N^{\theta_F}(s)+k_1\right)\right]^2ds.\end{aligned} \tag{3.41}$$

And, equation (3.20) becomes

$$g_2^F(t)=g_2^{Fb2}(t)\doteq-\kappa\bar{\nu}\int_T^t g_1^F(s)\,ds-\frac{(\theta_Fa)^2(T-t)}{2\sigma_F^2}. \tag{3.42}$$

– *Subcase (Lb3)* When $\theta_Fa<K\gamma_F\sigma_F^2\varphi^L(t)<\bar{\theta}a$, then, $p^*(t)=a+K\gamma_F\sigma_F^2\varphi^L(t)$. $q^*(t,p^*(t))=\frac{1}{1+k_1}(K+k_1)$. And $\frac{(p(t)-a)}{\gamma_F\varphi^F(t)\sigma_F^2(1+k_1)}+\frac{k_1}{1+k_1}<1$ becomes $\frac{1}{1+k_1}(K+k_1)<1$. From equation (3.37) and $g_2^L(T)=0$, we have

$$\begin{aligned}g_2^L(t)&=g_2^{Lb3}(t)\\&\doteq-k_2\gamma_L\theta_Fa\int_T^t\varphi^L(s)\,ds-\kappa\bar{\nu}\int_T^t g_1^L(s)\,ds\\&\quad+\left[\gamma_L\gamma_F\sigma_F^2\frac{(1+k_2)(1-K)K}{1+k_1}\right.\\&\quad\left.-\frac{(\gamma_L\sigma_F)^2}{2}\left(1-\frac{(1+k_2)(K+k_1)}{1+k_1}\right)^2\right]\int_T^t\left(\varphi^L(s)\right)^2ds.\end{aligned} \tag{3.43}$$

And, equation (3.20) becomes

$$\begin{aligned}g_2^F(t)&=g_2^{Fb3}(t)\\&\doteq\gamma_F\theta_Fa\int_T^t\varphi^F(s)\,ds-\kappa\bar{\nu}\int_T^t g_1^F(s)\,ds\\&\quad+K\gamma_F^2\sigma_F^2\left(-1+\frac{K}{2}\right)\int_T^t\left(\varphi^L(s)\right)^2ds.\end{aligned} \tag{3.44}$$

Therefore, the theorem is as follows.

Table 1 The optimal premium and reinsurance strategies

Cases	$p^*(t)$	$q^*(t)$
(1) $N^{\theta_F}(t) \geq 1$	$\forall p \in [c_F, \bar{c}]$	1
(2) $N^{\bar{\theta}}(t) \leq K$	$\bar{c}$	$\frac{1}{1+k_1}(N^{\bar{\theta}}(t) + k_1)$
(3) $K \leq N^{\theta_F}(t) < 1$	c_F	$\frac{1}{1+k_1}(N^{\theta_F}(t) + k_1)$
(4) $N^{\theta_F}(t) < K < N^{\bar{\theta}}(t)$	$a + K\gamma_F\sigma_F^2\varphi^L(t)$	$\frac{1}{1+k_1}(K + k_1)$

Theorem 1 *Assuming $k_1k_2 < 1$. The equilibrium strategy of the Stackelberg game problem 1 is $(p^*(\cdot), b^*_{L1}(\cdot), b^*_{L2}(\cdot), q^*(t), b^*_{F1}(\cdot), b^*_{F2}(\cdot))$, where $b^*_{L1}(\cdot)$ and $b^*_{L2}(\cdot)$ are given by (3.26) and (3.27), respectively; $b^*_{F1}(\cdot)$ and $b^*_{F2}(\cdot)$ are given by (3.28) and (3.29), respectively; $p^*(t)$ and $q^*(t)$ under various cases are represented in Table 1; where K, $N^{\theta_F}(t)$ and $N^{\bar{\theta}}(t)$ satisfy equation (3.34).*

For the reinsurer and the insurer, the value functions are respectively

$$J^L(t, \hat{x}_L, \nu) = -\frac{1}{\gamma_L} \exp\{-\gamma_L\varphi^L(t)\hat{x}_L + g_1^L(t)\nu + g_2^L(t)\}, \tag{3.45}$$

and

$$J^F(t, \hat{x}_F, \nu) = -\frac{1}{\gamma_F} \exp\{-\gamma_F\varphi^F(t)\hat{x}_F + g_1^F(t)\nu + g_2^F(t)\}, \tag{3.46}$$

where $\varphi^F(t)$ and $\varphi^L(t)$ are given by equations (3.16) and (3.31); $g_1^F(t)$ and $g_1^L(t)$ are given by equations (3.17) and (3.32); $g_2^L(t)$ and $g_2^F(t)$ under different cases are as follows

$$\begin{aligned} g_2^L(t) &= \begin{cases} g_2^{La}(t), & N^{\theta_F}(t) \geq 1; \\ g_2^{Lb1}(t), & N^{\bar{\theta}}(t) \leq K; \\ g_2^{Lb2}(t), & K \leq N^{\theta_F}(t) < 1; \\ g_2^{Lb3}(t), & N^{\theta_F}(t) < K < N^{\bar{\theta}}(t); \end{cases} \\ g_2^F(t) &= \begin{cases} g_2^{Fa}(t), & N^{\theta_F}(t) \geq 1; \\ g_2^{Fb1}(t), & N^{\bar{\theta}}(t) \leq K; \\ g_2^{Fb2}(t), & K \leq N^{\theta_F}(t) < 1; \\ g_2^{Fb3}(t), & N^{\theta_F}(t) < K < N^{\bar{\theta}}(t). \end{cases} \end{aligned} \tag{3.47}$$

$g_2^{La}(t)$, $g_2^{Lb1}(t)$, $g_2^{Lb2}(t)$, $g_2^{Lb3}(t)$, $g_2^{Fa}(t)$, $g_2^{Fb1}(t)$, $g_2^{Fb2}(t)$, and $g_2^{Fb3}(t)$ are given by equations (3.36), (3.39), (3.41), (3.43), (3.19), (3.40), (3.42), and (3.44).

Theorem 1 demonstrates the equilibrium reinsurance-investment strategy does not depend on the current wealth. Moreover, for the insurer, their investment strategy does not depend on its reinsurance strategy. Our conclusions are consistent with that of most existing related research, among them, Bensoussan et al. [7], Yan et al. [26], A et al. [1] and Deng et al. [14], etc. In addition, from the form of investment strategies of the insurer and the reinsurer, we can see that they imitate each other's investment strategies, showing a herd effect.

Remark 1 When the Stackelberg equilibrium is achieved in Case (4) of Theorem 1, the optimal reinsurance premium is given by the variance premium principle. More precisely, for per unit of risk, the total instantaneous reinsurance premium associated with the ceded proportion $(1-q^*(t))100\%$ is

$$p^*(t)\big(1-q^*(t)\big) = a\big(1-q^*(t)\big) + \big[(1+k)\gamma_F + \gamma_L\big]\varphi^F(t)\sigma_F^2\big(1-q^*(t)\big)^2.$$

3.2 Special cases

Special case 1 If there is no opportunity to trade derivatives, i.e., $b_{F2}(t) = b_{L2}(t) \equiv 0$, and we have

$$B^{F_1}(t) = b_{F1}(t), \qquad B^{L_1}(t) = b_{L1}(t), \qquad B^{F_2}(t) = B^{L_2}(t) = 0.$$

According to the procedure mentioned in Sect. 2.4, the optimal reinsurance strategy of the insurer, the optimal reinsurance premium strategy of the reinsurer, and their optimal investment strategies in the stock can be obtained successively. The optimal reinsurance strategy and the optimal reinsurance premium strategy are consistent with Table 1, and the optimal investment strategies are as follow:

$$b_{L_1}^*(t) = \frac{1}{1-k_1k_2}\left[\frac{\eta + \rho\sigma_1 g_1^L(t)}{\gamma_L\varphi^L(t)} + k_2\frac{\eta + \rho\sigma_1 g_1^F(t)}{\gamma_F\varphi^F(t)}\right],$$
$$b_{F_1}^*(t) = \frac{1}{1-k_1k_2}\left[\frac{\eta + \rho\sigma_1 g_1^F(t)}{\gamma_F\varphi^F(t)} + k_1\frac{\eta + \rho\sigma_1 g_1^L(t)}{\gamma_L\varphi^L(t)}\right],$$

where $\varphi^L(t) = \varphi^F(t) = \exp\{r_0(T-t)\} \doteq \varphi(t)$, $g_1^L(t) = g_1^F(t) \doteq g_1(t)$ satisfies

$$0 = \frac{dg_1(t)}{dt} - g_1(t)\kappa + \frac{1}{2}\sigma_1^2\big(g_1(t)\big)^2 - \frac{1}{2}\big(\eta + \rho\sigma_1 g_1(t)\big)^2.$$

Then, the optimal investment strategies can be simplified to:

$$b_{L_1}^*(t) = \frac{\eta + \rho\sigma_1 g_1(t)}{(1-k_1k_2)\varphi(t)}\left(\frac{1}{\gamma_L} + \frac{k_2}{\gamma_F}\right),$$
$$b_{F_1}^*(t) = \frac{\eta + \rho\sigma_1 g_1(t)}{(1-k_1k_2)\varphi(t)}\left(\frac{1}{\gamma_F} + \frac{k_1}{\gamma_L}\right).$$

Special case 2 When $k_1 = k_2 = 0$, there is no wealth gap between the insurer and reinsurer. The equilibrium strategy is $(p^*(\cdot), b_{L1}^*(\cdot), b_{L2}^*(\cdot), q^*(\cdot), b_{F1}^*(\cdot), b_{F2}^*(\cdot))$, where $p^*(t)$ and $q^*(t)$ under various cases are represented in Table 2; where $K^0 = \frac{\gamma_F+\gamma_L}{2\gamma_F+\gamma_L}$. And, $b_{L1}^*(\cdot), b_{L2}^*(\cdot), b_{F1}^*(\cdot)$, $b_{F2}^*(\cdot)$ are as follows

$$b_{L_1}^*(t) = \frac{\eta + \rho\sigma_1 g_1^L(t)}{\gamma_L\varphi^L(t)} - \frac{f_{s_1}S_1(t) + \sigma_1\rho f_v}{\sigma_1\sqrt{1-\rho^2}f_v}\frac{\xi + \sqrt{1-\rho^2}\sigma_1 g_1^L(t)}{\gamma_L\varphi^L(t)}, \tag{3.48}$$

$$b_{L_2}^*(t) = \frac{S_2(t)}{\sigma_1\sqrt{1-\rho^2}f_v}\frac{\xi + \sqrt{1-\rho^2}\sigma_1 g_1^L(t)}{\gamma_L\varphi^L(t)}, \tag{3.49}$$

$$b_{F_1}^*(t) = \frac{\sigma_1 f_v(\eta\sqrt{1-\rho^2} - \xi\rho) - f_{s_1}S_1(t)(\xi + \sqrt{1-\rho^2}\sigma_1 g_1^F(t))}{\gamma_F\varphi^F(t)\sigma_1\sqrt{1-\rho^2}f_v}, \tag{3.50}$$

$$b_{F_2}^*(t) = \frac{S_2(t)}{\sigma_1\sqrt{1-\rho^2}f_v}\frac{\xi + \sqrt{1-\rho^2}\sigma_1 g_1^F(t)}{\gamma_F\varphi^F(t)}. \tag{3.51}$$

Table 2 The optimal premium and reinsurance strategies

Cases	$p^*(t)$	$q^*(t)$
(1) $N^{\theta_F}(t) \geq 1$	$\forall p \in [c_F, \bar{c}]$	1
(2) $N^{\bar{\theta}}(t) \leq K^0$	$\bar{c}$	$N^{\bar{\theta}}(t)$
(3) $K^0 \leq N^{\theta_F}(t) < 1$	c_F	$N^{\theta_F}(t)$
(4) $N^{\theta_F}(t) < K^0 < N^{\bar{\theta}}(t)$	$a + K^0\gamma_F\sigma_F^2\varphi^L(t)$	K^0

The form of the optimal investment strategies in this special case is consistent with the results in the literature Xue et al. [25]. From equations (3.16), (3.31), (3.17), and (3.32), we get $\varphi^F(t) = \varphi^L(t)$ and $g_1^F(t) = g_1^L(t)$. Then, we have

$$b^*_{F_1}(t) = \frac{\gamma_L}{\gamma_F} b^*_{L_1}(t), \qquad b^*_{F_2}(t) = \frac{\gamma_L}{\gamma_F} b^*_{L_2}(t). \tag{3.52}$$

We find that for the optimal investment strategies, the insurer and the reinsurer imitate each other, which shows a herd effect.

4 Numerical analysis

This section conducts some numerical examples and analyzes the sensitivity of the equilibrium strategy $(p^*(\cdot), b^*_{L1}(\cdot), b^*_{L2}(\cdot), q^*(\cdot), b^*_{F1}(\cdot), b^*_{F2}(\cdot))$ to model parameters.

Referring to Heston [16] and Xue et al. [25], the pricing formula of a European call option with stochastic volatility is

$$C_t = c\big(S_1(t), V(t); \mathbb{P}, \tau\big) = S_1(t)\psi_1 - e^{-r_0\tau}\mathbb{P}\psi_2, \tag{4.1}$$

where τ is the expiration, and $\mathbb{P}$ is the strike price. Then, we have

$$\begin{aligned} f_{S_1} &= \psi_1 + \frac{1}{\pi}\int_0^\infty e^{a_1(u)+c_1(u)V(t)} \cos\big[b_1(u) + d_1(u)V(t) + uD\big]\,du \\ &\quad - \frac{e^{-r_0\tau\mathbb{P}}}{\pi S_1(t)}\int_0^\infty e^{a_2(u)+c_2(u)V(t)} \cos\big[b_2(u) + d_2(u)V(t) + uD\big]\,du, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} f_v &= -S_1(t)\frac{1}{\pi}\int_0^\infty \frac{c_1(u)}{u} e^{a_1(u)+c_1(u)V(t)} \sin\big[b_1(u) + d_1(u)V(t) + uD\big]\,du \\ &\quad + \frac{e^{-r_0\tau\mathbb{P}}}{\pi}\int_0^\infty \frac{c_2(u)}{u} e^{a_2(u)+c_2(u)V(t)} \sin\big[b_2(u) + d_2(u)V(t) + uD\big]\,du, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} \psi_1 &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{u} e^{a_1(u)+c_1(u)V(t)} \sin\big[b_1(u) + d_1(u)V(t) + uD\big]\,du, \\ \psi_2 &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{u} e^{a_2(u)+c_2(u)V(t)} \sin\big[b_2(u) + d_2(u)V(t) + uD\big]\,du, \\ e^{A(1-iu)} &= e^{a_1(u)+ib_1(u)}, \qquad e^{A(-iu)} = e^{a_2(u)+ib_2(u)}, \qquad D = \ln\mathbb{P} - \ln S_1(t) + r_0\tau, \\ e^{B(1-iu)} &= e^{c_1(u)+id_1(u)}, \qquad e^{B(-iu)} = e^{c_2(u)+id_2(u)}, \end{aligned}$$

Table 3 The values of the financial market parameters

T	τ	r_0	η	κ	$\bar{v}$	σ_1	ρ	ξ	$\sqrt{V_0}$	$\mathbb{P}$	$S_1(0)$
5	4	0.05	4	5	$(0.13)^2$	0.25	−0.4	−6	0.15	2	2

Table 4 The values of the insurer's and the reinsurer's parameters

λ_F	μ_F	σ_F	θ_F	γ_F	k_1	x_F^0	$\bar{\theta}$	γ_L	k_2	x_L^0
0.8	5	8	1	0.3	0.5	3	3	0.1	0.2	5

$$\begin{aligned} A(y) &= -\frac{\kappa^*\bar{v}^*}{\sigma_1^2}\left[(q+b)\tau + 2\ln\left[1-\frac{q+b}{2q}\left(1-e^{-q\tau}\right)\right]\right], \\ B(y) &= -\frac{a(1-e^{-q\tau})}{2q-(q+b)(1-e^{-q\tau})}, \end{aligned} \tag{4.4}$$

with $a = y(1-y)$, $b = \sigma_1\rho y - \kappa^*$, $q = \sqrt{b^2 + a\sigma_1^2}$, $\kappa^* = \kappa - \sigma_1(\rho\eta + \sqrt{1-\rho^2}\xi)$ and $\bar{v}^* = \frac{\kappa\bar{v}}{\kappa^*}$. Here, κ^* and $\bar{v}^*$ represent the risk-neutral mean-reversion rate and long-run mean, respectively; i denotes the imaginary unit.

To be consistent with Xue et al. [25], we use the same parameter values given in Table 3 and Table 4.

4.1 Sensitivity analysis of the equilibrium investment strategy

According to the above parameter settings, this subsection analyzes the sensitivity of the optimal investment strategies to model parameters, including the risk aversion coefficients and sensitivity coefficients of the insurer and the reinsurer, the volatility parameter and mean reversion rate parameter in Heston's stochastic volatility model and option pricing model, etc.

Figure 1 shows the effects of the risk aversion coefficients γ_L and γ_F on optimal investment strategies, respectively. The larger γ_L and γ_F, the more risk-averse the reinsurer and the insurer, and the smaller both the long position in stocks and the short position in options. For reinsurer, the optimal investment strategy is sensitivity to both γ_L and γ_F. For insurer, the conclusion is similar.

Figure 2 shows the effects of sensitivity parameters of the insurer and the reinsurer on optimal investment strategies. As k_1 increases, both the reinsurer and the insurer invest more in stocks and short more options. For the sensitivity parameter k_2, the conclusion is similar.

Figure 3 shows the effects of parameter η on the optimal investment strategy of the reinsurer and the insurer. As η increases, the reinsurer invests more in the stocks, and the short position in options first decreases and then increases. And we can get the similar conclusion for the insurer.

Figure 4 shows the sensitivity of the optimal investment strategy to the mean-reversion rate κ that depicts the persistency of volatility. As κ increases, both the reinsurer and the insurer invest more in the stocks and short more options.

Figure 5 shows the sensitivities of the optimal investment strategy to σ_1, i.e., the volatility of volatility. As σ_1 increases, the reinsurer invests less in the stocks and shorts less the option. For the insurer, we can get the similar conclusion.

Figure 6 shows the sensitivity of optimal investment strategy to the volatility $\sqrt{V_0}$. The larger the volatility, the more volatile the finance market, the reinsurer invests more in

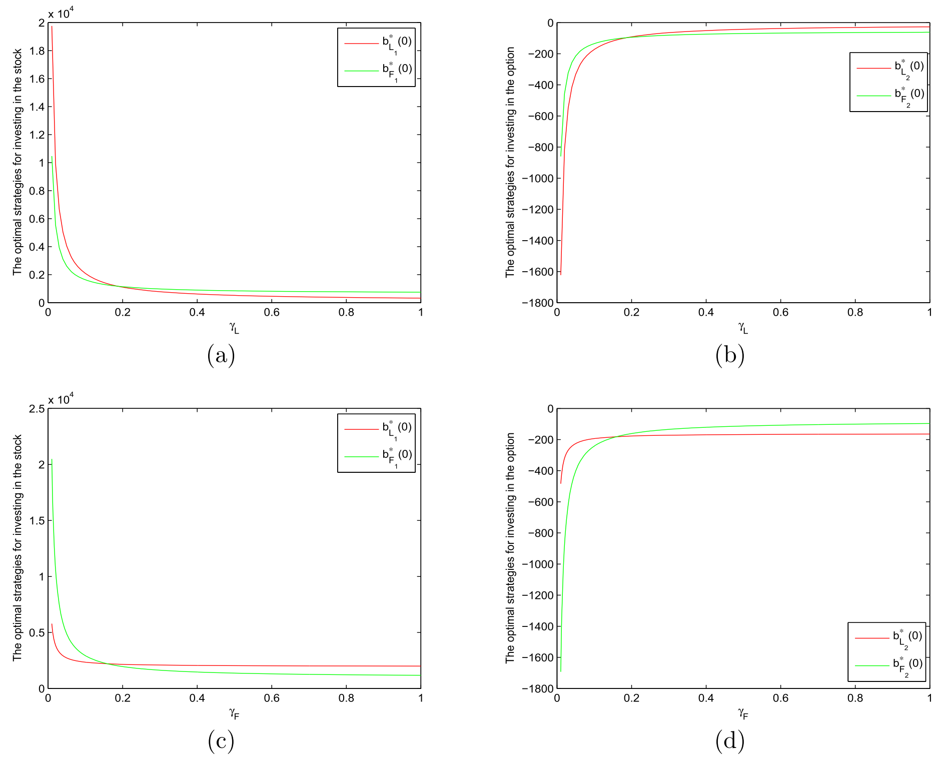

Figure 1 Sensitivities of optimal investment strategies to risk aversion coefficients

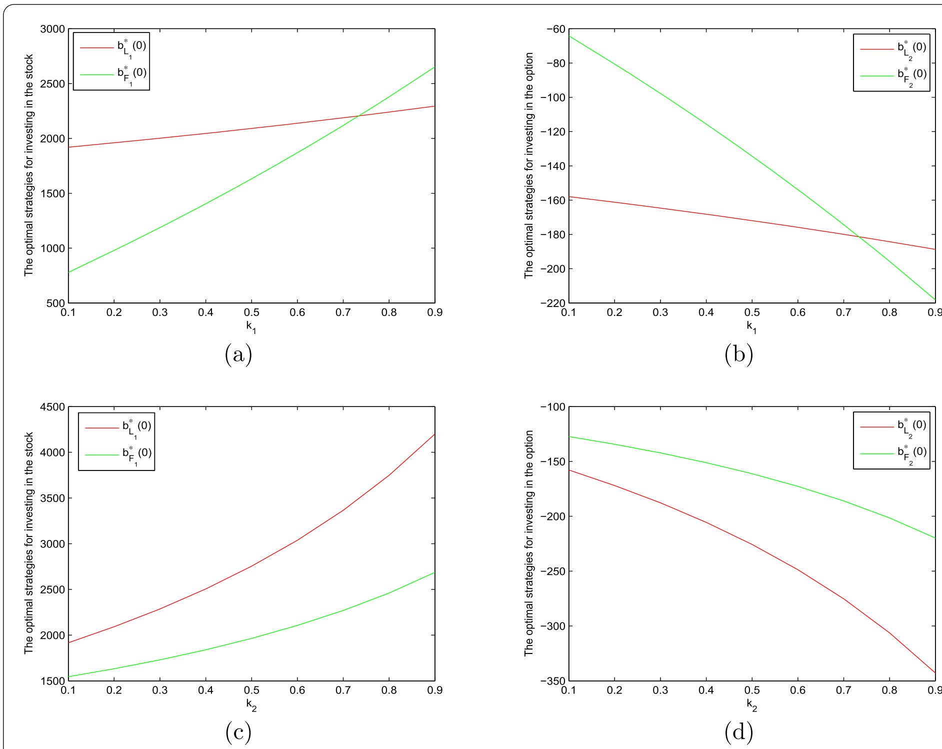

Figure 2 Sensitivities of optimal investment strategies to parameters k_1 and k_2

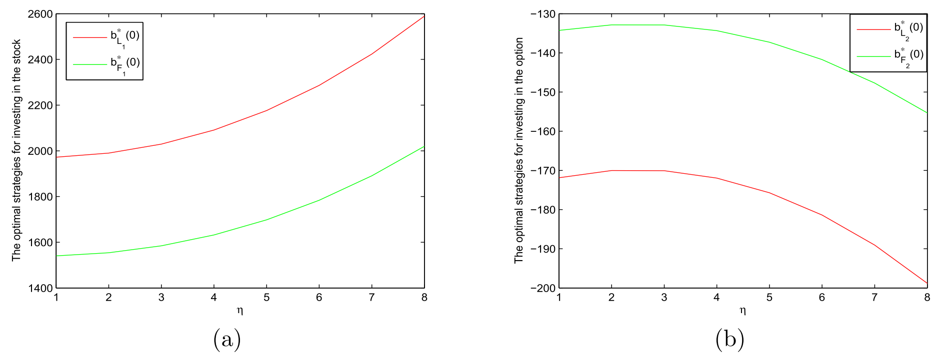

Figure 3 Sensitivities of optimal investment strategies to parameter η

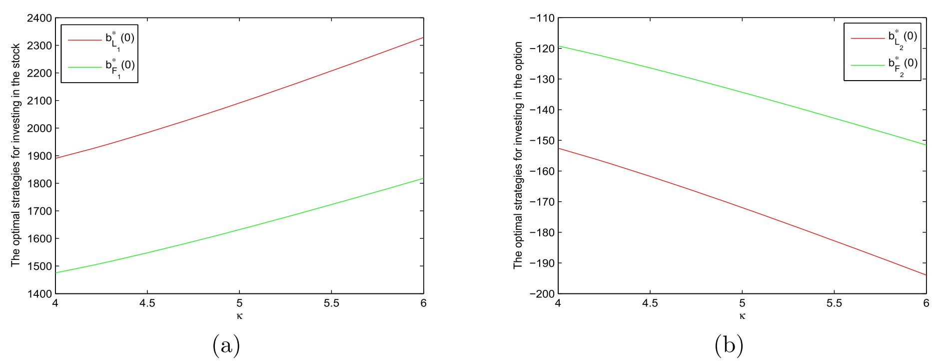

Figure 4 Sensitivities of optimal investment strategies to parameter κ

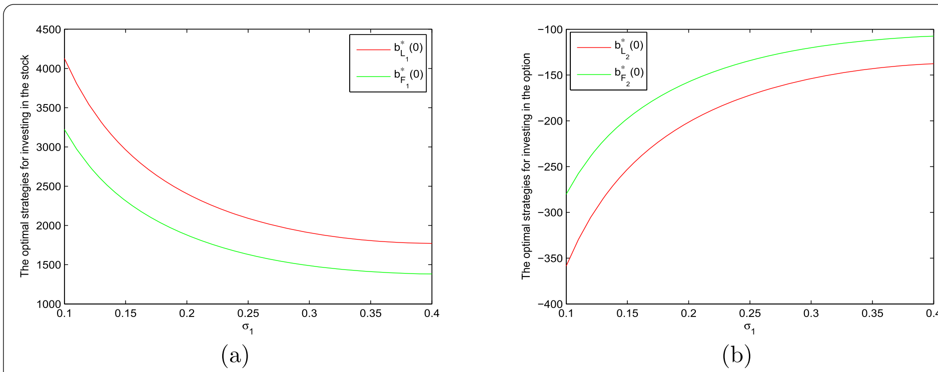

Figure 5 Sensitivities of optimal investment strategies to σ_1

the stocks and shorts more the option to achieve the optimal volatility exposure. For the insurer, we can get the similar conclusion, which is consistent with the sensitivity analysis in Xue et al. [25].

Figure 7 shows the sensitivity of the optimal investment strategy to the premium ξ of volatility risk. With ξ increasing, the reinsurer first takes a long (short) position and then switches to a short (long) position in the stock (option). A short position in the derivative may switch to a long position with parameters changing, indicating that the insurer

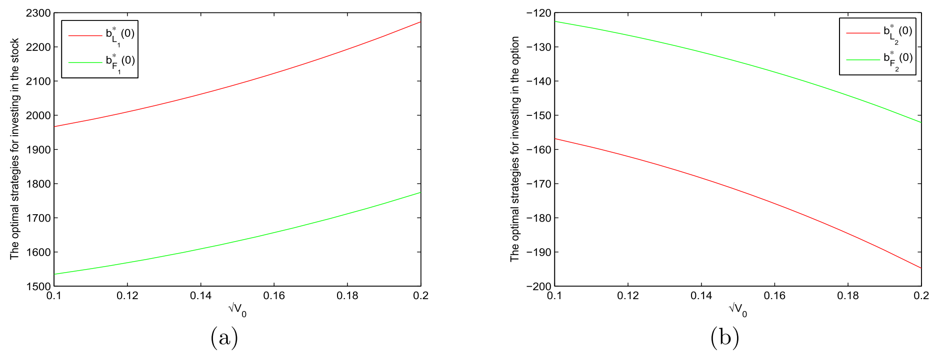

Figure 6 Sensitivities of optimal investment strategies to volatility $\sqrt{V}$

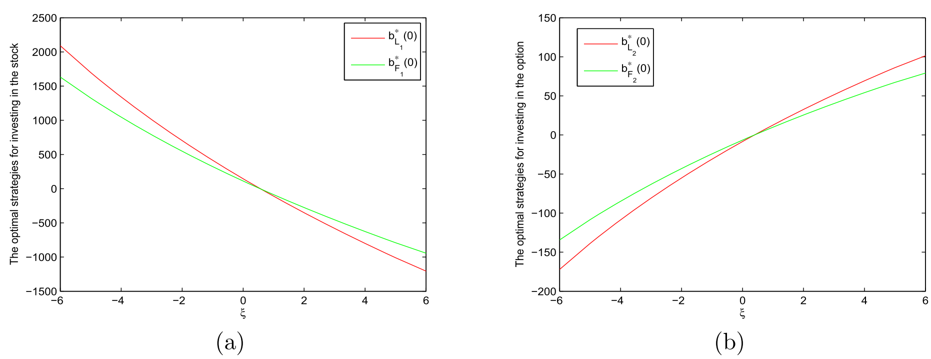

Figure 7 Sensitivities of optimal investment strategies to ξ

and reinsurer can flexibly adjust the position in the derivative to manage the market uncertainty risk. More precisely, the absolute value of the amount invested in the stock and the option firstly decreases and then increases with ξ increasing. For the insurer, we also get a similar conclusion, which is consistent with the sensitivity analysis in Xue et al. [25].

4.2 Sensitivity analysis of the equilibrium reinsurance strategy

Figure 8 shows the change of the optimal premium strategy of the reinsurer and the optimal reinsurance strategy of the insurer over time t. Meanwhile, the corresponding numerical results are represented in Table 5. The results show that condition $N^{\bar{\theta}}(t) \leq K$ is satisfied when $t \leq 1$, which is corresponding to Case 2 in Theorem 1; condition $N^{\theta_F}(t) < K < N^{\bar{\theta}}(t)$ is satisfied when $t \geq 2$, which is corresponding to Case 4 in Theorem 1. For more intuitively investigating the effects of model parameters on the optimal premium strategy and the optimal reinsurance strategy, we fully discuss the strategies in Case (4) in which all the optimal strategies fall within the feasible range. Thus, we give the strategies at $t = 4$ for sensitivity analysis.

Figure 9 shows the effect of risk aversion coefficients of the insurer and the reinsurer on optimal reinsurance premium strategy and optimal reinsurance strategy. The graph shows that the more risk averse the insurer is, the lower its own risk reserve level will be, and the higher the premium price charged by the reinsurer will be. In addition, the more risk averse

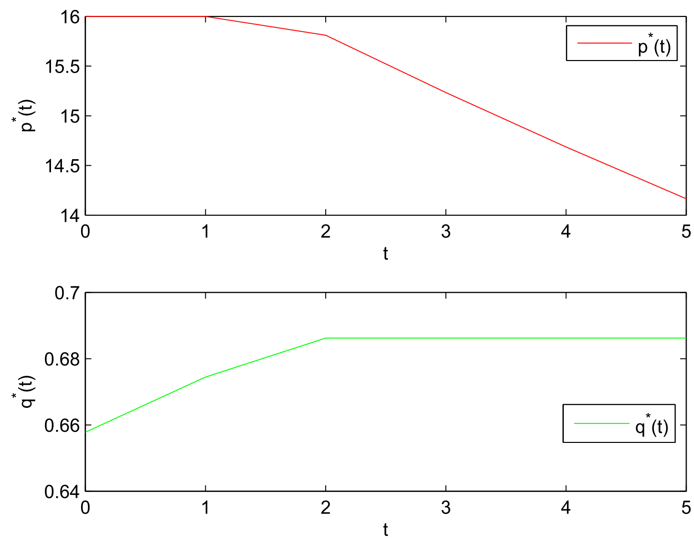

Figure 8 The effect of t on optimal premium strategy and reinsurance strategy

Table 5 Numerical results corresponding to Fig. 8

t	0	1	2	3	4	5
K	0.5294	0.5294	0.5294	0.5294	0.5294	0.5294
$N^{\theta_F}(t)$	0.1623	0.1706	0.1793	0.1885	0.1982	0.2083
$N^{\bar{\theta}}(t)$	0.4868	0.5117	0.5379	0.5655	0.5945	0.6250
$p^*(t)$	16.0000	16.0000	15.8097	15.2337	14.6859	14.1647
$q^*(t)$	0.6578	0.6745	0.6863	0.6863	0.6863	0.6863

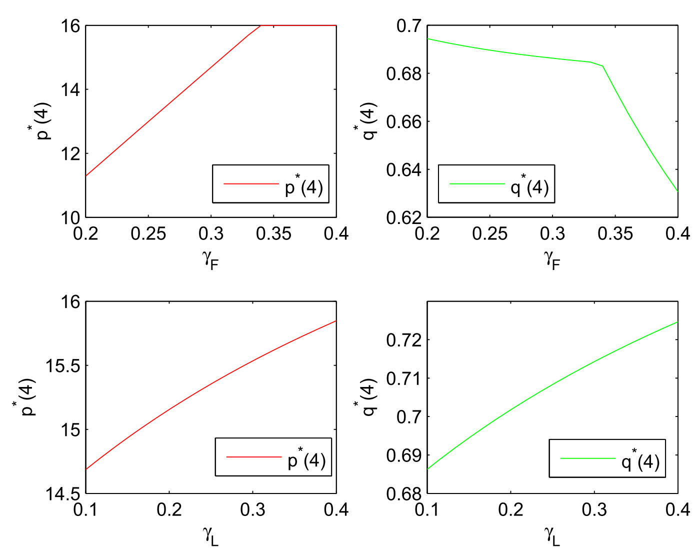

Figure 9 Effects of the risk aversion coefficients on $p^*(4)$ and $q^*(4)$

the reinsurer is, the more inclined it is to set a higher premium price to avoid risks. At this time, the insurer tends to purchase reinsurance contracts with a lower proportion to avoid more expenses. This is consistent with the phenomenon of the market economy.

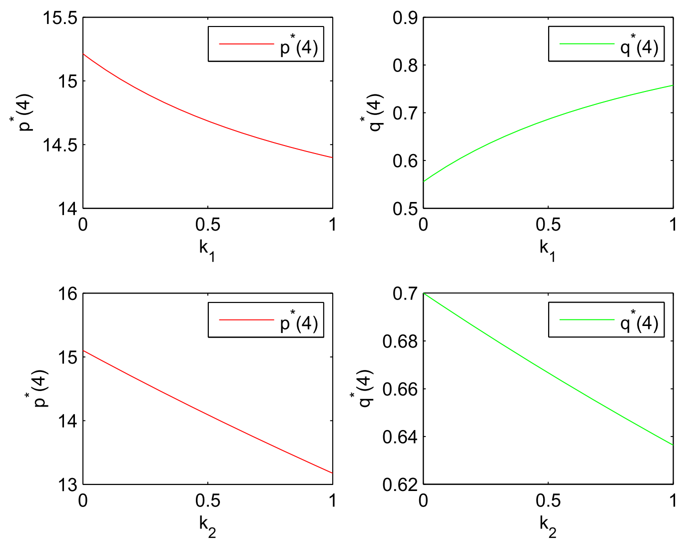

Figure 10 Effects of the sensitivity coefficients on $p^*(4)$ and $q^*(4)$

Figure 10 shows the effects of the sensitivity parameters of the insurer and the reinsurer on the optimal reinsurance premium price strategy and the optimal reinsurance strategy. The graph shows that the larger k_1 is, the higher $q^*(4)$ is, and the lower $p^*(4)$ is. That is to say, the more the insurer cares about the performance of the reinsurer, the more inclined it is to buy fewer reinsurance contracts, and the more willing it is to take more claims risks, the risk of more claims, which leads to the lower price of the reinsurance premium. The more the reinsurer cares about the performance level of the insurer, the more it tends to attract a higher proportion of reinsurance contracts by reducing the reinsurance premium price, which leads to the reduction of the risk reservation level of the insurer. In general, the optimal premium price strategy of the reinsurer and the optimal reinsurance strategy of the insurer are not only affected by their own sensitivity parameter but also by the sensitivity of the other player in the game.

5 Conclusion

In the paper, we investigate a stochastic Stackelberg differential reinsurance-investment game problem with derivative trading under a stochastic volatility model. The reinsurer who occupies a monopoly position can determine the price of the reinsurance premium and its asset allocation strategy invested in the stock and the derivative. The insurer, the follower of the Stackelberg game, can determine the proportion of reinsurance according to the price of reinsurance premium and its asset allocation strategy. The target of the reinsurer and the insurer is to find their own optimal strategy that maximizes the CARA utility of relative performance. The explicit equilibrium strategy for the game problem is deduced by solving HJB equations sequentially. The equilibrium investment strategy demonstrates that the insurer and the reinsurer imitate each other's investment strategies, showing a herd effect. The numerical experiment represents the sensitivity of the equilibrium strategy to model parameters. We find that a short position in the derivative may switch to a long position with parameters changing, which provides investors with important decision-making information.

The differential game in the insurance market is a hot topic in the current economic and financial field, and its future research can be very extensive and interesting, such as to study the Stackelberg reinsurance-investment game problem with derivatives trading under mean-variance criterion, to consider regime switching to better model market randomness and to consider parameter uncertainty in the stochastic Stackelberg differential game model.

Funding
This research is supported by the Natural Science Foundation of Shandong Province (Grant No. ZR2022QG060, Grant No. ZR2021QG036) and the National Natural Science Foundation of China (Grant No. 72171133).

Availability of data and materials
The data used in this paper is available upon the request.

Declarations

Ethics approval and consent to participate
The paper is not currently being considered for publication elsewhere.

Competing interests
The authors declare no competing interests.

Author contributions
Rui Gao: Methodology, Software, Writing—original draft, Funding acquisition. Yanfei Bai: Writing—review, Formal analysis, Funding acquisition. All authors reviewed the manuscript

Author details
[1]School of Statistics and Mathematics, Shandong University of Finance and Economics, Jinan 250014, China. [2]School of Insurance, Shandong University of Finance and Economics, Jinan 250014, China.

Received: 30 January 2023 Accepted: 6 April 2023 Published online: 19 April 2023

References

1. A, C., Lai, Y., Shao, Y.: Optimal excess-of-loss reinsurance and investment problem with delay and jump-diffusion risk process under the CEV model. J. Comput. Appl. Math. **342**, 317–336 (2018)
2. Ahn, D.-H., Boudoukh, J., Richardson, M., Whitelaw, R.: Optimal risk management using options. J. Finance **54**(1), 359–375 (1999)
3. Bai, Y., Zhou, Z., Gao, R., Xiao, H.: Nash equilibrium investment-reinsurance strategies for an insurer and a reinsurer with intertemporal restrictions and common interests. Mathematics **8**(1), 139 (2020)
4. Bai, Y., Zhou, Z., Xiao, H., Gao, R.: A Stackelberg reinsurance-investment game with asymmetric information and delay. Optimization **70**(10), 2131–2168 (2021)
5. Bai, Y., Zhou, Z., Xiao, H., Gao, R., Zhong, F.: A hybrid stochastic differential reinsurance and investment game with bounded memory. Eur. J. Oper. Res. **296**(2), 717–737 (2022)
6. Bakshi, G., Madan, D.: Spanning and derivative-security valuation. J. Financ. Econ. **55**(2), 205–238 (2000)
7. Bensoussan, A., Siu, C.C., Yam, S.C.P., Yang, H.: A class of non-zero-sum stochastic differential investment and reinsurance games. Automatica **50**(8), 2025–2037 (2014)
8. Bi, J., Meng, Q., Zhang, Y.: Dynamic mean-variance and optimal reinsurance problems under the no-bankruptcy constraint for an insurer. Ann. Oper. Res. **212**(1), 43–59 (2014)
9. Browne, S.: Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin. Math. Oper. Res. **20**(4), 937–958 (1995)
10. Chen, L., Shen, Y.: On a new paradigm of optimal reinsurance: a stochastic Stackelberg differential game between an insurer and a reinsurer. ASTIN Bull. **48**(02), 905–960 (2018)
11. Chen, L., Shen, Y.: Stochastic Stackelberg differential reinsurance games under time-inconsistent mean–variance framework. Insur. Math. Econ. **88**, 120–137 (2019)
12. Chen, S., Li, Z., Li, K.: Optimal investment-reinsurance policy for an insurance company with VaR constraint. Insur. Math. Econ. **47**(2), 144–153 (2010)
13. Cox, J., Ingersoll, J., Ross, S.: A theorey of term structure of interest rates. Econometrica **53**, 385–407 (1985)
14. Deng, C., Zeng, X., Zhu, H.: Non-zero-sum stochastic differential reinsurance and investment games with default risk. Eur. J. Oper. Res. **264**(3), 1144–1158 (2018)
15. Guan, G., Liang, Z.: A stochastic Nash equilibrium portfolio game between two DC pension funds. Insur. Math. Econ. **70**, 237–244 (2016)
16. Heston, S.L.: A closed-form solution for options with stochastic volatility with applications to bond and currency options. Rev. Financ. Stud. **6**(2), 327–343 (1993)
17. Huang, Y., Yang, X., Zhou, J.: Optimal investment and proportional reinsurance for a jump-diffusion risk model with constrained control variables. J. Comput. Appl. Math. **296**, 443–461 (2016)
18. Li, D., Rong, X., Zhao, H.: Stochastic differential game formulation on the reinsurance and investment problem. Int. J. Control **88**(9), 1861–1877 (2015)

19. Li, P., Zhao, W., Zhou, W.: Ruin probabilities and optimal investment when the stock price follows an exponential Lévy process. Appl. Math. Comput. **259**, 1030–1045 (2015)
20. Li, Z., Zeng, Y., Lai, Y.: Optimal time-consistent investment and reinsurance strategies for insurers under Heston's SV model. Insur. Math. Econ. **51**(1), 191–203 (2012)
21. Liu, H.-H., Chang, A., Shiu, Y.-M.: Interest rate derivatives and risk exposure: evidence from the life insurance industry. N. Am. J. Econ. Finance **51**, 100978 (2020)
22. Liu, J., Pan, J.: Dynamic derivative strategies. J. Financ. Econ. **69**(3), 401–430 (2003)
23. Meng, H., Li, S., Jin, Z.: A reinsurance game between two insurance companies with nonlinear risk processes. Insur. Math. Econ. **62**, 91–97 (2015)
24. Taksar, M., Zeng, X.: Optimal non-proportional reinsurance control and stochastic differential games. Insur. Math. Econ. **48**(1), 64–71 (2011)
25. Xue, X., Wei, P., Weng, C.: Derivatives trading for insurers. Insur. Math. Econ. **84**, 40–53 (2019)
26. Yan, M., Peng, F., Zhang, S.: A reinsurance and investment game between two insurance companies with the different opinions about some extra information. Insur. Math. Econ. **75**, 58–70 (2017)
27. Zeng, X.: A stochastic differential reinsurance game. J. Appl. Probab. **47**(2), 335–349 (2010)
28. Zhao, H., Rong, X.: On the constant elasticity of variance model for the utility maximization problem with multiple risky assets. IMA J. Manag. Math. **28**(2), 299–320 (2017)
29. Zhou, Z., Ren, T., Xiao, H., Liu, W.: Time-consistent investment and reinsurance strategies for insurers under multi-period mean-variance formulation with generalized correlated returns. J. Manag. Sci. Eng. **4**(2), 142–157 (2019)

Publisher's Note