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Solvability and Volterra property of nonlocal problems for mixed fractional-order diffusion-wave equation

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Abstract

The paper is devoted to the study of one class of problems with nonlocal conditions for a mixed diffusion-wave equation with two independent variables. The main results of the work are the proof of regular and strong solvability, as well as the Volterra property of three problems with conditions pointwise connecting the values of the tangent derivative of the desired solution on one of the characteristics with derivatives in various directions of the solution on an arbitrary curve lying inside the characteristic triangle for a fractional-order diffusion-hyperbolic equation.

Keywords: Diffusion-wave equation; Nonlocal conditions; Solvability; Volterra property; Fractional-order operator

1 Introduction

In recent years, there has been an increased interest in the study of fractional differential equations, in which an unknown function is contained under the sign of a fractional derivative. This is due to the development of fractional integration theory and differentiation itself, as well as applications in various fields of science: physics, mechanics, chemistry, engineering, anomalous diffusion processes, and other areas of natural science.

Since the fractional-order equations generalize the integer-order equations, and there are a relatively small number of systematized analytical and numerical methods for such equations, this direction is the priority of the general theory of differential equations.

The first fundamental studies in the theory of fractional calculus are works of B. Riemann, J. Liouville, Hj. Holmgren, A.V. Letnikov, A. Grünwald, H. Weyl, M.M. Djrbashian, A.B. Nersesyan, etc. After solving a number of local problems for fractional-order equations with various integro-differentiation operators of one argument, interest in the study of partial differential equations of fractional order has increased. In this direction, we refer to [2, 3, 9, 10, 14, 15, 17, 18, 21–23, 25].

The solvability issues of local and nonlocal problems for various fractional-order mixed-type equations are considered in [1, 4, 11, 16, 20].

As far as we know, the spectral properties, including the Volterra property of the mixed fractional equations, are almost not studied.

Note that the solvability issues and spectral properties of local and nonlocal problems for a mixed parabolic–hyperbolic equation of the second and third orders are studied in [5–8, 12, 13, 19].

This work is devoted to one of the most important problems, the study of the solvability and spectral properties (Volterra property) of three nonlocal problems for the diffusion–hyperbolic equation (of fractional order).

We consider the equation

$$Lz(x,y) = f(x,y), \tag{1.1}$$

where

$$Lz(x,y) = \begin{cases} {}_cD^{\alpha}_{0x} z(x,y) - z_{yy}(x,y), & (x,y) \in \Omega_0, \\ z_{xx}(x,y) - z_{yy}(x,y), & (x,y) \in \Omega_1, \end{cases} \tag{1.2}$$

$${}_cD^{\alpha}_{0x} z(x,y) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{z_x(t,y)}{(x-t)^{\alpha}}\, dt, \quad 0 < \alpha < 1,$$

where $\Gamma(x) = \int_0^{\infty} e^t t^{x-1}\, dt$, $x > 0$, is Euler's gamma function, (1.2) is an integral-differential operator of fractional order α in the sense of Caputo [22] in the domain $\Omega = \Omega_0 \cup \Omega_1 \cup AB$. Here Ω_0 is the rectangle ABB_0A_0 with vertices $A(0,0)$, $B(1,0)$, $B_0(1,1)$, and $A_0(0,1)$, Ω_1 is the domain bounded with segment AB and characteristics $AC : x + y = 0$, $BC : x - y = 1$ equation (1.1), and $f(x,y)$ is a given function.

Let $AD : y = -\gamma(x)$, $0 < x < l$, be a smooth curve, where $0,5 < l \le 1$, $\gamma(0) = 0$, $l + \gamma(l) = 1$ if $l < 1$ and $\gamma(l) = 0$ if $l = 1$, located inside the characteristic triangle $0 < x + y \le x - y < 1$.

We suppose that $\gamma(x)$ is twice continuously differentiable function, $x \pm \gamma(x)$ are monotonically increasing functions, and $0 < \gamma'(x) < 1$ and $\gamma(x) > 0$ for $x > 0$.

2 A problem with nonlocal conditions with derivatives in the same characteristic directions for a diffusion–hyperbolic equation

We consider a nonlocal problem for equation (1.1) in the domain Ω, where in the hyperbolic part of the mixed domain, the nonlocal condition pointwise connects the values of the tangent derivative of the desired solution on the characteristic AC with the derivatives in the direction of the characteristic AC of the desired function on an arbitrary curve AD lying inside the characteristic triangle ABC, with the ends at the origin and on the characteristic BC (at point B).

Problem M_1B Find a solution of equation (1.1) satisfying the following conditions:

$$z(0,y) = 0, \quad 0 \le y \le 1, \tag{2.1}$$

$$z(x,1) = 0, \quad 0 \le x \le 1, \tag{2.2}$$

$$[z_x - z_y]\big[\theta_0(t)\big] + \mu(t)[z_x - z_y]\big[\theta^*(t)\big] = 0, \quad 0 < t < 1, \tag{2.3}$$

where $\theta_0(t)(\theta^*(t))$ is an affix of the intersection point of the characteristic AC (curve AD) with the characteristic coming out of the point $(t,0)$, $0 < t < 1$, and $\mu(t)$ is a given function.

In the case $\alpha = 1$, problem M_1B coincides with a nonlocal problem for a mixed parabolic–hyperbolic equation with noncharacteristic line of changing type. In this case the issues of regular and strong solvability, as well as the Volterra property of problem M_1B, are investigated in [7, 8].

In the domain Ω_0 we consider the following auxiliary problem.

Problem C_1 Find a solution of equation (1.1) for $y > 0$ satisfying conditions (2.1), (2.2), and

$$z_x(x,0) - z_y(x,0) = \delta(x), \quad 0 < x < 1, \tag{2.4}$$

where $\delta(x)$ is a given function from $C^1[0,1]$.

Lemma 2.1 *Let $\delta(x) \in C^1[0,1]$. Then for any function $f(x,y) \in C^1(\bar{\Omega}_0)$, the solution of problem C_1 admits the a priori estimate*

$$\begin{aligned} &D_{0x}^{\alpha-1}\|z(x,y)\|^2_{L_2(0,1)} + 2\int_0^x \|z_y(t,y)\|^2_{L_2(0,1)}\,dt \\ &\quad \le C\left[\int_0^x \|f(t,y)\|^2_{L_2(0,1)}\,dt + \int_0^x \delta^2(t)\,dt\right], \end{aligned} \tag{2.5}$$

where $\|f(x,y)\|^2_{L_2(0,1)} = \int_0^1 f^2(x,y)\,dy$.

Hereafter symbol C will denote a positive constant that does not depend on $z(x,y)$, not necessarily the same.

Proof of Lemma 2.1 Multiplying equation (1.1) for $y > 0$ by $z(x,y)$, integrating from 0 to 1 over y, and taking into account conditions (2.1) and (2.2), after some transformations, we have

$$\int_0^1 z(x,y)D_{0x}^{\alpha}z(x,y)\,dy + \int_0^1 z_y^2(x,y)\,dy + \tau(x)\nu(x) = \int_0^1 f(x,y)z(x,y)\,dy, \tag{2.6}$$

where

$$\tau(x) = z(x,0), \quad 0 \le x \le 1, \tag{2.7}$$

$$\nu(x) = z_y(x,0), \quad 0 < x < 1. \tag{2.8}$$

It is known [3] that $\int_0^1 z(x,y)\cdot D_{0x}^{\alpha}z(x,y)\,dy \ge \frac{1}{2}\int_0^1 D_{0x}^{\alpha}z^2(x,y)\,dy$. By this inequality, from (2.6), taking into account (2.4) and notations (2.7) and (2.8), we obtain

$$\begin{aligned} &\int_0^1 D_{0x}^{\alpha}z^2(x,y)\,dy + 2\int_0^1 z_y^2(x,y)\,dy + 2\tau(x)\tau'(x) \\ &\quad \le 2\int_0^1 z(x,y)f(x,y)\,dy + 2\tau(x)\delta(x). \end{aligned} \tag{2.9}$$

Integrating (2.9) over t from 0 to x and taking into account

$$\int_0^x D_{ot}^{\alpha}\|z(t,y)\|^2_{L_2(0,1)}\,dt = D_{ox}^{\alpha-1}\|z(x,y)\|^2_{L_2(0,1)}$$

and $\tau(0) = 0$, we have

$$\int_0^1 D_{0x}^{\alpha} z^2(x,y)\,dy + 2\int_0^1 z_y^2(x,y)\,dy + 2\tau(x)\tau'(x) \\ \leq 2\int_0^1 z(x,y)f(x,y)\,dy + 2\tau(x)\delta(x).$$

In the latter, on the right side, applying the known inequalities, we obtain

$$D_{0x}^{\alpha-1}\|z(x,y)\|_{L_2(0,1)}^2 + 2\int_0^x \|z_y(t,y)\|_{L_2}^2\,dt + \tau^2(x) \\ \leq \int_0^x \left[\|z(t,y)\|_{L_2(0,1)}^2 + \|f(t,y)\|_{L_2(0,1)}^2 + \tau^2(t) + \delta^2(t)\right]dt. \tag{2.10}$$

In the left part of (2.10), omitting the first two terms and applying the Gronwall–Bellman inequality, we have

$$\int_0^x \tau^2(t)\,dt \leq C\int_0^x \left[\|z(t,y)\|_{L_2(0,1)}^2 + \|f(t,y)\|_{L_2(0,1)}^2 + \delta^2(t)\right]dt.$$

Taking into account the last term of (2.10), we get

$$D_{0x}^{\alpha-1}\|z(x,y)\|_{L_2(0,1)}^2 + 2\int_0^x \|z_y(t,y)\|_{L_2(0,1)}^2 \\ \leq C\int_0^x \left[\|z(t,y)\|_{L_2(0,1)}^2 + \|f(t,y)\|_{L_2(0,1)}^2 + \delta^2(t)\right]dt. \tag{2.11}$$

Similarly, omitting the second term of the left part in (2.11) and applying Lemma 2 in [3], we have

$$\int_0^x \|z(t,y)\|_{L_2(0,1)}^2\,dt \leq CD_{0x}^{-\alpha-1}\left[\|f(x,y)\|_{L_2(0,1)}^2 + \delta^2(x)\right],$$

from which, taking into account

$$D_{0x}^{-\alpha-1}\|f(x,y)\|_{L_2(0,1)}^2 \leq \frac{x^{\alpha}}{\Gamma(1+\alpha)}\int_0^x \|f(t,y)\|_{L_2(0,1)}^2\,dt,$$

we obtain

$$\int_0^x \|z(t,y)\|_{L_2(0,1)}^2\,dt \leq C\int_0^x \left[\|f(t,y)\|_{L_2(0,1)}^2 + \delta^2(t)\right]dt. \tag{2.12}$$

From (2.10)–(2.12) the validity of the a priori estimate (2.5) follows. Lemma 2.1 is proved. □

Now consider equation (1.1) in the domain Ω_1. By virtue of the unambiguous solvability of the Cauchy problem (1.1), (2.7), (2.8) for the wave equation, any regular solution of the M_1B problem in the domain Ω_1 is represented as

$$z(x,y) = \frac{1}{2}\left[\tau(\xi) + \tau(\eta) - \int_{\xi}^{\eta} \nu(t)\,dt\right] - \int_{\xi}^{\eta} d\xi_1 \int_{\xi_1}^{\eta} f_1(\xi_1,\eta_1)\,d\eta_1, \tag{2.13}$$

where

$$\xi = x + y, \qquad \eta = x - y, \qquad 4f_1(\xi,\eta) = f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right).$$

Due to the conditions imposed on the function $\gamma(x)$, equation of the curve AD in characteristic variables ξ, η allows the representation

$$\xi = \lambda(\eta), \quad 0 \le \eta \le 1, \quad \text{and} \quad 0 < \lambda'(0) < 1, \qquad \lambda(\eta) < \eta. \tag{2.14}$$

In (2.13) satisfying condition (2.3), after some simple transformations, we have

$$\nu(x) = \tau'(x) - \Phi(x), \quad 0 < x < 1, \tag{2.15}$$

where

$$\Phi(x) = \frac{2}{1+\mu(x)} \int_0^x f_1(\xi_1, x)\, d\xi_1 + \frac{2\mu(x)}{1+\mu(x)} \int_{\lambda(x)}^x f_1(\xi_1, x)\, d\xi_1. \tag{2.16}$$

The ratio (2.15) is the main functional relationship between $\tau(x)$ and $\nu(x)$ brought to the segment AB from the hyperbolic domain Ω_1.

Substituting the obtained expression of $\nu(x)$ into (2.13) and taking into account (2.16), after some transformations, we get the following presentation of the solution $z(\xi,\eta)$ in the domain Ω_1:

$$\begin{aligned} z(\xi,\eta) = \tau(\xi) &+ \int_\xi^\eta \frac{d\eta_1}{1+\mu(\eta_1)} \int_0^\xi f_1(\xi_1,\eta_1)\, d\xi_1 \\ &+ \int_\xi^\eta \frac{\mu(\eta_1)\, d\eta_1}{1+\mu(\eta_1)} \int_{\lambda(\eta)}^\xi f_1(\xi_1,\eta_1)\, d\xi_1. \end{aligned} \tag{2.17}$$

Taking into account (2.14) and (2.16), after some calculations, it is not difficult to establish the following estimate:

$$\int_0^x \Phi^2(t)\, dt \le C \int_0^x d\xi \int_\xi^x \left|f_1(\xi,t)\right|^2 dt. \tag{2.18}$$

Now in (2.5), assuming that $\delta(x) = \Phi(x)$ and taking into account (2.18), it is not difficult to verify the validity of the following lemma.

Lemma 2.2 *Let $\mu(x) \in C^1[0,1]$ and $\mu(x) \ne -1$. Then for any function $f(x,y) \in C^1(\bar{\Omega})$, $f(0,0) = 0$, the solution to problem M_1B admits the a priori estimate*

$$\begin{aligned} &D_{0x}^{\alpha-1} \left\|z(x,y)\right\|_{L_2(0,1)}^2 + \int_0^x \left\|z_y(t,y)\right\|_{L_2(0,1)}^2 dt \\ &\quad \le C\left[\int_0^x \left\|f(t,y)\right\|_{L_2(0,1)}^2 dt + \int_0^x d\xi \int_\xi^x \left|f(\xi,t)\right|^2 dt\right]. \end{aligned} \tag{2.19}$$

Lemma 2.2 implies the following estimate:

$$\left\|z(x,y)\right\|_{L_2(\Omega_0)} + \left\|z_y(x,y)\right\|_{L_2(\Omega_0)} \le C\left\|f(x,y)\right\|_{L_2(\Omega)}, \tag{2.20}$$

where $L_2(\Omega)$ is the space of square-summable functions in Ω. Consider the following auxiliary *problem* C_2: In the domain Ω_0, find a solution of equation (1.1) satisfying conditions (2.1), (2.2), and (2.7).

The solution of equation (1.1) satisfying conditions (2.1), (2.2), and (2.7) in the domain Ω_0 can be represented in the form [25]

$$\begin{aligned} z(x,y) &= \int_0^x E_{y_1}(x-x_1,y,0)\tau(x_1)\,dx_1 \\ &\quad + \int_0^x dx_1 \int_0^1 E(x-x_1,y,y_1)f(x_1,y_1)\,dy_1, \end{aligned} \tag{2.21}$$

where

$$\begin{aligned} &E(x,y,y_1) = \frac{x^{\beta-1}}{2}\sum_{n=-\infty}^{+\infty}\left[e_{1,\beta}^{1,\beta}\left(-\frac{|y-y_1+2n|}{x^\beta}\right) - e_{1,\beta}^{1,\beta}\left(-\frac{|y+y_1+2n|}{x^\beta}\right)\right], \\ &\beta = \frac{\alpha}{2}, \end{aligned} \tag{2.22}$$

with the Wright-type function $e_{1,\beta}^{1,\beta}(t) = \sum_{n=0}^{\infty}\frac{t^n}{n!\Gamma(\beta-\beta n)}$ [25]. Differentiating (2.21) over y, we have

$$z_y(x,y) = \int_0^x E_{y_1y}(x-x_1,y,0)\tau(x_1)\,dx_1 + \int_0^x dx_1\int_0^1 E_y(x-x_1,y,y_1)f(x_1,y_1)\,dy_1. \tag{2.23}$$

Using the known formulas [19, 25]

$$\begin{aligned} &\frac{d^n}{dt}t^{\mu-1}e_{\alpha,\beta}^{\mu,\delta}\left(ct^\alpha\right) = t^{\mu-n-1}e_{\alpha,\beta}^{\mu-n,\delta}\left(ct^\alpha\right), \\ &\frac{d^n}{dt^n}t^{\delta-1}e_{\alpha,\beta}^{\mu,\delta}\left(ct^{-\beta}\right) = t^{\delta-n-1}e_{\alpha,\beta}^{\mu,\delta-n}\left(ct^{-\beta}\right), \qquad \frac{1}{t}e_{\alpha,\beta}^{-k,\delta}(t) = e_{\alpha,\beta}^{\alpha-k,\delta-\beta}(t), \end{aligned}$$

after some calculations, from (2.22) it is not difficult to establish that

$$E_{y_1y}(x-x_1,y,0) = \frac{\partial}{\partial x_1}\left(\sum_{n=-\infty}^{+\infty}(x-x_1)^{-\beta}e_{1,\beta}^{1,1-\beta}\left(-\frac{(y+2n)}{(x-x_1)^\beta}\right)\right). \tag{2.24}$$

Further, from (2.24), taking into account $\tau(0) = 0$ and applying formulas [25]

$$-\beta t e_{1,\beta}^{1,\delta-\beta}(t) = e_{1,\beta}^{1,\delta-1}(t) + (1-\delta)e_{1,\beta}^{1,\beta}(t), \qquad \lim_{|t|\to\infty} e_{\alpha,\beta}^{\mu,\delta}(t) = 0,$$

we have

$$\begin{aligned} &\int_0^x E_{y_1y}(x-x_1,y,0)\tau(x_1)\,dx_1 \\ &\quad = -\int_0^x\left[\sum_{n=-\infty}^{\infty}\frac{1}{(x-x_1)^\beta}e_{1,\beta}^{1,1-\beta}\left(-\frac{|y+2n|}{(x-x_1)^\beta}\right)\right]\tau'(t)\,dt. \end{aligned} \tag{2.25}$$

Now taking into account (2.25), from (2.23), as $y \to 0$, we have

$$\nu(x) = -\int m(x-x_1)\tau'(x_1)\,dx_1 + \int_0^x dx_1\int_0^1 E_y(x-x_1,0,y_1)f(x_1,y_1)\,dy_1, \tag{2.26}$$

$$m(x) = \sum_{n=-\infty}^{+\infty} x^{-\beta} e_{1,\beta}^{1,1-\beta}\left(-\frac{|2n|}{x^{\beta}}\right) = \frac{1}{\Gamma(1-\beta)} x^{-\beta} + 2x^{-\beta} \sum_{n=1}^{+\infty} e_{1,\beta}^{1,1-\beta}\left(-\frac{2n}{x^{\beta}}\right). \tag{2.27}$$

Note that (2.26) is the main functional relation between $\tau'(x)$ and $\nu(x)$ brought to the segment AB from the domain Ω_0.

Excluding from the functional relations (2.15) and (2.26) the function $\nu(x)$, we obtain the equation with respect to $\tau'(x)$:

$$\tau'(x) + \int_0^x m(x-t)\tau'(t)\,dt = Q(x), \tag{2.28}$$

where

$$Q(x) = \Phi(x) + \int_0^x dx_1 \int_0^1 \mathrm{E}_y(x-x_1,0,y_1) f(x_1,y_1)\,dy_1. \tag{2.29}$$

Lemma 2.3 ([19]) *Let* $0 < \theta \le 1$. *Then for functions* $\mathrm{E}(x,y,y_1)$ *and* $\mathrm{E}_y(x,y,y_1)$, *we have the following estimates*:

$$\left|\mathrm{E}(x,y,y_1)\right| \le Cx^{(2+\theta)\beta-1}, \quad 0 < x \le 1, 0 \le y_1 < y \le 1, 0 < \theta \le 1, \tag{2.30}$$

$$\left|\mathrm{E}_y(x,y,y_1)\right| \le Cx^{\beta(1+\theta)-1}, \quad 0 < x \le 1, 0 \le y_1 < y \le 1, 0 < \theta \le 1. \tag{2.31}$$

Proof of Lemma 2.3 The proof is carried out using the inequality

$$\left|y^{p-1}t^{\delta-1}e_{\omega,\tau}^{p,\delta}\left(-y^{\omega}t^{-\tau}\right)\right| < Cy^{p-\omega\theta-1} \cdot t^{\delta+\theta\tau-1}, \quad 0 < \theta \le 1.$$

By Lemma 2.3 and $\gamma(x) \in C^2[0,l]$, $\mu(x) \in C^1[0,1], \mu(x) \ne -1, f(x,y) \in C^1(\bar{\Omega}), f(0,0) = 0$, from (2.29) we easily establish that

$$Q(x) \in C^1[0,1], \qquad Q(0) = 0. \tag{2.32}$$

Thus by (2.27) problem M_1B is equivalently (in the sense of unambiguous solvability) reduced to a Volterra-type integral equation of the second kind with weak singularity (2.28). Therefore by (2.32) there is a unique solution of equation (2.28) from the class $C^1[0,1]$, representable as

$$\tau'(x) = Q(x) + \int_0^x R(x-t)Q(t)\,dt, \tag{2.33}$$

where $R(x)$ is the resolvent of the integral equation (2.28),

$$R(x) = \sum_{n=1}^{\infty} (-1)^n m_n(x), \qquad m_1(x) = m(x), \qquad m_{n+1}(x) = \int_0^x m_1(x-t)m_n(t)\,dt.$$

From (2.33), taking into account $\tau(0) = 0$, we have

$$\tau(x) = \int_0^x R_1(x-t)Q(t)\,dt, \quad \text{where } R_1(x) = 1 + \int_0^x R(t)\,dt. \tag{2.34}$$

Substituting (2.34) into (2.21) and taking into account (2.16) and (2.29), after some transformations, we get

$$z(x,y) = \iint_{\Omega} \theta(x-x_1)M_{01}(x,y,x_1,y_1)f(x_1,y_1)\,dx_1\,dy_1, \quad y>0, \tag{2.35}$$

where

$$\begin{aligned} M_{01}(x,y,x_1,y_1) = \theta(y_1)\Bigg[&E(x-x_1,y,y_1) \\ &+ \int_{x_1}^{x} dz \int_{x_1}^{z} E_{y_1}(x-z,y,0)R_1(z-t)E_y(t-x_1,0,y_1)\,dt\Bigg] \\ &+ \frac{\theta(-y_1)}{1+\mu(\eta_1)}\Bigg[\int_{\eta_1}^{x} E_{y_1}(x-t,y,0)R_1(t-\eta_1)\,dt \\ &+ \theta\big(\xi_1-\lambda(\eta_1)\big)\mu(\eta_1)\int_{\eta_1}^{x} E_{y_1}(x-t,y,0)R_1(t-\eta_1)\,dt\Bigg], \end{aligned}$$

where $\xi_1 = x_1+y_1$, $\eta_1 = x_1-y_1$, $\theta(y)=1$, $y>0$, and $\theta(y)=0$, $y<0$.

Similarly, substituting (2.34) into (2.17) and taking into account (2.16) and (2.29), after some calculations, we get

$$z(x,y) = \iint_{\Omega} \theta(x-x_1)M_{11}(x,y,x_1,y_1)f(x_1,y_1)\,dx_1\,dy_1, \quad y<0, \tag{2.36}$$

where

$$\begin{aligned} M_{11}(x,y,x_1,y_1) = \theta(y_1)&\int_0^{\xi} R_1(\xi-t)E_y(t-x_1,0,y_1)\,dt \\ &+ \theta(-y_1)\theta(\xi-\eta_1)\frac{R_1(\xi-\eta_1)}{2(1+\mu(\eta_1))}\big[1+\mu(\eta_1)\theta\big(\xi_1-\lambda(\eta_1)\big)\big] \\ &+ \theta(-y_1)\theta(\eta-\eta_1)\theta(\eta_1-\xi)\theta(\xi-\xi_1)\frac{[1+\mu(\eta_1)\theta(\xi_1-\lambda(\eta_1))]}{2[1+\mu(\eta_1)]}, \\ \xi = x+y, &\qquad \eta = x-y. \end{aligned}$$

From (2.35) and (2.36) we have

$$z(x,y) = \iint_{\Omega} M_1(x,y,x_1,y_1)f(x_1,y_1)\,dx\,dy, \tag{2.37}$$

$$M_1(x,y,x_1,y_1) = \theta(x-x_1)\big[\theta(y)M_{01}(x,y,x_1,y_1)+\theta(-y)M_{11}(x,y,x_1,y_1)\big]. \tag{2.38}$$

Taking into account explicit types of functions

$$M_{01}(x,y,x_1,y_1), \qquad M_{11}(x,y,x_1,y_1), \quad \text{and} \quad \mu(x)\in C^2[0,1], \qquad \mu(x)\neq -1,$$

it is not difficult to establish that in (2.38) all terms are bounded, with the exception of the first, $M_{01}(x,y,x_1,y_1)$, in which by Lemma 2.3 the summand $E(x-x_1,y,y_1)$ may be not limited. Therefore it is sufficient to show that

$$\theta(x-x_1)\theta(y_1)\theta(y)E(x-x_1,y,y_1)\in L_2(\Omega\times\Omega).$$

By Lemma 2.3 from estimate (2.30) by direct calculation we have

$$\left\|\theta(x-x_1)\mathrm{E}(x-x_1,y,y_1)\right\|^2_{L_2(\Omega\times\Omega)} \le C\left\{(2+\theta)\beta\left[1+(2+\theta)\beta\right]\right\}^{-1}.$$

Therefore $M_1(x,y,x_1,y_1) \in L_2(\Omega\times\Omega)$. □

Lemma 2.4 *If $\mu(x) \in C^1[0,1]$, $\mu(x) \neq -1$, and $f(x,y) \in L_2(\Omega)$, then $Q(x) \in L_2[0,1]$, and*

$$\left\|Q(x)\right\|^2_{L_2(0,1)} \le C\left\|f(x,y)\right\|^2_{L_2(\Omega)}. \tag{2.39}$$

Proof of Lemma 2.4 Taking into account (2.16), (2.18), (2.29), and (2.31), is carried out by direct calculation using the well-known Cauchy–Bunyakovsky inequalities.

Therefore from (2.33) we have

$$\left\|\tau'(x)\right\|_{L_2(0,1)} \le C\left\|Q(x)\right\|_{L_2(0,1)} \le C\left\|f(x,y)\right\|_{L_2(\Omega)}. \tag{2.40}$$

From (2.17) by (2.40) and direct calculation it is not difficult to establish that

$$\left\|z(x,y)\right\|_{W_2^1(\Omega_1)} \le C\left\|f(x,y)\right\|_{L_2(\Omega)}, \tag{2.41}$$

where $W_2^1(\Omega)$ is the Sobolev space. From (2.19) and (2.41) we have

$$\begin{aligned} &D_{0x}^{\alpha-1}\left\|z(x,y)\right\|^2_{L_2(0,1)} + \int_0^x \left\|z_y(t,y)\right\|^2_{L_2(0,1)}\,dt + \left\|z(x,y\right\|^2_{W_2^1(\Omega_2)} \\ &\quad \le C\left[\int_0^x \left\|f(t,y)\right\|^2_{L_2(0,1)} + \int_0^x d\xi \int_\xi^1 \left|f(\xi,x)\right|^2 dt + \left\|f(x,y)\right\|^2_{L_2(\Omega)}\right]. \end{aligned} \tag{2.42}$$

We call a function $z(x,y) \in V$ a regular solution of problem M_1B in the domain Ω, where

$$\begin{aligned} V = \bigl\{&z(x,y) : z(x,y) \in C(\bar{\Omega}) \cap C^{1.1}(\Omega \cup AC), \\ &D_{0x}^{\alpha} z(x,y), z_{yy}(x,y) \in C(\Omega_0), z(x,y) \in C^{2.2}(\Omega_1)\bigr\}, \end{aligned}$$

if it satisfies equation (1.1) in $\Omega_0 \cup \Omega_1$ and conditions (2.1)–(2.3). □

Thus, summarizing the above statements, we have proved the following theorem.

Theorem 2.1 *Let $\mu(t) \in C^1[0,1]$ and $\mu(x) = -1$, $0 \le x \le 1$. Then for any function $f(x,y) \in C^1(\bar{\Omega})$, $f(A) = 0$, there exists a unique regular solution to problem M_1B (1.1), (2.1)–(2.3), and it is presented in the form (2.37) and satisfies inequality (2.42).*

From (2.42) or (2.20) and (2.41) the following estimate follows:

$$\left\|z(x,y)\right\|_{L_2(\Omega_0)} + \left\|z_y(x,y)\right\|_{L_2(\Omega_0)} + \left\|z(x,y)\right\|_{W_2^1(\Omega_1)} \le C\left\|f(x,y)\right\|_{L_2(\Omega)}. \tag{2.43}$$

The function $z(x,y) \in L_2(\Omega)$ is called a strong solution to problem M_1B if there exists the sequence of functions $\{z_n(x,y)\}$, $z_n(x,y) \in V$, satisfying conditions (2.1)–(2.3) such that

$$\left\|z_n(x,y) - z(x,y)\right\|_{L_2(\Omega)} \to 0, \qquad \left\|Lz_n(x,y) - f(x,y)\right\|_{L_2(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

Theorem 2.2 *Let the conditions of Theorem 2.1 be satisfied. Then for any function $f(x,y) \in L_2(\Omega)$, there exists a unique strong solution $z(x,y)$ to problem M_1B. This solution can be presented in the form (2.37) and satisfies estimate (2.43).*

Proof of Theorem 2.2 Now let us show that for $f(x,y) \in L_2(\Omega)$, the solution to problem M_1B is strong. Due to the density in $L_2(\Omega)$,

$$G = \{f(x,y) : f(x,y) \in C^1(\bar{\Omega}), f(A) = 0\}.$$

For any function $f(x,y) \in L_2(\Omega)$, there exists a sequence $\{f_n(x,y)\}, f_n(x,y) \in G$, such that $\|f_n(x,y) - f(x,y)\|_{L_2(\Omega)} \to 0$ as $n \to \infty$.

By $z_n(x,y)$ we denote a solution to problem M_1B (1.1), (2.1)–(2.3) with right-hand part $f_n(x,y)$ in equation (1.1).

From (2.32) it follows that if $f_n(x,y) \in G$, then $Q_n(x) \in C^1[0,1]$ and $Q_n(0) = 0$, where

$$\begin{aligned}
&Q_n(x) = \Phi_n(x) + \int_0^x dx_1 \int_0^1 E_y(x - x_1, 0, y_1) f_n(x_1, y_1)\, dy_1,\\
&\Phi_n(x) = \frac{2}{1+\mu(x)} \int_0^x f_{1n}(\xi_1, x)\, d\xi_1 + \frac{2\mu(x)}{1+\mu(x)} \int_{\lambda(x)}^x f_{1n}(\xi_1, x)\, d\xi_1,\\
&4f_{1n}(\xi, n) = f_n\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right), \qquad \xi = x + y, \eta = x - y.
\end{aligned}$$

Then equation (2.28) can be considered as an integral equation of the second kind in the space $C^1[0,1]$. It has a unique solution $\tau_n'(x) \in C^1[0,1]$. Since $\nu_n(x) = \tau_n'(x) - \Phi_n(x)$, we have $\nu_n(x) \in C^1[0,1]$. Therefore the function $z_n(x,y)$ defined by formulas (2.13) and (2.21) (here it is necessary to replace the functions $\tau(x)$, $\nu(x)$, $f(x,y)$ by $\tau_n(x)$, $\nu_n(x)$, $f_n(x,y)$, respectively) belongs to class V.

However, on the other hand, by Lemma 2.4, $Q(x) \in L_2(0,1)$ when $f(x,y) \in L_2(\Omega)$. Therefore equation (2.28) can be considered as a Volterra integral equation of the second kind in the space $L_2(0,1)$. Equation (2.28) in the space $L_2(0,1)$ is unambiguously solvable, $\tau'(x) \in L_2(0,1)$, and $\|\tau'(x)\|_{L_2(0,1)} \le C\|Q(x)\|_{L_2(0,1)}$. As before, by (2.15) we have $\nu(x) \in L_2(0,1)$.

In this case the function $z(x,y)$ defined by formulas (2.13) and (2.21) at least belongs to class $C(\bar{\Omega}) \cap W_2^{0,1}(\Omega_0) \cap W_2^{1,1}(\Omega_1)$.

By (2.39) it is also not difficult to verify estimate (2.43).

Now, due to the completeness of the space $L_2(\Omega)$, the sequence $\{f_n(x,y)\}$ we constructed above is fundamental. From the linearity of equation (1.1) and estimate (2.43) we obtain that $\|z_n(x,y) - z_m(x,y)\|_{L_2(\Omega)} \le C\|f_n(x,y) - f_m(x,y)\|_{L_2(\Omega)}$, i.e., the sequence $\{z_n(x,y)\}$ is fundamental in $L_2(\Omega)$. Taking into account the completeness of the space $L_2(\Omega)$, we get that there exists a limit $z(x,y) \in L_2(\Omega)$ of the sequence $z_n(x,y)$, which will be the desired strong solution to problem M_1B with the right-hand part $f(x,y) \in L_2(\Omega)$.

Analyzing the above facts, it is also not difficult to establish that a strong solution $z(x,y)$ to problem M_1B is representable as (2.37). Theorem 2.2 is proved.

Now let us establish the Volterra property of problem M_1B. By B_1 we denote the closure in space $L_2(\Omega)$ of the fractional differential operator satisfying conditions (2.1)–(2.3) and given on V by expression (1.2).

According to the definition of a strong solution to problem M_1B, $z(x,y)$ is a strong solution to problem M_1B if and only if $z(x,y) \in D(B_1)$, where $D(B_1)$ is the domain of the operator B_1.

From Theorem 2.2 it follows that the operator B_1 is closed and its domain is dense in $L_2(\Omega)$; the inverse operator B_1^{-1} exists, is defined on the whole $L_2(\Omega)$, and is completely continuous. In this regard, a natural question arises: is there an eigenvalue of the operator B_1^{-1} and hence of problem M_1B? The main result is the theorem on the absence of eigenvalues of the operator B_1^{-1}. □

Theorem 2.3 *Let $\mu(x) \neq -1$. Then the integral operator*

$$B_1^{-1}f(x,y) = \iint_\Omega M_1(x,y,x_1,y_1)f(x_1,y_1)\,dx_1\,dy_1, \tag{2.44}$$

where $M_1(x,y,x_1,y_1) \in L_2(\Omega \times \Omega)$, is Volterra in $L_2(\Omega)$.

Proof To prove Theorem 2.3, we need to show that the operator B_1^{-1} defined by formula (2.44) is completely continuous and quasinilpotent. Since the complete continuity of this operator follows from the fact that $M(x,y,x_1,y_1) \in L_2(\Omega \times \Omega)$, we show that B_1^{-1} quasinilpotent, i.e.,

$$\lim_{n\to\infty} \|B_1^{-1}\|^{\frac{1}{n}}_{L_2(\Omega)\to L_2(\Omega)} = 0, \tag{2.45}$$

where $B_1^{-n} = B_1^{-1}[B_1^{-(n-1)}]$, $n = 1,2,\ldots$.

From (2.44) by direct calculation, taking into account (2.35)–(2.38), it is not difficult to obtain that

$$B_1^{-n}f(x,y) = \iint_\Omega M_n(x,y,x_1,y_1)f(x_1,y_1)\,dx_1\,dy_1, \tag{2.46}$$

where

$$M_n(x,y,x_1,y_1) = \iint_\Omega M_1(x,y,x_2,y_2)M_{(n-1)}(x_2,y_2,x_1,y_1)\,dx_2\,dx_1, \quad n = 2,3,\ldots,$$

$$M_1(x,y,x_1,y_1) = M_1(x,y,x_1,y_1).$$

Lemma 2.5 *For the iterated kernels $M_n(x,y,x_1,y_1)$, we have the following estimate*:

$$\left|M_n(x,y,x_1,y_1)\right| \leq \left(\frac{3}{2}\right)^{n-1} N^n \frac{\Gamma^n(\gamma)}{\Gamma(n\gamma)}(x-x_1)^{n\gamma-1}, \tag{2.47}$$

where $\gamma = (2+\theta)\beta$, $N = Cd$, C is the coefficient from estimate (2.30),

$$d = \max_{\substack{(x,y)\in\Omega \\ (x_1,y_1)\in\Omega}} \left|(x-x_1)^{\gamma-1}M_1(x,y,x_1,y_1)\right| \quad \text{if } \gamma < 1,$$

and

$$d = \max_{\substack{(x,y)\in\Omega \\ (x_1,y_1)\in\Omega}} \left|M_1(x,y,x_1,y_1)\right| \quad \text{if } \gamma \geq 1.$$

Proof of Lemma 2.5 We use mathematical induction over n. For $n = 1$, the inequality

$$\left|M_1(x,y,x_1,y_1)\right| \le N(x-x_1)^{\gamma-1}$$

follows from representation (2.38) taking into account (2.30).

Let (2.47) be valid for $n = k-1$. Let us prove the validity of this formula for $n = k$. Using inequality (2.47) for $n = 1$ and $n = k-1$, we have

$$\begin{aligned}
&\left|M_k(x,y,x_1,y_1)\right| \\
&\quad= \left|\iint_\Omega M_1(x,y,x_2,y_2)\cdot M_{(k-1)}(x_2,y_2,x_1,y_1)\,dx_2\,dy_2\right| \\
&\quad\le \iint_\Omega \left|M_1(x,y,x_2,y_2)\right|\cdot\left|M_{(k-1)}(x_2,y_2,x_1,y_1)\right| dx_2\,dy_2 \\
&\quad\le \iint_\Omega \theta(x-x_2)N(x-x_2)^{\gamma-1}\theta(x_2-x_1)\left(\frac{3}{2}\right)^{k-2} \\
&\qquad\times N^{k-1}\frac{\Gamma^{k-1}(\gamma)}{\Gamma[(k-1)\gamma]}(x_2-x_1)^{(k-1)\gamma-1}\,dx_2\,dy_2 \\
&\quad\le \left(\frac{3}{2}\right)^{k-1} N^k\frac{\Gamma^{k-1}(\gamma)}{\Gamma[(k-1)\gamma]}\int_{x_1}^{x}(x-x_2)^{\gamma-1}(x_2-x_1)^{(k-1)\gamma-1}\,dx_2 \\
&\quad= \left(\frac{3}{2}\right)^{k-1} N^k\frac{\Gamma^{k-1}(\gamma)}{\Gamma[(k-1)\gamma]}(x-x_1)^{k\gamma-1}\int_0^1 \sigma^{\gamma-1}(1-\sigma)^{(k-1)\gamma-1}\,d\sigma \\
&\quad= \left(\frac{3}{2}\right)^{k-1} N^k\frac{\Gamma^{k}(\gamma)}{\Gamma(k\gamma)}(x-x_1)^{k\gamma-1},
\end{aligned}$$

which proves the lemma. □

Using the well-known Schwarz inequality and Lemma 2.5, from representation (2.46) we have

$$\begin{aligned}
&\left\|B_1^{-n}f(x,y)\right\|^2_{L_2(\Omega)} \\
&\quad= \iint_\Omega \left|B_1^{-n}f(x,y)\right|^2 dx\,dy \\
&\quad= \iint_\Omega \left[\iint_\Omega M_n(x,y,x_1,y_1)f(x_1,y_1)\,dx_1\,dy_1\right]^2 dx\,dy \\
&\quad\le \iint_\Omega \left[\left(\iint_\Omega \left|M_n(x,y,x_1,y_1)\right|^2 dx_1\,dy_1\right)\left(\iint_\Omega \left|f(x_1,y_1)\right|^2 dx_1\,dy_1\right)\right] dx\,dy \\
&\quad\le \left(\frac{3}{2}N\right)^{2n}\frac{\Gamma^{2n}(\gamma)}{[(2n\gamma-1)](2n\gamma)\Gamma^2(n\gamma)}\left\|f(x,y)\right\|^2_{L_2(\Omega)},
\end{aligned}$$

from which we obtain

$$\left\|B_1^{-n}\right\|_{L_2(\Omega)\to L_2(\Omega)} \le \left(\frac{3N}{2}\right)^n\left(4-\frac{2}{n\gamma}\right)^{-\frac{1}{2}}\frac{\Gamma^n(\gamma)}{\Gamma(1+n\gamma)}.$$

From the latter it is not difficult to establish equality (2.45). Theorem 2.3 is proved. □

Consequence 1 Problem M_1B is a Volterra problem.

Consequence 2 For any complex number λ, the equation $B_1z(x,y) - \lambda z(x,y) = f(x,y)$ unambiguously solvable for all $f(x,y) \in L_2(\Omega)$.

3 A problem for a diffusion–hyperbolic equation with a nonlocal condition with derivatives in different characteristic directions

This section is devoted to the study of a nonlocal problem with derivatives in different characteristic directions for equation (1.1).

The main goal is to show that for the correctness and Volterra property of problem M_2B considered in this section, in contrast to problem M_1B, it is essential to consider the ratio between the coefficient of "compression" $\mu(0)$ at the origin of the derivative in the direction of the characteristic BC and the polar angle ω formed by the curve AD and the abscissa axis.

Problem M_2B Find a solution to equation (1.1) satisfying the conditions (2.1), (2.2), and

$$[z_x - z_y]\big[\theta_0(t)\big] + \mu(t)[z_x + z_y]\big[\theta^*(t)\big] = 0, \tag{3.1}$$

where $\theta_0(t) = (\frac{t}{2}, -\frac{t}{2})$, $\theta^*(t) = (\frac{\lambda(t)+t}{2}, \frac{\lambda(t)-t}{2})$, $\xi = \lambda(\eta)$ is the equation of the curve AD in characteristic coordinates $\xi = x + y$, $\eta = x - y$, and $\mu(t)$ is a given function.

As in Sect. 2, by a regular solution to problem M_2B we mean a function $z(x,y) \in V$ satisfying equation (1.1) in $\Omega_0 \cup \Omega_1$ and conditions (2.1), (2.2), and (3.1).

Theorem 3.1 *Let $\mu(t) \in C^2[0,1]$, and suppose the following condition is satisfied*:

$$\big|\mu(0)\big|^2 < \operatorname{tg}\left(\omega + \frac{\pi}{4}\right), \quad -\frac{\pi}{4} < \omega < 0. \tag{3.2}$$

Then for any function $f(x,y) \in C^1(\bar{\Omega})$, $f(A) = 0$, there is a unique regular solution to problem M_2B, which satisfies inequality (2.43) *and can be represented in the form*

$$z(x,y) = \iint_\Omega M_2(x,y,x_1,y_1) f(x_1,y_1)\, dx_1\, dy_1, \tag{3.3}$$

where $M_2(x,y,x_1,y_1) \in L_2(\Omega \times \Omega)$.

Proof As before, denoting $z(x,0) = \tau(x)$, $0 \le x \le 1$, $z_y(x,0) = \nu(x)$, $0 \le x \le 1$, the solution to problem M_2B in the domain Ω_1 can be represented by d'Alembert's formula (2.13).

Using condition (3.1) in formula (2.13), we obtain

$$\tau'(t) + \mu(t)\tau'\big(\lambda(t)\big) - \nu(t) + \mu(t)\nu\big(\lambda(t)\big) = F_1(t), \tag{3.4}$$

where

$$F_1(t) = 2\int_0^t f_1(\xi_1,t)\, d\xi_1 - 2\mu(t)\int_{\lambda(t)}^t f_1\big(\lambda(t),\eta_1\big)\, d\eta_1. \tag{3.5}$$

Relation (3.4) is the main relation between $\tau'(x)$ and $\nu(x)$, brought to the segment AB from the hyperbolic part Ω_1.

The main functional relation between $\tau'(x)$ and $\nu(x)$, brought to the segment AB from the parabolic part of the domain, has the form (2.26).

Now, excluding the function $\nu(x)$ from relations (2.26) and (3.4), for $\tau'(x)$, we obtain the integro-differential equation

$$\begin{aligned} &\tau'(t) + \mu(t)\tau'\bigl(\lambda(t)\bigr) + \int_0^t m(t-\sigma)\tau'(\sigma)\,d\sigma \\ &\quad - \mu(t)\int_0^{\lambda(t)} m\bigl(\lambda(t)-\sigma\bigr)\tau'(\sigma)\,d\sigma = F(t), \quad 0 \le t \le 1, \end{aligned} \tag{3.6}$$

where

$$F(t) = F_1(t) + Q_0(t) - \mu(t)Q_0\bigl(\lambda(t)\bigr), \tag{3.7}$$

$$Q_0(t) = \int_0^t dx_1 \int_0^1 E_y(t-x_1,0,y_1)f(x_1,y_1)\,dy_1. \tag{3.8}$$

Thus, problem M_2B in the sense of unique solvability is equivalently reduced to integro-functional equation (3.6). Note that similar integro-functional equations have been studied in [7, 8].

Consider the equation

$$\varphi(x) + \mu(x)\varphi\bigl(\lambda(x)\bigr) = F_2(x). \tag{3.9}$$

First, we present the following lemma, which will be needed later.

Lemma 3.1 ([7]) *Let*

$$\bigl|\mu(0)\bigr|^2 < \lambda'(0). \tag{3.10}$$

Then for any function $F_2(x) \in L_2(0,1)$, there is a unique solution $\varphi(x) \in L_2(0,1)$ to equation (3.9), and it satisfies inequality

$$\bigl\|\varphi(x)\bigr\|_{L_2(0,1)} \le C\bigl\|F_2(x)\bigr\|_{L_2(0,1)}. \tag{3.11}$$

Proof The proof of the lemma is given in [7]. For ease of reading, we briefly outline it here. Let us consider the operator acting by the formula

$$A\varphi(x) = \mu(x)\varphi\bigl(\lambda(x)\bigr). \tag{3.12}$$

It is obvious that $A^n\varphi(x) = \prod_{k=0}^{n-1}\mu(\lambda^k(x))\cdot\varphi(\lambda^n(x)), n \ge 2$, where $\lambda^n(x) = \lambda[\lambda^{n-1}(x)]$, $\lambda^0(x) = x$.

Taking in to account (3.12), equation (3.9) can be presented in the form

$$(E + A)\varphi(x) = F_2(x),$$

where E is the identity operator.

It is easy to establish that the operator

$$B^{-1} = \sum_{k=0}^{\infty} (-1)^n A^n$$

is formally inverse to the operator $B = E + A$. Therefore let us show that the operator $B^{-1} = (E + A)^{-1}$ is bounded in the space $L_2(0,1)$.

We have

$$\begin{aligned}\|A^n\varphi(x)\|^2_{L_2(0,1)} &= \int_0^1 \left[\prod_{k=0}^{n-1} \mu\big(\lambda^k(x)\big)\right]^2 \big|\varphi\big(\lambda^n(x)\big)\big|^2\,dx \\ &= \int_0^1 \frac{[\prod_{k=0}^{n-1}\mu(\lambda^k(x))]^2}{\prod_{k=0}^{n-1}\lambda'(\lambda^k(x))} \cdot \big|\varphi\big(\lambda^n(x)\big)\big|^2 \cdot \prod_{k=0}^{n-1} \lambda'\big(\lambda^k(x)\big)\,dx.\end{aligned}$$

Further, substituting $\lambda^n(x) = t$, we obtain

$$\begin{aligned}\|A^n\varphi(x)\|^2_{L_2(0,1)} &\le \max_{0\le x\le 1} \prod_{k=0}^{n-1} \frac{[\mu(\lambda^k(x))]^2}{|\lambda'(\lambda^k(x))|} \int_0^{\lambda^n(1)} \big|\varphi(t)\big|^2\,dt \\ &\le \prod_{k=0}^{n-1} \max_{0\le t\le \lambda^k(1)} \frac{\mu^2(t)}{|\lambda'(t)|} \cdot \|\varphi(t)\|^2_{L_2(0,1)}.\end{aligned}$$

Therefore $\|A^n\|_{L_2(0,1)\to L_2(0,1)} \le a_n$, where $a_n = \prod_{k=0}^{n-1} \max_{0\le t\le\lambda^k(1)} \frac{|\mu(t)|}{\sqrt{|\lambda'(t)|}}$. Since $\lambda(x) < x$ for $x \ne 0$ and $\lambda(0) = 0$, the sequence $\lambda^n(1)$ steadily converges to zero as $n \to \infty$. Hence

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \max_{0\le t\le\lambda^n(1)} \frac{|\mu(t)|}{\sqrt{|\lambda'(t)|}} = \lim_{\alpha\to 0} \max_{0\le t\le\alpha} \frac{|\mu(t)|}{\sqrt{|\lambda'(t)|}} = \frac{|\mu(0)|}{\sqrt{\lambda'(0)}}.$$

Therefore by (3.10) the number series $\sum_{n=1}^{\infty} a_n$ converges, and

$$\|\mathrm{B}^{-1}\|_{L_2(0,1)\to L_2(0,1)} \le \sum_{n=1}^{\infty} a_n < \infty,$$

which shows the boundedness of operator B^{-1} in $L_2(0,1)$ and the correctness of estimate (3.11). This proves Lemma 3.1. □

Lemma 3.2 *Let*

$$\big|\mu(0)\big| \cdot \lambda'(0) < 1. \tag{3.13}$$

If $F_2(x) \in C^1[0,1]$ and $F_2(0) = 0$, then there is a unique solution to equation (3.9) from the class $C^1[0,1]$, and $\varphi(0) = 0$.

Proof Consider equation (3.9) in the class $C^1[0,1]$. It is obvious that if $F_2(x) \in C^1[0,1]$ and $F_2(0) = 0$, then $\varphi(0) = 0$. Therefore differentiating (3.9), for $\varphi'(x)$, we obtain the equation

$$[E + A_1]\varphi'(x) + T_1\varphi'(x) = F_2'(x), \tag{3.14}$$

where

$$A_1\varphi'(x) = \lambda'(x)\mu(x)\varphi'\big(\lambda(x)\big), \qquad T_1\varphi'(x) = \mu'(x)\int_0^{\lambda(x)} \varphi'(t)\,dt.$$

Since we consider equation (3.9) in a narrower class than $L_2(0,1)$, the solution of equation (3.9) and therefore the solution of (3.14) is unique. Also, it is obvious that T_1 is completely continuous in $C[0,1]$. Therefore solvability of equation (3.14) in $C[0,1]$ is equivalent to the existence of the operator

$$A_2^{-1} = (E + A_1)^{-1} = \sum_{n=0}^{\infty} (-1)^n A_1^n$$

continuous in $C[0,1]$.

It is easy to see that the operator A_2^{-1} exists, is bounded in $C[0,1]$, and $A_2^{-1}\cdot A_2 = A_2\cdot A_2^{-1} = E$, where $A_2 = E + A_1$.

Indeed, similarly to Lemma 3.1, we have

$$\big\|A_1^n\varphi'(x)\big\|_{C[0,1]} \le \big\|\varphi'(x)\big\|_{C[0,1]}\cdot\prod_{k=0}^{n-1}\max_{0\le t\le\lambda^k(1)}\big|\mu(t)\lambda'(t)\big|.$$

Hence $\|A_1^n\|_{C[0.1]\to C[0.1]} \le \prod_{k=0}^{n-1}\max_{0\le t\le\lambda^k(1)}|\mu(t)|\cdot|\lambda'(t)| \equiv b_n$. Taking into account that $\lambda^n(1)\to 0$ as $n\to\infty$, we get

$$\ell im_{n\to\infty}\frac{b_{n+1}}{b_n} = \ell im_{n\to\infty}\max_{0\le t\le\lambda^n(1)}\big|\mu(t)\cdot\lambda'(t)\big| = \big|\mu(0)\big|\cdot\lambda'(0).$$

Therefore by (3.13) the number series $\sum_{n=1}^{\infty} b_n$ converges, and

$$\big\|(E + A_1)^{-1}\big\|_{C[0,1]\to C[0,1]} \le \sum_{n=1}^{\infty} b_n < \infty,$$

which shows the continuity of operator A_2^{-1} in $C[0,1]$. Lemma 3.2 is proved. □

Lemma 3.3 *If $\mu(t)\in C^2[0,1]$ and $f(x,y)\in C^1(\bar{\Omega})$, $f(A) = 0$, then $F(t)\in C^1[0,1]$ and $F(0) = 0$.*

Proof of Lemma 3.3 Using the explicit form of the function $\mathrm{E}(x,y,y_1)$, by Lemma 2.3 it is not difficult to establish that the function $Q_0(x)$ defined by formula (3.8) belongs to class $C^1[0,1]$ and $Q_0(0) = 0$. From (3.5), taking into account the conditions imposed on the function $\mu(t)$, it is easy to establish that $F_1(x)\in C^1[0,1]$ and $F_1(0) = 0$. Hence the proof of Lemma 3.3 follows by (3.7). □

Lemma 3.4 *If $\mu(t)\in C^2[0,1]$ and $f(x,y)\in L_2(\Omega)$, then $F(t)\in L_2(0,1)$, and*

$$\big\|F(t)\big\|_{L_2(0,1)} \le C\big\|f(x,y)\big\|_{L_2(\Omega)}.$$

Proof of Lemma 2.4 To prove Lemma 3.4, taking into account (3.8) and applying the Cauchy–Bunyakovsky inequality, we establish the following chain of inequalities:

$$\begin{aligned}
\left|Q_0(t)\right|^2 &= \left|\int_0^t dx_1 \int_0^1 E_y(t-x_1,0,y_1)f(x_1,y_1)\,dy_1\right|^2 \\
&\le \int_0^1 \left[\int_0^t E_y(t-x_1,0,y_1)f(x_1,y_1)\,dx_1\right]^2 dy_1 \int_0^1 dy_1 \\
&\le \int_0^1 dy_1 \int_0^t \left|(t-x_1)^{\frac{1-\beta(1+\theta)}{2}} E_y(t-x_1,0,y_1)(t-x_1)^{\frac{\beta(1+\theta)-1}{2}} f(x_1,y_1)\right|^2 dx_1 \\
&\le \int_0^1 dy_1 \left[\int_0^t C^2(t-x_1)^{\beta(1+\theta)-1}\,dx_1 \int_0^t \left|(t-x_1)^{\frac{\beta(1+\theta)-1}{2}} f(x_1,y_1)\right|^2 dx_1\right] \\
&\le \frac{C^2 t^{\beta(1+\theta)}}{\beta^2(1+\theta)^2} \int_0^1 dy_1 \left[\int_0^t (t-x_1)^{\beta(1+\theta)-1}\,dx_1 \int_0^t \left|f(x_1,y_1)\right|^2 dx_1\right] \\
&\le \frac{C^2 t^{2\beta(1+\theta)}}{\beta^2(1+\theta)^2} \int_0^1 dy_1 \int_0^t \left|f(x_1,y_1)\right|^2 dx_1.
\end{aligned}$$

From this estimate, (3.5), and (3.7) by direct calculation we obtain the proof of Lemma 3.4. □

Lemma 3.5 *Let condition* (3.10) *be fulfilled. Then for any function* $F(x) \in L_2(\Omega)$, *there is a unique solution to equation* (3.6). *This solution belongs to the class* $L_2(0,1)$ *and satisfies the inequality*

$$\left\|\tau'(x)\right\|_{L_2(0,1)} \le C\left\|F(x)\right\|_{L_2(0,1)}. \tag{3.15}$$

Proof We introduce the integral operator T acting in $L_2(0,1)$ according to the formula

$$T\varphi(x) = \int_0^x m(x-t)\varphi(t)\,dt. \tag{3.16}$$

Since $x^\beta m(x)$ is a continuous function, it is obvious that T is a completely continuous operator in $L_2(0,1)$.

Taking into account (3.12) and (3.16), from (3.6), passing to the operator record, we obtain

$$[E+A]\tau'(x) + [E-A]T\tau'(x) = F(x). \tag{3.17}$$

By Lemma 3.1 the operator $(E+A)^{-1}$ is bounded. Applying the operator $(E+A)^{-1}$ to (3.17), we have

$$\tau'(x) = (E+A)^{-1}F(x) - (E+A)^{-1}(E-A)T\tau'(x). \tag{3.18}$$

Equation (3.18) will be solved by the method of successive approximations. Suppose that $\tau_0'(x) \equiv 0$,

$$\tau_n'(x) = (E+A)^{-1}F(x) - (E+A)^{-1}(E-A)T\tau_{n-1}'(x), \quad n = 1,2,\ldots. \tag{3.19}$$

For $n = 1$, it follows from Lemma 3.1 that

$$\|\tau'_1(x)\|_{L_2(0,1)} \le \|(E+A)^{-1}\|_{L_2(0,1)\to L_2(0,1)} \cdot \|F(x)\|_{L_2(0,1)} \le k\|F(x)\|_{L_2(0,1)}, \tag{3.20}$$

where $k = \sum_{n-1}^{\infty} a_n < \infty$.

By direct calculation we can prove the following estimates:

$$\|T\|_{L_2(0,1)\to L_2(0,1)} \le \frac{p}{\sqrt{2(1-2\beta)(1-\beta)}} = P, \tag{3.21}$$

$$\|E-A\|_{L_2(0,1)\to L_2(0,1)} \le L, \tag{3.22}$$

where $p = \max_{0\le t\le x\le 1}\{|(x-t)^{\beta} m(x-t)|\}$ and $L = 1 + \max_{0\le x\le 1} \frac{|\mu(x)|}{\sqrt{|\lambda'(x)|}}$.

Denote $\psi_n(x) = \tau'_n(x) - \tau'_{n-1}(x)$, $n = 1, 2, \ldots$. Then from (3.19) we have

$$\psi_n(x) = -(E+A)^{-1}(E-A)T\psi_{n-1}(x), \quad n = 2, 3, \ldots, \tag{3.23}$$

$$\psi_1(x) = (E+A)^{-1}F(x). \tag{3.24}$$

We claim that

$$\|\psi_n(x)\|_{L_2(0,1)} \le \frac{k^n (LP)^{n-1}}{n!} \|F(x)\|_{L_2(0,1)}.$$

The proof follows from estimates (3.20)–(3.22) and from equations (3.23) and (3.24). The latter implies the convergence in $L_2(0,1)$ of the series

$$\tau'(x) = \lim_{n\to\infty} \tau'_n(x) = \sum_{n=1}^{\infty} \psi_n(x), \tag{3.25}$$

which is majorized in $L_2(0,1)$ by the convergent numerical series

$$\|F(x)\|_{L_2(0,1)} \cdot \sum_{n=1}^{\infty} \frac{k^n (LP)^{n-1}}{n!}.$$

It is not difficult to make sure that the constructed function $\tau'(x)$ satisfies equation (3.18). In fact, summing up the recurrent relations (3.23) and (3.24) over n from 1 to k, we obtain

$$\sum_{n=1}^{k} \psi_n(x) = -(E+A)^{-1}(E-A)T \sum_{n=1}^{k} \psi_n(x) + (E+A)^{-1}F(x).$$

Passing to the limit as $k \to \infty$ in this equality, taking advantage of the limitations of operators A and T, due to the convergence of the series (3.25), we obtain equation (3.18).

Now let us show the uniqueness of the solution to equation (3.18). For this, as is known, it is sufficient to show that the corresponding homogeneous equation (3.18) has only a zero solution. Let $\bar{\tau}'(x) \in L_2(0.1)$ be a solution to homogeneous equation (3.18):

$$\bar{\tau}'(x) = -(E+A)^{-1}(E-A)T\bar{\tau}'(x). \tag{3.26}$$

We apply to (3.26) the method of successive approximations, taking $\bar{\tau}_0'(x) = \bar{\tau}'(x)$ and

$$\bar{\tau}_n'(x) = -(E+A)^{-1}(E-A)T\bar{\tau}_{n-1}'(x).$$

Since the function $\bar{\tau}'(x)$ is a solution of equation (3.26), then by the unambiguous solvability of equation (3.9) in $L_2(0,1)$ every next approximation will coincide with it $\bar{\tau}_n'(x) = \bar{\tau}'(x)\dots$.

Reasoning similarly, i.e., as in the derivation of the inequality for $\psi_n(x)$, we get

$$\|\bar{\tau}_n'(x)\|_{L_2(0,1)} \le \frac{k^{n-1}(LP)^{n-1}}{n!}\|\bar{\tau}_1'(x)\|_{L_2(0,1)}.$$

Taking into account that $\bar{\tau}_n'(x) \equiv \bar{\tau}'(x)$ and passing here to the limit as $n \to \infty$, we obtain that $\bar{\tau}'(x) \equiv 0$, as required.

Note that from the convergence of the series (3.25) in $L_2(0,1)$ we get inequality (3.15) or, more precisely,

$$\|\tau'(x)\|_{L_2(0,1)} \le \left(\sum_{n=1}^{\infty}\frac{k^n(LP)^{n-1}}{n!}\right)\|F(x)\|_{L_2(0,1)}.$$

Lemma 3.5 is proved. □

Lemma 3.6 *Let $F(x) \in C^1[0,1]$ and $F(0) = 0$. Then if $\tau'(x) \in L_2(0,1)$ is the solution to equation (3.6), then $\tau'(x) \in C^1[0,1]$ and $\tau'(0) = 0$.*

Proof It is clear that if $\tau'(x)$ is a solution to equation (3.6), then $\tau'(x)$ is a solution to equation (3.9), where

$$F_2(x) = F(x) - T\tau'(x) + AT\tau'(x).$$

Since T is an operator with weak singularity (see (3.16) and (2.27)) and is completely continuous as an operator from $L_2(0,1)$ to $C[0,1]$, and the operator A is bounded operator in $C[0,1]$, by direct calculation we obtain $F_2(x) \in C^1[0,1]$ and $F_2(0) = 0$. Next, applying Lemma 3.2, we get the statement of Lemma 3.6. □

Lemma 3.7 *Let the conditions of Lemma 3.2 be fulfilled. Then for any function $F(t) \in C^1[0,1]$, $F(0) = 0$, equation (3.6) has is a unique solution $\tau'(x) \in C^1[0,1]$, $\tau'(0) = 0$.*

Proof of Lemma 3.7 Proof follows from Lemmas 3.5 and 3.6. □

By Lemma 3.7 equation (3.6) has a unique solution $\tau'(x) \in C^1[0,1]$. From (2.26) by (3.8) of Lemma 3.3 we have $\nu(x) \in C^1[0,1]$.

Thus if $f(x,y) \in C^1(\bar{\Omega})$ and $f(0,0) = 0$, then $\tau(x) \in C^2[0,1]$ and $\nu(x) \in C^1[0,1]$. Then by formulas (2.13) and (2.21) the solution to problem M_2B belongs to V.

Now, acting as in Theorem 2.1, we obtain all the statements of Theorem 3.1 (estimate (2.43) and representations (3.3); see below).

To complete the proof of Theorem 3.1, we note that if conditions (3.10) (of Lemma 3.1) are met. Then conditions (3.13) (of Lemma 3.2) are also met, since $0 < \lambda'(0) < 1$.

Conditions (3.10) are equivalent to condition (3.2).

Indeed, it easily follows from the equation of the curve AD: $\xi = \lambda(\eta)$ in characteristic coordinates that

$$\lambda'(0) = \frac{1-\gamma'(0)}{1+\gamma'(0)} = \operatorname{tg}\left(\frac{\pi}{4}+\omega\right).$$

Theorem 3.1 is proved. □

Denote by B_2 the closure in $L_2(\Omega)$ of the operator given on the set of functions from V satisfying the conditions (2.1), (2.2), and (3.1) with expression (1.2).

The function $z(x,y) \in L_2(\Omega)$ is called a strong solution of problem M_2B if $z(x,y) \in D(B_2)$ and $B_2z(x,y) = f(x,y)$.

Theorem 3.2 *Let condition* (3.2) *be fulfilled. Then there is a unique strong solution to problem M_2B for any function $f(x,y) \in L_2(\Omega)$. This solution satisfies inequality* (2.43) *and can be represented by* (3.3).

Proof Note at once that by Lemmas 3.3–3.6 and representations (2.13) and (2.21) we get inequality (2.43) for all $f(x,y) \in L_2(\Omega)$.

Evaluation (2.43) also implies the uniqueness of a strong solution to problem M_2B.

Due to the density in $L_2(\Omega)$ of the set

$$C_0^1(\bar{\Omega}) = \left\{f(x,y) : f(x,y) \in C^1(\Omega), f(x,y)\big|_{\partial\Omega} = \frac{\partial f(x,y)}{\partial x}\bigg|_{\partial\Omega} = \frac{\partial f(x,y)}{\partial y}\bigg|_{\partial\Omega} = 0\right\},$$

for any function $f(x,y) \in L_2(\Omega)$, there is a sequence $f_n(x,y) \in C_0^1(\bar{\Omega})$ such that $\|f_n(x,y) - f(x,y)\|_0 \to 0$ as $n \to \infty$.

By $z_n(x,y)$ we denote a regular solution to problem M_2B for equation (1.1) with right-hand part $f_n(x,y)$ and initial conditions $\tau_n(x) = z_n(x,0)$, $\nu_n(x) = z_{ny}(x,0)$. By Lemma 3.7 we have $\tau_n(x) \in C^2[0,1]$, $\tau_n(0) = \tau_n{}'(0) = 0$, $\nu_n(x) \in C^1[0,1]$, and therefore by formulas (2.13) and (2.21) we get $z_n(x,y) \in V$ for all $f_n(x,y) \in C_0^1(\bar{\Omega})$.

By the completeness of the space $L_2(\Omega)$ the sequence $f_n(x,y)$ is fundamental. From the linearity of equation (1.1) and estimate (2.43) we obtain that

$$\left\|z_n(x,y) - z_m(x,y)\right\|_{L_2(\Omega)} \le C\left\|f_n(x,y) - f_m(x,y)\right\|_{L_2(\Omega)},$$

i.e., the sequence $\{z_n(x,y)\}$ is fundamental in $L_2(\Omega)$. Taking into account the completeness of the space $L_2(\Omega)$, we obtain that there is a unique limit $z(x,y) \in L_2(\Omega)$ of the sequence $\{z_n(x,y)\}$, which will be the desired strong solution of problem M_2B for equation (1.1) with the right part $f(x,y) \in L_2(\Omega)$.

To complete the proof of Theorem 3.2, we show that for any $f(x,y) \in L_2(\Omega)$, a strong solution to problem M_2B is represented by (3.3).

Since

$$(E+A)^{-1}(E-A) = -E + 2(E+A)^{-1},$$

taking into account (3.16), equation (3.18) can be represented as

$$\tau'(x) - \int_0^x M(x-t)\tau'(t)\,dt = \Phi(x), \tag{3.27}$$

where

$$\Phi(x) = (E+A)^{-1}F(x) = (E+A)^{-1}F_1(x) - Q_0(x) + 2(E+A)^{-1}Q_0(x), \tag{3.28}$$

$$\begin{aligned} \mathrm{M}(x-t) &= m(x-t) - 2\sum_{n=1}^{\infty}(-1)^n\theta\bigl(\lambda^n(x)-t\bigr) \\ &\quad\times\prod_{k=0}^{n-1}\mu\bigl(\lambda^k(x)-t\bigr)m\bigl(\lambda^n(x)-t\bigr), \end{aligned} \tag{3.29}$$

where $\theta(x) = 1$, $x > 0$, $\theta(x) = 0$, $x < 0$.

It is obvious that $\theta(x-t)M(x-t) \in L_2(\Omega\times\Omega)$. The solution of equation (3.27) is represented as

$$\tau'(x) = \Phi(x) + \int_0^x \Gamma_0(x,t)\Phi(t)\,dt,$$

where $\Gamma(x,t)$ is the resolvent of the kernel (3.29) of the integral equation (3.27). Taking into account that $\tau(0) = 0$, we have

$$\tau(x) = \int_0^x \Gamma_1(x,t)\Phi(t)\,dt, \tag{3.30}$$

where $\Gamma_1(x,t) = \int_t^x \Gamma(z,t)\,dz + 1$.

Now in (2.13) or (2.17), taking into account (2.15)–(2.16), (2.21), (3.5), (3.28), and (3.30), after making the necessary calculations, we get (3.3). In formula (3.3),

$$\begin{aligned} M_2(x,y,x_1,y_1) &= \theta(y)\theta(y_1)M_{00}(x,y,x_1,y_1) + \theta(y)\theta(-y_1)M_{01}(x,y,x_1,y_1) \\ &\quad + \theta(-y)\theta(y_1)M_{10}(x,y,x_1,y_1) + \theta(-y)\theta(-y_1)M_{11}(x,y,x_1,y_1), \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} M_{00}(x,y,x_1,y_1) &= \theta(x-x_1)E(x-x_1,y,y_1) \\ &\quad + \int_0^1 \theta(x-t)E_\eta(x-t,y,\eta)\big|_{\eta=0}P_1(t,x_1,y_1)\,dt, \\ M_{01}(x,y,x_1,y_1) &= \frac{1}{2}\int_0^1 \theta(x-t)E_\eta(x-t,y,\eta)\big|_{\eta=0}N_1(t,\xi_1,\eta_1)\,dt, \\ \xi_1 &= x_1 + y_1, \qquad \eta_1 = x_1 - \eta_1. \end{aligned}$$

$$\begin{aligned} &P_i(x,x_1,y_1) \\ &\quad = -\theta(x-x_1)\int_{x_1}^x \Gamma_i(x,t)E_y(t-x_1,0,y_1)\,dt \\ &\qquad + 2\sum_{n=1}^{\infty}(-1)^n\theta\bigl(\lambda^n(x)-t\bigr)\int_{x_1}^{\lambda(x)}\Gamma_i\bigl(x,\delta^n(t)\bigr)\prod_{k=0}^{n-1}\frac{\mu(\delta^{n-k}(t))}{\lambda'(\delta^{n-k}(t))}E_y(t-x,0,y_1)\,dt, \end{aligned}$$

$$
\begin{aligned}
&N_i(x,\xi_1,\eta_1)\\
&\quad= 2\sum_{n=0}^{\infty}(-1)^n\Bigg[\theta\big(\lambda^n(x)-\xi_1\big)\theta\big(\lambda^n(x)-\eta_1\big)\Gamma_i\big(x,\delta^n(\eta_1)\big)\prod_{k=0}^{n-1}\frac{\mu(\delta^{n-k}(\eta_1))}{\lambda'(\delta^{n-k}(\eta_1))}\\
&\qquad-\theta\big(\lambda^{n+1}(x)-\xi_1\big)\theta\big(\delta(\xi_1)-\eta_1\big)\Gamma_i\big(x,\delta^{n+1}(\xi_1)\big)\prod_{k=0}^{n}\frac{\mu(\delta^{n+1-k}(\xi_1))}{\lambda'(\delta^{n+1-k}(\xi_1))}\Bigg],\\
&\qquad i=0,1.
\end{aligned}
$$

where $\xi=\lambda(\eta)$, $0<\eta<1$, or $\eta=\delta(\xi)$, $0<\xi<\xi_0=\lambda(1)$ of the equation of the curve AD in characteristic coordinates $\xi=x+y$, $\eta=x-y$, $\delta^n(t)=\delta(\delta^{n-1}(t))$, $\delta^0(t)=t$,

$$
\begin{aligned}
&M_{10}(x,y,x_1,y_1)\\
&\quad= P_1(\xi,x_1,y_1)+\frac{1}{2}\int_0^{\lambda(\xi)}m_1(t,\xi,\eta)P_0(t,x_1,y_1)\,dt\\
&\qquad+\frac{1}{2}\int_{\lambda(\xi)}^{\lambda(\eta)}\Bigg[m_1\big(t,\delta(t),\eta\big)-\frac{\mu(\delta(t))}{\lambda'(\delta(t))}\Bigg]P_0(t,x_1,y_1)\,dt\\
&\qquad-\frac{1}{2}\int_0^{\lambda(\xi)}\theta(t-x_1)m_1(t,\xi,\eta)E_y(t-x_1,0,y_1)\,dt\\
&\qquad-\frac{1}{2}\int_{\lambda(\xi)}^{\lambda(\eta)}\theta(t-x_1)m_1\big(t,\delta(t),\eta\big)E_y(t-x_1,0,y_1)\,dt\\
&\qquad+\sum_{n=0}^{\infty}(-1)^n\Bigg\{\int_0^{\lambda^{n+1}(\xi)}\theta(t-x_1)m_1\big(\delta^n(t),\xi,\eta\big)\prod_{k=0}^{n-1}\frac{\mu(\delta^{n-k}(t))}{\lambda'(\delta^{n-k}(t))}E_y(t-x_1,0,y_1)\,dt\\
&\qquad+\int_{\lambda^{n+1}(\xi)}^{\lambda^{n+1}(\eta)}\theta(t-x_1)\Bigg[m_1\big(\delta^n(t),\delta^{n+1}(t),\eta\big)-\frac{\mu(\delta^{n+1}(t))}{\lambda'(\delta^{n+1}(t))}\Bigg]\\
&\qquad\times\prod_{k=0}^{n}\frac{\mu(\delta^{n-k}(t))}{\lambda'(\delta^{n-k}(t))}E_y(t-x_1,0,y_1)\,dt\Bigg\},\\
&m_1(t,\xi,\eta)=\int_{\xi}^{\eta}\mu(z)m\big(\lambda(z)-t\big)\,dt,\\
&M_{11}(x,y,x_1,y_1)\\
&\quad=\theta(\eta-\eta_1)(\eta_1-\xi)-\theta\big(\lambda(\eta)-\xi_1\big)\theta\big(\xi_1-\lambda(\xi)\big)\theta\big(\delta(\xi_1)-\eta_1\big)\\
&\qquad\times\frac{\mu(\delta(\xi_1))}{\lambda'(\delta(\xi_1))}+N_1(x,\xi_1,\eta_1)+\frac{1}{2}\int_0^{\lambda(\xi)}m_1(t,\xi,\eta)N_0(t,\xi_1,\eta)\,dt\\
&\qquad+\frac{1}{2}\int_{\lambda(\xi)}^{\lambda(\eta)}\Bigg[m_1\big(t,\delta(t),\eta\big)-\frac{\mu(\delta(t))}{\lambda'(\delta(t))}\Bigg]N_0(t,\xi_1,\eta_1)\,dt+\sum_{n=0}^{\infty}(-1)^n\\
&\qquad\times\Bigg\{\theta\big(\lambda^{n+1}(\xi)-\eta_1\big)m_1\big(\delta^n(\eta_1),\xi,\eta\big)\prod_{k=0}^{n-1}\frac{\mu(\delta^{n-k}(\eta_1))}{\lambda'(\delta^{n-k}(\eta_1))}\\
&\qquad-\theta\big(\delta(\xi_1)-\eta_1\big)\theta\big(\lambda^{n+2}(\xi)-\xi_1\big)m_1\big(\delta^{n+1}(\xi_1),\xi,\eta\big)\prod_{k=0}^{n}\frac{\mu(\delta^{n+1-k}(\xi_1))}{\lambda'(\delta^{n+1-k}(\xi_1))}\\
&\qquad+\theta\big(\lambda^{n+1}(\eta)-\eta_1\big)\theta\big(\eta_1-\lambda^{n+1}(\xi)\big)\Bigg[m_1\big(\delta^n(\eta_1),\delta^{n+1}(\eta_1),\eta\big)-\frac{\mu(\delta^{n+1}(\eta_1))}{\lambda'(\delta^{n+1}(\eta_1))}\Bigg]
\end{aligned}
$$

$$\times \prod_{k=0}^{n} \frac{\mu(\delta^{n-k}(\eta_1))}{\lambda'(\delta^{n-k}(\eta_1))} - \theta\big(\lambda^{n+2}(\eta) - \xi_1\big)\theta\big(\xi_1 - \lambda^{n+2}(\xi)\big)\theta\big(\delta(\xi_1) - \eta_1\big)$$
$$\times \left[m_1\big(\delta^{n+1}(\xi_1), \delta^{n+2}(\xi_1), \eta\big) - \frac{\mu(\delta^{n+2}(\xi_1))}{\lambda'(\delta^{n+2}(\xi_1))} \right] \prod_{k=0}^{n} \frac{\mu(\delta^{n+1-k}(\xi_1))}{\lambda'(\delta^{n+1-k}(\xi_1))} \Bigg\}.$$

Similarly, acting as in Sect. 2, it is not difficult to establish that

$$M_2(x, y, x_1, y_1) \in L_2(\Omega \times \Omega).$$

Theorem 3.2 is proved. □

As noted above, the operator corresponding to problem M_2B is denoted by B_2. The main result of this section is the following:

Theorem 3.3 *Let condition* (3.2) *be fulfilled. Then problem M_2B is Volterra, that is, for any complex number λ, the solution to the equation*

$$B_2 z(x, y) - \lambda z(x, y) = f(x, y) \tag{3.32}$$

exists and is unique for all $f(x, y) \in L_2(\Omega)$.

Proof By Theorem 3.2 the inverse operator B_2^{-1} of problem M_2B (of operator B_2) exists, is defined everywhere on $L_2(\Omega)$, is presented in the form

$$B_2^{-1} f(x, y) = \iint_{\Omega} M_2(x, y, x_1, y_1) f(x_1, y_1)\, dx_1\, dy_1$$

and thus is completely continuous. Therefore, to prove Theorem 3.3, it remains to show that B_2^{-1} is quasinilpotent in $L_2(\Omega)$. To do this, we will use the Volterra criterion of integral operators by Nersesyan [24]. We need the following concepts.

Definition 1 Let $S \subset \Omega \times \Omega$. $M(U, U_1)$ is called an S-kernel if $M(U, U_1) \in L_2(\Omega \times \Omega)$ and $M(U, U_1) = 0$ for $(U, U_1) \in S$.

Definition 2 An open set $S \subset \Omega \times \Omega$ is called a set of V type if any S-kernel does not have eigenvalues.

Let us introduce the notation:

$$U \xrightarrow{S} U_2 \quad \text{if } (U, U_1) \in S, \qquad U \xleftarrow{S} U_2 \quad \text{if } (U, U_1) \notin S.$$

Theorem ([24]) *In order for S to be a set of V type, it is necessary and sufficient that for any $k \geq 1$, from the condition*

$$U_1(x_1, y_1) \xrightarrow{S} U_2(x_2, y_2) \xrightarrow{S} U_3(x_3, y_3) \xrightarrow{S} \cdots \xrightarrow{S} U_k(x_k, y_k)$$

it follows that $U_k(x_k, y_k) \xleftarrow{S} U_1(x_1, y_1)$.

From (3.31) it is not difficult to see that $M_2(x,y,x_1,y_1) \equiv 0$ if $x < x_1$.

In our case the kernel $M_2(x,y,x_1,y_1)$ is an S-kernel for the set $S \subset \Omega \times \Omega$ defined by the relation $(U_1(x_1,y_1), U_2(x_2,y_2)) \in S$ if $x_1 < x_2$.

Let us consider the sequence of points $U_i(x_i,y_i) \in \Omega$, $i = 1,2,\dots,k$.

Let the conditions $U_1(x_1,y_1) \overset{S}{\to} U_2(x_2,y_2) \overset{S}{\to} \cdots \overset{S}{\to} U_k(x_k,y_k)$ be satisfied for any $k \geq 1$. Then we have a chain of inequalities $x_1 < x_2 < \cdots < x_k$. Since $x_1 < x_k$, $(U_k(x_k,y_k), U_1(x_1,y_1)) \notin S$. Therefore our set S is a set of type V. Thus the operator B_2^{-1} has no eigenvalues and by complete continuity is a Volterra operator. From this Theorem 3.3 easily follows.

Indeed, due to the reversibility of the operator B_2, the unambiguous solvability of equation (3.32) is equivalent to the unambiguous solvability of equation $z(x,y) - \lambda B_2^{-1} z(x,y) = B_2^{-1} f$, which is a Volterra-type equation of the second kind. Theorem 3.3 is proved. □

4 A problem with nonlocal conditions for a diffusion–hyperbolic equation

In the last section, we formulate a nonlocal problem for equation (1.1), the distinguishing feature of which (from the previously considered problems) is that in the hyperbolic part of the mixed domain, the nonlocal condition pointwise connects the tangent derivatives of the desired solution on the characteristic AC and on an arbitrary curve AD lying inside the characteristic triangle ABC.

Problem M_3B Find a solution of equation (1.1) satisfying conditions (2.1), (2.2), and

$$\frac{d}{dt} z\bigl[\theta_0(t)\bigr] + \mu(t) \frac{d}{dt} z\bigl[\theta^*(t)\bigr] = 0. \tag{4.1}$$

Note that if $\mu(t) = \mu = \text{const}$, then condition (4.1) is equivalent to

$$z\bigl[\theta_0(t)\bigr] + \mu z\bigl[\theta^*(t)\bigr] = 0,$$

which pointwise connects the values of the desired solution on the characteristic with the value of the solution on some curve lying strictly inside the domain.

In case where $\alpha = 1$ and $\mu(t) = \infty$ ($\mu^{-1}(t) = 0$), from problem M_3B: (1.1), (2.1), (2.2), and (4.1) we obtain an analogue of the generalized Tricomi problem (problem M in the terminology of A.V. Bitsadze) for a parabolic–hyperbolic equation with an uncharacteristic line of type change. The strong solvability and Volterra of problem M for equation (1.1) were first proved by Salakhitdinov and Berdyshev [8] (see Sect. 2).

The function $z(x,y) \in V$ is called a regular solution to problem M_3B if $z(x,y)$ satisfies conditions (2.1), (2.2), (4.1), and equation (1.1) in $\Omega_0 \cup \Omega_1$.

The function $z(x,y) \in L_2(\Omega)$ is called a strong solution to problem M_3B if there exists a sequence $\{z_n(x,y)\}$ satisfying conditions (2.1), (2.2), (4.1), and $z_n(x,y) \in V$ such that $z_n(x,y)$ and $Lz_n(x,y)$ converge in $L_2(\Omega)$, respectively, to $z(x,y)$ and $f(x,y)$. The following theorems on the regular and strong solvability of problem M_3B are valid.

Theorem 4.1 *Let* $\mu(t) \in C^2[0,1]$, $\mu(t) \neq -1$, $0 \leq t \leq 1$, *and*

$$\left| \frac{\mu(0)}{1+\mu(0)} \right|^2 < \operatorname{ctg}\left(\omega + \frac{\pi}{4}\right), \quad -\frac{\pi}{4} < \omega < 0. \tag{4.2}$$

Then for any function $f(x,y) \in C^1(\bar{\Omega})$, $f(0,0) = 0$, *there is a unique regular solution to problem* M_3B. *This solution satisfies inequality* (2.43) *and can be represented as*

$$z(x,y) = \iint_{\Omega} M_3(x,y,x_1,y_1) f(x_1,y_1)\,dx_1\,dy_1, \tag{4.3}$$

where $M_3(x,y,x_1,y_1) \in L_2(\Omega \times \Omega)$.

Theorem 4.2 *Let the conditions of Theorem* 4.1 *be fulfilled. Then for any function* $f(x,y) \in L_2(\Omega)$, *there is a unique strong solution to problem* M_3B. *This solution satisfies inequality* (2.43) *and can be represented as* (4.3).

As before, by B_3 we denote the closure in $L_2(\Omega)$ of the operator given by expression (1.2) on the set of functions V satisfying conditions (2.1), (2.2), and (4.1). The domain $D(B_3)$ of the operator B_3 obviously consists of strong solutions to problem M_3B. It follows from Theorem 4.2 that under condition (4.2), the operator B_3 is invertible, and the inverse operator B_3^{-1} is defined everywhere on $L_2(\Omega)$ and by evaluation (2.43) and representation (4.3) is completely continuous. Therefore, if there is a spectrum of operator B_3 (problem M_3B), then it can consist only of eigenvalues of finite multiplicity.

The purpose of the last section is to prove the following theorem, which states that when condition (4.2) is fulfilled, there are no eigenvalues of problem M_3B (of the operator B_3).

Theorem 4.3 *Let the conditions of Theorem* 4.1 *be fulfilled. Then the inverse operator*

$$B_3^{-1} f(x,y) = \iint_{\Omega} M_3(x,y,x_1,y_1) f(x_1,y_1)\,dx_1\,dy_1$$

to the operator of problem M_3B *is Volterra. This theorem easily implies the absence of eigenvalues of problem* M_3B.

Applying the same notations as in the previous section, satisfying condition (4.1) in the D'alembert formula (2.13), we obtain

$$\bigl(1+\mu(t)\bigr)\tau'(t) + \mu(t)\lambda'(t)\tau'\bigl(\lambda(t)\bigr) - \bigl(1+\mu(t)\bigr)\nu(t) + \mu(t)\lambda'(t)\nu\bigl(\lambda(t)\bigr) = F_3(t), \tag{4.4}$$

where

$$F_3(t) = 2\int_0^t f_1(\xi_1,t)\,d\xi_1 - 2\mu(t)\int_{\lambda(t)}^t f_1\bigl(\lambda(t),\eta_1\bigr)\,d\eta_1 + 2\mu(t)\int_{\lambda(t)}^t f_1(\xi_1,t)\,d\xi_1.$$

Relation (4.4) is the basic relation between $\tau'(x)$ and $\nu(x)$, brought to the segment AB from the hyperbolic part of the mixed domain Ω.

By the unambiguous solvability of the boundary value problem C_2 for equation (1.1) (with conditions (2.1)–(2.2) and $z|_{AB} = \tau(x)$), acting similarly as in the previous section, we obtain the basic functional relation between $\tau'(x)$ and $\nu(x)$, brought to the segment from the parabolic part of the mixed domain in the form (2.26).

Now excluding from (2.26) and (4.4) the function $\nu(x)$, for $\tau'(x)$, we obtain the integro-functional equation

$$\begin{aligned}&\tau'(t)+\frac{\mu(t)\lambda'(t)}{1+\mu(t)}\tau'\bigl(\lambda(t)\bigr)+\int_0^t m(t-z)\tau'(z)\,dz\\&\quad-\frac{\mu(t)\lambda'(t)}{1+\mu(t)}\int_0^{\lambda(t)} m\bigl(\lambda(t)-z\bigr)\tau'(z)\,dz=F_4(t),\end{aligned}\tag{4.5}$$

where

$$F_4(t)=\frac{F_3(t)}{1+\mu(t)}-\frac{\mu(t)\lambda'(t)}{1+\mu(t)}Q_0\bigl(\lambda(t)\bigr)-Q_0(t).$$

Now, in the presence of (4.5), the proofs of Theorems 4.1–4.3 are carried out in the same way as in Sect. 3, so we do not give them here, but only note that in this case the kernel $M_3(x,y,x_1,y_1)$ in (4.3) has the form

$$\begin{aligned}M_3(x,y,x_1,y_1)&=\theta(y)\theta(y_1)M_{00}(x,y,x_1,y_1)+\theta(y)\theta(-y_1)M_{01}(x,y,x_1,y_1)\\&\quad+\theta(-y)\theta(y_1)M_{10}(x,y,x_1,y_1)+\theta(-y)\theta(-y_1)M_{11}(x,y,x_1,y_1),\end{aligned}$$

where

$$\begin{aligned}&M_{00}(x,y,x_1,y_1)=\theta(x-x_1)E(x-x_1,y,y_1)\\&\qquad+\int_0^1\theta(x-t)E_\eta(x-t,y,\eta)\big|_{\eta=0}P_2(t,x_1,y_1)\,dt,\\&M_{01}(x,y;x_1,y_1)=\frac{1}{2}\int_0^1\theta(x-t)E_\eta(x-t,y,\eta)\big|_{\eta=0}N_2(t,\xi_1,\eta_1)\,dt,\\&\xi_1=x_1+y_1,\qquad\eta_1=x_1-y_1,\qquad\theta(x)=1,\quad x>0,\qquad\theta(x)=0,\quad x<0,\\&P_2(x,x_1,y_1)\\&\quad=-\theta(x-x_1)\int_{x_1}^x\Gamma_3(x,t)E_y(t-x_1,0,y_1)\,dt\\&\qquad+2\sum_{n=0}^\infty(-1)^n\theta\bigl(\lambda^n(x)-x_1\bigr)\int_{x_1}^{\lambda_n(x)}\frac{\Gamma_3(x,\delta^n(t))}{1+\mu(t)}E_y(t-x,0,y_1)\prod_{k=0}^{n-1}\frac{\mu(\delta^{n-k}(t))}{1+\mu(\delta^{n-k}(t))},\\&N_2(x,\xi_1,\eta_1)\\&\quad=2\sum_{n=0}^\infty(-1)^n\biggl\{\theta\bigl(\lambda^n(x)-\eta_1\bigr)\frac{\Gamma_3(x_1,\delta^n(\eta_1))}{1+\mu(\eta_1)}\\&\qquad\times\bigl[\theta\bigl(\lambda^n(x)-\xi_1\bigr)+\mu(\eta_1)\theta\bigl(\xi_1-\delta(\eta_1)\bigr)\bigr]\prod_{k=0}^{n-1}\frac{\mu(\delta^{n-k}(\eta_1))}{1+\mu(\delta^{n-k}(\eta_1))}\\&\qquad-\theta\bigl(\lambda^{n+1}(x)-\xi_1\bigr)\theta\bigl(\delta(\xi_1)-\eta_1\bigr)\frac{\mu(\delta(\xi_1))\Gamma_3(x,\delta^{n+1}(\xi_1))}{1+\mu(\delta(\xi_1))}\prod_{k=0}^{n}\frac{\mu(\delta^{n+1-k}(\xi_1))}{1+\mu(\delta^{n+1-k}(\xi_1))}\biggr\}.\end{aligned}$$

Here, as before, $\xi=\lambda(\eta)$, $0\le\eta\le1$, or $\eta=\delta(\xi)$, $0\le\xi\le\xi_0=\lambda(1)$, the equation of the curve AD in characteristic coordinates $\xi=x+y$, $\eta=x-y$, $\delta^n(t)=\delta(\delta^{n-1}(t))$, $\delta^0(t)=t$,

$\lambda^n(x) = \lambda(\lambda^{n-1}(x))$, and $E(x,y,y_1)$ is an analogue of the Green function of the first initial boundary value problem (problem C_2) for the diffusion equation in the square AA_0B_0B defined by formula (2.22), $\Gamma_4(x,t)$ is the resolvent of the integral equation kernel

$$\tau'(x) - \int_0^x M(x-t)\tau'(t)\,dt = \Phi(x),$$

$$M(x-t) = m(x-t) - 2\sum_{n=0}^{\infty}(-1)^n \prod_{k=0}^{n-1} \frac{\mu(\lambda^k(x))\lambda'(\lambda^k(x))}{1+\mu(\lambda^k(x))}\theta\bigl(\lambda^n(x)-t\bigr)m\bigl(\lambda^n(x)-t\bigr),$$

the function $m(x-t)$ is defined by (2.27),

$$\Gamma_3(x,t) = 1 + \int_t^x \Gamma_4(z,t)\,dz,$$

$$\begin{aligned}
&M_{10}(x,y;x_1,y_1)\\
&\quad = P_2(\xi,x_1,y_1) + \frac{1}{2}\int_0^{\lambda(\xi)} m_2(t,\xi,\eta)P_3(t,x_1,y_1)\,dt\\
&\qquad + \frac{1}{2}\int_{\lambda(\xi)}^{\lambda(\eta)} \left[m_2\bigl(t,\delta(t),\eta\bigr) - \frac{\mu(\delta(t))}{1+\mu(\delta(t))}\right]P_3(t,x_1,y_1)\,dt\\
&\qquad - \frac{1}{2}\int_0^{\lambda(\xi)} \theta(t-x_1)m_2(t,\xi,\eta)E_y(t-x_1,0,y_1)\,dt\\
&\qquad - \frac{1}{2}\int_{\lambda(\xi)}^{\lambda(\eta)} \theta(t-x_1)m_2\bigl(t,\delta(t),\eta\bigr)E_y(t-x_1,0,y_1)\,dt\\
&\qquad + \sum_{n=0}^{\infty}(-1)^n\left\{\int_0^{\lambda^{n+1}(\xi)} \theta(t-x_1)\frac{m_2(\delta^n(t),\xi,\eta)}{1+\mu(t)}E_y(t-x_1,0,y_1)\prod_{k=0}^{n-1}\frac{\mu(\delta^{n-k}(t))}{1+\mu(\delta^{n-k}(t))}\,dt\right.\\
&\qquad \left. + \int_{\lambda^{n+1}(\xi)}^{\lambda^{n+1}(\eta)} \theta(t-x_1)\frac{E_y(t-x,0,y_1)}{1+\mu(\delta(t))}\left[m_2\bigl(\delta^n(t),\delta^{n+1}(t),\eta\bigr) - \frac{\mu(\delta^{n+1}(t))}{1+\mu(\delta^{n+1}(t))}\right]\right\}\\
&\qquad \times \prod_{k=0}^{n-1}\frac{\mu(\delta^{n-k}(t))}{1+\mu(\delta^{n-k}(t))}\,dt,
\end{aligned}$$

where $m_2(t,\xi,\eta) = \int_\xi^\eta \frac{\mu(z)\lambda'(z)}{1+\mu(z)}m(\lambda(z)-t)\,dz$, $P_3(t,x,y)$ and $N_3(t,\xi,\eta)$ from $P_2(t,x,y)$ and $N_2(t,\xi,\eta)$ differ by that in expressions $P_2(t,\xi,\eta)$ and $N_2(t,\xi,\eta)$, instead $\Gamma_3(t,x)$, it is necessary to write $\Gamma_4(x,t)$;

$$\begin{aligned}
&M_{11}(x,y,x_1,y_1)\\
&\quad = \frac{\theta(\eta-\eta_1)\theta(\eta_1-\xi)\theta(\xi_1-\xi)}{1+\mu(\eta_1)}\bigl[1+\theta\bigl(\xi_1-\lambda(\eta_1)\bigr)\mu(\eta_1)\bigr]\\
&\qquad - \theta\bigl(\lambda(\eta)-\xi_1\bigr)\theta\bigl(\xi_1-\lambda(\xi_1)\bigr)\frac{\theta(\delta(\xi_1)-\eta_1)\theta(\eta_1-\xi_1)\mu(\delta(\xi_1))\delta'(\xi_1)}{1+\mu(\delta(\xi_1))}\\
&\qquad + N_2(\xi,\xi_1,\eta_1) + \frac{1}{2}\int_0^{\lambda(\xi)} m_2(t,\xi,\eta)N_3(t,\xi_1,\eta_1)\,dt\\
&\qquad + \frac{1}{2}\int_{\lambda(\xi)}^{\lambda(\eta)} \left[m_2\bigl(t,\delta(t),\eta\bigr) - \frac{\mu(\delta(t))}{1+\mu(\delta(t))}\right]N_3(t,\xi_1,\eta_1)\,dt
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=0}^{\infty}(-1)^n\Bigg\{\theta\big(\lambda^{n+1}(\xi)-\eta_1\big)\frac{m_2(\delta^n(\eta_1),\xi,\eta)}{1+\mu(\eta_1)}\big[1+\mu(\eta_1)\theta\big(\xi_1-\lambda(\eta_1)\big)\big]\\
&\times\prod_{k=0}^{n-1}\frac{\mu(\delta^{n-k}(\eta_1))}{1+\mu(\delta^{n-k}(\eta_1))}-\theta\big(\lambda^{n+2}(\xi)-\xi_1\big)\theta\big(\delta(\xi_1)-\eta_1\big)\frac{\mu(\delta(\xi_1))m_2(\delta^{n+1}(\xi_1),\xi,\eta)}{\lambda'(\delta(\xi_1))[1+\mu(\delta(\xi_1))]}\\
&\times\prod_{k=0}^{n-1}\frac{\mu(\delta^{n+1-k}(\xi_1))}{1+\mu(\delta^{n+1-k}(\xi_1))}+\theta\big(\lambda^{n+1}(\eta)-\eta_1\big)\theta\big(\eta_1-\lambda^{n+1}(\xi)\big)\frac{1}{1+\mu(\eta_1)}\\
&\times\Bigg[m_2\big(\delta^n(\eta_1),\delta^{n+1}(\eta_1),\eta\big)-\frac{\mu(\delta^{n+1}(\eta_1))}{1+\mu(\delta^{n+1}(\eta_1))}\Bigg]\big[1+\mu(\eta_1)\theta\big(\xi_1-\delta(\eta_1)\big)\big]\\
&\times\prod_{k=0}^{n-1}\frac{\mu(\delta^{n-k}(\eta_1))}{1+\mu(\delta^{n-k}(\eta_1))}-\frac{\theta(\lambda^{n+2}(\eta)-\xi_1)\theta(\xi_1-\lambda^{n+2}(\xi))\theta(\delta(\xi_1)-\eta_1)\mu(\delta(\xi_1))}{\lambda'(\delta(\xi_1))[1+\mu(\delta(\xi_1))]}\\
&\times\Bigg[m_2\big(\delta^{n+1}(\xi_1),\delta^{n+2}(\xi_1),\eta\big)-\frac{\mu(\delta^{n+2}(\xi_1))}{1+\mu(\delta^{n+2}(\xi_1))}\Bigg]\prod_{k=0}^{n-1}\frac{\mu(\delta^{n+1-k}(\xi_1))}{1+\mu(\delta^{n+1-k}(\xi_1))}\Bigg\}.
\end{aligned}
$$

In conclusion, we note that conditions (3.2) and (4.2) are essential for the correctness (Volterra property) of problems M_2B and M_3B discussed in Sects. 3 and 4. In [8], there is an example when, in violation of condition (3.2), the solution of problem M_2B is not unique, that is, zero is an eigenvalue of problem M_2B.

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