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Positive continuous solutions for some semilinear elliptic problems in the half space

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Abstract

The aim of this article is twofold. The first goal is to give a new characterization of the Kato class of functions $K^\infty(\mathbb{R}^d_+)$ that was defined in (Bachar et al. 2002:41, 2002) for $d = 2$ and in (Bachar and Mâagli 9(2):153–192, 2005) for $d \geq 3$ and adapted to study some nonlinear elliptic problems in the half space. The second goal is to prove the existence of positive continuous weak solutions, having the global behavior of the associated homogeneous problem, for sufficiently small values of the nonnegative constants λ and μ to the following system: $\Delta u = \lambda f(x,u,v)$, $\Delta v = \mu g(x,u,v)$ in $\mathbb{R}^d_+$, $\lim_{x\to(\xi,0)} u(x) = a_1\phi_1(\xi)$, $\lim_{x\to(\xi,0)} v(x) = a_2\phi_2(\xi)$ for all $\xi \in \mathbb{R}^{d-1}$, $\lim_{x_d\to\infty} \frac{u(x)}{x_d} = b_1$, $\lim_{x_d\to\infty} \frac{v(x)}{x_d} = b_2$, where ϕ_1 and ϕ_2 are nontrivial nonnegative continuous functions on $\partial\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \{0\}$, a_1, a_2, b_1, b_2 are nonnegative constants such that $(a_1 + b_1)(a_2 + b_2) > 0$. The functions f and g are nonnegative and belong to a class of functions containing in particular all functions of the type $f(x,u,v) = p(x)u^\alpha g_1(v)$ and $g(x,u,v) = q(x)g_2(u)v^\beta$ with $\alpha \geq 1$, $\beta \geq 1$, g_1, g_2 are continuous on $[0,\infty)$, and p, q are nonnegative functions in $K^\infty(\mathbb{R}^d_+)$.

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1 Introduction

In this paper, we study the existence of positive continuous solutions in the upper half space $\mathbb{R}^d_+ = \{x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_d > 0\}$, $d \geq 2$, for the following semilinear elliptic system:

$$\begin{cases} \Delta u = \lambda f(\cdot,u,v) & \text{in } \mathbb{R}^d_+ \text{ (in the sense of distributions)}, \\ \Delta v = \mu g(\cdot,u,v) & \text{in } \mathbb{R}^d_+ \text{ (in the sense of distributions)}, \\ \lim_{x\to(\xi,0)} u(x) = a_1\phi_1(\xi) \lim_{x\to(\xi,0)} v(x) = a_2\phi_2(\xi), & \forall \xi \in \mathbb{R}^{d-1}, \\ \lim_{x_d\to\infty} \frac{u(x)}{x_d} = b_1 \lim_{x_d\to\infty} \frac{v(x)}{x_d} = b_2, \end{cases} \tag{1.1}$$

where ϕ_1 and ϕ_2 are nontrivial nonnegative continuous functions on $\partial\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \{0\}$, a_1, a_2, b_1, b_2 are nonnegative constants such that $(a_1 + b_1)(c_2 + b_2) > 0$, $\lambda \geq 0$, $\mu \geq 0$, and f, g are two nontrivial nonnegative functions on $\mathbb{R}^d_+ \times [0,\infty) \times [0,\infty)$.

This problem has been investigated recently, in particular the cases of nonlinearities f, g, by many authors (see for example [17, 19, 20] and the references therein). In [20], the author considered the particular case where $f(x,u,v) = p(x)g_1(v)$ and $g(x,u,v) = q(x)g_2(u)$, where g_1, g_2 are nonnegative continuous functions that are both nondecreasing or both nonincreasing and p, q are nonnegative measurable functions belonging to the Kato class $K^\infty(\mathbb{R}^d_+)$ introduced and studied in [5] for $d = 2$ and in [4] for $d \geq 3$. Under some conditions on ϕ_1 and ϕ_2, the existence of positive continuous solutions having the global behavior of the associated homogeneous system is established. This also was done by investigating the properties of the Kato class $K^\infty(\mathbb{R}^d_+)$. System (1.1) has been also studied in [17] for the particular cases $\lambda = \mu = 1$, $f(x,u,v) = p(x)u^\alpha v^r$, and $g(x,u,v) = q(x)u^s v^\beta$, where $\alpha \geq 1$, $\beta \geq 1$, $r \geq 0$, $s \geq 0$ and p, q are two nonnegative measurable functions that belong to the class $K^\infty(\mathbb{R}^d_+)$, and some results of existence similar to those in [20] have been obtained.

Our aim in this paper is twofold. The first goal is to give a new characterization of the Kato class $K^\infty(\mathbb{R}^d_+)$, as it will be stated in Theorem 2.2 in the sequel. This explains in a certain manner the optimality of the 3G-inequality (2.5), satisfied by the Green function and established in [4] and [5]. The second goal is to extend the results of [17, 20] to a class of nonlinearities f and g, including in particular those where f is nondecreasing with respect to u but not necessarily monotone with respect to v and g is nondecreasing with respect to v but not necessarily monotone with respect to u. This will be done after establishing and exploiting an existence result of a positive continuous solution for the problem

$$\begin{cases} \Delta u = \lambda f(x,u) & \text{in } \mathbb{R}^d_+ \text{ (in the sense of distributions)}, \\ \lim_{x\to(\xi,0)} u(x) = a\phi(\xi), & \forall \xi \in \mathbb{R}^{d-1}, \\ \lim_{x_d\to\infty} \frac{u(x)}{x_d} = b, \end{cases} \tag{1.2}$$

where $\lambda \geq 0$, $a \geq 0$, $b \geq 0$ with $a + b > 0$, ϕ is a nontrivial nonnegative continuous function on $\partial\mathbb{R}^d_+$ and the function f belongs to a class of functions containing in particular those of the form $p(x)u^\alpha$ with $\alpha \geq 1$, and this will be an extension of the results of [17] established in the case where $f(x,u) = p(x)u^\alpha$. We note that elliptic equations have been extensively studied, we refer the readers to [1, 13, 15] and other papers in the literature.

Our paper is organized as follows. Section 2 is devoted to giving a new characterization of the Kato class $K^\infty(\mathbb{R}^d_+)$ and to recalling some properties of this class that will be used in the study of (1.2) and (1.1). In Sect. 3, we prove the existence of a positive continuous solution for (1.2). The last section is devoted to the study of the existence of positive continuous solutions for system (1.1).

Next, we give some notations that will be used in the sequel. We denote by $B(\mathbb{R}^d_+)$ the set of all Borel measurable functions in $\mathbb{R}^d_+$, by $B^+(\mathbb{R}^d_+)$ the set of nonnegative ones, by $B_b(\mathbb{R}^d_+)$ the set of bounded ones, and by $C(\mathbb{R}^d_+)$ the set of continuous functions u in $\mathbb{R}^d_+$. We denote also by $C_0(\mathbb{R}^d_+)$ the set of functions $u \in C(\mathbb{R}^d_+)$ satisfying $\lim_{x\to\xi\in\partial\mathbb{R}^d_+} u(x) = \lim_{|x|\to\infty} u(x) = 0$ and by $C_0(\overline{\mathbb{R}^d_+})$ the set of all functions $u \in B(\mathbb{R}^d_+)$ that are continuous in $\overline{\mathbb{R}^d_+}$ and satisfy $\lim_{|x|\to\infty} u(x) = 0$.

Let G be the Green function of the Laplace operator in $\mathbb{R}^d_+$ with Dirichlet boundary conditions. For any $p \in B^+(\mathbb{R}^d_+)$, we denote by Vp the Green potential of p defined on $\mathbb{R}^d_+$ by

$$Vp(x) = \int_{\mathbb{R}^d_+} G(x,y)p(y)\,dy,$$

and we recall that if $p \in L^1_{\mathrm{loc}}(\mathbb{R}^d_+)$ and $Vp \in L^1_{\mathrm{loc}}(\mathbb{R}^d_+)$, then we have in the sense of distributions (see [10] p. 52)

$$\Delta(Vp) = -p \quad \text{in } \mathbb{R}^d_+. \tag{1.3}$$

For any nonnegative bounded continuous function ϕ on $\mathbb{R}^{d-1}$, we denote by $H\phi$ the unique bounded continuous solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 \quad \text{in } \mathbb{R}^d_+, \\ \lim_{x \to (\xi,0)} u(x) = \phi(\xi), \quad \forall \xi \in \mathbb{R}^{d-1}. \end{cases}$$

It follows by the Herglotz representation theorem (see [2, 3, 12]) that

$$H\phi(x) = c_d \int_{\mathbb{R}^{d-1}} \frac{x_d}{|x-\xi|^d} \phi(\xi)\, d\xi \quad \text{for every } x \in \mathbb{R}^d_+.$$

Using the inequality $|x - \xi| \le |x| + |\xi| \le (1 + |x|)(1 + |\xi|)$, the fact that ϕ is nonnegative, bounded, and that $\int_{\mathbb{R}^{d-1}} \frac{d\xi}{(1+|\xi|)^d} < \infty$, we obtain

$$H\phi(x) \ge c_d \frac{x_d}{(1+|x|)^d} \int_{\mathbb{R}^{d-1}} \frac{\phi(\xi)}{(1+|\xi|)^d}\, d\xi = c \frac{x_d}{(1+|x|)^d}. \tag{1.4}$$

Let $(X_t)_{t \ge 0}$ be the canonical Brownian motion defined on $C([0,\infty); \mathbb{R}^d)$, P^x be the probability measure on the Brownian continuous paths starting at x, and $\tau = \inf\{t > 0 : X_t \notin \mathbb{R}^d_+\}$ be the first exist time of $(X_t)_{t \ge 0}$ from $\mathbb{R}^d_+$. For any $q \in B^+(\mathbb{R}^d_+)$, we define (see [9] or [10] p. 84) the subordinate q-Green potential kernel V_q by

$$V_q(p)(x) = \frac{1}{2} E^x \left(\int_0^\tau e^{-\frac{1}{2}\int_0^t q(X_s)\, ds} p(X_t)\, dt \right) \quad \text{for } p \in B\big(\mathbb{R}^d_+\big), \tag{1.5}$$

where E^x is the expectation on P^x. Moreover, for $q \in B^+(\mathbb{R}^d_+)$ such that $Vq < \infty$, we have, see [8, 10, 14], the resolvent equation

$$V = V_q + V_q(qV). \tag{1.6}$$

So, for each $u \in B(\mathbb{R}^d_+)$ such that $V(q|u|) < \infty$, we have

$$\big[I + V(q\cdot)\big]\big[I - V_q(q\cdot)\big]u = \big[I - V_q(q\cdot)\big]\big[I + V(q\cdot)\big]u = u, \tag{1.7}$$

$$\text{and for every } u \in B^+\big(\mathbb{R}^d_+\big) \text{ we have} \quad 0 \le V_q(u) \le V(u). \tag{1.8}$$

We close this section by adopting the following notation. If S is a nonempty set and f, g are two nonnegative functions defined on S, we write $f \sim g$ if there exists a positive constant C such that $\frac{1}{C} f(x) \le g(x) \le C f(x)$ for every $x \in S$. We note also that throughout this paper the positive constant C may vary from line to line.

2 The Kato class of functions

Let G be the Green function of the Dirichlet Laplacian in $\mathbb{R}^d_+$, $(d \geq 2)$. Then it was proved in [6] that G has the following integral representation:

$$G(x,y) = C_d |x-y|^{2-d} \int_1^{\frac{|x-\overline{y}|}{|x-y|}} \frac{dv}{v^{d-1}}, \tag{2.1}$$

where $\overline{y} = (y_1, y_2, \dots, y_{d-1}, -y_d)$ for $y = (y_1, y_2, \dots, y_{d-1}, y_d)$ and $C_d = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}}$. Moreover, the authors in [5] and [4] proved that G has the following global estimates:

$$G(x,y) \sim \begin{cases} \operatorname{Log}(1 + \frac{x_2 y_2}{|x-y|^2}) & \text{if } d = 2, \\ \frac{1}{|x-y|^{d-2}} \min(1, \frac{x_d y_d}{|x-y|^2}) & \text{if } d \geq 3. \end{cases} \tag{2.2}$$

Moreover, there exists $C > 0$ such that for every $x, y \in \mathbb{R}^d_+$ we have

$$\frac{x_d y_d}{(|x|+1)^d (|y|+1)^d} \leq CG(x,y). \tag{2.3}$$

Using the fact that $\frac{ab}{a+b} \leq \min(a,b) \leq \frac{2ab}{a+b}$ for $a > 0$ and $b > 0$, it follows from (2.2) that

$$G(x,y) \sim \begin{cases} \operatorname{Log}(1 + \frac{x_d y_d}{|x-y|^2}) & \text{if } d = 2, \\ \frac{x_d y_d}{|x-y|^{d-2}(|x-y|^2 + x_d y_d)} & \text{if } d \geq 3. \end{cases} \tag{2.4}$$

These estimates have been used to prove the following important 3G-inequality. Namely, there exists a positive constant C_0 such that for each $x, y, z \in \mathbb{R}^d_+$ we have

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leq C_0 \left[\frac{z_d}{x_d} G(x,z) + \frac{z_d}{y_d} G(y,z) \right]. \tag{2.5}$$

This 3G-inequality was exploited by the authors in [5] for $d = 2$ and in [4] for $d \geq 3$ to define a new Kato class on the half space $\mathbb{R}^d_+$, which has been adapted to study some semilinear elliptic boundary value problems using some potential theory tools. More precisely, this class was defined as follows.

Definition 2.1 ([5] and [4]) A measurable function q belongs to the Kato class $K^\infty(\mathbb{R}^d_+)$ if q satisfies the following conditions:

$$\lim_{\alpha \to 0} \left(\sup_{x \in \mathbb{R}^d_+} \int_{\mathbb{R}^d_+ \cap B(x,\alpha)} \frac{z_d}{x_d} G(x,z) \big| q(z) \big| \, dz \right) = 0 \tag{2.6}$$

and

$$\lim_{M \to \infty} \left(\sup_{x \in \mathbb{R}^d_+} \int_{\mathbb{R}^d_+ \cap [|z| \geq M]} \frac{z_d}{x_d} G(x,z) \big| q(z) \big| \, dz \right) = 0. \tag{2.7}$$

Our main goal in this section is to give a new characterization of this class of functions by means of the left-hand side term of inequality (2.5). This gives an affirmative answer to the question on the possibility of considering the left-hand term of inequality (2.5) in the definition of the Kato class. More precisely, we prove the following.

Theorem 2.2 *Let q be a Borel measurable function in $\mathbb{R}^d_+$. Then $q \in K^\infty(\mathbb{R}^d_+)$ if and only if*

$$\lim_{\alpha\to 0}\left(\sup_{(x,y)\in\mathbb{R}^d_+\times\mathbb{R}^d_+}\int_{\mathbb{R}^d_+\cap(B(x,\alpha)\cup B(y,\alpha))}\frac{G(x,z)G(z,y)}{G(x,y)}\left|q(z)\right|dz\right)=0 \tag{2.8}$$

and

$$\lim_{M\to\infty}\left(\sup_{(x,y)\in\mathbb{R}^d_+\times\mathbb{R}^d_+}\int_{\mathbb{R}^d_+\cap([|z|\ge M])}\frac{G(x,z)G(z,y)}{G(x,y)}\left|q(z)\right|dz\right)=0. \tag{2.9}$$

The following lemma will be also used in the proof.

Lemma 2.3 *Let $x,y \in \mathbb{R}^d_+$. Then we have the following properties*:

(1) *If $x_d y_d \le |x-y|^2$, then* $\max(x_d,y_d) \le \frac{1+\sqrt{5}}{2}|x-y|$.

(2) *If $|x-y|^2 \le x_d y_d$, then $\frac{3-\sqrt{5}}{2}y_d \le x_d \le \frac{3+\sqrt{5}}{2}y_d$.*

(3) $\frac{1}{2}(|x-y|^2+x_d^2+y_d^2) \le |x-y|^2+x_dy_d \le |x-y|^2+x_d^2+y_d^2$.

Proof (1) and (2) were proved in [4].

(3) Squaring the inequality $|x_d-y_d| \le |x-y|$, we obtain $x_d^2+y_d^2 \le |x-y|^2+2x_dy_d$. This together with the fact that $ab \le a^2+b^2$ gives

$$|x-y|^2+x_d^2+y_d^2 \le 2\left[|x-y|^2+x_dy_d\right] \le 2\left[|x-y|^2+x_d^2+y_d^2\right].$$

This achieves the proof. □

The following result is the key to the proof of Theorem 2.2.

Proposition 2.4 *There exists a constant $C>0$ such that for all $\alpha>0$ and all $x,y\in\mathbb{R}^d_+$ we have*

$$\begin{aligned}\int_{\mathbb{R}^d_+\cap(B(x,\alpha)\cup B(y,\alpha))}\frac{G(x,z)G(z,y)}{G(x,y)}\left|q(z)\right|dz &\le C\int_{\mathbb{R}^d_+\cap B(x,3\alpha)}\frac{z_d}{x_d}G(x,z)\left|q(z)\right|dz\\ &\quad + C\int_{\mathbb{R}^d_+\cap B(y,3\alpha)}\frac{z_d}{y_d}G(y,z)\left|q(z)\right|dz.\end{aligned}$$

Proof Let $\alpha>0$ and $x,y\in\mathbb{R}^d_+$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^d_+\cap(B(x,\alpha)\cup B(y,\alpha))}\frac{G(x,z)G(z,y)}{G(x,y)}\left|q(z)\right|dz &= \int_{\mathbb{R}^d_+\cap B(x,\alpha)\cap B(y,\alpha)}\frac{G(x,z)G(z,y)}{G(x,y)}\left|q(z)\right|dz\\ &\quad+\int_{\mathbb{R}^d_+\cap B(x,\alpha)\cap B^c(y,\alpha)}\frac{G(x,z)G(z,y)}{G(x,y)}\left|q(z)\right|dz\\ &\quad+\int_{\mathbb{R}^d_+\cap B(y,\alpha)\cap B^c(x,\alpha)}\frac{G(x,z)G(z,y)}{G(x,y)}\left|q(z)\right|dz\\ &= I_1(x,y)+I_2(x,y)+I_3(x,y).\end{aligned}$$

Using inequality (2.5), we obtain

$$
\begin{aligned}
I_1(x,y) &:= \int_{\mathbb{R}^d_+\cap B(x,\alpha)\cap B(y,\alpha)} \frac{G(x,z)G(z,y)}{G(x,y)}\big|q(z)\big|\,dz \\
&\le C_0 \int_{\mathbb{R}^d_+\cap B(x,\alpha)\cap B(y,\alpha)} \left[\frac{z_d}{x_d}G(x,z)+\frac{z_d}{y_d}G(y,z)\right]\big|q(z)\big|\,dz \\
&\le C_0\left[\int_{\mathbb{R}^d_+\cap B(x,\alpha)} \frac{z_d}{x_d}G(x,z)\big|q(z)\big|\,dz+\int_{\mathbb{R}^d_+\cap B(y,\alpha)} \frac{z_d}{y_d}G(y,z)\big|q(z)\big|\,dz\right].
\end{aligned}
$$

Next, we estimate $I_2(x,y)$ and $I_3(x,y)$. To this aim, we will discuss two cases as follows.

Case 1: $B(x,\alpha)\cap B(y,\alpha)\neq\emptyset$.

Choose $z_0\in B(x,\alpha)\cap B(y,\alpha)$. Then, for every $z\in B(x,\alpha)\cap B^c(y,\alpha)$, we have

$$
|z-y|\le|z-x|+|x-z_0|+|z_0-y|\le 3\alpha.
$$

Similarly, for every $z\in B(y,\alpha)\cap B^c(x,\alpha)$, we have

$$
|z-x|\le|z-y|+|y-z_0|+|z_0-x|\le 3\alpha.
$$

Hence $B(x,\alpha)\cap B^c(y,\alpha)\subset B(x,\alpha)\cap B(y,3\alpha)$ and $B(y,\alpha)\cap B^c(x,\alpha)\subset B(y,\alpha)\cap B(x,3\alpha)$. So we obtain

$$
\begin{aligned}
I_2(x,y) &:= \int_{\mathbb{R}^d_+\cap B(x,\alpha)\cap B^c(y,\alpha)} \frac{G(x,z)G(z,y)}{G(x,y)}\big|q(z)\big|\,dz \\
&\le \int_{\mathbb{R}^d_+\cap B(x,\alpha)\cap B(y,3\alpha)} \frac{G(x,z)G(z,y)}{G(x,y)}\big|q(z)\big|\,dz \\
&\le C_0 \int_{\mathbb{R}^d_+\cap B(x,\alpha)\cap B(y,3\alpha)} \left[\frac{z_d}{x_d}G(x,z)+\frac{z_d}{y_d}G(y,z)\right]\big|q(z)\big|\,dz \\
&\le C_0\left[\int_{\mathbb{R}^d_+\cap B(x,\alpha)} \frac{z_d}{x_d}G(x,z)\big|q(z)\big|\,dz+\int_{\mathbb{R}^d_+\cap B(y,3\alpha)} \frac{z_d}{y_d}G(y,z)\big|q(z)\big|\,dz\right]
\end{aligned}
$$

and

$$
\begin{aligned}
I_3(x,y) &:= \int_{\mathbb{R}^d_+\cap B(y,\alpha)\cap B^c(x,\alpha)} \frac{G(x,z)G(z,y)}{G(x,y)}\big|q(z)\big|\,dz \\
&\le \int_{\mathbb{R}^d_+\cap B(y,\alpha)\cap B(x,3\alpha)} \frac{G(x,z)G(z,y)}{G(x,y)}\big|q(z)\big|\,dz \\
&\le C_0 \int_{\mathbb{R}^d_+\cap B(y,\alpha)\cap B(x,3\alpha)} \left[\frac{z_d}{x_d}G(x,z)+\frac{z_d}{y_d}G(y,z)\right]\big|q(z)\big|\,dz \\
&\le C_0\left[\int_{\mathbb{R}^d_+\cap B(x,3\alpha)} \frac{z_d}{x_d}G(x,z)\big|q(z)\big|\,dz+\int_{\mathbb{R}^d_+\cap B(y,\alpha)} \frac{z_d}{y_d}G(y,z)\big|q(z)\big|\,dz\right].
\end{aligned}
$$

Case 2: $B(x,\alpha)\cap B(y,\alpha)=\emptyset$.

In this case $B(x,\alpha)\subset B^c(y,\alpha)$ and $B(y,\alpha)\subset B^c(x,\alpha)$. For every $z\in B(x,\alpha)$, we have

$$
|y-z|\le|y-x|+|x-z|\le|y-x|+\alpha\le 2|x-y|
$$

and

$$|x-y| \leq |x-z| + |y-z| \leq \alpha + |y-z| \leq 2|y-z|.$$

So, in this case

$$\frac{1}{2}|y-z| \leq |x-y| \leq 2|y-z|. \tag{2.10}$$

Similarly, for every $z \in B(y,\alpha)$, we have

$$|x-z| \leq |x-y| + |y-z| \leq |x-y| + \alpha \leq 2|x-y|$$

and

$$|x-y| \leq |x-z| + |y-z| \leq |x-z| + \alpha \leq 2|x-z|.$$

Also, in this case

$$\frac{1}{2}|x-z| \leq |x-y| \leq 2|x-z|. \tag{2.11}$$

Now, using (2.4) we obtain

$$\frac{G(x,z)G(z,y)}{G(x,y)} \sim \begin{cases} \frac{\operatorname{Log}(1+\frac{y_d z_d}{|z-y|^2})}{\operatorname{Log}(1+\frac{x_d y_d}{|x-y|^2})} G(x,z) & \text{if } d=2, \\ \frac{|x-y|^{d-2}}{|z-y|^{d-2}} \frac{(|x-y|^2+x_d y_d)}{(|z-y|^2+z_d y_d)} \frac{z_d}{x_d} G(x,z) & \text{if } d \geq 3, \end{cases}$$

and

$$\frac{G(x,z)G(z,y)}{G(x,y)} \sim \begin{cases} \frac{\operatorname{Log}(1+\frac{x_d z_d}{|z-x|^2})}{\operatorname{Log}(1+\frac{x_d y_d}{|x-y|^2})} G(y,z) & \text{if } d=2, \\ \frac{|x-y|^{d-2}}{|z-x|^{d-2}} \frac{(|x-y|^2+x_d y_d)}{(|z-x|^2+z_d x_d)} \frac{z_d}{y_d} G(y,z) & \text{if } d \geq 3. \end{cases}$$

So we will discuss two subcases.

Subcase 1: If $x_d y_d \leq |x-y|^2$.

In this case we have $|x-y|^2 + x_d y_d \leq 2|x-y|^2$. So, for $d \geq 3$, we use this fact and (2.10) to obtain

$$\frac{|x-y|^{d-2}}{|z-y|^{d-2}} \frac{(|x-y|^2+x_d y_d)}{(|z-y|^2+z_d y_d)} \leq \frac{|x-y|^{d-2}(|x-y|^2+x_d y_d)}{|z-y|^d} \leq 2\frac{|x-y|^d}{|z-y|^d} \leq 2^{d+1}.$$

On the other hand, for $d=2$ we use (2.10), the inequalities $\frac{1}{2}t \leq \operatorname{Log}(1+t)$ for $t \in [0,1]$ and $\operatorname{Log}(1+t) \leq t$ for $t \geq 0$ to obtain

$$\frac{\operatorname{Log}(1+\frac{y_d z_d}{|z-y|^2})}{\operatorname{Log}(1+\frac{x_d y_d}{|x-y|^2})} \leq 2\frac{|x-y|^2}{x_d y_d} \frac{y_d z_d}{|z-y|^2} \leq 8\frac{z_d}{x_d}.$$

Consequently, for every $z \in B(x,\alpha)$, we have

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leq C\frac{z_d}{x_d}G(x,z)$$

and

$$I_2(x,y) = \int_{\mathbb{R}^d_+ \cap B(x,\alpha)} \frac{G(x,z)G(z,y)}{G(x,y)}\big|q(z)\big|\,dz \leq C\int_{\mathbb{R}^d_+ \cap B(x,\alpha)} \frac{z_d}{x_d}G(x,z)\big|q(z)\big|\,dz.$$

Similarly, for every $z \in B(y,\alpha)$, we obtain by using (2.11) that

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leq C\frac{z_d}{y_d}G(y,z)$$

and

$$I_3(x,y) = \int_{\mathbb{R}^d_+ \cap B(y,\alpha)} \frac{G(x,z)G(z,y)}{G(x,y)}\big|q(z)\big|\,dz \leq C\int_{\mathbb{R}^d_+ \cap B(y,\alpha)} \frac{z_d}{y_d}G(y,z)\big|q(z)\big|\,dz.$$

Subcase 2: If $|x-y|^2 \leq x_d y_d$.

In this case we obtain from Lemma 2.3 that

$$\frac{3-\sqrt{5}}{2}y_d \leq x_d \leq \frac{3+\sqrt{5}}{2}y_d. \tag{2.12}$$

Next we will treat the cases $d \geq 3$ and $d = 2$ separately. If $d \geq 3$, then we deduce from (2.12), (2.10) and property 3 of Lemma 2.3 that for every $z \in B(x,\alpha)$ we have

$$\begin{aligned}
\frac{|x-y|^{d-2}}{|z-y|^{d-2}}\frac{(|x-y|^2+x_d y_d)}{(|z-y|^2+z_d y_d)} &\leq 2^{d-2}\frac{|x-y|^2+x_d y_d}{|z-y|^2+z_d y_d} \\
&\leq 2^d\frac{|x-y|^2+x_d^2+y_d^2}{|z-y|^2+z_d^2+y_d^2} \\
&\leq 2^d\frac{(1+(\frac{3+\sqrt{5}}{2})^2)(|x-y|^2+y_d^2)}{|z-y|^2+z_d^2+y_d^2} \\
&\leq 2^d\left(\frac{9+3\sqrt{5}}{2}\right)\frac{|x-y|^2+y_d^2}{|z-y|^2+y_d^2} \\
&\leq 2^{d+1}(9+3\sqrt{5}).
\end{aligned}$$

Consequently, for every $z \in B(x,\alpha)$, we have

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leq C\frac{z_d}{x_d}G(x,z)$$

and

$$I_2(x,y) = \int_{\mathbb{R}^d_+ \cap B(x,\alpha)} \frac{G(x,z)G(z,y)}{G(x,y)}\big|q(z)\big|\,dz \leq C\int_{\mathbb{R}^d_+ \cap B(x,\alpha)} \frac{z_d}{x_d}G(x,z)\big|q(z)\big|\,dz.$$

Similarly, for $z \in B(y,\alpha)$, we use (2.11) and similar arguments as above to obtain

$$\frac{G(x,z)G(z,y)}{G(x,y)} \le C\frac{z_d}{y_d}G(y,z)$$

and

$$I_3(x,y) = \int_{\mathbb{R}^d_+\cap B(y,\alpha)} \frac{G(x,z)G(z,y)}{G(x,y)}\big|q(z)\big|\,dz \le C\int_{\mathbb{R}^d_+\cap B(y,\alpha)} \frac{z_d}{y_d}G(y,z)\big|q(z)\big|\,dz.$$

Finally, for $d = 2$ we will discuss two subcases:

(i) If $|x-z|^2 \le x_d z_d$ or $|y-z|^2 \le y_d z_d$. Then, taking into account (2.12) and using Lemma 2.3, we obtain in this case that

$$\begin{aligned}&\frac{3-\sqrt{5}}{2}x_d \le z_d \le \frac{3+\sqrt{5}}{2}x_d \quad \text{and}\\ &\left(\frac{3-\sqrt{5}}{2}\right)^2 y_d \le z_d \le \left(\frac{3+\sqrt{5}}{2}\right)^2 y_d,\end{aligned}$$

or

$$\begin{aligned}&\frac{3-\sqrt{5}}{2}y_d \le z_d \le \frac{3+\sqrt{5}}{2}y_d \quad \text{and}\\ &\left(\frac{3-\sqrt{5}}{2}\right)^2 x_d \le z_d \le \left(\frac{3+\sqrt{5}}{2}\right)^2 x_d.\end{aligned}$$

Using the above facts, (2.10), and the fact that for $\lambda > 0$ and $t \ge 0$ we have

$$\min(1,\lambda)\operatorname{Log}(1+t) \le \operatorname{Log}(1+\lambda t) \le \max(1,\lambda)\operatorname{Log}(1+t),$$

we obtain for $z \in B(x,\alpha)$ that

$$\begin{aligned}\frac{\operatorname{Log}(1+\frac{y_d z_d}{|z-y|^2})}{\operatorname{Log}(1+\frac{x_d y_d}{|x-y|^2})} &\le \frac{\operatorname{Log}(1+(\frac{3+\sqrt{5}}{2})\frac{x_d z_d}{|z-y|^2})}{\operatorname{Log}(1+(\frac{3-\sqrt{5}}{2})^2\frac{x_d z_d}{4|z-y|^2})}\\ &\le \left(\frac{3+\sqrt{5}}{2}\right)\frac{16}{(3-\sqrt{5})^2}\frac{\operatorname{Log}(1+\frac{x_d z_d}{|z-y|^2})}{\operatorname{Log}(1+\frac{x_d z_d}{|z-y|^2})}\\ &\le (3+\sqrt{5})^3\\ &\le (3+\sqrt{5})^3\left(\frac{3+\sqrt{5}}{2}\right)^2\frac{z_d}{x_d}.\end{aligned}$$

Hence, for every $z \in B(x,\alpha)$, we obtain

$$\frac{G(x,z)G(z,y)}{G(x,y)} \le C\frac{z_d}{x_d}G(x,z)$$

and

$$I_2(x,y) = \int_{\mathbb{R}^d_+\cap B(x,\alpha)} \frac{G(x,z)G(z,y)}{G(x,y)}\big|q(z)\big|\,dz \le C\int_{\mathbb{R}^d_+\cap B(x,\alpha)} \frac{z_d}{x_d}G(x,z)\big|q(z)\big|\,dz.$$

Similarly, for $z \in B(y,\alpha)$, we use (2.11) to obtain

$$\frac{G(x,z)G(z,y)}{G(x,y)} \le C\frac{z_d}{y_d}G(y,z)$$

and

$$\begin{aligned} I_3(x,y) &= \int_{\mathbb{R}^d_+ \cap B(y,\alpha)} \frac{G(x,z)G(z,y)}{G(x,y)}\big|q(z)\big|\,dz \\ &\le C\int_{\mathbb{R}^d_+ \cap B(y,\alpha)} \frac{z_d}{y_d}G(y,z)\big|q(z)\big|\,dz. \end{aligned}$$

(ii) If $|x-z|^2 \ge x_d z_d$ and $|y-z|^2 \ge y_d z_d$, then in this case we have $\max(x_d,z_d) \le |x-z|$ and $\max(y_d,z_d) \le |y-z|$. Hence it follows from the inequalities $\frac{t}{1+t} \le \operatorname{Log}(1+t) \le t$ for $t \ge 0$ that

$$\frac{x_d y_d}{|x-y|^2 + x_d y_d} \le \operatorname{Log}\left(1 + \frac{x_d y_d}{|x-y|^2}\right).$$

Hence

$$\begin{aligned} \frac{\operatorname{Log}(1+\frac{y_d z_d}{|z-y|^2})}{\operatorname{Log}(1+\frac{x_d y_d}{|x-y|^2})} &\le \frac{|x-y|^2 + x_d y_d}{|y-z|^2}\frac{z_d}{x_d} \\ &\le \frac{|x-y|^2 + (\frac{3+\sqrt{5}}{2})y_d^2}{|y-z|^2}\frac{z_d}{x_d} \\ &\le \left(\frac{3+\sqrt{5}}{2}\right)\frac{|x-y|^2 + y_d^2}{|y-z|^2}\frac{z_d}{x_d} \\ &\le \left(\frac{3+\sqrt{5}}{2}\right)\frac{|x-y|^2 + |y-z|^2}{|y-z|^2}\frac{z_d}{x_d}, \end{aligned}$$

and similarly

$$\frac{\operatorname{Log}(1+\frac{x_d z_d}{|x-z|^2})}{\operatorname{Log}(1+\frac{x_d y_d}{|x-y|^2})} \le \left(\frac{3+\sqrt{5}}{2}\right)\frac{|x-y|^2 + |x-z|^2}{|x-z|^2}\frac{z_d}{y_d}.$$

So, using (2.10), for $z \in B(x,\alpha)$, we get

$$\frac{G(x,z)G(z,y)}{G(x,y)} \le C\frac{z_d}{x_d}G(x,z)$$

and

$$I_2(x,y) = \int_{\mathbb{R}^d_+ \cap B(x,\alpha)} \frac{G(x,z)G(z,y)}{G(x,y)}\big|q(z)\big|\,dz \le C\int_{\mathbb{R}^d_+ \cap B(x,\alpha)} \frac{z_d}{x_d}G(x,z)\big|q(z)\big|\,dz.$$

Now, for $z \in B(y,\alpha)$, we use (2.11) and similar arguments as above to obtain

$$\frac{G(x,z)G(z,y)}{G(x,y)} \le C\frac{z_d}{y_d}G(y,z)$$

and

$$I_3(x,y) = \int_{\mathbb{R}^d_+\cap B(y,\alpha)} \frac{G(x,z)G(z,y)}{G(x,y)} \big|q(z)\big|\,dz \le C \int_{\mathbb{R}^d_+\cap B(y,\alpha)} \frac{z_d}{y_d} G(y,z)\big|q(z)\big|\,dz.$$

This achieves the proof of the proposition. □

Proof of Theorem 2.2 Assume that $q \in K^\infty(\mathbb{R}^d_+)$. Clearly, we deduce from (2.5) and (2.7) that (2.9) is satisfied. Moreover, using Proposition 2.4 and equation (2.6), we deduce that (2.8) is also satisfied. To prove the converse, we remark that by considering in (2.1) the substitution

$$v^2 = 1 + \frac{4x_d y_d}{|x-y|^2}(1-t) = \frac{|x-\bar{y}|^2}{|x-y|^2} - 4\frac{x_d y_d}{|x-y|^2} t,$$

we obtain

$$G(x,y) = 2C_d \frac{x_d y_d}{|x-y|^d} \int_0^1 \frac{dt}{(\frac{|x-\bar{y}|^2}{|x y|^2} - 4\frac{x_d y_d}{|x-y|^2} t)^{\frac{n}{2}}}.$$

Hence, for each $\xi \in \partial\mathbb{R}^d_+$ and $x, z \in \mathbb{R}^d_+$, we have

$$\lim_{y\to\xi} \frac{G(z,y)}{G(x,y)} = \frac{z_d}{x_d} \frac{|x-\xi|^d}{|z-\xi|^d}.$$

Now, if we choose $\alpha > 0$ and $x \in \mathbb{R}^d_+$, then we deduce from the Fatou lemma that

$$\begin{aligned}
&\int_{\mathbb{R}^d_+\cap B(x,\alpha)} \frac{z_d}{x_d} \frac{|x-\xi|^d}{|z-\xi|^d} G(x,z)\big|q(z)\big|\,dz \\
&\quad \le \liminf_{y\to\xi} \int_{\mathbb{R}^d_+\cap B(x,\alpha)} \frac{G(x,z)G(z,y)}{G(x,y)}\big|q(z)\big|\,dz \\
&\quad \le \liminf_{y\to\xi} \int_{\mathbb{R}^d_+\cap (B(\chi,\alpha)\cup B(\zeta,\alpha))} \frac{G(x,z)G(z,y)}{G(x,y)}\big|q(z)\big|\,dz \\
&\quad \le \sup_{(\chi,\zeta)\in\mathbb{R}^d_+\times\mathbb{R}^d_+} \int_{\mathbb{R}^d_+\cap (B(\chi,\alpha)\cup B(\zeta,\alpha))} \frac{G(\chi,z)G(z,\zeta)}{G(\chi,\zeta)}\big|q(z)\big|\,dz.
\end{aligned}$$

Using this fact and the Fatou lemma again, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d_+\cap B(x,\alpha)} \frac{z_d}{x_d} G(x,z)\big|q(z)\big|\,dz &= \int_{\mathbb{R}^d_+\cap B(x,\alpha)} \lim_{\substack{|\xi|\to\infty \\ \xi\in\partial\mathbb{R}^d_+}} \frac{|x-\xi|^d}{|z-\xi|^d} \frac{z_d}{x_d} G(x,z)\big|q(z)\big|\,dz \\
&\le \liminf_{\substack{|\xi|\to\infty \\ \xi\in\partial\mathbb{R}^d_+}} \int_{\mathbb{R}^d_+\cap B(x,\alpha)} \frac{z_d}{x_d} \frac{|x-\xi|^d}{|z-\xi|^d} G(x,z)\big|q(z)\big|\,dz \\
&\le \sup_{(\chi,\zeta)\in\mathbb{R}^d_+\times\mathbb{R}^d_+} \int_{\mathbb{R}^d_+\cap (B(\chi,\alpha)\cup B(\zeta,\alpha))} \frac{G(\chi,z)G(z,\zeta)}{G(\chi,\zeta)}\big|q(z)\big|\,dz.
\end{aligned}$$

This shows that if (2.8) is satisfied then (2.6) is also satisfied. In the same manner, we prove that if (2.9) is satisfied then (2.7) is also satisfied. This achieves the proof. □

Next, we recall some important properties that will be used in the study of the boundary value problems (1.2) and (1.1). The proofs of these properties can be found in references [4, 5], and [7].

Proposition 2.5 *Let $q \in K^\infty(\mathbb{R}^d_+)$. Then the following statements hold*:

(1)

$$\alpha_q := \sup_{(x,y)\in\mathbb{R}^d_+\times\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} \frac{G(x,z)G(z,y)}{G(x,y)} \left|q(z)\right| dz < \infty. \tag{2.13}$$

(2) *For any nonnegative superharmonic function ω and every $x \in \mathbb{R}^d_+$, we have*

$$\int_{\mathbb{R}^d_+} G(x,z)\omega(z)\left|q(z)\right| dz \le \alpha_q \omega(x). \tag{2.14}$$

(3) *The function $y \longrightarrow \frac{y_d}{(|y|+1)^d} q(y) \in L^1(\mathbb{R}^d_+)$. In particular, $q \in L^1_{\mathrm{loc}}(\mathbb{R}^d_+)$.*

(4) *The Green potential Vq belongs to $C_0(\mathbb{R}^d_+)$.*

The following results are also stated in [4, 5, 7], and [17], and they will also play an important role in the sequel.

Proposition 2.6 *Let ω be a nonnegative superharmonic function in $\mathbb{R}^d_+$ and q be a nonnegative function in $K^\infty(\mathbb{R}^d_+)$. Then, for each $x \in \mathbb{R}^d_+$ such that $0 < \omega(x) < \infty$, we have*

$$\exp(-\alpha_q)\omega(x) \le \omega(x) - V_q(q\omega)(x) \le \omega(x). \tag{2.15}$$

Proposition 2.7 *Let q be a nonnegative function in $K^\infty(\mathbb{R}^d_+)$ and let $\widetilde{h}(x) = bx_d + a$ for $a \ge 0, b \ge 0$ with $a + b > 0$. Then*:

(1) *The family of functions*

$$\mathcal{E}_q = \left\{Vp; p \in B\left(\mathbb{R}^d_+\right) \text{ with } |p| \le q\right\}$$

is equicontinuous in $\overline{\mathbb{R}^d_+} \cup \{\infty\}$ and consequently it is relatively compact in $C_0(\mathbb{R}^d_+)$.

(2) *The family of functions*

$$\mathcal{F}_q = \left\{x \to \int_{\mathbb{R}^d_+} \frac{y_d}{x_d} G(x,y)p(y)\, dy; p \in B\left(\mathbb{R}^d_+\right) \text{ with } |p| \le q\right\}$$

is relatively compact in $C_0(\overline{\mathbb{R}^d_+})$.

(3) *The family of functions*

$$\mathcal{G}_q = \left\{x \to \int_{\mathbb{R}^d_+} \frac{\widetilde{h}(y)}{\widetilde{h}(x)} G(x,y)p(y)\, dy; p \in B\left(\mathbb{R}^d_+\right) \text{ with } |p| \le q\right\}$$

is relatively compact in $C_0(\overline{\mathbb{R}^d_+})$.

(4) $\lim_{x\to\xi} V(\widetilde{h}q)(x) = 0, \forall \xi \in \partial\mathbb{R}^d_+$.

Next, we recall a fundamental example of functions in $K^\infty(\mathbb{R}^d_+)$ studied in [4] and [5].

Example 2.1 Let $\beta,\delta\in\mathbb{R}$ and define $q(x)=\frac{1}{x_d^{\beta}(|x|+1)^{\delta-\beta}}$ for $x\in\mathbb{R}^d_+$. Then

$$q\in K^{\infty}\big(\mathbb{R}^d_+\big)\quad\text{if and only if}\quad\beta<2<\delta.$$

3 Existence of positive solutions for some semilinear elliptic equations

The aim of this section is to study the existence of positive continuous weak solutions for problem (1.2). First, we define the notion of continuous weak solutions for this problem.

Definition 3.1 A function u is called a continuous weak solution of (1.2) if

(i) $u\in C(\overline{\mathbb{R}^d_+},\mathbb{R})$.

(ii) $\int_{\mathbb{R}^d_+}u(x)\Delta\varphi(x)-\lambda f(x,u(x))\varphi\,dx=0$ for every $\varphi\in C_c^{\infty}(\mathbb{R}^d_+)$: the set of all infinitely differentiable functions in $\mathbb{R}^d_+$ with compact support in $\mathbb{R}^d_+$.

(iii) $\lim_{\substack{x\to\xi\in\partial\mathbb{R}^d_+\\ x\in\mathbb{R}^d_+}}u(x)=a\phi(\xi)$ and $\lim_{x_d\to\infty}\frac{u(x)}{x_d}=b$.

To state an existence result for (1.2) for λ sufficiently small, we define $h(x)=bx_d+aH\phi(x)$ and $\widetilde{h}(x)=bx_d+a$ for $x\in\mathbb{R}^d_+$, and we assume that f satisfies the following hypotheses:

(H_1) $f(\cdot,0)\in K^{\infty}(\mathbb{R}^d_+)$.

(H_2) $f:\mathbb{R}^d_+\times[0,\infty)\longrightarrow[0,\infty)$ is a Borel measurable function such that for each $x\in\mathbb{R}^d_+$ the map $t\to f(x,t)$ is continuous and satisfies the following condition: For each $M>0$, there exists a nonnegative function $q_M\in K^{\infty}(\mathbb{R}^d_+)$ such that for each $x\in\mathbb{R}^d_+$ the map $t\to t\widetilde{h}(x)q_M(x)-f(x,t\widetilde{h}(x))$ is continuous and nondecreasing on $[0,M]$.

(H_3) $\sigma_0:=\inf_{x\in\overline{\mathbb{R}^d_+}}[\frac{h(x)}{Vf(\cdot,0)(x)}]>0$.

Remarks 3.2 (1) Conditions (H_1) and (H_2) are satisfied in the particular case $f(x,t)=p(x)g(t)$, where $p\in K^{\infty}(\mathbb{R}^d_+)$ and $g(t)=t^{\alpha}$, $\alpha\geq 1$ or more generally $g:[0,\infty)\to[0,\infty)$ is continuous and satisfying for each $M>0$, there exists a constant $b=b(M)\geq 0$ such that $g(t)-g(s)\leq b(t-s)$ for $0\leq s<t\leq M$. Indeed in this case (H_2) is satisfied with $q_M=b(M)p$.

(2) Hypothesis (H_3) is satisfied in the particular case where $f(\cdot,0)=0$ with $\sigma_0=\infty$.

Under conditions (H_1)–(H_2), we will prove in the next that continuous weak solutions u of (1.2) in $\mathbb{R}^d_+$ satisfying $0\leq u\leq h$ are those satisfying the integral equation (3.1).

Lemma 3.3 (see [17]) *Let p_1 and p_2 be two nonnegative measurable functions in $\mathbb{R}^d_+$ such that $p_1\leq p_2$ and Vp_2 is continuous in $\mathbb{R}^d_+$. Then Vp_1 is also continuous in $\mathbb{R}^d_+$*

Lemma 3.4 *Assume that hypotheses (H_1)–(H_2) are satisfied, let $u\in B^+(\mathbb{R}^d_+)$ satisfying $0\leq u(x)\leq h(x)$ for $x\in\mathbb{R}^d_+$, and assume that $\lambda>0$. Then u is a continuous weak solution of* (1.2) *if and only if*

$$u(x)=h(x)-\lambda V\big(f(\cdot,u)\big)(x)\quad\textit{for } x\in\mathbb{R}^d_+.\tag{3.1}$$

Proof Assume that u is a continuous weak solution of (1.2). We define $|\phi|_\infty=\sup_{\xi\in\partial\mathbb{R}^d_+}\phi(\xi)$ and $M=\max(1,|\phi|_\infty)$. Then $0\leq u\leq h\leq M\widetilde{h}$. From hypothesis (H_2), there exists $q=q_M\in K^{\infty}(\mathbb{R}^d_+)$ such that for each $x\in\mathbb{R}^d_+$ the map $t\to t\widetilde{h}(x)q(x)-f(x,t\widetilde{h}(x))$ is nondecreasing on $[0,M]$. Hence

$$-f(\cdot,0)\leq qu-f(\cdot,u)\leq M\widetilde{h}q-f(\cdot,M\widetilde{h}).\tag{3.2}$$

In particular, we obtain

$$0 \le f(\cdot,u) \le M\widetilde{h}q + f(\cdot,0). \tag{3.3}$$

Since the functions $q, f(\cdot,0) \in K^\infty(\mathbb{R}^d_+)$, then it follows from Proposition 2.5 that $V(f(\cdot,0)) \in C_0(\mathbb{R}^d_+)$ and $\frac{1}{\widetilde{h}}V(\widetilde{h}q) \in C_0(\overline{\mathbb{R}^d_+})$. This implies that $V(\widetilde{h}q) \in C(\mathbb{R}^d_+)$ and $V(M\widetilde{h}q + f(\cdot,0)) \in C(\mathbb{R}^d_+)$. This together with Lemma 3.3 implies that $V(f(\cdot,u)) \in C(\mathbb{R}^d_+)$. Put $v(x) = u + \lambda V(f(\cdot,u)) - aH\phi$. Then $v \in C(\mathbb{R}^d_+)$ and is harmonic in the sense of distributions in $\mathbb{R}^d_+$. It follows from Weyl's theorem (see [11] p.250) that v is a harmonic function in $\mathbb{R}^d_+$. Moreover, $v \ge -a|\phi|_\infty$ in $\mathbb{R}^d_+$ and $\lim_{x\to\xi\in\partial\mathbb{R}^d_+} v(x) = 0$. Using Theorem 1.1 in [18], we deduce that there exists $C_* \ge 0$ such that $v(x) = C_* x_d$ in $\mathbb{R}^d_+$. Since $\lim_{x_d\to\infty} \frac{v(x)}{x_d} = \lim_{x_d\to\infty} \frac{v(x)}{x_d} = b$, then $C_* = b$, and consequently u satisfies (3.1).

Conversely, since $q, f(\cdot,0) \in K^\infty(\mathbb{R}^d_+)$, then $M\widetilde{h}q + f(\cdot,0) \in L^1_{\mathrm{loc}}(\mathbb{R}^d_+)$. So from (3.3) we obtain $f(\cdot,u) \in L^1_{\mathrm{loc}}(\mathbb{R}^d_+)$. Again from (3.3) and the fact that $V(M\widetilde{h}q + f(\cdot 0)) \in C(\mathbb{R}^d_+)$ we deduce from Lemma 3.3 that $V(f(\cdot,u)) \in C(\mathbb{R}^d_+)$ and from (3.1) that $u \in C(\mathbb{R}^d_+)$. Using (1.3) we obtain $\Delta u = \Delta h - \lambda\Delta V(f(\cdot,u)) = \lambda f(\cdot,u)$ in the sense of distributions. On the other hand, using (3.3) we obtain

$$0 \le V\big(f(\cdot,u)\big) \le MV(\widetilde{h}q) + V\big(f(\cdot,0)\big). \tag{3.4}$$

Hence it follows from property 4 of Propositions 2.5 and 2.7 that $\lim_{x\to\xi\in\partial\mathbb{R}^d_+} V(f(\cdot,u))(x) = 0$, and consequently $\lim_{x\to\xi\in\partial\mathbb{R}^d_+} u(x) = a\phi(\xi)$. Finally, using (3.4), the fact that $\frac{1}{\widetilde{h}}V(\widetilde{h}q) \in C_0(\overline{\mathbb{R}^d_+})$, and that $\lim_{|x|\to\infty} \frac{1}{\widetilde{h}(x)} V(f(\cdot,0))(x) = 0$, we obtain $\lim_{|x|\to\infty} \frac{1}{\widetilde{h}(x)} V(f(\cdot,u))(x) = 0$. In particular, $\lim_{x_d\to\infty} \frac{1}{\widetilde{h}(x)} V(f(\cdot,u))(x) = 0$. Consequently, $\lim_{x_d\to\infty} \frac{u(x)}{x_d} = \lim_{x_d\to\infty} \frac{h(x)}{x_d} - \lambda \lim_{x_d\to\infty} \frac{V(f(\cdot,u))(x)}{x_d} = b - \lambda \lim_{x_d\to\infty} \frac{V(f(\cdot,u))(x)}{\widetilde{h}(x)} \frac{\widetilde{h}(x)}{x_d} = b$. This achieves the proof. □

Next we establish a uniqueness result for an eventual continuous weak solution u, satisfying $0 \le u \le h$ for (1.2) in the case where $\lambda \ge 0$ and the nonlinearity f is nonnegative, nondecreasing, and continuous with respect to the second variable.

Proposition 3.5 *Let $f : \mathbb{R}^d_+ \times [0,\infty) \longrightarrow [0,\infty)$ be a Borel measurable function satisfying (H_1)–(H_2) and assume further that for each $x \in \mathbb{R}^d_+$ the function $t \to f(x,t)$ is nondecreasing on $[0,\infty)$. Then, for any nontrivial nonnegative continuous bounded function ϕ on the boundary $\partial\mathbb{R}^d_+$, any nonnegative real numbers a, b with $a + b > 0$ and $\lambda \ge 0$, problem (1.2) has at most one nonnegative continuous weak solution satisfying $0 \le u \le h$.*

Proof Assume that there exist two nonnegative continuous weak solutions u_1, u_2 of (1.2) with $0 \le u_1 \le h$ and $0 \le u_1 \le h$. Let $M = \max(1, |\phi|_\infty)$. Then $0 \le u_1 \le h \le M\widetilde{h}$ and $0 \le u_2 \le h \le M\widetilde{h}$.

Since f satisfies (H_1)–(H_2), it follows from Lemma 3.4 that

$$(u_2 - u_1) + \lambda V\big(f(\cdot,u_2) - f(\cdot,u_1)\big) = 0. \tag{3.5}$$

Let $q = q_M \in K^\infty(\mathbb{R}^d_+)$ be the function given in hypothesis (H_2) and define

$$k(x) = \begin{cases} \frac{f(x,u_2(x))-f(x,u_1(x))}{u_2(x)-u_1(x)} & \text{if } u_2(x) \neq u_1(x), \\ 0 & \text{if } u_2(x) = u_1(x). \end{cases}$$

Then we have $0 \le k(x) \le q(x)$ for every $x \in \mathbb{R}^d_+$. Hence $k \in K^\infty(\mathbb{R}^d_+)$ and using (H_1), (3.3), and properties 2 and 4 of Proposition 2.5, we obtain

$$\begin{aligned} V\big(\lambda k|u_2 - u_1|\big) &\le \lambda V\big(f(\cdot,u_2)\big) + \lambda V\big(f(\cdot,u_1)\big) \le 2\lambda M V(\widetilde{h}q) + 2\lambda V\big(f(\cdot,0)\big) \\ &\le 2\lambda M \alpha_q \widetilde{h} + 2\lambda V\big(f(\cdot,0)\big) < \infty. \end{aligned}$$

Applying $(I - V_{\lambda k}(\lambda k\cdot))$ on both sides of equality (3.5), we obtain from (1.7) that $u_2 = u_1$. □

The second main result of this paper is the following.

Theorem 3.6 *Let ϕ be a nontrivial nonnegative bounded continuous function on $\partial\mathbb{R}^d_+$ and assume that hypotheses (H_1), (H_2), and (H_3) are satisfied. Then there exists $\lambda_0 > 0$ such that for $\lambda \in [0,\lambda_0)$ problem (1.2) has a positive continuous weak solution u satisfying the following global behavior*:

$$c_\lambda h(x) \le u(x) \le h(x) \quad \textit{for each } x \in \overline{\mathbb{R}^d_+}, \tag{3.6}$$

where $c_\lambda \in (0,1]$.

Proof We will adapt the proof in [7]. Put $M = \max(1, |H\phi|_\infty)$. Since $H\phi$ is harmonic and bounded in $\mathbb{R}^d_+$ with boundary value ϕ, it follows from the maximum principle that $M = \max(1, |\phi|_\infty)$. From hypothesis (H_2), there exists $q = q_M \in K^\infty(\mathbb{R}^d_+)$ such that for each $x \in \mathbb{R}^d_+$ we have

$$\frac{f(x,t\widetilde{h}(x)) - f(x,s\widetilde{h}(x))}{t-s} \le \widetilde{h}(x)q(x) \quad \text{for every } 0 \le s < t \le M. \tag{3.7}$$

Consider the function $\theta : \lambda \to \lambda \exp(\lambda\alpha_q)$. Then θ is a bijection from $[0,\infty)$ to $[0,\infty)$. Put $\lambda_0 = \theta^{-1}(\sigma_0) > 0$, with the convention that $\lambda_0 = \infty$ if $\sigma_0 = \infty$. For $\lambda \in [0,\lambda_0)$, we define the nonempty closed convex set

$$\Lambda = \left\{ u \in B^+\big(\mathbb{R}^d_+\big) : \left(1 - \frac{\theta(\lambda)}{\sigma_0}\right) \exp(-\lambda\alpha_q) h(x) \le u(x) \le h(x) \right\}.$$

We mention that for $u \in \Lambda$ we have $u \le h \le M\widetilde{h}$. So it follows from (3.7) that

$$0 \le f(\cdot u) \le qu + f(\cdot,0). \tag{3.8}$$

Let T be the operator defined on Λ by

$$Tu = h - V_{\lambda q}(\lambda q h) + \lambda V_{\lambda q}\big(qu - f(,u)\big).$$

We will prove that Λ is invariant under T and T has a fixed point in Λ, which is a solution of the integral equation (3.1).

For each $u \in \Lambda$, we have

$$\begin{aligned} Tu &= h - \lambda V_{\lambda q}(qh) + \lambda V_{\lambda q}\big(qu - f(\cdot,u)\big) \\ &\leq h - \lambda V_{\lambda q}(qh) + \lambda V_{\lambda q}(qu) \\ &\leq h. \end{aligned}$$

Using Proposition 2.6, hypothesis (H_1), and (1.8) we get

$$\begin{aligned} Tu &= h - V_{\lambda q}(\lambda qh) - \lambda V_{\lambda q}\big(f(\cdot,0)\big) + \lambda V_{\lambda q}\big(qu + f(\cdot,0) - f(\cdot,u)\big) \\ &\geq e^{-\lambda\alpha_q} h - \lambda V_{\lambda q}\big(f(\cdot,0)\big) \\ &\geq e^{-\lambda\alpha_q} h - \lambda V\big(f(\cdot,0)\big) \\ &\geq e^{-\lambda\alpha_q} h - \lambda \frac{V(f(\cdot,0))}{h} h \\ &\geq e^{-\lambda\alpha_q} h - \lambda \sup_{x\in\overline{\mathbb{R}^d_+}} \left[\frac{Vf(\cdot,0)(x)}{h(x)}\right] h \\ &\geq e^{-\lambda\alpha_q} h - \frac{\lambda}{\inf_{x\in\overline{\mathbb{R}^d_+}}[\frac{h(x)}{Vf(\cdot,0)(x)}]} h \\ &\geq \exp(-\lambda\alpha_q)\left[1 - \frac{\theta(\lambda)}{\sigma_0}\right] h. \end{aligned}$$

Consequently, $T\Lambda \subset \Lambda$.

Next, we prove that T is a nondecreasing operator on Λ. For this aim, we consider $u, v \in \Lambda$ such that $u \leq v$. Then, using hypothesis (H_2), we get

$$\begin{aligned} Tu - Tv &= \lambda V_{\lambda q}\big(qu - f(\cdot,u) - qv + f(\cdot,v)\big) \\ &= \lambda V_{\lambda q}\big(f(\cdot,v) - f(\cdot,u) - q(v-u)\big) \leq 0. \end{aligned}$$

Next, we consider the sequence $(u_n)_{n\geq 0}$ defined by

$$u_0 = h - \lambda V_{\lambda q}(qh) - \lambda V_{\lambda q}\big(f(\cdot,0)\big) \quad \text{and} \quad u_{n+1} = Tu_n \text{ for } n \geq 0.$$

Using the monotonicity of T, we obtain

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq u_{n+1} \leq h.$$

It follows from (3.7) and the dominated convergence theorem that the sequence $(u_n)_{n\geq 0}$ converges to a function $u \in \Lambda$ satisfying $Tu = u$, or equivalently

$$u = h - V_{\lambda q}(\lambda qh) + \lambda V_{\lambda q}\big(qu - f(\cdot,u)\big).$$

This implies that

$$\big(I - V_{\lambda q}(\lambda q\cdot)\big)u = \big(I - V_{\lambda q}(\lambda q\cdot)\big)h - V_{\lambda q}\big(\lambda f(\cdot,u)\big).$$

Applying the operator $(I + V(\lambda q\cdot))$ on the last equation, we deduce by (1.6) and (1.7) that u is a solution of the integral equation (3.1). Hence it follows from Lemma 3.4 that u is a continuous weak solution of (1.2). □

Example 3.1 Let $\alpha \geq 1$ and $\beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned}&\beta - \big(1-\operatorname{sgn}(a)\big)(\alpha+\gamma-1) - \big(1-\operatorname{sgn}(b)\big)\min(0,\gamma)\\&\quad - \operatorname{sgn}(a)\operatorname{sgn}(b)\min(0,\alpha+\gamma-1) < 2 < \delta - \gamma - \operatorname{sgn}(b)(\alpha-1),\end{aligned}$$

where

$$\operatorname{sgn}(r) = \begin{cases} 1 & \text{if } r > 0,\\ 0 & \text{if } r = 0.\end{cases}$$

Define $f(x,t) = \frac{1}{x_d^{\beta}(|x|+1)^{\delta-\beta}}(x_d+t)^{\gamma}t^{\alpha}$ for $(x,t) \in \mathbb{R}^d_+ \times [0,\infty)$. Then f satisfies hypotheses (H_1)–(H_3). Indeed, since $f(x,0) = 0$, then (H_1) and (H_3) are satisfied with $\sigma_0 = \infty$. To prove (H_2), we consider for every $M > 0$ and $0 \leq s \leq t \leq M$. It follows by the mean value theorem that there exists $\eta \in [s,t]$ such that

$$\begin{aligned}&\frac{f(x,t\widetilde{h}(x)) - f(x,s\widetilde{h}(x))}{(t-s)\widetilde{h}(x)}\\&\quad = \frac{1}{x_d^{\beta}(|x|+1)^{\delta-\beta}}\left(\frac{(x_d+t\widetilde{h}(x))^{\gamma}(t\widetilde{h}(x))^{\alpha} - (x_d+s\widetilde{h}(x))^{\gamma}(s\widetilde{h}(x))^{\alpha}}{(t-s)\widetilde{h}(x)}\right)\\&\quad = \frac{1}{x_d^{\beta}(1+|x|)^{\delta-\beta}}\Big(\gamma\big(x_d+\eta\widetilde{h}(x)\big)^{\gamma-1}\big(\eta\widetilde{h}(x)\big)^{\alpha} + \alpha\big(x_d+\eta\widetilde{h}(x)\big)^{\gamma}\big(\eta\widetilde{h}(x)\big)^{\alpha-1}\Big).\end{aligned} \tag{3.9}$$

We will discuss two cases as follows.

Case 1. $a = 0$. Since $a + b > 0$, we obtain $b > 0$, and so $\widetilde{h}(x) = b\theta(x) = bx_d$. Since $\alpha - 1 \geq 0$,

$$\gamma\big(x_d+\eta b\theta(x)\big)^{\gamma-1}\big(\eta b\theta(x)\big)^{\alpha} + \alpha\big(x_d+\eta b\theta(x)\big)^{\gamma}\big(\eta b\theta(x)\big)^{\alpha-1} \leq c(x_d)^{\alpha+\gamma-1}.$$

So, we deduce by (3.9) that

$$\frac{f(x,t\widetilde{h}(x)) - f(x,s\widetilde{h}(x))}{(t-s)\widetilde{h}(x)} \leq \frac{1}{x_d^{\beta-(\alpha+\gamma-1)}(1+|x|)^{\delta-\beta}}.$$

We conclude by Example 2.1 that f satisfies $(\mathbf{H}_2)$ if $\beta - (\alpha+\gamma-1) < 2 < \delta - (\alpha+\gamma-1)$.

Case 2. $a > 0$. Since $b \geq 0$, we discuss the following subcases.

Subcase 1. $b = 0$. So $\widetilde{h} = a$. Hence, if $\gamma \geq 0$, then we have

$$\gamma(x_d+\eta a)^{\gamma-1}(\eta a)^{\alpha} + \alpha(x_d+\eta a)^{\gamma}(\eta a)^{\alpha-1} \leq c(x_d+\eta a)^{\gamma}(\eta a)^{\alpha-1}$$

$$\begin{aligned}&\leq c(x_d+1)^\gamma\\&\leq c\big(1+|x|\big)^\gamma\end{aligned}$$

and if $\gamma<0$, then we have

$$\begin{aligned}\gamma(x_d+\eta a)^{\gamma-1}(\eta a)^\alpha+\alpha(x_d+\eta a)^\gamma(\eta a)^{\alpha-1}&\leq(\eta a)^{\alpha-1}(x_d+\eta a)^{\gamma-1}\big((\alpha+\gamma)\eta a+\alpha x_d\big)\\&\leq c(\eta a)^{\alpha-1}(x_d+\eta a)^\gamma\\&\leq cx_d^\gamma.\end{aligned}$$

Then by (3.9) we obtain

$$\frac{f(x,t\widetilde{h}(x))-f(x,s\widetilde{h}(x))}{(t-s)\widetilde{h}(x)}\leq\frac{1}{x_d^{\beta-\min(0,\gamma)}(1+|x|)^{\delta-\beta-\max(0,\gamma)}}.$$

We conclude by Example 2.1 that f satisfies $(\mathbf{H}_2)$ if $\beta-\min(0,\gamma)<2<\delta-\gamma$.

Subcase 2. $b>0$. So $\widetilde{h}(x)=a+bx_d$. Hence if $\gamma<1-\alpha$, then we have

$$\begin{aligned}&\gamma\big(x_d+\eta\widetilde{h}(x)\big)^{\gamma-1}\big(\eta\widetilde{h}(x)\big)^\alpha+\alpha\big(x_d+\eta\widetilde{h}(x)\big)^\gamma\big(\eta\widetilde{h}(x)\big)^{\alpha-1}\\&\quad\leq\big(x_d+\eta\widetilde{h}(x)\big)^{\alpha-1}\big(x_d+\eta\widetilde{h}(x)\big)^{\gamma-1}\big((\alpha+\gamma)\eta\widetilde{h}(x)+\alpha x_d\big)\\&\quad\leq\alpha(x_d+\eta a+\eta bx_d)^{\alpha+\gamma-1}\\&\quad\leq\alpha x_d^{\alpha+\beta-1},\end{aligned}$$

and if $1-\alpha\leq\gamma$, then we have

$$\begin{aligned}&\gamma\big(x_d+\eta\widetilde{h}(x)\big)^{\gamma-1}\big(\eta\widetilde{h}(x)\big)^\alpha+\alpha\big(x_d+\eta\widetilde{h}(x)\big)^\gamma\big(\eta\widetilde{h}(x)\big)^{\alpha-1}\\&\quad\leq\big(x_d+\eta\widetilde{h}(x)\big)^{\gamma-1}\big(\eta\widetilde{h}(x)\big)^{\alpha-1}\big((\alpha+\gamma)\eta\widetilde{h}(x)+\alpha x_d\big)\\&\quad\leq\max(\alpha,\alpha+\gamma)\big(x_d+\eta\widetilde{h}(x)\big)^{\gamma+\alpha-1}\\&\quad\leq c(1+x_d)^{\gamma+\alpha-1}\\&\quad\leq c\big(1+|x|\big)^{\gamma+\alpha-1}.\end{aligned}$$

Then by (3.9) we obtain

$$\frac{f(x,t\widetilde{h}(x))-f(x,s\widetilde{h}(x))}{(t-s)\widetilde{h}(x)}\leq\frac{1}{x_d^{\beta-\min(0,\gamma+\alpha-1)}(1+|x|)^{\delta-\beta-\max(0,\gamma+\alpha-1)}}.$$

We conclude by Example 2.1 that f satisfies $(\mathbf{H}_2)$ if $\beta-\min(0,\gamma+\alpha-1)<2<\delta-(\alpha+\gamma-1)$.

Example 3.2 Let $a\geq 0$, $b\geq 0$ with $a+b>0$ and $\delta,\beta,\gamma\in\mathbb{R}$ satisfying

$$\beta<1+\gamma-\operatorname{sgn}(a)\max(\gamma-1,0)\quad\text{and}\quad\delta>\gamma+1+\big(1-\operatorname{sgn}(b)\big)d.$$

Let f be the positive function defined on $\mathbb{R}^d_+ \times [0,\infty)$ by

$$f(x,t) = \frac{1}{x_d^{\beta}(1+|x|)^{\delta-\beta}}(x_d+t)^{\gamma}.$$

Then f satisfies hypotheses $(\mathbf{H}_1)$–$(\mathbf{H}_3)$. Indeed,

$$\begin{aligned}\frac{f(x,0)}{\widetilde{h}(x)} &= \frac{1}{x_d^{\beta-\gamma}(1+|x|)^{\delta-\beta}}\frac{1}{a+bx_d}\\ &\leq c\frac{1}{x_d^{\beta-\gamma+\operatorname{sgn}(b)}(1+|x|)^{\delta-\beta}}.\end{aligned}$$

Since $\beta-\gamma+\operatorname{sgn}(b) \leq \beta-\gamma+1+\operatorname{sgn}(a)\max(\gamma-1,0) < 2$ and $\delta-\gamma+\operatorname{sgn}(b) > 1+\operatorname{sgn}(b)+(1-\operatorname{sgn}(b))d \geq 2$, we conclude by Example 2.1 that f satisfies $(\mathbf{H}_1)$. Now we verify $(\mathbf{H}_3)$. Using (1.4) and the fact that $a+b>0$, we obtain

$$h(x) \geq c\frac{x_d}{(1+|x|)^{(1-\operatorname{sgn}(b))d}}.$$

• If $b=0$. Since $\beta-\gamma \leq \beta-\gamma+\operatorname{sgn}(a)\max(0,\gamma-1) < 1$ and $\delta-\gamma > 1+d$, then

$$f(x,0) = \frac{1}{x_d^{\beta-\gamma}(1+|x|)^{\delta-\beta}} \quad \text{belongs to } K^{\infty}\bigl(\mathbb{R}^d_+\bigr),$$

and it was proved in [4, 5] that

$$V\bigl(f(\cdot,0)\bigr)(x) \leq C\frac{x_d}{(1+|x|)^d}.$$

So

$$\frac{h(x)}{V(f(\cdot,0))(x)} \geq \frac{1}{C}h(x)\frac{(1+|x|)^d}{x_d} \geq \frac{c}{C} > 0.$$

• If $b>0$. Using the fact that $\beta-\gamma+1 \leq \beta-\gamma+1+\operatorname{sgn}(a)\max(0,\gamma-1) < 2$ and $\delta-\gamma+1>2$, we conclude by Example 2.1 that the function $p(x)=\frac{f(x,0)}{x_d}$ belongs to $K^{\infty}(\mathbb{R}^d_+)$, and using assertion (2) of Proposition 2.13, we obtain

$$V\bigl(f(\cdot,0)\bigr)(x) \leq \alpha_p x_d.$$

Hence

$$\frac{h(x)}{V(f(\cdot,0))(x)} \geq \frac{1}{\alpha_p}\frac{h(x)}{x_d} \geq \frac{b}{\alpha_p} > 0.$$

This proves that $\sigma_0 > 0$ and $(\mathbf{H}_3)$ is satisfied.

Finally, we will verify $(\mathbf{H}_2)$. Let $M>0$ and $0 \leq s \leq t \leq M$. By the mean value theorem, we deduce that there exists $\eta \in [s,t]$ such that

$$\frac{f(x,t\widetilde{h}(x))-f(x,s\widetilde{h}(x))}{(t-s)\widetilde{h}(x)} = \frac{1}{x_d^{\beta}(1+|x|)^{\delta-\beta}}\left(\frac{(x_d+t\widetilde{h}(x))^{\gamma}-(x_d+s\widetilde{h}(x))^{\gamma}}{(t-s)\widetilde{h}(x)}\right)$$

$$= \frac{\gamma(x_d + \eta\widetilde{h}(x))^{\gamma-1}}{x_d^{\beta}(1+|x|)^{\delta-\beta}}. \tag{3.10}$$

So we will distinguish the following cases.

*Case 1.*If $\gamma \le 0$, then

$$\frac{f(x,t\widetilde{h}(x)) - f(x,s\widetilde{h}(x))}{(t-s)\widetilde{h}(x)} \le 0,$$

and we can take $q_M = 0$. So $(\mathbf{H}_2)$ is satisfied.

Case 2. If $0 < \gamma \le 1$, then

$$\frac{f(x,t\widetilde{h}(x)) - f(x,s\widetilde{h}(x))}{(t-s)\widetilde{h}(x)} \le c\frac{x_d^{\gamma-1}}{x_d^{\beta}(1+|x|)^{\delta-\beta}} = \frac{c}{x_d^{\beta-\gamma+1}(1+|x|)^{\delta-\beta}}.$$

Since $\beta - \gamma < 1$ and $\delta - \gamma + 1 > 2 + (1 - \operatorname{sgn}(b))d \ge 2$, we conclude by Example 2.1 that $(\mathbf{H}_2)$ is satisfied.

Case 3. If $\gamma > 1$, we consider the following subcases.

- If $a = 0$. Then $b > 0$, $\widetilde{h}(x) = bx_d$, and

$$\begin{aligned}\frac{f(x,t\widetilde{h}(x)) - f(x,s\widetilde{h}(x))}{(t-s)\widetilde{h}(x)} &\le \frac{\gamma(x_d + \eta b x_d)^{\gamma-1}}{x_d^{\beta}(1+|x|)^{\delta-\beta}} \\ &\le c\frac{x_d^{\gamma-1}}{x_d^{\beta}(1+|x|)^{\delta-\beta}} \\ &\le \frac{c}{x_d^{\beta-\gamma+1}(1+|x|)^{\delta-\beta}}.\end{aligned}$$

Since $\beta - \gamma < 1$ and $\delta - \gamma + 1 > 2 + (1 - \operatorname{sgn}(b))d \ge 2$, we conclude by Example 2.1 that $(\mathbf{H}_2)$ is satisfied.

- If $a > 0$. Then $\widetilde{h}(x) = a + bx_d$ and

$$\frac{f(x,t\widetilde{h}(x)) - f(x,s\widetilde{h}(x))}{(t-s)\widetilde{h}(x)} \le c\frac{(1+x_d)^{\gamma-1}}{x_d^{\beta}(1+|x|)^{\delta-\beta}} \le \frac{c(1+|x|)^{\gamma-1}}{x_d^{\beta}(1+|x|)^{\delta-\beta}} = \frac{c}{x_d^{\beta}(1+|x|)^{\delta-\beta-\gamma+1}}.$$

Since $\beta < 1 + \gamma - \max(\gamma - 1, 0) = 2$ and $\delta - \gamma + 1 > 2 + (1 - \operatorname{sgn}(b))d \ge 2$, we conclude by Example 2.1 that $(\mathbf{H}_2)$ is satisfied.

4 Existence of positive solutions for some semilinear elliptic systems

In this section we deal with the existence of positive weak solutions that are continuous in $\overline{\mathbb{R}^d_+}$ for the semilinear elliptic system (1.1). We adopt the following notations: $h_1(x) := a_1 H\phi_1(x) + b_1 x_d$, $\widetilde{h}_1(x) := a_1 + b_1 x_d$, $h_2(x) := a_2 H\phi_2(x) + b_2 x_d$, and $\widetilde{h}_2(x) := a_2 + b_2 x_d$ for $x \in \mathbb{R}^d_+$. We assume that the functions f, g satisfy the following hypotheses:

$(\mathbf{H}_4)$ The map $(u,v) \longrightarrow (f(x,u,v), g(x,u,v))$ is continuous on $[0,\infty) \times [0,\infty)$ for every fixed $x \in \mathbb{R}^d_+$, the map $u \longrightarrow f(x,u,v)$ is nondecreasing for every fixed $(x,v) \in \mathbb{R}^d_+ \times [0,\infty)$, and the map $v \longrightarrow g(x,u,v)$ is nondecreasing for every fixed $(x,u) \in \mathbb{R}^d_+ \times [0,\infty)$.

$(\mathbf{H}_5)$ The functions $\frac{f(\cdot,0,0)}{\widetilde{h}_1}$ and $\frac{g(\cdot,0,0)}{\widetilde{h}_2}$ are in $K^{\infty}(\mathbb{R}^d_+)$.

($\mathbf{H}_6$) For every $M > 0$, there exist a nonnegative function $p = p_M \in K^\infty(D)$ and two Borel measurable functions $g_M, f_M : \mathbb{R}^d_+ \times [0,\infty) \longrightarrow [0,\infty)$ continuous with respect to the second variable such that for every $0 \le t_1 \le t_2 \le M$, $0 \le s_1 \le s_2 \le m$, and $x \in \mathbb{R}^d_+$, we have

$$\begin{aligned}&\bigl|f\bigl(x, t_2\widetilde{h}_1(x), s_2\widetilde{h}_2(x)\bigr) - f\bigl(x, t_1\widetilde{h}_1(x), s_1\widetilde{h}_2(x)\bigr)\bigr| \\ &\quad \le p(x)\widetilde{h}_1(x)(t_2 - t_1) + \bigl|g_M(x, s_2) - g_M(x, s_1)\bigr|\end{aligned}$$

and

$$\begin{aligned}&\bigl|g\bigl(x, t_2\widetilde{h}_1(x), s_2\widetilde{h}_2(x)\bigr) - g\bigl(x, t_1\widetilde{h}_1(x), s_1\widetilde{h}_2(x)\bigr)\bigr| \\ &\quad \le p(x)\widetilde{h}_2(x)(s_2 - s_1) + \bigl|f_M(x, t_2) - f_M(x, t_1)\bigr|.\end{aligned}$$

Moreover, the functions $\sup_{s\in[0,M]} \frac{g_M(\cdot,s)}{\widetilde{h}_1}$ and $\sup_{t\in[0,M]} \frac{f_M(\cdot,t)}{\widetilde{h}_2}$ belong to $K^\infty(\mathbb{R}^d_+)$.

($\mathbf{H}_7$) We have

$$\sigma_1 = \inf_{x\in\mathbb{R}^d_+} \frac{h_1(x)}{V\varrho_1(x)} > 0 \quad \text{and} \quad \sigma_2 = \inf_{x\in\mathbb{R}^d_+} \frac{h_2(x)}{V\varrho_2(x)} > 0,$$

where

$$\varrho_1(x) = f(x,0,0) + \max_{0\le s\le M} g_M(x,s) \quad \text{and} \quad \varrho_2(x) = g(x,0,0) + \max_{0\le t\le M} f_M(x,t),$$

with g_M, f_M given in hypothesis ($\mathbf{H}_6$) for $M = \max(1, \|H\phi_1\|_\infty, \|H\phi_2\|_\infty)$.

Our third main result in this paper is the following.

Theorem 4.1 *Assume that f, g satisfy* ($\mathbf{H}_4$)–($\mathbf{H}_7$). *Then there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that for each $\lambda \in [0, \lambda_0)$ and $\mu \in [0, \mu_0)$ system* (1.1) *has a positive continuous solution satisfying*

$$c_\lambda h_1 \le u \le h_1 \quad \textit{and} \quad c_\mu h_2 \le v \le h_2,$$

where $c_\lambda, c_\mu \in [0, 1)$.

Proof Proof of Theorem 4.1 Let $M = \max(1, \|H\phi_1\|_\infty, \|H\phi_2\|_\infty)$, then we have $h_1 \le M\widetilde{h}_1$ and $h_2 \le M\widetilde{h}_2$. From (H_6), there exist a nonnegative function $p \in K^\infty(\mathbb{R}^d_+)$ and two Borel measurable functions $g_M, f_M : \mathbb{R}^d_+ \times [0,\infty) \longrightarrow [0,\infty)$ continuous with respect to the second variable such that for any $0 \le t_1 \le t_2 \le M$, $0 \le s_1 \le s_2 \le M$, and $x \in \mathbb{R}^d_+$ we have

$$\begin{aligned}&\bigl|f\bigl(x, t_2\widetilde{h}_1(x), s_2\widetilde{h}_2(x)\bigr) - f\bigl(x, t_1\widetilde{h}_1(x), s_1\widetilde{h}_2(x)\bigr)\bigr| \\ &\quad \le p(x)\widetilde{h}_1(x)(t_2 - t_1) + \bigl|g_M(x, s_2) - g_M(x, s_1)\bigr|\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}&\bigl|g\bigl(x, t_2\widetilde{h}_1(x), s_2\widetilde{h}_2(x)\bigr) - g\bigl(x, t_1\widetilde{h}_1(x), s_1\widetilde{h}_2(x)\bigr)\bigr| \\ &\quad \le p(x)\widetilde{h}_2(x)(s_2 - s_1) + \bigl|f_M(x, t_2) - f_M(x, t_1)\bigr|.\end{aligned}$$

Define $\theta(\lambda) = \lambda\exp(\lambda\alpha_p)$. Then θ is an increasing bijection from $[0,\infty)$ to itself. Let $\lambda_0 = \theta^{-1}(\sigma_1) > 0$ and $\mu_0 = \theta^{-1}(\sigma_2) > 0$, with convention that $\theta^{-1}(\infty) = \infty$.

For $\lambda \in [0,\lambda_0)$ and $\mu \in [0,\mu_0)$, we consider the nonempty closed bounded convex set given by

$$\Gamma = \left\{(\phi,\psi) \in C_0\left(\overline{\mathbb{R}^d_+}\right) \times C_0\left(\overline{\mathbb{R}^d_+}\right), 0 \le \phi \le \left(1-\left(1-\frac{\theta(\lambda)}{\sigma_1}\right)\exp(-\lambda\alpha_p)\right)\frac{h_1}{\widetilde{h}_1} \text{ and } \right.$$
$$\left. 0 \le \psi \le \left(1-\left(1-\frac{\theta(\mu)}{\sigma_2}\right)\exp(-\mu\alpha_p)\right)\frac{h_2}{\widetilde{h}_2}\right\}.$$

For $(\phi,\psi) \in \Gamma$, we consider the following problems:

$$\begin{cases} \Delta y = \lambda f(\cdot,y,h_2-\widetilde{h}_2\psi) & \text{in } \mathbb{R}^d_+ \text{ (in the distributional sense)},\\ y = a_1\phi_1 & \text{in } \partial\mathbb{R}^d_+,\\ \lim_{|x|\to\infty}\frac{y(x)}{x_d} = b_1, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \Delta z = \mu g(\cdot,h_1-\widetilde{h}_1\phi,z) & \text{in } \mathbb{R}^d_+ \text{ (in the distributional sense)},\\ z = a_2\phi_2 & \text{in } \partial\mathbb{R}^d_+,\\ \lim_{x_d\to\infty}\frac{z(x)}{x_d} = b_2. \end{cases} \tag{4.3}$$

Next, we claim that the previous problem (4.2) has a unique positive continuous weak solution. To do this, we start by proving that the function $(x,y) \longmapsto f(x,y,h_2-\widetilde{h}_2\psi)$ verifies $(\mathbf{H}_1)$–$(\mathbf{H}_3)$. Indeed, using the fact that $0 \le (1-\frac{\theta(\mu)}{\sigma_2})\exp(-\mu\alpha_p)\frac{h_2(x)}{\widetilde{h}_2(x)} \le \frac{h_2(x)}{\widetilde{h}_2(x)} - \psi(x) \le \frac{h_2(x)}{\widetilde{h}_2(x)} \le M$, we obtain by taking $t_1 = t_2 = 0$, $s_2 = \frac{h_2}{\widetilde{h}_2} - \psi$, and $s_1 = 0$ in inequality (4.1) that

$$\begin{aligned}\frac{f(x,0,(h_2-\widetilde{h}_2\psi)(x))}{\widetilde{h}_1(x)} &\le \frac{f(x,0,0)}{\widetilde{h}_1(x)} + \frac{|g_M(x,\frac{h_2(x)}{\widetilde{h}_2(x)}-\psi(x))-g_M(x,0)|}{\widetilde{h}_1(x)}\\ &\le \frac{f(x,0,0)}{\widetilde{h}_1(x)} + \max_{0\le s\le M}\frac{g_M(x,s)}{\widetilde{h}_1(x)}.\end{aligned}$$

Since the functions $\frac{f(\cdot,0,0)}{\widetilde{h}_1}$ and $\max_{0\le s\le M}\frac{g_M(\cdot,s)}{\widetilde{h}_1}$ are in $K^\infty(\mathbb{R}^d_+)$, then $\frac{f(\cdot,0,h_2-\widetilde{h}_2\psi)}{\widetilde{h}_1} \in K^\infty(\mathbb{R}^d_+)$.

By taking $s_1 = s_2 = \frac{h_2}{\widetilde{h}_2} - \psi$ in hypothesis $(\mathbf{H}_6)$, it is easy to see that $(x,y) \longmapsto f(x,y,h_2-\widetilde{h}_2\psi)$ verifies $(\mathbf{H}_2)$. Using the previous inequality, we deduce that

$$\sigma_1' = \inf_{x\in\mathbb{R}^d_+}\frac{h_1(x)}{V(f(\cdot,0,h_2-\widetilde{h}_2\psi))(x)} \ge \inf_{x\in\mathbb{R}^d_+}\frac{h_1(x)}{V\varrho_1(x)} = \sigma_1 > 0.$$

Then, by Proposition 3.5 and Theorem 3.6, we deduce that (4.2) has a unique positive continuous weak solution y satisfying

$$y(x) = h_1(x) - \lambda V\big(f(\cdot,y,h_2-\widetilde{h}_2\psi)\big)(x), \tag{4.4}$$
$$\left(1-\frac{\theta(\lambda)}{\sigma_1}\right)\exp(-\lambda\alpha_p)h_1 \le \left(1-\frac{\theta(\lambda)}{\sigma_1'}\right)\exp(-\lambda\alpha_p)h_1 \le y \le h_1 \le M\widetilde{h}_1,$$

and

$$\begin{aligned}\frac{f(x,y(x),(h_2-\widetilde{h}_2\psi)(x))}{\widetilde{h}_1(x)} &\leq \frac{f(x,M\widetilde{h}_1(x),(h_2-\widetilde{h}_2\psi)(x))}{\widetilde{h}_1(x)} \\ &\leq \frac{f(x,0,0)}{\widetilde{h}_1(x)} + Mp(x) + \max_{0\leq s\leq M}\frac{g_M(x,s)}{\widetilde{h}_1(x)}.\end{aligned}$$

Similarly, we prove that (4.3) has a unique positive continuous solution z satisfying

$$\begin{aligned} &z(x) = h_2(x) - \mu V\big(g(\cdot,h_1-\widetilde{h}_1\phi,z)\big)(x), \\ &\left(1-\frac{\theta(\mu)}{\sigma_2}\right)\exp(-\mu\alpha_p)h_2 \leq z \leq h_2 \leq M\widetilde{h}_2, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\frac{g(x,(h_1-\widetilde{h}_1\phi)(x),z(x))}{\widetilde{h}_2(x)} &\leq \frac{g(x,(h_1-\widetilde{h}_1\phi)(x),M\widetilde{h}_2(x))}{\widetilde{h}_2(x)} \\ &\leq \frac{g(x,0,0)}{\widetilde{h}_2(x)} + Mp(x) + \max_{0\leq t\leq M}\frac{f_M(x,t)}{\widetilde{h}_2(x)}.\end{aligned}$$

Let T be the operator defined on Γ by

$$T(\phi,\psi) = \left(\frac{h_1-y}{\widetilde{h}_1},\frac{h_2-z}{\widetilde{h}_2}\right).$$

Using the fact that $\frac{f(\cdot,0,0)}{\widetilde{h}_1}$, $\frac{g(\cdot,0,0)}{\widetilde{h}_2}$, p, $\max_{0\leq s\leq M}\frac{g_M(\cdot,s)}{\widetilde{h}_1}$, and $\max_{0\leq t\leq M}\frac{f_M(\cdot,t)}{\widetilde{h}_2}$ are in $K^\infty(\mathbb{R}^d_+)$, we deduce by assertion (3) of Proposition 2.7 that

$$T\Gamma = \left\{\left(\frac{\lambda}{\widetilde{h}_1}V\big(f(\cdot,y,h_2-\widetilde{h}_2\psi)\big),\frac{\mu}{\widetilde{h}_2}V\big(g(\cdot,h_1-\widetilde{h}_1\phi,z)\big)\right);(\phi,\psi)\in\Gamma\right\}$$

is relatively compact in $C_0(\overline{\mathbb{R}^d_+})\times C_0(\overline{\mathbb{R}^d_+})$. Next, we will prove the continuity of T with respect to the norm $\|.\|$ defined on Γ by $\|(\phi,\psi)\| = \|\phi\|_\infty + \|\psi\|_\infty$. Let (ϕ_n,ψ_n) be a sequence in Γ that converges to $(\phi,\psi)\in\Gamma$ with respect to $\|.\|$, and let $y_n, z_n, y, z \in \Gamma$ such that

$$T(\phi_n,\psi_n) = \left(\frac{h_1-y_n}{\widetilde{h}_1},\frac{h_2-z_n}{\widetilde{h}_2}\right) \quad \text{and} \quad T(\phi,\psi) = \left(\frac{h_1-y}{\widetilde{h}_1},\frac{h_2-z}{\widetilde{h}_2}\right).$$

Then we have

$$\big|T(\phi_n,\psi_n)-T(\phi,\psi)\big| = \left|\frac{y-y_n}{\widetilde{h}_1}\right|_\infty + \left|\frac{z-z_n}{\widetilde{h}_2}\right|_\infty.$$

Using equation (4.4), we obtain

$$y-y_n = \lambda(V(f(\cdot,y_n,h_2-\widetilde{h}_2\psi_n)-V\big(f(\cdot,y,h_2-\widetilde{h}_2\psi)\big).$$

So

$$y-y_n+\lambda V\big(f(\cdot,y,h_2-\widetilde{h}_2\psi)-f(\cdot,y_n,h_2-\widetilde{h}_2\psi)\big)$$

$$= \lambda V\big(f(\cdot, y_n, h_2 - \widetilde{h}_2\psi_n) - f(\cdot, y_n, h_2 - \widetilde{h}_2\psi)\big).$$

Thus the last equation can be written

$$y - y_n + V\big(\lambda k_n(y - y_n)\big) = \lambda V\big(f(\cdot, y_n, h_2 - \widetilde{h}_2\psi_n) - f(\cdot, y_n, h_2 - \widetilde{h}_2\psi)\big), \tag{4.6}$$

where

$$k_n(x) = \begin{cases} \frac{f(x,y(x),(h_2-\widetilde{h}_2\psi)(x))-f(x,y_n(x),(h_2-\widetilde{h}_2\psi)(x))}{y(x)-y_n(x)} & \text{if } y(x) \neq y_n(x), \\ 0 & \text{if } y(x) = y_n(x). \end{cases}$$

From hypotheses ($\mathbf{H}_4$) and ($\mathbf{H}_6$), we deduce that $0 \leq k_n(x) \leq \widetilde{h}_1(x)p(x)$ for every $n \in \mathbb{N}$ and $x \in \mathbb{R}^d_+$. Using ($\mathbf{H}_6$), assertion (2) of Proposition 2.5, and the fact that $y \leq M\widetilde{h}_1$, $y_n \leq M\widetilde{h}_1$, we obtain

$$\begin{aligned} V\big(\lambda k_n|y - y_n|\big) &\leq \lambda V\big(\big|f(\cdot, y, h_2 - \widetilde{h}_2\psi) - f(x, y_n, h_2 - \widetilde{h}_2\psi)\big|\big) \\ &\leq \lambda V\big(p|y - y_n|\big) \\ &\leq 2M\lambda V(p\widetilde{h}_1) \\ &\leq 2M\lambda\alpha_p\widetilde{h}_1 \\ &< \infty. \end{aligned}$$

Applying $(I - V_{\lambda k_n}(\lambda k_n \cdot))$ on both sides of equation (4.6), we deduce by (1.6) and (1.7) that

$$y - y_n = \lambda V_{\lambda k_n}\big(f(\cdot, y_n, h_2 - \widetilde{h}_2\psi_n) - f(\cdot, y_n, h_2 - \widetilde{h}_2\psi)\big). \tag{4.7}$$

On the other hand, we have by hypothesis ($\mathbf{H}_6$)

$$\begin{aligned} \big|f(\cdot, y_n, h_2 - \widetilde{h}_2\psi_n) - f(\cdot, y_n, h_2 - \widetilde{h}_2\psi)\big|(x) &\leq \max_{0\leq s\leq M} g_M(x,s) \\ &= a_1 \max_{0\leq s\leq M} \frac{g_M(x,s)}{\widetilde{h}_1} + b_1 x_d \max_{0\leq s\leq M} \frac{g_M(x,s)}{\widetilde{h}_1}. \end{aligned}$$

So, again from hypotheses ($\mathbf{H}_6$), ($\mathbf{H}_4$) and the assertions of Propositions 2.5 and 2.7, we deduce by the dominated convergence theorem that for each $x \in \mathbb{R}^d_+$,

$$\lim_{n\longrightarrow\infty} \lambda V\big(f(\cdot, y_n, h_2 - \widetilde{h}_2\psi_n) - f(\cdot, y_n, h_2 - \widetilde{h}_2\psi)\big)(x) = 0,$$

which implies by (1.8) and (4.7) that for $x \in \mathbb{R}^d_+$, $(y_n(x))_n$ converges to $y(x)$ as n tends to ∞. Similarly, we prove that for $x \in \mathbb{R}^d_+$, $(z_n(x))_n$ converges to $z(x)$ as n tends to ∞. So $(T(\phi_n, \psi_n))_n$ converges to $T(\phi, \psi)$ as n tends to ∞. Now, using the fact that $T\Gamma$ is relatively compact in $C_0(\overline{\mathbb{R}^d_+}) \times C_0(\overline{\mathbb{R}^d_+})$, the pointwise convergence implies the uniform convergence. That is,

$$\big\| T(\phi_n, \psi_n) - T(\phi, \psi)\big\| = \left|\frac{y - y_n}{\widetilde{h}_1}\right|_\infty + \left|\frac{z - z_n}{\widetilde{h}_2}\right|_\infty \longrightarrow 0$$

as n tends to ∞. Applying the Schauder fixed point theorem (see [16]), we deduce that there exists $(\phi,\psi)\in\Gamma$ such that $T(\phi,\psi)=(\phi,\psi)$, which gives

$$\begin{aligned}(\phi,\psi)&=\left(\frac{h_1-y}{\widetilde{h}_1},\frac{h_2-z}{\widetilde{h}_2}\right)\\&=\left(\frac{\lambda V(f(\cdot,h_1-\widetilde{h}_1\phi,h_2-\widetilde{h}_2\psi))}{\widetilde{h}_1},\frac{\mu V(g(\cdot,h_1-\widetilde{h}_1\phi,h_2-\widetilde{h}_2\psi))}{\widetilde{h}_2}\right).\end{aligned}$$

Put $u=h_1-\widetilde{h}_1\phi$ and $v=h_2-\widetilde{h}_2\psi$, then u,v are solutions in $\mathbb{R}^d_+$ of the integral equations

$$u=h_1-\lambda V\bigl(f(\cdot,u,v)\bigr)\quad\text{and}\quad v=h_2-\mu V\bigl(g(\cdot,u,v)\bigr).$$

Since $\phi,\psi\in C_0(\overline{\mathbb{R}^d_+})$, then $u,v\in C(\overline{\mathbb{R}}^d_+)$. From $(\mathbf{H}_6)$, we have

$$\begin{aligned}f(\cdot,u,v)&\le f(\cdot,0,0)+pu+\max_{0\le s\le M}g_M(x,s)\\&\le f(\cdot,0,0)+Mp\widetilde{h}_1+\max_{0\le s\le M}g_M(x,s).\end{aligned}$$

Since $\frac{f(\cdot,0,0)}{\widetilde{h}_1}$, p, $\max_{0\le s\le M}\frac{g_M(\cdot,s)}{\widetilde{h}_1}\in K^\infty(\mathbb{R}^d_+)$, then $\frac{f(\cdot,u,v)}{\widetilde{h}_1}\in K^\infty(\mathbb{R}^d_+)$. Moreover, we have by Proposition 2.5 that

$$\begin{aligned}0\le V\bigl(f(\cdot,u,v)\bigr)(x)&\le a_1V\left(\frac{f(\cdot,u,v)}{\widetilde{h}_1}\right)(x)+b_1V\left(\frac{f(\cdot,u,v)}{\widetilde{h}_1}\vartheta\right)(x)\\&\le a_1V\left(\frac{f(\cdot,u,v)}{\widetilde{h}_1}\right)(x)+c\alpha_{\frac{f(\cdot,u,v)}{\widetilde{h}_1}}\vartheta(x),\end{aligned}$$

where $\vartheta(x)=x_d$. Using this inequality, the fact that $\frac{f(\cdot,u,v)}{\widetilde{h}_1}\in K^\infty(\mathbb{R}^d_+)$, we deduce from assertion (1) of Proposition 2.7 that

$$\lim_{x\longrightarrow\xi\in\partial\mathbb{R}^d_+}V\bigl(f(\cdot,u,v)\bigr)(x)=0\quad\text{for } d\ge 2.$$

On the other hand, we have

$$\frac{V(f(\cdot,u,v))}{\vartheta}\le a_1\frac{V(\frac{f(\cdot,u,v)}{\widetilde{h}_1})}{\vartheta}+b_1\frac{V(\frac{f(\cdot,u,v)}{\widetilde{h}_1}\vartheta)}{\vartheta}.$$

Since $\frac{f(\cdot,u,v)}{\widetilde{h}_1}\in K^\infty(\mathbb{R}^d_+)$, we obtain by using assertion (2) of Proposition 2.7 that

$$\lim_{x_d\longrightarrow\infty}\frac{V(\frac{f(\cdot,u,v)}{\widetilde{h}_1}\vartheta)(x)}{\vartheta(x)}=0.$$

Using this fact, the fact $V(\frac{f(\cdot,u,v)}{\widetilde{h}_1})$ is bounded in $\overline{\mathbb{R}^d_+}$, we obtain from the last inequality that

$$\lim_{x_d\longrightarrow\infty}\frac{V(\frac{f(\cdot,u,v)}{\widetilde{h}_1})(x)}{x_d}=0.$$

Similarly, we prove that

$$\lim_{x\longrightarrow\xi\in\partial\mathbb{R}^d_+} V\big(g(\cdot,u,v)\big)(x)=0 \quad \text{and} \quad \lim_{x_d\longrightarrow\infty}\frac{V(g(\cdot,u,v))(x)}{x_d}=0.$$

So (u,v) is a positive continuous solution of system (1.1) in the sense of distributions satisfying

$$\begin{aligned}&\left(1-\frac{\theta(\lambda)}{\sigma_1}\right)\exp(-\lambda\alpha_p)h_1\le u\le h_1 \quad \text{and}\\ &\left(1-\frac{\theta(\mu)}{\sigma_2}\right)\exp(-\mu\alpha_p)h_2\le v\le h_2.\end{aligned}$$ □

Example 4.1 Let $\beta_1,\delta_1,\sigma_1,\gamma_1,\eta_1\in\mathbb{R}$ such that $\gamma_1\ge 0$ and $\gamma_1+\sigma_1\ge 0$. Define the nonnegative function f on $\mathbb{R}^d_+\times[0,\infty)\times[0,\infty)$ by

$$f(x,t,s)=\frac{1}{x_d^{\beta_1}(1+|x|)^{\delta_1-\beta_1}}(x_d+t+s)^{\sigma_1}(x_d+t)^{\gamma_1}(x_d+s)^{\eta_1}.$$

Consider the function $H(t)=(x_d+t+s)^{\sigma_1}(x_d+t)^{\gamma_1}$ for $(x,s)\in\mathbb{R}^d_+\times[0,\infty)$. We note $H'(t)=(x_d+t+s)^{\sigma_1-1}(x_d+t)^{\gamma_1-1}[(\gamma_1+\sigma_1)(t+x_d)+\gamma_1 s]\ge 0$ for all $(x,t,s)\in\mathbb{R}^d_+\times[0,\infty)\times[0,\infty)$ if and only if $\gamma_1+\sigma_1\ge 0$ and $\gamma_1\ge 0$. Hence $(\mathbf{H}_4)$ is satisfied.

Assume that the following conditions are satisfied:

$$\begin{aligned}&\beta_1-\eta_1-\big(1-\operatorname{sgn}(a_1)\big)\gamma_1-\big(1-\operatorname{sgn}(a_1+a_2)\big)\sigma_1+\operatorname{sgn}(a_2)\\ &\quad+\max\big[0,-\operatorname{sgn}(a_1+a_2)\sigma_1,\operatorname{sgn}(a_2)(\eta_1-1),\operatorname{sgn}(a_2)(\eta_1-\sigma_1)\big]<1 \quad \text{and} \qquad (4.8)\\ &\delta_1-\eta_1-\gamma_1-\sigma_1>\operatorname{sgn}(a_1)+\operatorname{sgn}(b_1)+\big(1-\operatorname{sgn}(b_1)\big)d. \qquad (4.9)\end{aligned}$$

Then f satisfies hypotheses $(\mathbf{H}_5)$, $(\mathbf{H}_6)$, and $(\mathbf{H}_7)$. Indeed, using the fact that $\frac{1}{\widetilde{h}_1(x)}\le\frac{1}{a_1}$ if $a_1>0$ and $\frac{1}{\widetilde{h}_1(x)}=\frac{1}{b_1x_d}$ if $a_1=0$, we obtain

$$\frac{1}{\widetilde{h}_1(x)}\le\frac{c}{x_d^{(1-\operatorname{sgn}(a_1))}}. \qquad (4.10)$$

Hence

$$\frac{f(x,0,0)}{\widetilde{h}_1(x)}\le\frac{c}{x_d^{\beta_1-\sigma_1-\gamma_1-\eta_1+(1-\operatorname{sgn}(a_1))}(1+|x|)^{\delta_1-\beta_1}}.$$

Since $\gamma_1\ge 0$ and $(1-\operatorname{sgn}(a_1))\le 1$, we obtain by conditions (4.8) and (4.9) that

$$\begin{aligned}&\beta_1-\sigma_1-\gamma_1-\eta_1+\big(1-\operatorname{sgn}(a_1)\big)\\ &\quad\le\beta_1-\gamma_1-\eta_1-\big(1-\operatorname{sgn}(a_1+a_2)\big)\sigma_1-\operatorname{sgn}(a_1+a_2)\sigma_1+\big(1-\operatorname{sgn}(a_1)\big)\\ &\quad<\beta_1-\big(1-\operatorname{sgn}(a_1)\big)\gamma_1-\eta_1-\big(1-\operatorname{sgn}(a_1+a_2)\big)\sigma_1\\ &\qquad+\max\big[0,-\operatorname{sgn}(a_1+a_2)\sigma_1,\operatorname{sgn}(a_2)(\eta_1-1),\operatorname{sgn}(a_2)(\eta_1-\sigma_1)\big]+1\\ &\quad<2\end{aligned}$$

and

$$\delta_1 - \sigma_1 - \gamma_1 - \eta_1 + \left(1 - \operatorname{sgn}(a_1)\right) > 1 + \operatorname{sgn}(b_1) + \left(1 - \operatorname{sgn}(b_1)\right)d \geq 2.$$

From Example 2.1, we deduce that $\frac{f(\cdot,0,0)}{\widetilde{h}_1} \in K^\infty(\mathbb{R}^d_+)$ and $(\mathbf{H}_5)$ is satisfied.

To verify $(\mathbf{H}_6)$ and $(\mathbf{H}_7)$, we consider $M > 0$, $0 \leq t_1 \leq t_2 \leq M$, and $0 \leq s_1 \leq s_2 \leq M$. Then there exist $\tau_1, \tau_3 \in (t_1, t_2)$ and $\tau_2, \tau_4 \in (s_1, s_2)$ such that

$$\begin{aligned} &f\left(x, t_2\widetilde{h}_1(x), s_2\widetilde{h}_2(x)\right) - f\left(x, t_1\widetilde{h}_1(x), s_1\widetilde{h}_2(x)\right) \\ &\quad = \frac{1}{x_d^{\beta_1}(1+|x|)^{\delta_1-\beta_1}}\left[(t_2 - t_1)\widetilde{h}_1(x)A + (s_2 - s_1)\widetilde{h}_2(x)B\right], \end{aligned}$$

where

$$\begin{aligned} A := {}&\sigma_1\left(x_d + \tau_1\widetilde{h}_1(x) + s_2\widetilde{h}_2(x)\right)^{\sigma_1-1}\frac{(x_d + t_2\widetilde{h}_1(x))^{\gamma_1}}{(x_d + s_2\widetilde{h}_2(x))^{-\eta_1}} \\ &+ \gamma_1\left(x_d + \tau_3\widetilde{h}_1(x)\right)^{\gamma_1-1}\frac{(x_d + t_1\widetilde{h}_1(x) + s_1\widetilde{h}_2(x))^{\sigma_1}}{(x_d + s_2\widetilde{h}_2(x))^{-\eta_1}} \end{aligned}$$

and

$$\begin{aligned} B := {}&\sigma_1\left(x_d + t_1\widetilde{h}_1(x) + \tau_2\widetilde{h}_2(x)\right)^{\sigma_1-1}\frac{(x_d + t_2\widetilde{h}_1(x))^{\gamma_1}}{(x_d + s_2\widetilde{h}_2(x))^{-\eta_1}} \\ &+ \eta_1\left(x_d + \tau_4\widetilde{h}_2(x)\right)^{\eta_1-1}\left(x_d + t_1\widetilde{h}_1(x)\right)^{\gamma_1}\left(x_d + t_1\widetilde{h}_1(x) + s_1\widetilde{h}_2(x)\right)^{\sigma_1}. \end{aligned}$$

Next, we will dominate $|A|$ and $|B|$. For this aim, we distinguish the following cases.

Case 1. $b_1 = 0$. In this case, we have $a_1 > 0$, $h_1 = a_1H\phi_1$, $\widetilde{h}_1 = a_1$, $\frac{1}{h_1(x)} \leq c$, and condition (4.9) writes as

$$\delta_1 - \gamma_1 - \eta_1 - \sigma_1 > d + 1. \tag{4.11}$$

This case will be divided into two subcases.

Subcase 1. $a_2 = 0$. In this case, we have $b_2 > 0$, $h_2 = \widetilde{h}_2 = b_2x_d$, and condition (4.8) becomes

$$\beta_1 - \eta_1 + \max(0, -\sigma_1) < 1. \tag{4.12}$$

By discussing six sub-subcases ($0 \leq \gamma_1 < 1$ or $\gamma_1 \geq 1$) and($\sigma_1 < 0$ or $0 \leq \sigma_1 < 1$ or $\sigma_1 \geq 1$) and the fact that $x_d^{\sigma_1}(1+|x|)^{\gamma_1} + x_d^{\gamma_1}(1+|x|)^{\sigma_1} \leq 2x_d^{\min(\gamma_1,\sigma_1)}(1+|x|)^{\max(\gamma_1,\sigma_1)}$, we obtain

$$\begin{aligned} &\left|f\left(x, t_2\widetilde{h}_1(x), s_2\widetilde{h}_2(x)\right) - f\left(x, t_1\widetilde{h}_1(x), s_1\widetilde{h}_2(x)\right)\right| \\ &\quad \leq p_M(x)(t_2 - t_1)\widetilde{h}_1(x) + \left|g_M(x, s_2) - g_M(x, s_1)\right|, \end{aligned}$$

where

$$p_M(x) := \frac{c}{x_d^{\beta_1-\eta_1+\max(1-\gamma_1,1-\sigma_1,0)}(1+|x|)^{\delta_1-\beta_1-1+\min(1-\gamma_1,1-\sigma_1,2-\gamma_1-\sigma_1)}}$$

and

$$g_M(x,s) := \frac{cs\widetilde{h}_2(x)}{x_d^{\beta_1-\eta_1+1+\max(-\sigma_1,0)}(1+|x|)^{\delta_1-\beta_1-\gamma_1+\min(-\sigma_1,0)}}.$$

Since $\gamma_1 \geq 0$, we deduce by (4.12) that

$$\begin{aligned}\beta_1-\eta_1+\max(1-\gamma_1,1-\sigma_1,0) &\leq \beta_1-\eta_1+\max(1-\sigma_1,1)\\ &= \beta_1-\eta_1+1+\max(-\sigma_1,0)<2.\end{aligned}$$

On the other hand, using the fact that $\max(a,b,0)+\min(a,b,a+b)=a+b$, we obtain by using (4.11) that

$$\begin{aligned}&\delta_1-\eta_1-1+\max(1-\gamma_1,1-\sigma_1,0)+\min(1-\gamma_1,1-\sigma_1,2-\gamma_1-\sigma_1)\\ &\quad=\delta_1-\eta_1-\gamma_1-\sigma_1+1>2+d>2.\end{aligned}$$

Hence the function $p_M \in K^\infty(\mathbb{R}^d_+)$. Now, since $\widetilde{h}_2(x)=b_2x_d$ and $\frac{1}{\widetilde{h}_1(x)} \leq c$, we obtain

$$\max_{0\leq s\leq M}\frac{g_M(x,s)}{\widetilde{h}_1(x)} \leq \frac{cM}{x_d^{\beta_1-\eta_1+\max(-\sigma_1,0)}(1+|x|)^{\delta_1-\beta_1-\gamma_1+\min(-\sigma_1,0)}}.$$

Using condition (4.12), we obtain

$$\beta_1-\eta_1+\max(-\sigma_1,0)<\beta_1-\eta_1+1+\max(-\sigma_1,0)<2.$$

This together with the fact that

$$\delta_1-\eta_1-\gamma_1+\min(-\sigma_1,0)+\max(-\sigma_1,0)=\delta_1-\eta_1-\gamma_1-\sigma_1>d\geq 2$$

implies that the function $\max_{0\leq s\leq M}\frac{g_M(x,s)}{\widetilde{h}_1(x)} \in K^\infty(\mathbb{R}^d_+)$. Hence f satisfies $(\mathbf{H}_6)$. Now, we have

$$\begin{aligned}\varrho_1(x) &= f(x,0,0)+\max_{0\leq s\leq M}g_M(x,s)\\ &\leq \frac{1}{x_d^{\beta_1-\eta_1-\sigma_1-\gamma_1}(1+|x|)^{\delta_1-\beta_1}}+\frac{c}{x_d^{\beta_1-\eta_1+\max(-\sigma_1,0)}(1+|x|)^{\delta_1-\beta_1-\gamma_1+\min(-\sigma_1,0)}}.\end{aligned}$$

Since $\sigma_1+\gamma_1 \geq 0$, we deduce by conditions (4.11) and (4.12) that

$$\begin{aligned}&\beta_1-\eta_1-\sigma_1-\gamma_1<\beta_1-\eta_1<\beta_1-\eta_1+\max(-\sigma_1,0)<1 \quad \text{and}\\ &\delta_1-\eta_1-\sigma_1-\gamma_1>d+1.\end{aligned}$$

Hence from [4, 5] we obtain

$$V(\varrho_1)(x) \leq c\frac{x_d}{(1+|x|)^d}.$$

This together with (1.4) implies that f satisfies $(\mathbf{H}_7)$.

Subcase 2. $a_2 > 0$. In this case, we have $h_2(x) = a_2 H\phi_2(x) + b_2 x_d$ and $\widetilde{h}_2 = a_2 + b_2 x_d$, and condition (4.8) becomes

$$\beta_1 - \eta_1 + \max[0, -\sigma_1, \eta_1 - 1, \eta_1 - \sigma_1] < 0. \tag{4.13}$$

By discussing the eighteen sub-subcases($0 \le \gamma_1 < 1$ or $\gamma_1 \ge 1$),($\eta_1 < 0$ or $0 \le \eta_1 < 1$ or $\eta_1 \ge 1$) and($\sigma_1 < 0$ or $0 \le \sigma_1 < 1$ or $\sigma_1 \ge 1$), we obtain

$$\begin{aligned}&\left|f\left(x, t_2\widetilde{h}_1(x), s_2\widetilde{h}_2(x)\right) - f\left(x, t_1\widetilde{h}_1(x), s_1\widetilde{h}_2(x)\right)\right| \\&\quad \le p_M(x)(t_2 - t_1)\widetilde{h}_1(x) + \left|g_M(x, s_2) - g_M(x, s_1)\right|,\end{aligned}$$

where

$$p_M(x) := \frac{c}{x_d^{\beta_1 + \max(0,-\eta_1) + \max(0,1-\gamma_1,1-\sigma_1)}(1 + |x|)^{\delta_1 - \beta_1 + \min(0,-\eta_1) + \min(1-\gamma_1-\sigma_1,-\gamma_1,-\sigma_1)}}$$

and

$$g_M(x, s) := \frac{cs\widetilde{h}_2(x)}{x_d^{\beta_1 + \max(1-\eta_1,1-\sigma_1,1-\eta_1-\sigma_1,0)}(1 + |x|)^{\delta_1 - \beta_1 - \gamma_1 + \min(1-\eta_1-\sigma_1,-\eta_1,-\sigma_1,0)}}.$$

Since $\gamma_1 \ge 0$, we deduce by conditions (4.11) and (4.13) that

$$\begin{aligned}&\beta_1 + \max(0, -\eta_1) + \max(0, 1 - \gamma_1, 1 - \sigma_1) \\&\quad \le \beta_1 + \max(0, -\eta_1) + \max(1, 1 - \sigma_1) \\&\quad \le \beta_1 + \max(0, -\eta_1, -\sigma_1, -\sigma_1 - \eta_1) + 1 \\&\quad \le \beta_1 - \eta_1 + \max(\eta_1, 0, -\sigma_1 + \eta_1, -\sigma_1) + 1 \\&\quad \le \beta_1 - \eta_1 + \max(\eta_1, 1, 1 - \sigma_1 + \eta_1, 1 - \sigma_1) + 1 \\&\quad \le \beta_1 - \eta_1 + \max(\eta_1 - 1, 0, -\sigma_1, \eta_1 - \sigma_1) + 2 < 2\end{aligned}$$

and

$$\begin{aligned}&\delta_1 - \eta_1 + \min(1 - \gamma_1 - \sigma_1, -\gamma_1, -\sigma_1) + \max(0, 1 - \gamma_1, 1 - \sigma_1) \\&\quad = \delta_1 - \eta_1 + 1 - \gamma_1 - \sigma_1 + \min(0, \gamma_1 - 1, \sigma_1 - 1) + \max(0, 1 - \gamma_1, 1 - \sigma_1) \\&\quad = \delta_1 - \eta_1 + 1 - \gamma_1 - \sigma_1 > 2 + d > 2.\end{aligned}$$

Hence p_M belongs to $K^\infty(\mathbb{R}^d_+)$. Now, since $\widetilde{h}_2(x) \le c(1 + |x|)^{\operatorname{sgn}(b_2)}$, we get

$$\max_{0 \le s \le M} \frac{g_M(x, s)}{\widetilde{h}_1(x)} \le \frac{c}{x_d^{\beta_1 + \max(1-\eta_1,1-\sigma_1,1-\eta_1-\sigma_1,0)}(1 + |x|)^{\delta_1 - \beta_1 - \gamma_1 + \min(1-\eta_1-\sigma_1,-\eta_1,-\sigma_1,0) - \operatorname{sgn}(b_2)}}.$$

Using (4.11) and (4.13), we obtain

$$\beta_1 + \max(1 - \eta_1, 1 - \sigma_1, 1 - \eta_1 - \sigma_1, 0) = \beta_1 - \eta_1 + \max(0, \eta_1 - \sigma_1, -\sigma_1, \eta_1 - 1) + 1 < 2$$

and

$$\begin{aligned}
&\delta_1-\gamma_1+\min(1-\eta_1-\sigma_1,-\eta_1,-\sigma_1,0)-\operatorname{sgn}(b_2)+\max(1-\eta_1,1-\sigma_1,1-\eta_1-\sigma_1,0)\\
&\quad=\delta_1-\gamma_1-\eta_1-\sigma_1+1-\operatorname{sgn}(b_2)+\min(0,\sigma_1-1,\eta_1-1,\eta_1+\sigma_1-1)\\
&\qquad+\max(1-\eta_1,1-\sigma_1,1-\eta_1-\sigma_1,0)\\
&\quad=\delta_1-\gamma_1-\eta_1-\sigma_1+1-\operatorname{sgn}(b_2)>2-\operatorname{sgn}(b_2)+d>2.
\end{aligned}$$

This proves that $\max_{0\le s\le M}\frac{g_M(x,s)}{\widetilde{h}_1(x)}\in K^\infty(\mathbb{R}^d_+)$ and f satisfies $(\mathbf{H}_6)$. Next, we verify $(\mathbf{H}_7)$. Let $\varrho_1(x)=f(x,0,0)+\max_{0\le s\le M}g_M(x,s)$. Then

$$\begin{aligned}
\varrho_1(x)&\le\frac{1}{x_d^{\beta_1-\sigma_1-\gamma_1-\eta_1}(1+|x|)^{\delta_1-\beta_1}}\\
&\quad+\frac{c}{x_d^{\beta_1+\max(1-\eta_1,1-\sigma_1,1-\eta_1-\sigma_1,0)}(1+|x|)^{\delta_1-\beta_1-\gamma_1+\min(1-\eta_1-\sigma_1,-\eta_1,-\sigma_1,0)-\operatorname{sgn}(b_2)}}.
\end{aligned}$$

Since $\sigma_1+\gamma_1\ge 0$, then

$$\begin{aligned}
&\beta_1-\sigma_1-\gamma_1-\eta_1<\beta_1-\eta_1\le\beta_1-\eta_1+\max(0,-\sigma_1,\eta_1-1,\eta_1-\sigma_1)<0,\\
&\beta_1+\max(1-\eta_1,1-\sigma_1,1-\eta_1-\sigma_1,0)=\beta_1-\eta_1+\max(1,1+\eta_1-\sigma_1,1-\sigma_1,\eta_1)\\
&\quad=\beta_1-\eta_1+\max(0,\eta_1-\sigma_1,-\sigma_1,\eta_1-1)+1<1
\end{aligned}$$

and

$$\begin{aligned}
&\delta_1-\gamma_1+\min(1-\eta_1-\sigma_1,-\eta_1,-\sigma_1,0)-\operatorname{sgn}(b_2)+\max(1-\eta_1,1-\sigma_1,1-\eta_1-\sigma_1,0)\\
&\quad=\delta_1-\gamma_1-\eta_1-\sigma_1+1-\operatorname{sgn}(b_2)+\min(0,\sigma_1-1,\eta_1-1,\eta_1+\sigma_1-1)\\
&\qquad+\max(1-\eta_1,1-\sigma_1,1-\eta_1-\sigma_1,0)\\
&\quad=\delta_1-\gamma_1-\eta_1-\sigma_1+1-\operatorname{sgn}(b_2)>2-\operatorname{sgn}(b_2)+d\\
&\quad>1+d.
\end{aligned}$$

As in subcase 1, we obtain from [4, 5] that

$$V\big(\varrho_1(x)\big)\le c\frac{x_d}{(1+|x|)^d}.$$

This together with (1.4) implies that f satisfies $(\mathbf{H}_7)$.

Case 2. $b_1>0$. In this case, we have $a_1\ge 0$ and condition (4.9) will write as

$$\delta_1-\eta_1-\gamma_1-\sigma_1>1+\operatorname{sgn}(a_1). \tag{4.14}$$

We will also discuss two subcases.

Subcase 1. $a_2=0$. In this case, we have $h_2(x)=\widetilde{h}_2(x)=b_2x_d$, and condition (4.8) becomes

$$\beta_1-\eta_1-\sigma_1-\big(1-\operatorname{sgn}(a_1)\big)\gamma_1+1+\operatorname{sgn}(a_1)\max[0,\sigma_1]<2. \tag{4.15}$$

By discussing the six sub-subcases$(0 \leq \gamma_1 < 1$ or $\gamma_1 \geq 1)$ and$(\sigma_1 < 0$ or $0 \leq \sigma_1 < 1$ or $\sigma_1 \geq 1)$ and using the fact that $x_d^{\sigma_1}(1+|x|)^{\gamma_1} + x_d^{\gamma_1}(1+|x|)^{\sigma_1} \leq 2x_d^{\min(\gamma_1,\sigma_1)}(1+|x|)^{\max(\gamma_1,\sigma_1)}$, we obtain

$$\begin{aligned}&\bigl|f\bigl(x, t_2\widetilde{h}_1(x), s_2\widetilde{h}_2(x)\bigr) - f\bigl(x, t_1\widetilde{h}_1(x), s_1\widetilde{h}_2(x)\bigr)\bigr|\\&\quad\leq p_M(x)(t_2-t_1)\widetilde{h}_1(x) + \bigl|g_M(x,s_2) - g_M(x,s_1)\bigr|,\end{aligned}$$

where

$$p_M(x) = \frac{c}{x_d^{\beta_1-\eta_1-\gamma_1-\sigma_1+1+\operatorname{sgn}(a_1)\max(\gamma_1+\sigma_1-1,\sigma_1,\gamma_1)}(1+|x|)^{\delta_1-\beta_1+\operatorname{sgn}(a_1)\min(1-\gamma_1-\sigma_1,-\gamma_1,-\sigma_1)}}$$

and

$$g_M(x,s) = \frac{c\widetilde{h}_2(x)s}{x_d^{\beta_1-\eta_1-\gamma_1-\sigma_1+1+\operatorname{sgn}(a_1)(\max(\sigma_1,0)+\gamma_1)}(1+|x|)^{\delta_1-\beta_1+\operatorname{sgn}(a_1)(\min(0,-\sigma_1)-\gamma_1)}}.$$

Since $\gamma_1 \geq 0$, using (4.14) and (4.15), we obtain

$$\begin{aligned}&\beta_1-\eta_1-\gamma_1-\sigma_1+1+\operatorname{sgn}(a_1)\max(\gamma_1+\sigma_1-1,\gamma_1,\sigma_1)\\&\quad\leq \beta_1-\eta_1-\gamma_1-\sigma_1+1+\operatorname{sgn}(a_1)\max(\gamma_1+\sigma_1,\gamma_1)\\&\quad= \beta_1-\eta_1-\sigma_1+1-\bigl(1-\operatorname{sgn}(a_1)\bigr)\gamma_1+\operatorname{sgn}(a_1)\max(\sigma_1,0) < 2\end{aligned}$$

and

$$\begin{aligned}&\delta_1-\eta_1-\gamma_1-\sigma_1+1+\operatorname{sgn}(a_1)\bigl[\max(\gamma_1+\sigma_1-1,\gamma_1,\sigma_1)+\min(1-\gamma_1-\sigma_1,-\gamma_1,-\sigma_1)\bigr]\\&\quad= \delta_1-\eta_1-\gamma_1-\sigma_1+1 > 2+\operatorname{sgn}(a_1) \geq 2.\end{aligned}$$

Hence p_M belongs to $K^\infty(\mathbb{R}_+^d)$. Now, using (4.10) we get

$$\begin{aligned}&\max_{0\leq s\leq M}\frac{g_M(x,s)}{\widetilde{h}_1(x)}\\&\quad\leq \frac{cM\widetilde{h}_2(x)}{x_d^{\beta_1-\eta_1-\gamma_1-\sigma_1+1+(1-\operatorname{sgn}(a_1))+\operatorname{sgn}(a_1)(\max(\sigma_1,0)+\gamma_1)}(1+|x|)^{\delta_1-\beta_1+\operatorname{sgn}(a_1)(\min(0,-\sigma_1)-\gamma_1)}}\\&\quad\leq \frac{cM}{x_d^{\beta_1-\eta_1-\gamma_1-\sigma_1+(1-\operatorname{sgn}(a_1))+\operatorname{sgn}(a_1)(\max(\sigma_1,0)+\gamma_1)}(1+|x|)^{\delta_1-\beta_1+\operatorname{sgn}(a_1)(\min(0,-\sigma_1)-\gamma_1)}}.\end{aligned}$$

Using the fact that $0 \leq 1-\operatorname{sgn}(a_1) \leq 1$, we obtain

$$\begin{aligned}&\beta_1-\eta_1-\gamma_1-\sigma_1+\bigl(1-\operatorname{sgn}(a_1)\bigr)+\operatorname{sgn}(a_1)\bigl(\max(0,\sigma_1)+\gamma_1\bigr)\\&\leq \beta_1-\eta_1-\sigma_1+1-\bigl(1-\operatorname{sgn}(a_1)\bigr)\gamma_1+\operatorname{sgn}(a_1)\max(0,\sigma_1) < 2\end{aligned}$$

and

$$\begin{aligned}&\delta_1-\eta_1-\gamma_1-\sigma_1+\bigl(1-\operatorname{sgn}(a_1)\bigr)+\operatorname{sgn}(a_1)\bigl(\max(0,\sigma_1)+\gamma_1\bigr)\\&\quad+\operatorname{sgn}(a_1)\bigl(\min(0,-\sigma_1)-\gamma_1\bigr)\end{aligned}$$

$$= \delta_1 - \eta_1 - \gamma_1 - \sigma_1 + \big(1 - \operatorname{sgn}(a_1)\big) > 1 + \operatorname{sgn}(a_1) + \big(1 - \operatorname{sgn}(a_1)\big) = 2.$$

Hence f satisfies $(\mathbf{H}_6)$. Now we prove that f satisfies $(\mathbf{H}_7)$. Put $\theta(x) = x_d$ and $\Psi_1(x) = \frac{\varrho_1(x)}{\theta(x)} = \frac{\varrho_1(x)}{x_d}$. Since $\widetilde{h}_2(x) = b_2 x_d$, then we have

$$\begin{aligned}
\Psi_1(x) &= \frac{f(x,0,0)}{x_d} + \max_{0\le s\le m} \frac{g_m(x,s)}{x_d} \\
&\le \frac{1}{x_d^{\beta_1-\sigma_1-\gamma_1-\eta_1+1}(1+|x|)^{\delta_1-\beta_1}} \\
&\quad + \frac{cb_2M}{x_d^{\beta_1-\eta_1-\gamma_1-\sigma_1+1+\operatorname{sgn}(a_1)(\max(\sigma_1,0)+\gamma_1)}(1+|x|)^{\delta_1-\beta_1+\operatorname{sgn}(a_1)(\min(0,-\sigma_1)-\gamma_1)}}.
\end{aligned}$$

Since $\gamma_1 \ge 0$, we deduce by (4.14) and (4.15) that

$$\beta_1 - \sigma_1 - \gamma_1 - \eta_1 + 1 < \beta_1 - \eta_1 - \gamma_1 - \sigma_1 + 1 + \operatorname{sgn}(a_1)\big(\max(\sigma_1,0) + \gamma_1\big) < 2$$

and

$$\begin{aligned}
&\delta_1 - \eta_1 - \gamma_1 - \sigma_1 + 1 + \operatorname{sgn}(a_1)\big(\min(0,-\sigma_1) + \max(0,\sigma_1)\big) \\
&\quad = \delta_1 - \eta_1 - \gamma_1 - \sigma_1 + 1 > 2 + \operatorname{sgn}(a_1) \ge 2.
\end{aligned}$$

Hence $\Psi_1 \in K^\infty(\mathbb{R}^d_+)$, and consequently from Proposition 2.5 we deduce that

$$V(\rho_1)(x) = V(\Psi_1\theta)(x) \le \alpha_{\Psi_1} x_d \le \frac{\alpha_{\Psi_1}}{b_1} h_1(x).$$

This implies that $(\mathbf{H}_7)$ is satisfied.

Subcase 2. $a_2 > 0$. In this case, we have $h_2(x) = a_2 H\phi_2(x) + b_2 x_d$, $\widetilde{h}_2(x) = a_2 + b_2 x_d$ and condition (4.8) becomes

$$\beta_1 - \eta_1 - \sigma_1 - \big(1 - \operatorname{sgn}(a_1)\big)\gamma_1 + \max[0,\sigma_1,\eta_1,\eta_1+\sigma_1-1] < 0. \tag{4.16}$$

By discussing the nine subcases($\sigma_1 < 0$ or $0 \le \sigma_1 < 1$ or $\sigma_1 \ge 1$) and($\eta_1 < 0$ or $0 \le \eta_1 < 1$ or $\eta_1 \ge 1$) if $a_1 = 0$ and the eighteen sub-subcases($0 \le \gamma_1 < 1$ or $\gamma_1 \ge 1$), ($\sigma_1 < 0$ or $0 \le \sigma_1 < 1$ or $\sigma_1 \ge 1$) and ($\eta_1 < 0$ or $0 \le \eta_1 < 1$ or $\eta_1 \ge 1$) if $a_1 > 0$ and using the fact that $x_d^\kappa(1+|x|)^r + x_d^r(1+|x|)^\kappa \le 2x_d^{\min(r,\kappa)}(1+|x|)^{\max(r,\kappa)}$ for $\kappa, r \in \mathbb{R}$, we obtain

$$\begin{aligned}
&\big|f\big(x, t_2\widetilde{h}_1(x), s_2\widetilde{h}_2(x)\big) - f\big(x, t_1\widetilde{h}_1(x), s_1\widetilde{h}_2(x)\big)\big| \\
&\quad \le p_M(x)(t_2 - t_1)\widetilde{h}_1(x) + \big|g_M(x,s_2) - g_M(x,s_1)\big|,
\end{aligned}$$

where

$$p_M(x) := \frac{c x_d^{-[\beta_1-\eta_1-\sigma_1-(1-\operatorname{sgn}(a_1))\gamma_1+1+\max(0,\eta_1)+\max(0,\sigma_1-\operatorname{sgn}(a_1)\gamma_1,\operatorname{sgn}(a_1)(\sigma_1-1))]}}{(1+|x|)^{\delta_1-\beta_1-\operatorname{sgn}(a_1)\gamma_1+\min(0,-\eta_1)+\min(\operatorname{sgn}(a_1)(1-\sigma_1),\operatorname{sgn}(a_1)\gamma_1-\sigma_1,0)}}$$

and

$$g_M(x,s) := \frac{c\widetilde{h}_2(x) s x_d^{-[\beta_1-\eta_1-\sigma_1+1-(1-\operatorname{sgn}(a_1))\gamma_1+\max(0,\eta_1,\sigma_1,\eta_1+\sigma_1-1)]}}{(1+|x|)^{\delta_1-\beta_1-\operatorname{sgn}(a_1)\gamma_1+\min(0,-\eta_1,-\sigma_1,1-\eta_1-\sigma_1)}}.$$

Since $\gamma_1 \geq 0$, by using (4.14) and (4.16), we obtain

$$\begin{aligned}
&\beta_1 - \eta_1 - \sigma_1 - \big(1 - \operatorname{sgn}(a_1)\big)\gamma_1 + 1 + \max(0, \eta_1) \\
&\quad + \max\big[0, \sigma_1 - \operatorname{sgn}(a_1)\gamma_1, \operatorname{sgn}(a_1)(\sigma_1 - 1)\big] \\
&\leq \beta_1 - \eta_1 - \sigma_1 - \big(1 - \operatorname{sgn}(a_1)\big)\gamma_1 + 1 + \max(0, \eta_1) + \max[0, \sigma_1] \\
&\leq \beta_1 - \eta_1 - \sigma_1 - \big(1 - \operatorname{sgn}(a_1)\big)\gamma_1 + 1 + \max[0, \eta_1, \sigma_1, \eta_1 + \sigma_1] \\
&\leq \beta_1 - \eta_1 - \sigma_1 - \big(1 - \operatorname{sgn}(a_1)\big)\gamma_1 + \max[0, \eta_1, \sigma_1, \eta_1 + \sigma_1 - 1] + 2 < 2
\end{aligned}$$

and

$$\begin{aligned}
&\delta_1 - \beta_1 - \operatorname{sgn}(a_1)\gamma_1 + \min(0, -\eta_1) + \min\big[\operatorname{sgn}(a_1)(1 - \sigma_1), \operatorname{sgn}(a_1)\gamma_1 - \sigma_1, 0\big] + \beta_1 - \eta_1 \\
&\quad - \sigma_1 - \big(1 - \operatorname{sgn}(a_1)\big)\gamma_1 + 1 + \max(0, \eta_1) + \max\big[0, \sigma_1 - \operatorname{sgn}(a_1)\gamma_1, \operatorname{sgn}(a_1)(\sigma_1 - 1)\big] \\
&= \delta_1 - \eta_1 - \gamma_1 - \sigma_1 + 1 > 2 + \operatorname{sgn}(a_1) \geq 2,
\end{aligned}$$

which proves that $p_M \in K^\infty(\mathbb{R}^d_+)$. Now, since $\widetilde{h}_2(x) \leq (a_2 + b_2)(1 + |x|)^{\operatorname{sgn}(b_2)}$, using (4.10) we obtain

$$\max_{0\leq s\leq M} \frac{g_M(x,s)}{\widetilde{h}_1(x)} \leq \frac{c x_d^{-[\beta_1-\eta_1-\sigma_1+1+(1-\operatorname{sgn}(a_1))-(1-\operatorname{sgn}(a_1))\gamma_1+\max(0,\eta_1,\sigma_1,\eta_1+\sigma_1-1)]}}{(1 + |x|)^{\delta_1-\beta_1-\operatorname{sgn}(a_1)\gamma_1-\operatorname{sgn}(b_2)+\min(0,-\eta_1,-\sigma_1,1-\eta_1-\sigma_1)}}.$$

Using (4.14) and (4.16), we obtain

$$\begin{aligned}
&\beta_1 - \eta_1 - \sigma_1 + 1 + \big(1 - \operatorname{sgn}(a_1)\big) - \big(1 - \operatorname{sgn}(a_1)\big)\gamma_1 + \max(0, \eta_1, \sigma_1, \eta_1 + \sigma_1 - 1) \\
&\leq 2 + \beta_1 - \eta_1 - \sigma_1 - \big(1 - \operatorname{sgn}(a_1)\big)\gamma_1 + \max(0, \eta_1, \sigma_1, \eta_1 + \sigma_1 - 1) < 2
\end{aligned}$$

and

$$\begin{aligned}
&\delta_1 - \beta_1 - \operatorname{sgn}(a_1)\gamma_1 - \operatorname{sgn}(b_2) + \min(0, -\eta_1, -\sigma_1, 1 - \eta_1 - \sigma_1) + \beta_1 - \eta_1 \\
&\quad - \sigma_1 + 1 + \big(1 - \operatorname{sgn}(a_1)\big) - \big(1 - \operatorname{sgn}(a_1)\big)\gamma_1 + \max(0, \eta_1, \sigma_1, \eta_1 + \sigma_1 - 1) \\
&= \delta_1 - \eta_1 - \gamma_1 - \sigma_1 + 2 - \operatorname{sgn}(a_1) - \operatorname{sgn}(b_2) > 3 - \operatorname{sgn}(b_2) \geq 2.
\end{aligned}$$

Hence $\max_{0\leq s\leq M} \frac{g_M(x,s)}{\widetilde{h}_1(x)} \in K^\infty(\mathbb{R}^d_+)$, and so f satisfies $(\mathbf{H}_6)$. Finally, we verify $(\mathbf{H}_7)$. Put $\theta(x) = x_d$ and $\Psi_1(x) = \frac{\varrho_1(x)}{\theta(x)} = \frac{\varrho_1(x)}{x_d}$. Since $\widetilde{h}_2(x) \leq (a_2 + b_2)(1 + |x|)^{\operatorname{sgn}(b_2)}$, then we have

$$\begin{aligned}
\Psi_1(x) &= \frac{f(x,0,0)}{x_d} + \max_{0\leq s\leq M} \frac{g_M(x,s)}{x_d} \\
&\leq \frac{1}{x_d^{\beta_1-\sigma_1-\gamma_1-\eta_1+1}(1 + |x|)^{\delta_1-\beta_1}} \\
&\quad + \frac{c(a_2 + b_2) M x_d^{-[\beta_1-\eta_1-\sigma_1+2-(1-\operatorname{sgn}(a_1))\gamma_1+\max(0,\eta_1,\sigma_1,\eta_1+\sigma_1-1)]}}{(1 + |x|)^{\delta_1-\beta_1-\operatorname{sgn}(a_1)\gamma_1-\operatorname{sgn}(b_2)+\min(0,-\eta_1,-\sigma_1,1-\eta_1-\sigma_1)}}.
\end{aligned}$$

Since $\gamma_1 \geq 0$, we deduce by (4.14) and (4.16) that

$$\beta_1 - \sigma_1 - \gamma_1 - \eta_1 + 1$$

$$< \beta_1 - \eta_1 - \sigma_1 + 2 - \bigl(1 - \operatorname{sgn}(a_1)\bigr)\gamma_1 + \max(0, \eta_1, \sigma_1, \eta_1 + \sigma_1 - 1) < 2$$

and

$$\begin{aligned}
&\delta_1 - \beta_1 - \operatorname{sgn}(a_1)\gamma_1 - \operatorname{sgn}(b_2) + \min(0, -\eta_1, -\sigma_1, 1 - \eta_1 - \sigma_1) \\
&\quad + \beta_1 - \eta_1 - \sigma_1 + 2 - \bigl(1 - \operatorname{sgn}(a_1)\bigr)\gamma_1 + \max(0, \eta_1, \sigma_1, \eta_1 + \sigma_1 - 1) \\
&= \delta_1 - \eta_1 - \gamma_1 - \sigma_1 + 2 - \operatorname{sgn}(b_2) > 3 + \operatorname{sgn}(a_1) - \operatorname{sgn}(b_2) \geq 2.
\end{aligned}$$

Hence $\frac{\varrho_1}{\theta} \in K^\infty(\mathbb{R}^d_+)$, and consequently from Proposition 2.5 we deduce that

$$V(\rho_1)(x) = V(\Psi_1\theta)(x) \leq \alpha_{\Psi_1} x_d \leq \frac{\alpha_{\Psi_1}}{b_1} h_1(x).$$

This implies that $(\mathbf{H}_7)$ is satisfied.

As a consequence of Theorem 4.1 and the above example, we obtain the following.

Corollary 4.2 *Let a_1, a_2, b_1, b_2 be nonnegative constants with $(a_1 + b_1)(a_2 + b_2) > 0$, $\lambda \geq 0$, $\mu \geq 0$, and ϕ_1, ϕ_2 are nonnegative nontrivial continuous functions on $\partial\mathbb{R}^d_+$, $d \geq 2$. Let $\gamma_1, \gamma_2, \beta_1, \beta_2, \sigma_1, \sigma_2, \eta_1, \eta_2, \delta_1, \delta_2$ be real constants such that $\gamma_1 \geq 0$, $\gamma_2 \geq 0$, $\gamma_1 + \sigma_1 \geq 0$ and $\gamma_2 + \sigma_2 \geq 0$ and satisfying*

$$\begin{aligned}
&\beta_i - \eta_i - \bigl(1 - \operatorname{sgn}(a_i)\bigr)\gamma_i - \bigl(1 - \operatorname{sgn}(a_1 + a_2)\bigr)\sigma_i + \operatorname{sgn}(a_{i+1}) \\
&\quad + \max\bigl[0, -\operatorname{sgn}(a_1 + a_2)\sigma_i, \operatorname{sgn}(a_{i+1})(\eta_i - 1), \operatorname{sgn}(a_{i+1})(\eta_i - \sigma_i)\bigr] < 1 \quad \textit{and} \\
&\delta_i - \eta_i - \gamma_i - \sigma_i > \operatorname{sgn}(a_i) + \operatorname{sgn}(b_i) + \bigl(1 - \operatorname{sgn}(b_i)\bigr)d.
\end{aligned}$$

for $i \in \{1, 2\}$, where $a_3 = a_1$. Then there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that for each $\lambda \in [0, \lambda_0)$ and $\mu \in [0, \mu_0)$ the system

$$\begin{cases}
\Delta u = \frac{\lambda}{x_d^{\beta_1}(1+|x|)^{\delta_1 - \beta_1}} (x_d + u + v)^{\sigma_1} (x_d + u)^{\gamma_1} (x_d + v)^{\eta_1} \quad \textit{in } \mathbb{R}^d_+, \\
\Delta v = \frac{\mu}{x_d^{\beta_2}(1+|x|)^{\delta_2 - \beta_2}} (x_d + u + v)^{\sigma_2} (x_d + u)^{\gamma_2} (x_d + v)^{\eta_2} \quad \textit{in } \mathbb{R}^d_+, \\
u = a_1\phi_1, \qquad v = a_2\phi_2, \quad \textit{in } \partial\mathbb{R}^d_+, \\
\lim_{x_d \to \infty} \frac{u(x)}{x_d} = b_1 \quad \textit{and} \quad \lim_{x_d \to \infty} \frac{v(x)}{x_d} = b_2,
\end{cases}$$

has a positive continuous solution (*in the sense of distributions*) *satisfying*

$$\begin{aligned}
&c_\lambda [a_1 H\phi_1 + b_1 x_d] \leq u \leq [a_1 H\phi_1 + b_1 x_d] \quad \textit{and} \\
&c_\mu [a_2 H\phi_2 + b_2 x_d] \leq v \leq [a_2 H\phi_2 + b_2 x_d],
\end{aligned}$$

where $c_\lambda, c_\mu \in [0, 1)$.

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