Benner et al. *Boundary Value Problems* (2023) 2023:50
https://doi.org/10.1186/s13661-023-01734-1

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

A periodic boundary value problem with switchings under nonlinear perturbations

Peter Benner[1,2], Sergey Chuiko[1,3] and Alexander Zuyev[1,2,4*]

*Correspondence:
zuyev@mpi-magdeburg.mpg.de
[1] Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg, Germany
[2] Otto von Guericke University Magdeburg, Magdeburg, Germany
Full list of author information is available at the end of the article

Abstract

We investigate the existence of solutions of weakly nonlinear periodic boundary value problems for systems of ordinary differential equations with switchings and the construction of these solutions. We consider the critical case where the equation for the generating constants of a weakly nonlinear periodic boundary-value problem with switchings does not turn into an identity. We improve the classification of critical and noncritical cases and construct an iterative algorithm for finding solutions of weakly nonlinear periodic boundary value problems with switchings in the critical case. As examples of application of the constructed iterative scheme, we obtain approximations to the solutions of a periodic boundary value problem for the mathematical model of nonisothermal chemical reactions. To check the accuracy of the proposed approximations, we evaluate discrepancies in the original equation.

Keywords: Periodic boundary value problem; Equation for the generating constants; Critical case; Nonlinear chemical reaction model

1 Introduction

A classical framework for studying periodic solutions of nonlinear ordinary differential equations, originating from the works by H. Poincaré and A.M. Lyapunov, is based on the perturbation analysis in a neighborhood of a periodic solution of the generating linear problem. The essence of this approach was summarized in [1] for nonlinear systems containing a small parameter. In particular, under certain nonsingularity assumptions for systems with analytic right-hand sides, it was shown that there exists a unique periodic solution of the perturbed nonlinear system, which depends analytically on the small parameter. The method of small parameter has been developed in [2] for an n-dimensional nonautonomous system of ordinary differential equations on $t \in [a,b]$ with n-dimensional boundary form depending on values of the state vector at $t = a$ and $t = b$. The crucial assumption of this work concerns the solvability of the shortened boundary value problem obtained by putting the small parameter ε to be zero. Then the existence of solutions of the original boundary value problem is proved for sufficiently small $\varepsilon > 0$, and the convergence of this solution to the shortened one is established as $\varepsilon \to 0$. The above result is obtained for systems whose vector fields are continuous in time t and continuously differentiable with respect to the state vector and small parameter ε.

Perturbation theory is proved to be a powerful tool in the qualitative and quantitative study of periodic solutions to the Hill, Mathieu, van der Pol equations, and many other important mathematical models in nonlinear mechanics and physics (see, e.g., [3]). Without pretending to be complete, we also mention the existence results concerning periodic boundary value problems for nonlinear differential equations with singularities [4, 5], second-order nonautonomous nonlinear equations on the positive cone [6], superlinear second-order equations with positive solutions [7], first-order problem with resonance and nonlinear impulses [8], Hamiltonian systems with nonsmooth potentials [9], and second-order equations with a convection term [10].

The present work addresses the issue of solution existence for nonlinear periodic boundary value problems on $t \in [a, b]$ with switchings at given time instants $a = \tau_0 < \tau_1 < \cdots < \tau_p < \tau_{p+1} = b$. Such problems naturally arise in the dynamic optimization of a variety of mathematical models in natural science and engineering. Indeed, a typical framework in optimal control problems with nonconvex costs and input constraints results in bang-bang extremal controls because of Pontryagin's maximum principle. Then the characterization of optimal trajectories satisfying prescribed boundary conditions becomes a nontrivial issue due to the coupled nonlinear structure of the corresponding Hamiltonian system. As an example, we refer to [11], where an isoperimetric optimal control problem has been studied for nonlinear chemical reaction models under periodic boundary conditions. A class of switching controls satisfying necessary optimality conditions has been obtained in that paper, and it is shown that the proposed control strategy improves the performance of nonlinear chemical reactions in comparison to the steady-state operation. It has been noted by several authors (see, e.g., [12] and references therein) that the periodic operation of chemical reactions has a rich potential for applications in chemical engineering, and the performance of periodic controllers has been validated experimentally [13].

In [14] a procedure for evaluating periodic trajectories with switchings has been proposed based on the Chen–Fliess expansion of periodic solutions corresponding to bang-bang control inputs. This procedure gives attractive algebraic relations of the initial data and the switching times in case of small periods; however, to the best of our knowledge, the construction of periodic trajectories of arbitrary periods remains open for nonlinear systems with switching controls. The present work aims to fill this gap by proposing a general approach for characterizing the existence of solutions and an iterative computation scheme for periodic boundary value problems with switchings under nonlinear perturbations. Our study extends the methodology developed in [15] for boundary value problems with impulses at given time instants τ_j. In contrast to previous publications on periodic boundary value problems for nonlinear autonomous systems, dealing with parameter-dependent periods (and thus parameter-dependent intervals $[a, b(\varepsilon)]$, see [16] and references therein), we assume the endpoint b to be fixed.

The rest of this paper is organized as follows. The periodic boundary value problem is formulated in Sect. 2 for a class of systems of ordinary differential equations in $\mathbb{R}^n$ with nonlinear and discontinuous perturbations of the right-hand side depending on a small parameter. The main theoretical contribution is summarized in Sect. 3 in the form of necessary solvability conditions (Lemma 1) and an iterative scheme for the approximation of solutions (Theorem 1). These results are applied to a nonlinear chemical reaction model in Sect. 4 to justify possible computational benefits of the developed iterative scheme.

2 Problem statement

Consider the nonlinear system of ordinary differential equations

$$\frac{dz}{dt} = Az + \varepsilon h(\varepsilon) + \varepsilon Z(z,\varepsilon), \quad t \in [a,b], \tag{1}$$

under the periodic boundary condition

$$\ell_0 z(\cdot,\varepsilon) := z(a,\varepsilon) - z(b,\varepsilon) = 0, \tag{2}$$

where $z = z(t,\varepsilon) \in \mathbb{R}^n$ depends on time t and the small parameter $\varepsilon \in [0,\varepsilon_0]$, A is a constant $n \times n$ matrix, $h : [0,\varepsilon_0] \to \mathbb{R}^n$ and $Z : \mathbb{R}^n \times [0,\varepsilon_0] \to \mathbb{R}^n$ are continuous functions, and, moreover, $Z(z,\varepsilon)$ is continuously differentiable with respect to z for each fixed $\varepsilon \in [0,\varepsilon_0]$. Throughout the text, we treat the norm of a vector $z = (z_1,\dots,z_n)^T \in \mathbb{R}^n$ in the sense of the maximum norm, $\|z\|_\infty := \max_{1\le i\le n} |z_i|$. The latter induces the norm of an $m \times n$ matrix $Q = (q_{ij})$ as $\|Q\|_\infty := \max_{1\le i\le m} \sum_{j=1}^n |q_{ij}|$. For a continuous vector function $\zeta : [a,b] \to \mathbb{R}^n$, the norm is defined as $\|\zeta\|_{C([a,b];\mathbb{R}^n)} := \max_{t\in[a,b]} \|\zeta(t)\|_\infty$, and the norm of a continuous matrix-valued function $M : [a,b] \to \mathbb{R}^{m\times n}$ is $\|M\|_{C([a,b];\mathbb{R}^{m\times n})} := \max_{t\in[a,b]} \|M(t)\|_\infty$.

Let us first analyze the solvability of the boundary value problem (1)–(2) in a small neighborhood of a solution $z_0(t)$ of the generating linear problem

$$\frac{dz_0}{dt} = Az_0, \qquad \ell_0 z_0(\cdot) := z_0(a) - z_0(b) = 0. \tag{3}$$

Let us denote by $X(t) = e^{(t-a)A}$ the fundamental matrix of (3) and consider the *noncritical case*, i.e.,

$$\det Q_0 \neq 0, \qquad Q_0 := \ell_0 X(\cdot) = X(a) - X(b).$$

In this case, problem (3) admits only the trivial solution, so that all solutions of the inhomogeneous periodic boundary value problem (1)–(2) are equilibrium points:

$$z(t,\varepsilon) := \tilde{z}(\varepsilon), \qquad A\tilde{z}(\varepsilon) + \varepsilon h(\varepsilon) + \varepsilon Z\bigl(\tilde{z}(\varepsilon),\varepsilon\bigr) = 0.$$

The existence of such equilibria $\tilde{z}(\varepsilon)$ for $0 \le \varepsilon \le \varepsilon^*$ with some small enough $\varepsilon^* \in (0,\varepsilon_0]$ follows from the implicit function theorem for

$$\Phi(\tilde{z},\varepsilon) := A\tilde{z} + \varepsilon h(\varepsilon) + \varepsilon Z(\tilde{z},\varepsilon) = 0$$

and the conditions $\Phi(0,0) = 0$ and $\det \Phi'_{\tilde{z}}(0,0) \neq 0$.

Starting from this observation, we pose the question about the existence of nonequilibrium solution of the periodic boundary value problem under a time-varying discontinuous perturbation of the right-hand side of (1). To be more precise, we introduce a partition of $[a,b]$

$$a = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_p < \tau_{p+1} = b$$

and consider a switching scenario

$$f(t,\varepsilon) := \begin{cases} \mu_0(\varepsilon), & t \in [a,\tau_1[, \\ \ldots\ldots & \ldots\ldots\ldots, \\ \mu_p(\varepsilon), & t \in [\tau_p, b], \end{cases} \tag{4}$$

where the functions $\mu_k : [0,\varepsilon_0] \to \mathbb{R}^n$, $k = 0,1,\ldots,p$, are continuous.

We treat $f(t,\varepsilon)$ as the disturbance in the boundary value problem and rewrite the resulting differential equation in the following way:

$$\frac{dz}{dt} = Az + \varepsilon f(t,\varepsilon) + \varepsilon Z(z,\varepsilon), \quad t \in [a,b]. \tag{5}$$

As it follows from the literature review, the above-introduced class of switching functions $f(t,\varepsilon)$ has a straightforward relation to bang-bang controls in optimal control problems. In this paper, we aim to develop efficient tools for the analysis of such type problems. Note that the solutions of differential equation (5) with discontinuous right-hand side can be treated in the sense of Carathéodory (see [17, Chap. 1]); however, due to well-developed techniques in the theory of boundary value problems with continuous right-hand sides [18], we will "glue" piecewise-differentiable periodic solutions $z(t,\varepsilon)$ by imposing the following set of boundary and interface conditions:

$$\ell z(\cdot,\varepsilon) := \begin{pmatrix} z(a,\varepsilon) - z(b,\varepsilon) \\ z(\tau_1 + 0,\varepsilon) - z(\tau_1 - 0,\varepsilon) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ z(\tau_p + 0,\varepsilon) - z(\tau_p - 0,\varepsilon) \end{pmatrix} = 0. \tag{6}$$

Thus by *a solution of the periodic boundary value problem* (5)–(6) we mean a function $z : [a,b] \times [0,\varepsilon_0] \to \mathbb{R}^n$ such that, for each fixed $\varepsilon \in [0,\varepsilon_0]$, $z(t,\varepsilon)$ satisfies (5) on each interval (τ_j, τ_{j+1}), $j = 0,1,\ldots,p$, and $\ell z(\cdot,\varepsilon) = 0$. In the subsequent study, we will focus on the solutions $z(t,\varepsilon)$ that are continuous in $\varepsilon \in [0,\varepsilon_0]$ at each fixed $t \in [a,b]$.

The main problem under consideration in this paper is formulated as follows: *describe solvability conditions of the nonlinear periodic boundary value problem* (5)–(6) *and develop an iterative scheme for computing its solutions.*

3 Solutions existence

Let us introduce the matrix

$$Q := \begin{pmatrix} Q_0 \\ O_n \\ \ldots \\ O_n \end{pmatrix} \in \mathbb{R}^{n(p+1)\times n}$$

and orthogonal projection matrices [18]

$$P_Q : \mathbb{R}^n \to \operatorname{Ker} Q \quad \text{and} \quad P_{Q^*} : \mathbb{R}^{n(p+1)} \to \operatorname{Ker} Q^*,$$

where

$$P_{Q^*} = \begin{pmatrix} O_n & O_{n\times d} \\ O_{d\times n} & I_d \end{pmatrix}, \qquad P_{Q_d^*} = \begin{pmatrix} O_{d\times n} & I_d \end{pmatrix}, \qquad P_Q = O_n, \qquad d := np,$$

I_d is the identity matrix of size $d \times d$, O_n and $O_{n\times d}$ are zero matrices of sizes $n \times n$ and $n \times d$, respectively. As $P_{Q^*} \neq 0$, the perturbed nonlinear boundary value problem (5)–(6) exhibits a critical case. We introduce the Cauchy-type operator

$$K\bigl[g(\cdot)\bigr](t) := \begin{cases} X(t)\int_a^t X^{-1}(s)g(s)\,ds, & t \in [a, \tau_1[, \\ X(t)\int_{\tau_1}^t X^{-1}(s)g(s)\,ds, & t \in [\tau_1, \tau_2[, \\ \dots\dots\dots\dots\dots\dots\dots & \dots\dots\dots, \\ X(t)\int_{\tau_p}^t X^{-1}(s)g(s)\,ds, & t \in [\tau_p, b], \end{cases} \tag{7}$$

for the initial value problems

$$dz/dt = Az + g(t), \quad t \in [\tau_i, \tau_{i+1}[, \qquad z(\tau_i) = 0, \quad i = 0, 1, 2, \dots, p,$$

and define the generalized Green's operator [18, 19]

$$G\bigl[g(\cdot)\bigr](t) = K\bigl[g(\cdot)\bigr](t) - X(t)Q^+\ell K\bigl[g(\cdot)\bigr]$$

for the linear periodic problem

$$dz/dt = Az + g(t), \qquad z(a) - z(b) = 0.$$

Here Q^+ is the Moore–Penrose pseudoinverse matrix [18, 20].

Let $z(t,\varepsilon)$ satisfy the differential equation (5) on each interval (τ_j, τ_{j+1}), $j = 0, \dots, p$, for $\varepsilon \in [0, \varepsilon_0]$. Then by applying the functional (6) to z we obtain the following necessary and sufficient condition for z to be a solution of the boundary value problem (5)–(6):

$$P_{Q_d^*}\ell K\bigl[f(\cdot,\varepsilon) + Z\bigl(z(\cdot,\varepsilon),\varepsilon\bigr)\bigr] = 0. \tag{8}$$

Recall that the values of $f(t,\varepsilon)$ in (4) are defined in terms of the vector function

$$\lambda(\varepsilon) := \begin{pmatrix} \mu_0(\varepsilon) \\ \cdots \\ \mu_p(\varepsilon) \end{pmatrix} \in \mathbb{R}^{n(p+1)}$$

and define $\lambda_0 := \lambda(0)$, $f_0(t) := f(t,0)$, and

$$F(\lambda_0) := P_{Q_d^*}\ell K\bigl[f_0(\cdot) + Z(0,0)\bigr]. \tag{9}$$

As we look for the solutions $z(t,\varepsilon)$ of (5)–(6) continuous with respect to ε, by passing to the limit as $\varepsilon \to 0$ in (8), we obtain $F(\lambda_0) = 0$. Thus we have proved the following statement.

Lemma 1 *Let* $\det Q_0 \neq 0$, *and let the perturbed nonlinear boundary value problem* (5)–(6) *with* f *given by* (4) *admit a solution* $z \in C([a,b] \times [0,\varepsilon_0])$ *such that* $z(\cdot,\varepsilon) \in C^1([a,b] \setminus \{\tau_1,\tau_2,\dots,\tau_p\})$ *for each* $\varepsilon \in [0,\varepsilon_0]$. *Then*

$$F(\lambda_0) = 0, \tag{10}$$

where $F(\lambda_0)$ *is given by* (9), *and*

$$\lambda_0 = \begin{pmatrix} \mu_0(0) \\ \vdots \\ \mu_p(0) \end{pmatrix}.$$

Similarly to the weakly nonlinear periodic problems in critical cases [18], we call (10) the *equation for the generating constants* for problem (5)–(6). We further assume that (10) does not turn into an identity and has real roots. By fixing a solution $\lambda_0 \in \mathbb{R}^{n(p+1)}$ of (10) we can define the first approximation of a solution of (5)–(6):

$$z_1(t,\varepsilon) = \varepsilon G\big[f_0(\cdot) + Z(0,0)\big](t).$$

The obtained solution $\lambda_0 \in \mathbb{R}^{n(p+1)}$ of equation (10) as well as the first approximation $z_1(t,\varepsilon)$ of a solution of the original boundary value problem (5)–(6) are analogous to the generating solution of a regular periodic boundary value problem in the critical case [18], in a small neighborhood of which the solutions of the original boundary value problem may exist.

By formal substitution of the identity matrix I_n in place of $g(s)$ in (7) we adopt the notation $K[I_n](t)$, $t \in [a,b]$, for the $n \times n$ matrix obtained from (7). Then we introduce the constant matrix

$$C_0 := P_{Q_d^*}\ell K[I_n] \in \mathbb{R}^{d\times n(p+1)}$$

and orthogonal projection matrices [18, 19]

$$P_{C_0} : \mathbb{R}^{n(p+1)} \to \operatorname{Ker} C_0, \quad \text{and} \quad P_{C_0^*} : \mathbb{R}^d \to \operatorname{Ker} C_0^*.$$

In the considered case, the solvability condition (8) leads to the equation

$$C_0\lambda(\varepsilon) = -P_{Q_d^*}\ell K\big[Z\big(z(\cdot,\varepsilon),\varepsilon\big)\big],$$

which is solvable iff

$$P_{C_0^*}P_{Q_d^*} = 0. \tag{11}$$

Thus, under condition (11), the perturbed boundary value problem (5)–(6) has at least one solution represented by the operator system

$$z(t,\varepsilon) = \varepsilon G\big[f(\cdot,\varepsilon) + Z\big(z(\cdot,\varepsilon),\varepsilon\big)\big](t), \qquad \lambda(\varepsilon) = -C_0^+P_{Q_d^*}\ell K\big[Z\big(z(\cdot,\varepsilon),\varepsilon\big)\big]. \tag{12}$$

We denote the vectors

$$\lambda_k(\varepsilon) := \begin{pmatrix} \mu_0^{(k)}(\varepsilon) \\ \cdots\cdots \\ \mu_p^{(k)}(\varepsilon) \end{pmatrix} \in \mathbb{R}^{n(p+1)}, \qquad f_k(t,\varepsilon) := \begin{cases} \mu_0^{(k)}(\varepsilon), & t \in [a,\tau_1[, \\ \cdots\cdots & \cdots\cdots\cdots, \\ \mu_p^{(k)}(\varepsilon), & t \in [\tau_p, b], \end{cases} \qquad k = 0,1,\ldots,$$

and apply the simple-iteration method [18] for constructing an approximate solution of (12) under condition (11). We summarize the result as follows.

Theorem 1 *Let* $\det Q_0 \neq 0$, *let* $\lambda_0 \in \mathbb{R}^{n(p+1)}$ *be a solution of* (10) *under condition* (11), *and let the Jacobian matrix* $\frac{\partial Z(z,\varepsilon)}{\partial z}$ *be bounded for all* z *and small enough* $\varepsilon > 0$. *Then, for some small enough* $\varepsilon_* > 0$, *there exist a function* $\lambda \in C([0,\varepsilon_*])$, $\lambda(0) = \lambda_0$, *and a solution* $z \in C([a,b] \times [0,\varepsilon_*])$, $z(\cdot,\varepsilon) \in C^1([a,b] \setminus \{\tau_1,\ldots,\tau_p\})$ *of the boundary value problem* (5)–(6) *with* f *given by* (4) *in which* $\mu_0,\ldots,\mu_p$ *are the corresponding components of* λ. *The solution* $z(t,\varepsilon)$ *is defined by the operator system* (12) *and can be obtained as the limit of the following iterative scheme with* $\varepsilon \in [0,\varepsilon_*]$:

$$\begin{aligned} z_{k+1}(t,\varepsilon) &= \varepsilon G\big[f_k(\cdot,\varepsilon) + Z\big(z_k(\cdot,\varepsilon),\varepsilon\big)\big](t), \\ \lambda_{k+1}(\varepsilon) &= -C_0^+ P_{Q_d^*} \ell K\big[Z\big(z_k(\cdot,\varepsilon),\varepsilon\big)\big], \quad k = 0,1,2,\ldots. \end{aligned} \tag{13}$$

Proof The idea of the proof is analogous to that in [18, 19, 21]. Let us define the operator $\Phi_\varepsilon : C([a,b];\mathbb{R}^n) \to C([a,b];\mathbb{R}^n)$ acting on a vector function $\zeta(t)$ by the following rule:

$$\Phi_\varepsilon[\zeta] := \varepsilon G\big[f_\lambda(\cdot,\varepsilon) + Z\big(\zeta(\cdot),\varepsilon\big)\big], \tag{14}$$

where $f_\lambda(t,\varepsilon)$ is defined by (4) with $(\mu_0,\ldots,\mu_p)^T = \lambda = -C_0^+ P_{Q_d^*} \ell K[Z(\zeta(\cdot),\varepsilon)]$. The above Φ_ε is well-defined. Indeed, as we can see, the function $\Phi_\varepsilon[\zeta(\cdot)](t)$, $t \in [a,b]$ is continuous if $\zeta(t)$ is continuous, provided that (11) holds. Because of the linearity of G, the differential of $\Phi_\varepsilon[\zeta]$ in the direction $\delta\zeta$ can be written as

$$\delta\Phi_\varepsilon[\zeta](\delta\zeta) = \varepsilon G\big[\delta f_\lambda(\cdot,\varepsilon)\big] + \varepsilon G\big[g(\cdot)\big], \tag{15}$$

where

$$\delta f_\lambda(\cdot,\varepsilon) = \begin{cases} \delta\mu_0, & t \in [a,\tau_1), \\ \cdots\cdots & \cdots\cdots\cdots, \\ \delta\mu_p, & t \in [\tau_p, b], \end{cases}$$

$(\delta\mu_0,\ldots,\delta\mu_p)^T = \delta\lambda$, $\delta\lambda = -C_0^+ P_{Q_d^*} \ell K[g(\cdot)]$, $g(t) = \frac{\partial Z(z,\varepsilon)}{\partial z}|_{z=\zeta(t)}\delta\zeta(t)$, and $\frac{\partial Z(z,\varepsilon)}{\partial z}$ is the Jacobian matrix of $Z(z,\varepsilon)$ with respect to z. As this Jacobian matrix is bounded, the Fréchet derivative $D\Phi_\varepsilon[\zeta]$ at ζ exists, and its operator norm is defined by

$$\big\|D\Phi_\varepsilon[\zeta]\big\| = \sup_{\delta\zeta\neq 0} \frac{\|\delta\Phi_\varepsilon[\zeta](\delta\zeta)\|_{C([a,b];\mathbb{R}^n)}}{\|\delta\zeta\|_{C([a,b];\mathbb{R}^n)}}.$$

Moreover, as the right-hand side of (15) contains the factor ε, there exist $\lambda_* \in (0,1)$ and $\varepsilon_* \in (0,\varepsilon_*]$ such that representation (15) guarantees the contraction property

$$\left\| D\Phi_\varepsilon[\zeta] \right\| \le \lambda_* < 1 \quad \text{for all } \varepsilon \in [0,\varepsilon_*]. \tag{16}$$

The Banach fixed-point theorem implies the existence of a unique fixed point $z(\cdot,\varepsilon)$ of Φ_ε, i.e., $\Phi_\varepsilon[z(\cdot,\varepsilon)] = z(\cdot,\varepsilon)$ for each $\varepsilon \in [0,\varepsilon_*]$. Because of the definition of Φ_ε in (14), the above function $z(t,\varepsilon)$ can be obtained as the limit of the iterative process (13) in the case $\det Q_0 \neq 0$. □

4 Application to a nonlinear chemical reaction

As an application of the proposed theoretical framework, we consider an example of non-isothermal chemical reaction with τ-periodic controls presented in [11, 14]. The model is described by a boundary value problem for nonlinear differential equations with respect to the reactant concentration and the temperature, which, after a suitable rescaling in the case $\tau = 2$, can be represented as follows:

$$\frac{dz}{dt} = Az + \varepsilon f(t,\varepsilon) + \varepsilon Z(z,\varepsilon), \qquad \ell z(\cdot,\varepsilon) := z(0,\varepsilon) - z(2,\varepsilon) = 0, \tag{17}$$

where

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad Z(z,\varepsilon) = (1+x)e^{-\frac{\varepsilon}{1+y}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Here A is a constant 2×2 matrix whose eigenvalues λ_a and λ_b are distinct negative real numbers, and the vector function $f(t,\varepsilon)$ corresponds to switching controls. We assume that all variables in (17) are dimensionless, and for simplicity, the kinetic parameters in $Z(z,\varepsilon)$ are taken to be 1. For simplicity, we also assume that the matrix A is diagonal, and for further computations we take $\lambda_a = -1$ and $\lambda_b = -2$.

In the considered case the linear homogeneous problem (generating problem (3)) admits only the trivial solution $z_0(t) \equiv 0$ because the matrix

$$Q_0 = \begin{pmatrix} 1 - \frac{1}{e^2} & 0 \\ 0 & 1 - \frac{1}{e^4} \end{pmatrix}$$

is nonsingular. Moreover, as it follows from the consideration in Sect. 2 for a time-invariant term $f(t,\varepsilon) \equiv h(\varepsilon)$, problem (17) has only equilibrium solutions. In particular, if

$$h(\varepsilon) = \begin{pmatrix} 1+\varepsilon \\ 1-\varepsilon \end{pmatrix},$$

then the solutions of (17) can be represented as follows:

$$x(t,\varepsilon) = 2\varepsilon + 2\varepsilon^2 + \frac{3\varepsilon^3}{2} + \frac{\varepsilon^4}{3} + \frac{31\varepsilon^5}{24} + \frac{\varepsilon^6}{5} + \frac{337\varepsilon^7}{180} - \frac{2801\varepsilon^8}{5040} + \cdots,$$
$$y(t,\varepsilon) = \varepsilon + \frac{3\varepsilon^3}{4} + \frac{\varepsilon^4}{6} + \frac{31\varepsilon^5}{48} + \frac{\varepsilon^6}{10} + \frac{337\varepsilon^7}{360} - \frac{2801\varepsilon^8}{10{,}080} + \cdots.$$

The asymptotic expansions and numerical simulations in this paper have been carried out using Wolfram Mathematica 8.

It is natural to address the question of existence of nonequilibrium solutions under switching perturbations $f(t,\varepsilon)$. For this purpose, we take

$$f(t,\varepsilon) = \begin{cases} \mu_0(\varepsilon), & t \in [0,\tau_1), \\ \mu_1(\varepsilon), & t \in [\tau_1, 2], \tau_1 = 1, \end{cases} \tag{18}$$

and investigate the solvability of the boundary value problem (17) in the class of functions $z(t,\varepsilon)$ such that $z \in C([0,2]\times[0,\varepsilon])$, $z(\cdot,\varepsilon) \in C^1([0,2]\setminus\{1\})$. According to the methodology of Sect. 3, we compute the following matrices:

$$Q := \begin{pmatrix} Q_0 \\ O_2 \end{pmatrix}, \qquad P_{Q^*} = \begin{pmatrix} O & O \\ O & I_2 \end{pmatrix} \neq 0, \qquad P_{Q_d^*} = \begin{pmatrix} O & I_2 \end{pmatrix}, \qquad P_Q = O.$$

Equation (10) is not a trivial identity and has a real solution

$$\lambda_0 = \frac{1}{10}\begin{pmatrix} -10 \\ -10 \\ 1 \\ 0 \end{pmatrix},$$

which defines the matrix

$$C_0 = \frac{1}{e}\begin{pmatrix} e-1 & 0 & 0 & 0 \\ 0 & \sinh 1 & 0 & 0 \end{pmatrix}.$$

Since condition (11) is satisfied, the perturbed boundary value problem (17) has at least one solution for each $\varepsilon > 0$ small enough and some switching signal $f(\cdot,\varepsilon)$. Let us denote the vectors

$$\mu_0^{(1)}(\varepsilon) := \begin{pmatrix} \mu_0^{(1a)}(\varepsilon) \\ \mu_0^{(1b)}(\varepsilon) \end{pmatrix}, \qquad \mu_1^{(1)}(\varepsilon) := \begin{pmatrix} \mu_1^{(1a)}(\varepsilon) \\ \mu_1^{(1b)}(\varepsilon) \end{pmatrix}.$$

The iterative scheme (13) determines the first approximation of a solution of (17) in a neighborhood of $z_0 = 0$:

$$x_1(t,\varepsilon) = \frac{11\varepsilon e^{1-t}}{10(1+e)}, \qquad y_1(t,\varepsilon) = \frac{\varepsilon e^{2(1-t)}}{2(1+e^2)}, \quad t \in [0,1],$$
$$x_1(t,\varepsilon) = \frac{11\varepsilon(1+e-e^{2-t})}{10(1+e)}, \qquad y_1(t,\varepsilon) = \frac{(1+e^2-e^{4-2t})\varepsilon}{2(1+e^2)}, \quad t \in [1,2].$$

In addition, the iterative scheme (13) defines a first approximation to the components of (18):

$$\mu_0^{(1a)}(\varepsilon) \approx -1 + \frac{(-10-11e+10e^2)\varepsilon}{10(-1+e)(1+e)} + \frac{(5+16e+6e^3-5e^4)\varepsilon^2}{10(-1+e)(1+e)(1+e^2)} + \frac{(-20-103e-30e^2-132e^3+30e^4-29e^5+20e^6)\varepsilon^3}{120(-1+e)(1+e)(1+e^2)^2},$$

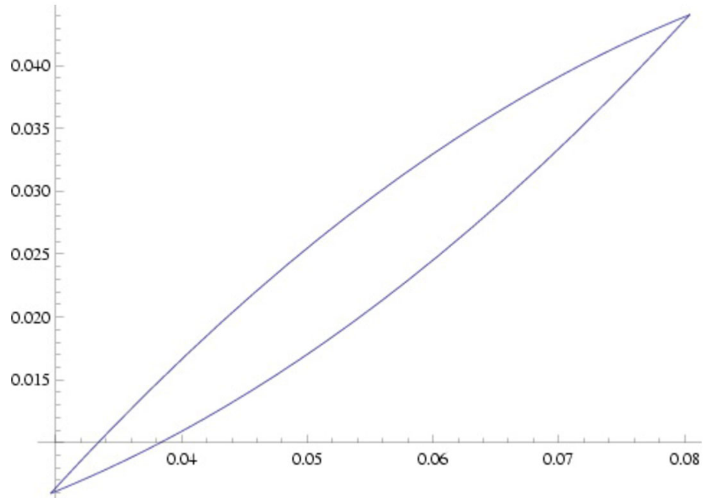

Figure 1 Phase portrait of an approximate solution of (17) in the *xy*-plane

$$\begin{aligned}
\mu_0^{(1b)}(\varepsilon) \approx{}& -\frac{(-1+e)(1+e)\cosh 1}{2e} + \frac{\varepsilon(-1+e)(5-e+5e^2)\cosh 1}{10e(1+e)} \\
&+ \frac{\varepsilon^2(5-17e+12e^2-32e^3+17e^4-5e^5)\cosh 1}{20e(1+e)(1+e^2)} \\
&+ \frac{\varepsilon^3(-10+56e+35e^2+101e^3+19e^4+85e^5-56e^6+10e^7)\cosh 1}{(120e(1+e)(1+e^2))^2} + \cdots, \\
\mu_1^{(1a)}(\varepsilon) ={}& \mu_1^{(0a)}(\varepsilon), \qquad \mu_1^{(1b)}(\varepsilon) = \mu_1^{(0b)}(\varepsilon).
\end{aligned}$$

The corresponding first approximation of the trajectory of the boundary value problem (17) is illustrated by Fig. 1.

Since $Z(z,\varepsilon)$ is continuously differentiable with respect to z in a neighborhood of $z_0 = 0$, we are able to check the contraction condition (16). In this case,

$$\begin{aligned}
\frac{\partial Z(z,\varepsilon)}{\partial z}\bigg|_{z=z_0(t,\varepsilon)} &\approx \begin{pmatrix} e^{-1-t-\varepsilon}(-1+e^{1+t}) & e^{-1-t-\varepsilon}(-1+e^{1+t})\varepsilon \\ \frac{e^{-2-2t-\varepsilon}}{4}(-1-e^2+2e^{2+2t}) & \frac{e^{-2-2t-\varepsilon}}{4}(-1-e^2+2e^{2+2t})\varepsilon \end{pmatrix}, \quad t<1, \\
\frac{\partial Z(z,\varepsilon)}{\partial z}\bigg|_{z=z_0(t,\varepsilon)} &\approx \begin{pmatrix} e^{-1-2t-\varepsilon}(e^2-e^t-e^{2+t}+e^{1+2t}) & e^{-1-2t-\varepsilon}(e^2-e^t-e^{2+t}+e^{1+2t})\varepsilon \\ \frac{e^{-2-4t-\varepsilon}}{4}(e^4-e^{2t}-e^{4+2t}+e^{2+4t}) & \frac{e^{-2-4t-\varepsilon}}{4}(e^4-e^{2t}-e^{4+2t}+e^{2+4t})\varepsilon \end{pmatrix}, \\
&\quad t\in[1,2], \\
\frac{\partial Z(z,\varepsilon)}{\partial z}\bigg|_{z=z_1(t,\varepsilon)} &\approx \begin{pmatrix} -\frac{e^{-t}(-1+e^t+e^{1+t})(-1+\varepsilon)}{1+e} & \frac{e^{-t}(-1+e^t+e^{1+t})\varepsilon}{1+e} \\ -\frac{e^{-2t}(-1+e^{2t}+e^{2+2t})(-1+\varepsilon)}{2(1+e^2)} & \frac{(1+e^2-e^{-2t})\varepsilon}{2(1+e^2)} \end{pmatrix}, \quad t<1, \\
\frac{\partial Z(z,\varepsilon)}{\partial z}\bigg|_{z=z_1(t,\varepsilon)} &\approx \begin{pmatrix} -\frac{e^{-t}(-e^2+e^t+e^{1+t})(-1+\varepsilon)}{1+e} & \frac{(1+e-e^{2-t})\varepsilon}{1+e} \\ -\frac{e^{-2t}(-e^4+e^{2t}+e^{2+2t})(-1+\varepsilon)}{2(1+e^2)} & \frac{(1+e^2-e^{4-2t})\varepsilon}{2(1+e^2)} \end{pmatrix}, \quad t\in[1,2].
\end{aligned}$$

The contraction condition (16) holds for $\varepsilon \in [0,\varepsilon_*]$; the value of $\varepsilon_* > 0$ for which the iterative scheme (13) is applicable can be found numerically. For the considered problem, practical convergence is preserved up to $\varepsilon_* \approx 1.09$:

$$\begin{aligned}
&\bigl\| D\Phi_\varepsilon\bigl[z_0(\cdot,\varepsilon)\bigr]\bigr\| \le 0.00864622 < 1, \quad \varepsilon \in [0, 0.01],\\
&\bigl\| D\Phi_\varepsilon\bigl[z_1(\cdot,\varepsilon)\bigr]\bigr\| \le 0.00901042 < 1, \quad \varepsilon \in [0, 0.01],\\
&\bigl\| D\Phi_\varepsilon\bigl[z_0(\cdot,\varepsilon)\bigr]\bigr\| \le 0.0629163 < 1, \quad \varepsilon \in [0, 0.1],\\
&\bigl\| D\Phi_\varepsilon\bigl[z_1(\cdot,\varepsilon)\bigr]\bigr\| \le 0.0730063 < 1, \quad \varepsilon \in [0, 0.1],\\
&\bigl\| D\Phi_\varepsilon\bigl[z_0(\cdot,\varepsilon)\bigr]\bigr\| \le 0.662227 < 1, \quad \varepsilon \in [0, 1.09],\\
&\bigl\| D\Phi_\varepsilon\bigl[z_1(\cdot,\varepsilon)\bigr]\bigr\| \le 0.983498 < 1, \quad \varepsilon \in [0, 1.09].
\end{aligned}$$

The computed approximations of the solution to the periodic boundary value problem (17) are characterized by the discrepancies

$$\Delta_k(\varepsilon) = \max_{t\in[0,2]} \bigl\| \dot{z}_k(t,\varepsilon) - Az_k(t,\varepsilon) - \varepsilon f_k(t,\varepsilon) - \varepsilon Z\bigl(z_k(t,\varepsilon),\varepsilon\bigr)\bigr\|_\infty, \quad k = 0, 1.$$

In particular, we have

$$\begin{aligned}
&\Delta_0(0.1) \approx 0.148661, \qquad \Delta_1(0.1) \approx 0.0478\,012,\\
&\Delta_0(0.01) \approx 0.0148661, \qquad \Delta_1(0.01) \approx 0.0044412.
\end{aligned}$$

5 Conclusion and future work

The above simulation results confirm that the proposed iterative scheme can be used for approximating periodic solutions of the perturbed boundary value problem (17) with acceptable accuracy. Note that the main contribution of this paper (Theorem 1) allows constructing approximate solutions of the nonlinear problem (5)–(6) in case of discontinuous perturbations $f(t,\varepsilon)$ with an arbitrary number of switchings. Although the applicability of our theoretical framework has been illustrated with a two-dimensional model of controlled chemical reaction, the efficiency of this approach for higher-dimensional systems with complicated switching scenario is considered as a topic for further numerical analysis.

Acknowledgements
Not applicable.

Funding
The third author was supported by the German Research Foundation (DFG) under Grant ZU 359/2-1. Open Access funding enabled and organized by Projekt DEAL.

Availability of data and materials
The datasets used and analyzed during the current study are available from the second author on reasonable request.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
All authors contributed to the study conception and technical content. The first draft of the manuscript was written by Sergey Chuiko and Alexander Zuyev, and all authors commented on further versions of the manuscript. All authors have read and approved the final manuscript.

Author details
[1]Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg, Germany. [2]Otto von Guericke University Magdeburg, Magdeburg, Germany. [3]Donbass State Pedagogical University, Sloviansk, Ukraine. [4]Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, Sloviansk, Ukraine.

Received: 4 November 2022 Accepted: 16 April 2023 Published online: 27 April 2023

References

1. Malkin, I.G.: Some Problems in the Theory of Nonlinear Oscillations. US Atomic Energy Commission, Technical Information Service, Maryland (1959)
2. Vejvoda, O.: On perturbed nonlinear boundary value problems. Czechoslov. Math. J. **11**(3), 323–364 (1961)
3. Nayfeh, A.H., Mook, D.T.: Nonlinear Oscillations. Wiley, Weinheim (1995)
4. Chu, J., Torres, P.J., Zhang, M.: Periodic solutions of second order non-autonomous singular dynamical systems. J. Differ. Equ. **239**(1), 196–212 (2007)
5. Chu, J., Sun, J., Wong, P.J.: Existence for singular periodic problems: a survey of recent results. Abstr. Appl. Anal. **2013**, Article ID 420835 (2013)
6. Graef, J.R., Kong, L., Wang, H.: Existence, multiplicity, and dependence on a parameter for a periodic boundary value problem. J. Differ. Equ. **245**(5), 1185–1197 (2008)
7. Ma, R., Xu, J., Han, X.: Global structure of positive solutions for superlinear second-order periodic boundary value problems. Appl. Math. Comput. **218**(10), 5982–5988 (2012)
8. Drábek, P., Langerová, M.: First order periodic problem at resonance with nonlinear impulses. Bound. Value Probl. **2014**(1), 186 (2014)
9. Ning, Y., An, T.: Periodic solutions of a class of nonautonomous second-order Hamiltonian systems with nonsmooth potentials. Bound. Value Probl. **2015**(1), 34 (2015)
10. Candito, P., Livrea, R.: Existence results for periodic boundary value problems with a convection term. In: International Conference on Differential & Difference Equations and Applications, pp. 593–602. Springer, Berlin (2019)
11. Zuyev, A., Seidel-Morgenstern, A., Benner, P.: An isoperimetric optimal control problem for a non-isothermal chemical reactor with periodic inputs. Chem. Eng. Sci. **161**, 206–214 (2017)
12. Silveston, P.L., Hudgins, R.R.: Periodic Operation of Chemical Reactors. Butterworth–Heinemann, Oxford (2013)
13. Felischak, M., Kaps, L., Hamel, C., Nikolic, D., Petkovska, M., Seidel-Morgenstern, A.: Analysis and experimental demonstration of forced periodic operation of an adiabatic stirred tank reactor: simultaneous modulation of inlet concentration and total flow-rate. Chem. Eng. J. **410**, 128197 (2021)
14. Benner, P., Seidel-Morgenstern, A., Zuyev, A.: Periodic switching strategies for an isoperimetric control problem with application to nonlinear chemical reactions. Appl. Math. Model. **69**, 287–300 (2019)
15. Boichuk, A., Chuiko, S.: Generalized Green operator for an impulsive boundary-value problem with switchings. Nonlinear Oscil. **10**(1), 46–61 (2007)
16. Aisagaliev, S.: On periodic solutions of autonomous systems. J. Math. Sci. **229**(4), 335–353 (2018)
17. Filippov, A.F.: Differential Equations with Discontinuous Righthand Sides: Control Systems. Springer, Dordrecht (1988)
18. Boichuk, A.A., Samoilenko, A.M.: Generalized Inverse Operators and Fredholm Boundary-Value Problems. de Gruyter, Berlin (2016)
19. Boichuk, A., Chuiko, S.: On approximate solutions of nonlinear boundary-value problems by the Newton–Kantorovich method. J. Math. Sci. **258**(5), 594–617 (2021)
20. Campbell, S.L., Meyer, C.D.: Generalized Inverses of Linear Transformations. SIAM, Philadelphia (2009)
21. Chuiko, S.: Nonlinear Noetherian boundary-value problem in the case of parametric resonance. J. Math. Sci. **205**(6), 859–870 (2015)

Publisher's Note