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# A higher-order extension of Atangana–Baleanu fractional operators with respect to another function and a Gronwall-type inequality

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**Abstract**

This paper aims to extend the Caputo–Atangana–Baleanu (*ABC*) and Riemann–Atangana–Baleanu (*ABR*) fractional derivatives with respect to another function, from fractional order $\mu \in (0,1]$ to an arbitrary order $\mu \in (n,n+1]$, $n = 0,1,2,\dots$. Also, their corresponding Atangana–Baleanu (*AB*) fractional integral is extended. Additionally, several properties of such definitions are proved. Moreover, the generalization of Gronwall's inequality in the framework of the *AB* fractional integral with respect to another function is introduced. Furthermore, Picard's iterative method is employed to discuss the existence and uniqueness of the solution for a higher-order initial fractional differential equation involving an *ABC* operator with respect to another function. Finally, examples are given to illustrate the effectiveness of the main findings. The idea of this work may attract many researchers in the future to study some inequalities and fractional differential equations that are related to *AB* fractional calculus with respect to another function.

**Keywords:** Fractional differential equations; Fractional calculus; Nonsingular fractional operators; Picard's iterative method

## 1 Introduction

In the last three decades, fractional calculus has been attracting the interest of many authors in several fields for the sake of a better description of chaotic complex systems, for example, dynamic systems, rheology, electrical networks, blood-flow phenomena, biophysics, and qualitative theories; see for more details [1, 14, 19, 22, 26–29]. In order to realize and describe the real phenomena in the fields of science and engineering, some researchers have developed fractional calculus to singular *and* nonsingular kernels. Caputo and Fabrizio [13], investigated a new definition of a fractional operator with an exponential kernel. Atangana and Baleanu [11], introduced a new interesting definition of a fractional operator with a Mittag–Leffler kernel that is called the *Atangana–Baleanu* (*AB*) fractional operator. Abdeljawad [2], generalized the *AB* fractional operator to higher arbitrary order. Then, many researchers studied the qualitative properties and approximate solutions of fractional differential equations involving *ABC fractional operators and Caputo–Fabrizio derivatives*, we refer the readers to [4–9, 12, 15, 24, 25]. *In particular, the authors of [17] studied a two-step reversible enzymatic reaction Dynamics system via ABC-fractional derivatives. Khan et al. [18] established open-channel flow of grease as a Maxwell fluid with $MoS_2$ by utilizing the Caputo–Fabrizio fractional model.*

Very recently, Fernandez and Baleanu [13], presented a fractional derivative of a function with respect to another function with a Mittag–Leffler kernel, which is in fact considered as a generalized *AB* fractional operator. Mohammed and Abdeljawad [21], constructed a connection between the *AB fractional operator and the Riemann–Liouville fractional integral with respect to another function* by formulating the corresponding *AB*-fractional integral of a function with respect to another function. Then, Kashuri [16], presented a fractional integral operator called the *Atangana–Baleanu–Kashuri* (*ABK*) fractional integral.

*Motivated by what has been mentioned above, the novelty and contributions of this paper are to extend the ABC, ABR fractional derivatives with respect to another increasing positive function $\phi$ (which are mostly called $\phi - ABC$, $\phi - ABR$, respectively), and their corresponding AB fractional integrals with respect to another function $\phi$ ($\phi - AB$), from order $\mu \in (0,1]$ to an arbitrary order $\mu \in (n, n+1]$. In fact, these provide a more reasonable extension than those of Abdeljawad in [21]. Indeed, the extension we obtain still shows that the ABR differential operator and AB integral operators are inverses of each other for orders more than 1 (see parts (i) and (ii) of Proposition 3.6). Also, several properties and applications of these definitions are investigated. Moreover, a new generalized Gronwall inequality in the framework of the $\phi - AB$ fractional integrals is introduced. The existence and uniqueness results of a higher-order $\phi - ABC$ fractional problem under initial boundary conditions are established by Picard's iterative method.*

Our paper is structured as follows: Some interesting preliminaries are provided in Sect. 2. In Sect. 3, the $\phi - AB$ fractional operator is extended to a higher order and a new generalized Gronwall inequality in the sense of the $\phi - AB$ fractional integral is investigated, moreover the existence and uniqueness results of $\phi - ABC$ initial fractional differential equation are established. Then, examples that represent the validity of the main findings are provided in Sect. 4. Finally, we conclude our results in Sect. 5.

## 2 Preliminaries

In this section, we present some essential preliminaries related to fractional calculus. Let us denote by $\mathcal{C}^n(\mathrm{J}, \mathbb{R})$ the Banach space of all the $n$th continuously differentiable functions $\omega$ equipped with usual norm $\|\omega\| = \sup\{|\omega(\mathrm{u})| : \mathrm{u} \in \mathrm{J} = [\iota, \tau]\}$.

**Definition 2.1** ([10]) Let $\phi : [\iota, \tau] \to \mathbb{R}$ be an increasing function $\forall \mathrm{u} \in [\iota, \tau]$. For $\mu > 0$, the $\mu$th left-sided $\phi$-Riemann–Liouville fractional integral for an integrable function $\omega : [\iota, \tau] \to \mathbb{R}$ with respect to another function $\phi(\mathrm{u})$ is given by

$$^{RL}\mathfrak{I}_{\iota}^{\mu,\phi}\omega(\mathrm{u}) = \frac{1}{\Gamma(\mu)} \int_{\iota}^{\mathrm{u}} \big(\phi(\mathrm{u}) - \phi(\mathrm{v})\big)^{\mu-1} \phi'(\mathrm{v})\omega(\mathrm{v})\,d\mathrm{v},$$

where $\Gamma(\mu) = \int_0^{+\infty} e^{-\mathrm{u}}\mathrm{u}^{\mu-1}\,d\mathrm{u}$, $\mu > 0$.

**Definition 2.2** ([3, 11]) Consider $\mu \in (0,1]$ and $\omega \in \mathcal{H}^1(\iota, \tau)$. The $\mu$th left-sided Riemann–Liouville fractional derivative in the sense of Atangana–Baleanu for a function $\omega$ is given by

$$\left({}^{ABR}\mathfrak{D}_{\iota}^{\mu}\omega\right)(\mathrm{u}) = \frac{\Delta(\mu)}{1-\mu}\frac{d}{d\mathrm{u}}\int_{\iota}^{\mathrm{u}}\mathbb{E}_{\mu}\left(\frac{-\mu}{1-\mu}(\mathrm{u}-\mathrm{v})^{\mu}\right)\omega(\mathrm{v})\,d\mathrm{v}, \quad \mathrm{u}\in[\iota,\tau],$$

where $\Delta(\mu)$ is the normalization function with $\Delta(0) = \Delta(1) = 1$, and $\mathbb{E}_{\mu}$ is called the Mittag–Leffler function defined by

$$\mathbb{E}_{\mu}(\mathrm{r}) = \sum_{i=0}^{\infty}\frac{\mathrm{r}^{i}}{\Gamma(\mu i+1)},$$

where $Re(\mu) > 0$, $\mathrm{r}\in\mathbb{C}$.

**Definition 2.3** ([3, 11]) Consider $\mu\in(0,1]$ and $\omega\in\mathcal{H}^{1}(\iota,\tau)$. The $\mu$th left-sided Caputo fractional derivative in the sense of Atangana–Baleanu for a function $\omega$ is given by

$$\left({}^{ABC}\mathfrak{D}_{\iota}^{\mu}\omega\right)(\mathrm{u}) = \frac{\Delta(\mu)}{1-\mu}\int_{\iota}^{\mathrm{u}}\mathbb{E}_{\mu}\left(\frac{-\mu}{1-\mu}(\mathrm{u}-\mathrm{v})^{\mu}\right)\omega'(\mathrm{v})\,d\mathrm{v}, \quad \mathrm{u}\in[\iota,\tau].$$

**Definition 2.4** ([3, 11]) Consider $\mu\in(0,1]$ and $\omega\in\mathcal{H}^{1}(\iota,\tau)$. The $\mu$th left-sided Riemann–Liouville fractional integral in the sense of Atangana–Baleanu for a function $\omega$ is given by

$$\left({}^{AB}\mathfrak{I}_{\iota}^{\mu}\omega\right)(\mathrm{u}) = \frac{1-\mu}{\Delta(\mu)}\omega(\mathrm{u}) + \frac{\mu}{\Delta(\mu)}{}^{RL}\mathfrak{I}_{\iota}^{\mu}\omega(\mathrm{u}), \quad \mathrm{u}\in[\iota,\tau].$$

**Definition 2.5** ([16]) Consider $\mu\in(0,1]$, $\rho>0$ and $\omega\in\mathcal{H}_{c}^{p}(\iota,\tau)$. The $\mu$th left-sided Kashuri fractional integral in the sense of Atangana–Baleanu for a function $\omega$ is given by

$$\left({}^{ABK}\mathfrak{I}_{\iota}^{\mu,\rho}\omega\right)(\mathrm{u}) = \frac{1-\mu}{\Delta(\mu)}\omega(\mathrm{u}) + \frac{\mu}{\Delta(\mu)}\frac{1}{\Gamma(\mu)}\int_{\iota}^{\mathrm{u}}\mathrm{v}^{\rho-1}\left(\frac{\mathrm{u}^{\rho}-\mathrm{v}^{\rho}}{\rho}\right)^{\mu-1}\omega(\mathrm{v})\,d\mathrm{v}, \quad \mathrm{u}\in[\iota,\tau].$$

**Definition 2.6** ([13, 21]) Let $\phi:[\iota,\tau]\to\mathbb{R}$ be an increasing function with $\phi'(\mathrm{u})\neq 0$, $\forall\mathrm{u}\in[\iota,\tau]$. Consider $\mu\in(0,1]$ and $\omega\in\mathcal{H}^{1}(\iota,\tau)$. The $\mu$th left-sided Riemann–Liouville fractional derivative in the sense of Atangana–Baleanu of a function $\omega$ with respect to another function $\phi(\mathrm{u})$ is given by

$$\left({}^{ABR}\mathfrak{D}_{\iota}^{\mu,\phi}\omega\right)(\mathrm{u}) = \frac{\Delta(\mu)}{(1-\mu)\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\int_{\iota}^{\mathrm{u}}\phi'(\mathrm{v})\mathbb{E}_{\mu}\left(\frac{-\mu}{1-\mu}\left(\phi(\mathrm{u})-\phi(\mathrm{v})\right)^{\mu}\right)\omega(\mathrm{v})\,d\mathrm{v},$$
$$\mathrm{u}\in[\iota,\tau].$$

**Definition 2.7** ([13, 21]) Let $\phi:[\iota,\tau]\to\mathbb{R}$ be an increasing function with $\phi'(\mathrm{u})\neq 0$, $\forall\mathrm{u}\in[\iota,\tau]$. Consider $\mu\in(0,1]$ and $\omega\in\mathcal{H}^{1}(\iota,\tau)$. The $\mu$th left-sided Caputo fractional derivative in the sense of Atangana–Baleanu of a function $\omega$ with respect to another function $\phi(\mathrm{u})$ is given by

$$\left({}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\omega\right)(\mathrm{u}) = \frac{\Delta(\mu)}{1-\mu}\int_{\iota}^{\mathrm{u}}\phi'(\mathrm{v})\mathbb{E}_{\mu}\left(\frac{-\mu}{1-\mu}\left(\phi(\mathrm{u})-\phi(\mathrm{v})\right)^{\mu}\right)\omega'_{\phi}(\mathrm{v})\,d\mathrm{v}, \quad \mathrm{u}\in[\iota,\tau],$$

where $\omega'_{\phi}(\mathrm{u}) = \frac{\omega'(\mathrm{u})}{\phi'(\mathrm{u})}$.

**Definition 2.8** ([21]) Let $\phi : [\iota, \tau] \to \mathbb{R}$ be an increasing function $\forall \mathrm{u} \in [\iota, \tau]$. Consider $\mu \in (0,1]$ and $\omega \in \mathcal{H}^1(\iota, \tau)$. The $\mu$th left-sided Riemann–Liouville fractional integral in the sense of Atangana–Baleanu of a function $\omega$ with respect to another function $\phi(\mathrm{u})$ is given by

$$\left({}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\omega\right)(\mathrm{u}) = \frac{1-\mu}{\Delta(\mu)}\omega(\mathrm{u}) + \frac{\mu}{\Delta(\mu)}\,{}^{RL}\mathfrak{I}_{\iota}^{\mu,\phi}\omega(\mathrm{u}), \quad \mathrm{u} \in [\iota, \tau].$$

*Remark* 2.9 We note that,

- By putting $\phi(\mathrm{u}) = \mathrm{u}$ in Definitions 2.6, 2.7, and 2.8, then we have Definitions 2.2, 2.3, and 2.4, respectively.
- By putting $\phi(\mathrm{u}) = \frac{\mathrm{u}^{\rho}}{\rho}$ in Definition 2.8, then we have Definition 2.5.

**Lemma 2.10** ([10]) *Let* $\mu, \varrho > 0$ *and* $\omega : [\iota, \tau] \to \mathbb{R}$. *Then,*

(i) ${}^{RL}\mathfrak{I}_{\iota}^{\mu,\phi}[\phi(\mathrm{u}) - \phi(\iota)]^{\varrho-1} = \frac{\Gamma(\varrho)}{\Gamma(\mu+\varrho)}[\phi(\mathrm{u}) - \phi(\iota)]^{\mu+\varrho-1}$;

(ii) ${}^{RL}\mathfrak{I}_{\iota}^{\mu,\phi}\,{}^{RL}\mathfrak{I}_{\iota}^{\varrho,\phi}\omega(\mathrm{u}) = {}^{RL}\mathfrak{I}_{\iota}^{\mu+\varrho,\phi}\omega(\mathrm{u})$;

(iii) $((\frac{1}{\phi(\mathrm{u})}\frac{d}{d\mathrm{u}})^n\,{}^{RL}\mathfrak{I}_{\iota}^{n,\phi}\omega)(\mathrm{u}) = \omega(\mathrm{u})$, $n \in \mathbb{N}$.

**Lemma 2.11** ([21]) *For* $\mu \in (0,1]$, *the following relations hold*:

(i) $({}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\,{}^{ABR}\mathfrak{D}_{\iota}^{\mu,\phi}\omega)(\mathrm{u}) = \omega(\mathrm{u})$;

(ii) $({}^{ABR}\mathfrak{D}_{\iota}^{\mu,\phi}\,{}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\omega)(\mathrm{u}) = \omega(\mathrm{u})$.

## 3 Main results

### 3.1 Higher order of fractional derivatives and integrals

In this subsection, we will introduce the definitions of higher order of fractional derivatives and integrals in the framework of Atangana–Baleanu with respect to another function $\phi$.

**Definition 3.1** Consider $\phi : [\iota, \tau] \to \mathbb{R}^+$ to be an increasing function with $\phi'(\mathrm{u}) \neq 0$, $\forall \mathrm{u} \in [\iota, \tau]$, $\mathfrak{g} \in \mathcal{H}^1(\iota, \tau)$ and $\mu \in (n, n+1]$, $\vartheta = \mu - n$, $n = 0, 1, 2, \ldots$. Then, the $\mu$th left-sided $\phi - ABR$ fractional derivative is given by

$$\begin{aligned}
&\left({}^{ABR}\mathfrak{D}_{\iota}^{\mu,\phi}\mathfrak{g}\right)(\mathrm{u}) \\
&\quad= \left(\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)^n \left({}^{ABR}\mathfrak{D}_{\iota}^{\vartheta,\phi}\mathfrak{g}(\mathrm{u})\right) \\
&\quad= \left(\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)^n \frac{\Delta(\vartheta)}{(1-\vartheta)\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\int_{\iota}^{\mathrm{u}} \phi'(\mathrm{v})\mathbb{E}_{\vartheta}\left(\frac{-\vartheta}{1-\vartheta}\big(\phi(\mathrm{u}) - \phi(\mathrm{v})\big)^{\vartheta}\right)\mathfrak{g}(\mathrm{v})\,d\mathrm{v} \\
&\quad= \frac{\Delta(\mu-n)}{(n+1-\mu)}\left(\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)^{n+1}\int_{\iota}^{\mathrm{u}} \phi'(\mathrm{v})\mathbb{E}_{\mu-n}\left(\frac{-(\mu-n)}{n+1-\mu}\big(\phi(\mathrm{u}) - \phi(\mathrm{v})\big)^{\mu-n}\right)\mathfrak{g}(\mathrm{v})\,d\mathrm{v}.
\end{aligned}$$

**Definition 3.2** Consider $\phi : [\iota, \tau] \to \mathbb{R}^+$ to be an increasing function with $\phi'(\mathrm{u}) \neq 0$, $\forall \mathrm{u} \in [\iota, \tau]$, $\mathfrak{g}^{(n)} \in \mathcal{H}^1(\iota, \tau)$ and $\mu \in (n, n+1]$, $\vartheta = \mu - n$, $n = 0, 1, 2, \ldots$. Then, the $\mu$th left-sided $\phi - ABC$ fractional derivative is given by

$$\begin{aligned}
\left({}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\mathfrak{g}\right)(\mathrm{u}) &= \left({}^{ABC}\mathfrak{D}_{\iota}^{\vartheta,\phi}\mathfrak{g}_{\phi}^{(n)}\right)(\mathrm{u}) \\
&= \frac{\Delta(\vartheta)}{1-\vartheta}\int_{\iota}^{\mathrm{u}} \phi'(\mathrm{v})\mathbb{E}_{\vartheta}\left(\frac{-\vartheta}{1-\vartheta}\big(\phi(\mathrm{u}) - \phi(\mathrm{v})\big)^{\vartheta}\right)\mathfrak{g}_{\phi}^{(n+1)}(\mathrm{v})\,d\mathrm{v}
\end{aligned}$$

$$= \frac{\Delta(\mu-n)}{n+1-\mu}\int_{\iota}^{\mathrm{u}} \phi'(\mathrm{v})\mathbb{E}_{\mu-n}\left(\frac{-(\mu-n)}{n+1-\mu}\big(\phi(\mathrm{u})-\phi(\mathrm{v})\big)^{\mu-n}\right)\mathfrak{g}_{\phi}^{(n+1)}(\mathrm{v})\,d\mathrm{v},$$

where $\mathfrak{g}_{\phi}^{(n)}(\mathrm{u}) = (\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}})^{n}\mathfrak{g}(\mathrm{u})$ and $\mathfrak{g}_{\phi}^{(0)}(\mathrm{u}) = \mathfrak{g}(\mathrm{u})$. If $\mu = m \in \mathbb{N}$ then $({}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\mathfrak{g})(\mathrm{u}) = \mathfrak{g}_{\phi}^{(m)}(\mathrm{u})$.

**Definition 3.3** Consider $\mathfrak{g} \in \mathcal{H}^{1}(\iota,\tau)$ and $\mu \in (n,n+1]$, $\vartheta = \mu - n$, $n = 0,1,2,\dots$. Then, the $\mu$th left-sided $\phi$-AB fractional integral is given by

$$\begin{aligned}\left({}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\mathfrak{g}\right)(\mathrm{u}) &= \left({}^{RL}\mathfrak{I}_{\iota}^{n,\phi}\,{}^{AB}\mathfrak{I}_{\iota}^{\vartheta,\phi}\mathfrak{g}\right)(\mathrm{u}) = \left({}^{AB}\mathfrak{I}_{\iota}^{\vartheta,\phi}\,{}^{RL}\mathfrak{I}_{\iota}^{n,\phi}\mathfrak{g}\right)(\mathrm{u}) \\ &= \frac{n+1-\mu}{\Delta(\mu-n)}\,{}^{RL}\mathfrak{I}_{\iota}^{n,\phi}\mathfrak{g}(\mathrm{u}) + \frac{(\mu-n)}{\Delta(\mu-n)}\,{}^{RL}\mathfrak{I}_{\iota}^{\mu,\phi}\mathfrak{g}(\mathrm{u}),\end{aligned}$$

where ${}^{RL}\mathfrak{I}_{\iota}^{n,\phi}$ is defined as:

$$\left({}^{RL}\mathfrak{I}_{\iota}^{n,\phi}\mathfrak{g}\right)(\mathrm{u}) = \frac{1}{\Gamma(n)}\int_{\iota}^{\mathrm{u}}\phi'(\mathrm{v})\big(\phi(\mathrm{u})-\phi(\mathrm{v})\big)^{n-1}\mathfrak{g}(\mathrm{v})\,d\mathrm{v}, \quad \mathrm{u} > \iota.$$

*Remark* 3.4 We remark that,

- In the Definitions 3.1, 3.2, and 3.3, if $\mu \in (0,1]$ we return to Definitions 2.6, 2.7, and 2.8, respectively.
- If $\mu = n+1$, then $\vartheta = 1$ and thus our generalization to the higher-order cases hold as follows:

$$\begin{aligned}&\left({}^{ABR}\mathfrak{D}_{\iota}^{\mu,\phi}\mathfrak{g}\right)(\mathrm{u}) = \left(\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)^{n}\left({}^{ABR}\mathfrak{D}_{\iota}^{1,\phi}\mathfrak{g}(\mathrm{u})\right) = \mathfrak{g}_{\phi}^{(n+1)}(\mathrm{u});\\ &\left({}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\mathfrak{g}\right)(\mathrm{u}) = \left({}^{ABC}\mathfrak{D}_{\iota}^{1,\phi}\mathfrak{g}_{\phi}^{(n)}\right)(\mathrm{u}) = \mathfrak{g}_{\phi}^{(n+1)}(\mathrm{u}); \quad \text{and}\\ &\left({}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\mathfrak{g}\right)(\mathrm{u}) = \left({}^{RL}\mathfrak{I}_{\iota}^{n,\phi}\,{}^{AB}\mathfrak{I}_{\iota}^{1,\phi}\mathfrak{g}\right)(\mathrm{u}) = \left({}^{RL}\mathfrak{I}_{\iota}^{n+1,\phi}\mathfrak{g}\right)(\mathrm{u}).\end{aligned}$$

**Proposition 3.5** *For $\mu \in (0,1]$, the following relations hold*:

(i) $({}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\,{}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\omega)(\mathrm{u}) = \omega(\mathrm{u}) - \omega(\iota)$;

(ii) $({}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\,{}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\omega)(\mathrm{u}) = \omega(\mathrm{u}) - \omega(\iota)\mathbb{E}_{\mu}(\frac{-\mu}{1-\mu}(\phi(\mathrm{u})-\phi(\iota))^{\mu})$.

*Proof*

(i) By using Definitions 2.7 and 2.8, we have

$$\begin{aligned}&\left({}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\,{}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\omega\right)(\mathrm{u})\\ &\quad= \frac{1-\mu}{\Delta(\mu)}\left({}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\omega\right)(\mathrm{u}) + \frac{\mu}{\Delta(\mu)}\left({}^{RL}\mathfrak{I}_{\iota}^{\mu,\phi}\,{}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\omega\right)(\mathrm{u})\\ &\quad= \sum_{i=0}^{\infty}\left(\frac{-\mu}{1-\mu}\right)^{i}\int_{\iota}^{\mathrm{u}}\phi'(\mathrm{v})\frac{(\phi(\mathrm{u})-\phi(\mathrm{v}))^{i\mu}}{\Gamma(i\mu+1)}\omega_{\phi}'(\mathrm{v})\,d\mathrm{v}\\ &\qquad+ \frac{\mu}{1-\mu}\,{}^{RL}\mathfrak{I}_{\iota}^{\mu,\phi}\sum_{i=0}^{\infty}\left(\frac{-\mu}{1-\mu}\right)^{i}\int_{\iota}^{\mathrm{u}}\phi'(\mathrm{v})\frac{(\phi(\mathrm{u})-\phi(\mathrm{v}))^{i\mu}}{\Gamma(i\mu+1)}\omega_{\phi}'(\mathrm{v})\,d\mathrm{v}\\ &\quad= \sum_{i=0}^{\infty}\left(\frac{-\mu}{1-\mu}\right)^{i}{}^{RL}\mathfrak{I}_{\iota}^{i\mu+1,\phi}\frac{\omega'(\mathrm{u})}{\phi'(\mathrm{u})} + \frac{\mu}{1-\mu}\,{}^{RL}\mathfrak{I}_{\iota}^{\mu,\phi}\sum_{i=0}^{\infty}\left(\frac{-\mu}{1-\mu}\right)^{i}{}^{RL}\mathfrak{I}_{\iota}^{i\mu+1,\phi}\frac{\omega'(\mathrm{u})}{\phi'(\mathrm{u})}\\ &\quad= \sum_{i=0}^{\infty}\left(\frac{-\mu}{1-\mu}\right)^{i}{}^{RL}\mathfrak{I}_{\iota}^{i\mu+1,\phi}\frac{\omega'(\mathrm{u})}{\phi'(\mathrm{u})} - \sum_{i=0}^{\infty}\left(\frac{-\mu}{1-\mu}\right)^{i+1}{}^{RL}\mathfrak{I}_{\iota}^{i\mu+\mu+1,\phi}\frac{\omega'(\mathrm{u})}{\phi'(\mathrm{u})}\end{aligned}$$

$$= {}^{RL}\mathfrak{I}_{\iota}^{1,\phi}\frac{\omega'(\mathrm{u})}{\phi'(\mathrm{u})} = \int_{\iota}^{\mathrm{u}} \omega'(\mathrm{v})\,d\mathrm{v} = \omega(\mathrm{u}) - \omega(\iota).$$

(ii) By using Definitions 2.7 and 2.8, and the identity

$${}^{RL}\mathfrak{I}_{\iota}^{\alpha+1,\phi}\left(\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)\omega(\mathrm{u}) = {}^{RL}\mathfrak{I}_{\iota}^{\alpha,\phi}\omega(\mathrm{u}) - \omega(\iota)\frac{(\phi(\mathrm{u})-\phi(\iota))^{\alpha}}{\Gamma(\alpha+1)}, \quad Re(\alpha) > 0,$$

we have

$$\begin{aligned}
&\left({}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\,{}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\omega\right)(\mathrm{u}) \\
&= {}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\left(\frac{1-\mu}{\Delta(\mu)}\omega(\mathrm{u}) + \frac{\mu}{\Delta(\mu)}\left({}^{RL}\mathfrak{I}_{\iota}^{\mu,\phi}\omega\right)(\mathrm{u})\right) \\
&= \frac{1-\mu}{\Delta(\mu)}\left({}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\omega\right)(\mathrm{u}) + \frac{\mu}{\Delta(\mu)}\,{}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\left({}^{RL}\mathfrak{I}_{\iota}^{\mu,\phi}\omega(\mathrm{u})\right) \\
&= \sum_{i=0}^{\infty}\left(\frac{-\mu}{1-\mu}\right)^{i}\,{}^{RL}\mathfrak{I}_{\iota}^{i\mu+1,\phi}\left(\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)\omega(\mathrm{u}) \\
&\quad + \frac{\mu}{1-\mu}\sum_{i=0}^{\infty}\left(\frac{-\mu}{1-\mu}\right)^{i}\,{}^{RL}\mathfrak{I}_{\iota}^{i\mu+1,\phi}\left(\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)\left({}^{RL}\mathfrak{I}_{\iota}^{\mu,\phi}\omega(\mathrm{u})\right) \\
&= \sum_{i=0}^{\infty}\left(\frac{-\mu}{1-\mu}\right)^{i}\left[{}^{RL}\mathfrak{I}_{\iota}^{i\mu,\phi}\omega(\mathrm{u}) - \frac{\omega(\iota)(\phi(\mathrm{u})-\phi(\iota))^{i\mu}}{\Gamma(i\mu+1)}\right] \\
&\quad - \sum_{i=0}^{\infty}\left(\frac{-\mu}{1-\mu}\right)^{i+1}\,{}^{RL}\mathfrak{I}_{\iota}^{i\mu+\mu,\phi}\omega(\mathrm{u}) \\
&= \omega(\mathrm{u}) - \sum_{i=0}^{\infty}\left(\frac{-\mu}{1-\mu}\right)^{i}\frac{\omega(\iota)(\phi(\mathrm{u})-\phi(\iota))^{i\mu}}{\Gamma(i\mu+1)} \\
&= \omega(\mathrm{u}) - \omega(\iota)\mathbb{E}_{\mu}\left(\frac{-\mu}{1-\mu}\left(\phi(\mathrm{u})-\phi(\iota)\right)^{\mu}\right).
\end{aligned}$$

□

**Proposition 3.6** *Let* $\omega \in \mathcal{C}^{n}(\mathrm{J},\mathbb{R})$, *and* $\phi \in \mathcal{C}^{n}(\mathrm{J},\mathbb{R}^{+})$. *For* $\mu \in (n, n+1]$, $\vartheta = \mu - n$, $n = 0,1,2,\ldots$, *the following relations hold*:

(i) $({}^{ABR}\mathfrak{D}_{\iota}^{\mu,\phi}\,{}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\omega)(\mathrm{u}) = \omega(\mathrm{u})$;

(ii) $({}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\,{}^{ABR}\mathfrak{D}_{\iota}^{\mu,\phi}\omega)(\mathrm{u}) = \omega(\mathrm{u})$;

(iii) $({}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\,{}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\omega)(\mathrm{u}) = \omega(\mathrm{u}) - \omega(\iota)\mathbb{E}_{\mu-n}(\frac{-(\mu-n)}{1-(\mu-n)}(\phi(\mathrm{u})-\phi(\iota))^{\mu-n})$;

(iv) $({}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\,{}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\omega)(\mathrm{u}) = \omega(\mathrm{u}) - \sum_{k=0}^{n}\frac{\omega_{\phi}^{(k)}(\iota)}{k!}(\phi(\mathrm{u})-\phi(\iota))^{k}$.

*Proof*

(i) In view of Definitions 3.1 and 3.3 and Lemmas 2.10 and 2.11, we have

$$\begin{aligned}
\left({}^{ABR}\mathfrak{D}_{\iota}^{\mu,\phi}\,{}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\omega\right)(\mathrm{u}) &= \left(\left(\frac{1}{\phi(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)^{n}{}^{ABR}\mathfrak{D}_{\iota}^{\vartheta,\phi}\,{}^{AB}\mathfrak{I}_{\iota}^{\vartheta,\phi}\,{}^{RL}\mathfrak{I}_{\iota}^{n,\phi}\omega\right)(\mathrm{u}) \\
&= \left(\left(\frac{1}{\phi(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)^{n}{}^{RL}\mathfrak{I}_{\iota}^{n,\phi}\omega\right)(\mathrm{u}) = \omega(\mathrm{u}).
\end{aligned}$$

(ii) In view of Definitions 3.1 and 3.3, we obtain that

$$\begin{aligned}
&\left({}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\,{}^{ABR}\mathfrak{D}_{\iota}^{\mu,\phi}\omega\right)(\mathrm{u})\\
&= \frac{(n+1-\mu)}{\Delta(\mu-n)}\,{}^{RL}\mathfrak{I}_{\iota}^{n,\phi}\left({}^{ABR}\mathfrak{D}_{\iota}^{\mu,\phi}\omega(\mathrm{u})\right) + \frac{(\mu-n)}{\Delta(\mu-n)}\,{}^{RL}\mathfrak{I}_{\iota}^{\mu,\phi}\left({}^{ABR}\mathfrak{D}_{\iota}^{\mu,\phi}\omega(\mathrm{u})\right)\\
&= {}^{RL}\mathfrak{I}_{\iota}^{n,\phi}\left(\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)^{n+1}\sum_{i=0}^{\infty}\left(\frac{-(\mu-n)}{n+1-\mu}\right)^{i}\int_{\iota}^{\mathrm{u}}\phi'(\mathrm{v})\frac{(\phi(\mathrm{u})-\phi(\mathrm{v}))^{i(\mu-n)}}{\Gamma(i(\mu-n)+1)}\omega(\mathrm{v})\,d\mathrm{v}\\
&\quad+\frac{(\mu-n)}{(n+1-\mu)}\,{}^{RL}\mathfrak{I}_{\iota}^{\mu,\phi}\left(\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)^{n+1}\sum_{i=0}^{\infty}\left(\frac{-(\mu-n)}{n+1-\mu}\right)^{i}\\
&\quad\times\int_{\iota}^{\mathrm{u}}\frac{\phi'(\mathrm{v})(\phi(\mathrm{u})-\phi(\mathrm{v}))^{i(\mu-n)}}{\Gamma(i(\mu-n)+1)}\omega(\mathrm{v})\,d\mathrm{v}\\
&= \sum_{i=0}^{\infty}\left(\frac{-(\mu-n)}{n+1-\mu}\right)^{i}\,{}^{RL}\mathfrak{I}_{\iota}^{n,\phi}\left(\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)^{n+1}\,{}^{RL}\mathfrak{I}_{\iota}^{i(\mu-n)+1,\phi}\omega(\mathrm{u})\\
&\quad-\sum_{i=0}^{\infty}\left(\frac{-(\mu-n)}{n+1-\mu}\right)^{i+1}\,{}^{RL}\mathfrak{I}_{\iota}^{\mu,\phi}\left(\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)^{n+1}\,{}^{RL}\mathfrak{I}_{\iota}^{i(\mu-n)+1,\phi}\omega(\mathrm{u})\\
&= \sum_{i=0}^{\infty}\left(\frac{-(\mu-n)}{n+1-\mu}\right)^{i}\,{}^{RL}\mathfrak{I}_{\iota}^{i(\mu-n),\phi}\omega(\mathrm{u}) - \sum_{i=0}^{\infty}\left(\frac{-(\mu-n)}{n+1-\mu}\right)^{i+1}\,{}^{RL}\mathfrak{I}_{\iota}^{i(\mu-n)+(\mu-n),\phi}\omega(\mathrm{u})\\
&= \omega(\mathrm{u}).
\end{aligned}$$

(iii) Due to Definitions 3.2 and 3.3, Lemma 2.10, and Proposition 3.5, we obtain

$$\begin{aligned}
&\left({}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\,{}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\omega\right)(\mathrm{u})\\
&= \left({}^{ABC}\mathfrak{D}_{\iota}^{\vartheta,\phi}\left(\frac{1}{\phi(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)^{n}\,{}^{RL}\mathfrak{I}_{\iota}^{n,\phi}\,{}^{AB}\mathfrak{I}_{\iota}^{\vartheta,\phi}\omega\right)(\mathrm{u})\\
&= \left({}^{ABC}\mathfrak{D}_{\iota}^{\vartheta,\phi}\,{}^{AB}\mathfrak{I}_{\iota}^{\vartheta,\phi}\omega\right)(\mathrm{u}) = \omega(\mathrm{u}) - \omega(\iota)\mathbb{E}_{\vartheta}\left(\frac{-\vartheta}{1-\vartheta}\left(\phi(\mathrm{u})-\phi(\iota)\right)^{\vartheta}\right)\\
&= \omega(\mathrm{u}) - \omega(\iota)\mathbb{E}_{\mu-n}\left(\frac{-(\mu-n)}{1-(\mu-n)}\left(\phi(\mathrm{u})-\phi(\iota)\right)^{\mu-n}\right).
\end{aligned}$$

(iv) Due to Definitions 3.2 and 3.3 and Proposition 3.5, we obtain that

$$\begin{aligned}
&\left({}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\,{}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\omega\right)(\mathrm{u})\\
&= \left({}^{RL}\mathfrak{I}_{\iota}^{n,\phi}\,{}^{AB}\mathfrak{I}_{\iota}^{\vartheta,\phi}\,{}^{ABC}\mathfrak{D}_{\iota}^{\vartheta,\phi}\omega_{\phi}^{(n)}\right)(\mathrm{u}) = {}^{RL}\mathfrak{I}_{\iota}^{n,\phi}\left(\omega_{\phi}^{(n)}(\mathrm{u})-\omega_{\phi}^{(n)}(\iota)\right)\\
&= \omega(\mathrm{u}) - \sum_{k=0}^{n-1}\frac{\omega_{\phi}^{(k)}(\iota)}{k!}\left(\phi(\mathrm{u})-\phi(\iota)\right)^{k} - \frac{\omega_{\phi}^{(n)}(\iota)}{n!}\left(\phi(\mathrm{u})-\phi(\iota)\right)^{n}\\
&= \omega(\mathrm{u}) - \sum_{k=0}^{n}\frac{\omega_{\phi}^{(k)}(\iota)}{k!}\left(\phi(\mathrm{u})-\phi(\iota)\right)^{k}.
\end{aligned}$$

□

**Proposition 3.7** *Let $\omega \in \mathcal{C}^n(\mathrm{J},\mathbb{R})$, $\phi \in \mathcal{C}^n(\mathrm{J},\mathbb{R}^+)$, $\phi'(\mathrm{u}) \neq 0$. For $\mu \in (n, n+1]$, $\vartheta = \mu - n$, $n = 0,1,2,\dots$, $\beta \geq n+1$ and $\varrho > 0$. Then, the following relations hold*:

(i) ${}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}[\phi(\mathrm{u})-\phi(\iota)]^{\varrho} = \frac{(n+1-\mu)\Gamma(\varrho+1)[\phi(\mathrm{u})-\phi(\iota)]^{\varrho+n}}{\Delta(\mu-n)\Gamma(n+\varrho+1)} + \frac{(\mu-n)\Gamma(\varrho+1)[\phi(\mathrm{u})-\phi(\iota)]^{\varrho+\mu}}{\Delta(\mu-n)\Gamma(\mu+\varrho+1)}$.

(ii) ${}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}[\phi(\mathrm{u})-\phi(\iota)]^{\beta} = \frac{\Delta(\mu-n)}{(n+1-\mu)}\sum_{i=0}^{\infty}(\frac{-(\mu-n)}{n+1-\mu})^{i}\frac{\Gamma(\beta+1)[\phi(\mathrm{u})-\phi(\iota)]^{i(\mu-n)+\beta-n}}{\Gamma(i(\mu-n)+\beta-n+1)}$.

(iii) ${}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}[\phi(\mathrm{u})-\phi(\iota)]^{\zeta} = 0$, $\zeta = 0,1,\dots,n$.

(iv) $({}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}1)(\mathrm{u}) = \frac{(n+1-\mu)[\phi(\mathrm{u})-\phi(\iota)]^{n}}{\Delta(\mu-n)\Gamma(n+1)} + \frac{(\mu-n)[\phi(\mathrm{u})-\phi(\iota)]^{\mu}}{\Delta(\mu-n)\Gamma(\mu+1)}$.

(v) $({}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}1)(\mathrm{u}) = 0$.

*Proof*

(i) For this, we apply Definition 3.3 and Lemma 2.10, and we obtain

$$\begin{aligned}
&{}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\big[\phi(\mathrm{u})-\phi(\iota)\big]^{\varrho} \\
&= \frac{(n+1-\mu)}{\Delta(\mu-n)}\,{}^{RL}\mathfrak{I}_{\iota}^{n,\phi}\big[\phi(\mathrm{u})-\phi(\iota)\big]^{\varrho} + \frac{(\mu-n)}{\Delta(\mu-n)}\,{}^{RL}\mathfrak{I}_{\iota}^{\mu,\phi}\big[\phi(\mathrm{u})-\phi(\iota)\big]^{\varrho} \\
&= \frac{(n+1-\mu)}{\Delta(\mu-n)}\frac{\Gamma(\varrho+1)}{\Gamma(n+\varrho+1)}\big[\phi(\mathrm{u})-\phi(\iota)\big]^{\varrho+n} \\
&\quad + \frac{(\mu-n)}{\Delta(\mu-n)}\frac{\Gamma(\varrho+1)[\phi(\mathrm{u})-\phi(\iota)]^{\varrho+\mu}}{\Gamma(\mu+\varrho+1)}.
\end{aligned}$$

(ii) By applying Definitions 2.3 and 3.2 and Lemma 2.10, we have

$$\begin{aligned}
&{}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\big[\phi(\mathrm{u})-\phi(\iota)\big]^{\beta} \\
&= {}^{ABC}\mathfrak{D}_{\iota}^{\vartheta,\phi}\left(\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)^{n}\big[\phi(\mathrm{u})-\phi(\iota)\big]^{\beta} \\
&= {}^{ABC}\mathfrak{D}_{\iota}^{\vartheta,\phi}\frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}\big[\phi(\mathrm{u})-\phi(\iota)\big]^{\beta-n} \\
&= \frac{\Delta(\vartheta)}{1-\vartheta}\int_{\iota}^{\mathrm{u}}\sum_{i=0}^{\infty}\left(\frac{-\vartheta}{1-\vartheta}\right)^{i}\frac{\Gamma(\beta+1)[\phi(\mathrm{v})-\phi(\iota)]^{\beta-(n+1)}}{\Gamma(\beta-n)\Gamma(i\vartheta+1)}\phi'(\mathrm{v})\big(\phi(\mathrm{u})-\phi(\mathrm{v})\big)^{i\vartheta}\,d\mathrm{v} \\
&= \frac{\Gamma(\beta+1)\Delta(\vartheta)}{\Gamma(\beta-n)(1-\vartheta)}\sum_{i=0}^{\infty}\left(\frac{-\vartheta}{1-\vartheta}\right)^{i}{}^{RL}\mathfrak{I}_{\iota}^{i\vartheta+1,\phi}\big[\phi(\mathrm{u})-\phi(\iota)\big]^{\beta-(n+1)} \\
&= \frac{\Delta(\vartheta)}{1-\vartheta}\sum_{i=0}^{\infty}\left(\frac{-\vartheta}{1-\vartheta}\right)^{i}\frac{\Gamma(\beta+1)}{\Gamma(i\vartheta+\beta-n+1)}\big[\phi(\mathrm{u})-\phi(\iota)\big]^{i\vartheta+\beta-n} \\
&= \frac{\Delta(\mu-n)}{n+1-\mu}\sum_{i=0}^{\infty}\left(\frac{-(\mu-n)}{n+1-\mu}\right)^{i}\frac{\Gamma(\beta+1)}{\Gamma(i(\mu-n)+\beta-n+1)}\big[\phi(\mathrm{u})-\phi(\iota)\big]^{i(\mu-n)+\beta-n}.
\end{aligned}$$

(iii) By applying Definitions 2.7 and 3.2, we obtain

$$\begin{aligned}
&{}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\big[\phi(\mathrm{u})-\phi(\iota)\big]^{\zeta} \\
&= {}^{ABC}\mathfrak{D}_{\iota}^{\vartheta,\phi}\left(\frac{1}{\phi'(\mathrm{u})}\frac{d}{d\mathrm{u}}\right)^{n}\big[\phi(\mathrm{u})-\phi(\iota)\big]^{\zeta} \\
&= {}^{ABC}\mathfrak{D}_{\iota}^{\vartheta,\phi}\frac{\Gamma(\zeta+1)}{\Gamma(\zeta-n+1)}\big[\phi(\mathrm{u})-\phi(\iota)\big]^{\zeta-n} \\
&= \frac{\Delta(\vartheta)}{1-\vartheta}\int_{\iota}^{\mathrm{u}}\mathbb{E}_{\vartheta}\left(\frac{-\vartheta}{1-\vartheta}\big(\phi(\mathrm{u})-\phi(\mathrm{v})\big)^{\vartheta}\right)\frac{\Gamma(\zeta+1)}{\Gamma(\zeta-n+1)}\frac{d}{d\mathrm{v}}\big[\phi(\mathrm{v})-\phi(\iota)\big]^{\zeta-n}\,d\mathrm{v} \\
&= 0.
\end{aligned}$$

(iv) and (v) can be concluding by putting $\varrho = \zeta = 0$ in parts (*i*) and (*iii*), respectively. □

### 3.2 Gronwall's inequality

At the beginning of this subsection, we will state the following generalization of Gronwall's inequality.

**Lemma 3.8** ([23]) *Consider $\mu > 0$ and $\phi \in \mathcal{C}^1(\mathrm{J}, \mathbb{R}^+)$ to be an increasing function such that $\phi'(\mathrm{u}) \neq 0$, $\forall \mathrm{u} \in \mathrm{J}$. Suppose that $\mathcal{V}(\mathrm{u})$ is a nonnegative function locally integrable on* J *and $\mathcal{U}(\mathrm{u})$ is nonnegative and nondecreasing, and also assume that $\omega$ is nonnegative and locally integrable on* J, *such that*

$$\omega(\mathrm{u}) \le \mathcal{V}(\mathrm{u}) + \mathcal{U}(\mathrm{u}) \int_\iota^{\mathrm{u}} \phi'(\mathrm{v})\big(\phi(\mathrm{u}) - \phi(\mathrm{v})\big)^{\mu-1} \omega(\mathrm{v})\, d\mathrm{v}, \quad \mathrm{u} \in \mathrm{J}, \tag{3.1}$$

*then, for every* $\mathrm{u} \in \mathrm{J}$, *we obtain*

$$\omega(\mathrm{u}) \le \mathcal{V}(\mathrm{u}) + \int_\iota^{\mathrm{u}} \sum_{i=1}^{\infty} \frac{[\mathcal{U}(\mathrm{u})\Gamma(\mu)]^i}{\Gamma(i\mu)} \phi'(\mathrm{v})\big(\phi(\mathrm{u}) - \phi(\mathrm{v})\big)^{i\mu-1} \mathcal{V}(\mathrm{v})\, d\mathrm{v}. \tag{3.2}$$

**Lemma 3.9** ([23]) *Under the conditions of Lemma 3.8, if $\mathcal{V}(\mathrm{u})$ is a nondecreasing function on $J$. Then, we have*

$$\omega(\mathrm{u}) \le \mathcal{V}(\mathrm{u}) \mathbb{E}_\mu\big[\mathcal{U}(\mathrm{u})\Gamma(\mu)\big(\phi(\mathrm{u}) - \phi(\iota)\big)^\mu\big], \quad \mathrm{u} \in \mathrm{J}. \tag{3.3}$$

In this position, we will introduce a new Gronwall inequality in the *framework* of the $\phi - AB$ fractional operator.

**Lemma 3.10** *Let $\mu \in (0,1]$ and $\phi \in \mathcal{C}^1(\mathrm{J}, \mathbb{R}^+)$ be an increasing function such that $\phi'(\mathrm{u}) \neq 0$, $\forall \mathrm{u} \in \mathrm{J}$. Suppose that $\mathcal{X}(\mathrm{u}) = \frac{\mathcal{G}(\mathrm{u})\Delta(\mu)}{\Delta(\mu)-(1-\mu)\mathcal{H}(\mathrm{u})}$ is a nonnegative function locally integrable on* J, *$\mathcal{Y}(\mathrm{u}) = \frac{\mu\mathcal{H}(\mathrm{u})}{\Delta(\mu)-(1-\mu)\mathcal{H}(\mathrm{u})}$ is nonnegative and nondecreasing, and assume also that $\omega$ is nonnegative and locally integrable on* J, *such that*

$$\omega(\mathrm{u}) \le \mathcal{G}(\mathrm{u}) + \mathcal{H}(\mathrm{u})\, {}^{AB}\mathfrak{I}_\iota^{\mu,\phi} \omega(\mathrm{u}), \quad \mathrm{u} \in \mathrm{J}, \tag{3.4}$$

*then, for every* $\mathrm{u} \in \mathrm{J}$, *we obtain*

$$\omega(\mathrm{u}) \le \mathcal{X}(\mathrm{u}) + \int_\iota^{\mathrm{u}} \sum_{i=1}^{\infty} \frac{[\mathcal{Y}(\mathrm{u})]^i}{\Gamma(i\mu)} \phi'(\mathrm{v})\big(\phi(\mathrm{u}) - \phi(\mathrm{v})\big)^{i\mu-1} \mathcal{X}(\mathrm{v})\, d\mathrm{v}. \tag{3.5}$$

*Proof* From Inequality (3.4) and Definition 2.8, we have

$$\begin{aligned}
\omega(\mathrm{u}) &\le \mathcal{G}(\mathrm{u}) + \mathcal{H}(\mathrm{u})\big({}^{AB}\mathfrak{I}_\iota^{\mu,\phi}\omega\big)(\mathrm{u}) \\
&\le \mathcal{G}(\mathrm{u}) + \mathcal{H}(\mathrm{u})\left(\frac{1-\mu}{\Delta(\mu)}\omega(\mathrm{u}) + \frac{\mu}{\Delta(\mu)}\frac{1}{\Gamma(\mu)}\int_\iota^{\mathrm{u}} \phi'(\mathrm{v})\big(\phi(\mathrm{u}) - \phi(\mathrm{v})\big)^{\mu-1}\omega(\mathrm{v})\, d\mathrm{v}\right).
\end{aligned}$$

Hence,

$$\omega(\mathrm{u}) \le \frac{\mathcal{G}(\mathrm{u})\Delta(\mu)}{\Delta(\mu) - (1-\mu)\mathcal{H}(\mathrm{u})}$$

$$+\frac{\mu\mathcal{H}(\mathrm{u})}{\Delta(\mu)-(1-\mu)\mathcal{H}(\mathrm{u})}\frac{1}{\Gamma(\mu)}\int_{\iota}^{\mathrm{u}}\phi'(\mathrm{v})\big(\phi(\mathrm{u})-\phi(\mathrm{v})\big)^{\mu-1}\omega(\mathrm{v})\,d\mathrm{v}.$$

Due to Lemma 3.8, we obtain

$$\begin{aligned}\omega(\mathrm{u})\le{}&\frac{\mathcal{G}(\mathrm{u})\Delta(\mu)}{\Delta(\mu)-(1-\mu)\mathcal{H}(\mathrm{u})}\\&+\int_{\iota}^{\mathrm{u}}\sum_{i=1}^{\infty}\frac{1}{\Gamma(i\mu)}\left(\frac{\mu\mathcal{H}(\mathrm{u})}{\Delta(\mu)-(1-\mu)\mathcal{H}(\mathrm{u})}\right)^{i}\frac{\mathcal{G}(\mathrm{v})\Delta(\mu)\phi'(\mathrm{v})(\phi(\mathrm{u})-\phi(\mathrm{v}))^{i\mu-1}}{\Delta(\mu)-(1-\mu)\mathcal{H}(\mathrm{v})}\,d\mathrm{v}.\end{aligned}$$

Therefore, Inequality (3.5) holds. □

**Corollary 3.11** *Under the conditions of Lemma 3.10, if $\mathcal{X}(\mathrm{u})$ is a nondecreasing function on $J$, then we have*

$$\omega(\mathrm{u})\le\mathcal{X}(\mathrm{u})\mathbb{E}_{\mu}\big[\mathcal{Y}(\mathrm{u})\big(\phi(\mathrm{u})-\phi(\iota)\big)^{\mu}\big],\quad \mathrm{u}\in\mathrm{J}. \tag{3.6}$$

*Proof* In view of Lemma 3.9, we have

$$\omega(\mathrm{u})\le\frac{\mathcal{G}(\mathrm{u})\Delta(\mu)}{\Delta(\mu)-(1-\mu)\mathcal{H}(\mathrm{u})}\mathbb{E}_{\mu}\left(\frac{\mu\mathcal{H}(\mathrm{u})(\phi(\mathrm{u})-\phi(\iota))^{\mu}}{\Delta(\mu)-(1-\mu)\mathcal{H}(\mathrm{u})}\right)^{i},$$

which satisfies Inequality (3.6). □

Herein, we will conclude a new Gronwall inequality in the *sense* of the $\phi-ABK$ fractional operator.

**Corollary 3.12** *For $\mu>0$, suppose that $\mathcal{X}(\mathrm{u})=\frac{\mathcal{G}(\mathrm{u})\Delta(\mu)}{\Delta(\mu)-(1-\mu)\mathcal{H}(\mathrm{u})}$ is a nonnegative function locally integrable on* J, *$\mathcal{Y}(\mathrm{u})=\frac{\mu\mathcal{H}(\mathrm{u})}{\Delta(\mu)-(1-\mu)\mathcal{H}(\mathrm{u})}$ is nonnegative and nondecreasing, and assume also that $\omega$ is nonnegative and locally integrable on* J, *such that*

$$\omega(\mathrm{u})\le\mathcal{G}(\mathrm{u})+\mathcal{H}(\mathrm{u})\,{}^{ABK}\mathfrak{I}_{\iota}^{\mu,\rho}\omega(\mathrm{u}),\quad \mathrm{u}\in\mathrm{J}, \tag{3.7}$$

*then, for every* $\mathrm{u}\in\mathrm{J}$, *we obtain*

$$\omega(\mathrm{u})\le\mathcal{X}(\mathrm{u})+\int_{\iota}^{\mathrm{u}}\sum_{i=1}^{\infty}\frac{\rho^{1-i\mu}\mathrm{v}^{\rho-1}[\mathcal{Y}(\mathrm{u})]^{i}}{\Gamma(i\mu)}\big(\mathrm{u}^{\rho}-\mathrm{v}^{\rho}\big)^{i\mu-1}\mathcal{X}(\mathrm{v})\,d\mathrm{v}. \tag{3.8}$$

*Proof* By putting $\phi(\mathrm{u})=\frac{\mathrm{u}^{\rho}}{\rho}$ in Lemma 3.10, the proof is finished. □

**Corollary 3.13** *Under conditions of Corollary 3.12, if $\mathcal{X}(\mathrm{u})$ is a nondecreasing function on $J$, then we have*

$$\omega(\mathrm{u})\le\mathcal{X}(\mathrm{u})\mathbb{E}_{\mu}\left[\mathcal{Y}(\mathrm{u})\left(\frac{\mathrm{u}^{\rho}-\mathrm{v}^{\rho}}{\rho}\right)^{\mu}\right],\quad \mathrm{u}\in\mathrm{J}. \tag{3.9}$$

*Proof* By putting $\phi(\mathrm{u})=\frac{\mathrm{u}^{\rho}}{\rho}$ in Corollary 3.11, the proof is finished. □

### 3.3 Existence and uniqueness results

In this subsection, we will study the existence and uniqueness of solution of the following initial fractional differential equation:

$$ {}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}\omega(\mathrm{u}) = \mathfrak{h}\big(\mathrm{u},\omega(\mathrm{u})\big), \quad \mathrm{u}\in \mathrm{J}=[\iota,\tau], \tag{3.10} $$

$$ \omega_{\phi}^{(k)}(\iota) = \lambda_k, \quad k=0,1,\dots,n, \tag{3.11} $$

where ${}^{ABC}\mathfrak{D}_{\iota}^{\mu,\phi}$ denotes the $\mu$th $\phi-ABC$ fractional derivative such that $\mu\in(n,n+1]$. The constants $\lambda_k\in\mathbb{R}$ $(k=0,1,\dots,n)$, $\mathfrak{h}:\mathrm{J}\times\mathbb{R}\to\mathbb{R}$ is a continuous function, and $\omega(\mathrm{u})\in\mathcal{C}^n(\mathrm{J},\mathbb{R})$ is a known function such that $\omega_{\phi}^{(k)}(u) = (\frac{1}{\phi'(u)}\frac{\mathrm{d}}{\mathrm{d}u})^k\omega(u)$ and $\omega_{\phi}^{(0)}(\mathrm{u})=\omega(\mathrm{u})$. Furthermore, $\phi:[\iota,\tau]\to\mathbb{R}^+$ *is an increasing function with* $\phi'(u)\in\mathcal{C}^n(\mathrm{J},\mathbb{R}^+)$ *and* $\phi'(\mathrm{u})\neq 0$, $\forall\mathrm{u}\in\mathrm{J}$.

Indeed, by applying the $\mu$th left-sided $\phi-AB$ fractional integral operator on both sides of (3.10) and by using Proposition 3.6 *along with* the initial boundary condition (3.11), we have

$$ \omega(\mathrm{u}) = \sum_{k=0}^{n}\frac{\lambda_k}{k!}\big(\phi(\mathrm{u})-\phi(\iota)\big)^k + {}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\mathfrak{h}\big(\mathrm{u},\omega(\mathrm{u})\big). \tag{3.12} $$

Now, we will prove the existence and uniqueness of the solution for the system (3.10) and (3.11) by using Picard's iterative method [20].

**Theorem 3.14** *Assume that there is a constant* $\mathcal{M}>0$ *such that* $\sup_{\mathrm{u}\in\mathrm{J}}|\mathfrak{h}(\mathrm{u},\omega_0(\mathrm{u}))|\le\mathcal{M}$, *and there is a constant* $\ell>0$ *such that* $|\mathfrak{h}(\mathrm{u},\omega_1)-\mathfrak{h}(\mathrm{u},\omega_2)|\le\ell|\omega_1-\omega_2|$, *for all* $\mathrm{u}\in\mathrm{J}$, $\omega_1,\omega_2\in\mathcal{C}^n(\mathrm{J},\mathbb{R})$. *Then, there is one and only one solution* $\omega(\mathrm{u})$ *of the system* (3.10) *and* (3.11) *on* $\mathrm{J}$, *provided that*

$$ \ell\left(\frac{(n+1-\mu)[\phi(\tau)-\phi(\iota)]^n}{\Delta(\mu-n)\Gamma(n+1)} + \frac{(\mu-n)[\phi(\tau)-\phi(\iota)]^\mu}{\Delta(\mu-n)\Gamma(\mu+1)}\right) < 1. \tag{3.13} $$

*Proof* Clearly, the system (3.10) and (3.11) has a solution equivalent to the solution of the fractional integral equation (3.12). Set

$$ \omega_0(\mathrm{u}) = \sum_{k=0}^{n}\frac{\lambda_k}{k!}\big(\phi(\mathrm{u})-\phi(\iota)\big)^k, \tag{3.14} $$

and

$$ \omega_i(\mathrm{u}) = \sum_{k=0}^{n}\frac{\lambda_k}{k!}\big(\phi(\mathrm{u})-\phi(\iota)\big)^k + {}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\mathfrak{h}\big(\mathrm{u},\omega_{i-1}(\mathrm{u})\big), \quad i\in\mathbb{N}. \tag{3.15} $$

Obviously, the series $\omega_0(\mathrm{u})+\sum_{j=1}^{\infty}(\omega_j(\mathrm{u})-\omega_{j-1}(\mathrm{u}))$ has the partial sum $\omega_i(\mathrm{u})=\omega_0(\mathrm{u})+\sum_{j=1}^{i}(\omega_j(\mathrm{u})-\omega_{j-1}(\mathrm{u}))$. Our aim is to show that the sequence $\{\omega_i(\mathrm{u})\}$ converges to $\omega(\mathrm{u})$.

Due to mathematical induction, for all $\mathrm{u}\in[\iota,\tau]$, we investigate that

$$ \begin{aligned} &\|\omega_i-\omega_{i-1}\| \\ &\quad\le\mathcal{M}\ell^{i-1}\left(\frac{(n+1-\mu)[\phi(\tau)-\phi(\iota)]^n}{\Delta(\mu-n)\Gamma(n+1)} + \frac{(\mu-n)[\phi(\tau)-\phi(\iota)]^\mu}{\Delta(\mu-n)\Gamma(\mu+1)}\right)^i, \quad i\in\mathbb{N}. \end{aligned} \tag{3.16} $$

According to the equations (3.14) and (3.15) and Proposition 3.7 part (iv), we have

$$\begin{aligned}\|\omega_1-\omega_0\| &= \sup_{\mathrm{u}\in\mathrm{J}}\left|{}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\mathfrak{h}\big(\mathrm{u},\omega_0(\mathrm{u})\big)\right| \\ &\leq \mathcal{M}\left(\frac{(n+1-\mu)[\phi(\tau)-\phi(\iota)]^n}{\Delta(\mu-n)\Gamma(n+1)}+\frac{(\mu-n)[\phi(\tau)-\phi(\iota)]^\mu}{\Delta(\mu-n)\Gamma(\mu+1)}\right).\end{aligned}$$

Therefore, for $i=1$ the inequality (3.16) holds. Next, we assume that the inequality (3.16) is fulfilled for $i=r$. Then,

$$\begin{aligned}\|\omega_{r+1}-\omega_r\| &= \sup_{\mathrm{u}\in\mathrm{J}}\left|{}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\mathfrak{h}\big(\mathrm{u},\omega_r(\mathrm{u})\big)-{}^{AB}\,\mathfrak{I}_{\iota}^{\mu,\phi}\mathfrak{h}\big(\mathrm{u},\omega_{r-1}(\mathrm{u})\big)\right| \\ &= \sup_{\mathrm{u}\in\mathrm{J}}\left|{}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\left[\mathfrak{h}\big(\mathrm{u},\omega_r(\mathrm{u})\big)-\mathfrak{h}\big(\mathrm{u},\omega_{r-1}(\mathrm{u})\big)\right]\right| \\ &\leq {}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\left[\ell\|\omega_r-\omega_{r-1}\|\right] \\ &\leq {}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\left[\mathcal{M}\ell^r\left(\frac{(n+1-\mu)[\phi(\tau)-\phi(\iota)]^n}{\Delta(\mu-n)\Gamma(n+1)}+\frac{(\mu-n)[\phi(\tau)-\phi(\iota)]^\mu}{\Delta(\mu-n)\Gamma(\mu+1)}\right)^r\right] \\ &\leq \mathcal{M}\ell^{(r+1)-1}\left(\frac{(n+1-\mu)[\phi(\tau)-\phi(\iota)]^n}{\Delta(\mu-n)\Gamma(n+1)}+\frac{(\mu-n)[\phi(\tau)-\phi(\iota)]^\mu}{\Delta(\mu-n)\Gamma(\mu+1)}\right)^{r+1}.\end{aligned}$$

Thus, the identity (3.16) holds for $i=r+1$. Then, in view of mathematical induction the relation (3.16) holds for every $i\in\mathbb{N}$ and all $\mathrm{u}\in[\iota,\tau]$. Hence, we obtain

$$\begin{aligned}&\sum_{i=1}^{\infty}\|\omega_i-\omega_{i-1}\| \\ &\quad\leq \sum_{i=1}^{\infty}\mathcal{M}\ell^{i-1}\left(\frac{(n+1-\mu)[\phi(\tau)-\phi(\iota)]^n}{\Delta(\mu-n)\Gamma(n+1)}+\frac{(\mu-n)[\phi(\tau)-\phi(\iota)]^\mu}{\Delta(\mu-n)\Gamma(\mu+1)}\right)^i.\end{aligned}\tag{3.17}$$

Due to the condition (3.13), the series on the right-hand side of the above inequality is convergent, and so $\sum_{i=1}^{\infty}\|\omega_i-\omega_{i-1}\|$ is also convergent, which proves that $\omega_0+\sum_{i=1}^{\infty}\|\omega_i-\omega_{i-1}\|$ converges.

Let us set $\omega=\omega_0+\sum_{i=1}^{\infty}\|\omega_i-\omega_{i-1}\|$, it follows that

$$\|\omega_i-\omega\|\longrightarrow 0\quad\text{as } i\longrightarrow\infty,\tag{3.18}$$

which shows that the solution of the system (3.10) and (3.11) exists. In fact, by (3.18), we have

$$\left\|\mathfrak{h}\big(\cdot,\omega_{i-1}(\cdot)\big)-\mathfrak{h}\big(\cdot,\omega(\cdot)\big)\right\|\leq\ell\|\omega_{i-1}-\omega\|\longrightarrow 0\quad\text{as } i\longrightarrow\infty.$$

Thus,

$$\lim_{i\to\infty}\mathfrak{h}\big(\mathrm{u},\omega_{i-1}(\mathrm{u})\big)=\mathfrak{h}\big(\mathrm{u},\omega(\mathrm{u})\big).\tag{3.19}$$

Therefore, by taking the limits on both sides of (3.15) as $i \to \infty$ and applying (3.19), we deduce that

$$\omega(\mathrm{u}) = \sum_{k=0}^{n} \frac{\lambda_k}{k!}\big(\phi(\mathrm{u}) - \phi(\iota)\big)^k + {}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\mathfrak{h}\big(\mathrm{u}, \omega(\mathrm{u})\big), \tag{3.20}$$

which represent the solution of the system (3.10) and (3.11).

Lastly, in order to show the solution $\omega$ is unique, we suppose that $\tilde{\omega}$ is another solution of the system (3.10) and (3.11). Thus, we have

$$\begin{aligned}
\|\omega - \tilde{\omega}\| &= \sup_{\mathrm{u}\in \mathrm{J}} \big| {}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\mathfrak{h}\big(\mathrm{u}, \omega(\mathrm{u})\big) - {}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\mathfrak{h}\big(\mathrm{u}, \tilde{\omega}(\mathrm{u})\big)\big| \\
&= \sup_{\mathrm{u}\in \mathrm{J}} \big| {}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\big[\mathfrak{h}\big(\mathrm{u}, \omega(\mathrm{u})\big) - \mathfrak{h}\big(\mathrm{u}, \tilde{\omega}(\mathrm{u})\big)\big]\big| \\
&\leq {}^{AB}\mathfrak{I}_{\iota}^{\mu,\phi}\big[\ell\|\omega - \tilde{\omega}\|\big] \\
&\leq \ell\left(\frac{(n+1-\mu)[\phi(\tau) - \phi(\iota)]^n}{\Delta(\mu - n)\Gamma(n+1)} + \frac{(\mu - n)[\phi(\tau) - \phi(\iota)]^{\mu}}{\Delta(\mu - n)\Gamma(\mu+1)}\right)\|\omega - \tilde{\omega}\|.
\end{aligned}$$

In view of the condition (3.13), we conclude that should be $\|\omega - \tilde{\omega}\| = 0$, it follows that, $\omega(\mathrm{u}) = \tilde{\omega}(\mathrm{u})$. Thus, the proof is completed. □

## 4 Examples

Here, we examine validating the main results by the following illustrative examples:

*Example* 4.1 Consider the following initial fractional differential equation:

$${}^{ABC}\mathfrak{D}_1^{1.7,\phi}\omega(\mathrm{u}) = \sin\big(\mathrm{u}^2\big) - \frac{\omega(\mathrm{u})}{\frac{1}{2} + \omega(\mathrm{u})}, \quad \mathrm{u} \in \mathrm{J} = [1, e], \tag{4.1}$$

$$\omega(1) = 1, \qquad \omega'_{\phi}(1) = 1, \tag{4.2}$$

where $\mu = 1.7 \in (1, 2]$, $\phi(\mathrm{u}) = ln(\mathrm{u})$.

Now, we will check the conditions of Theorem 3.14 as follows:

$$\begin{aligned}
\big|\mathfrak{h}(\mathrm{u}, \omega_1) - \mathfrak{h}(\mathrm{u}, \omega_2)\big| &= \left|\sin\big(\mathrm{u}^2\big) - \frac{\omega_1(\mathrm{u})}{\frac{1}{2} + \omega_1(\mathrm{u})} - \sin\big(\mathrm{u}^2\big) + \frac{\omega_2(\mathrm{u})}{\frac{1}{2} + \omega_2(\mathrm{u})}\right| \\
&\leq \frac{1}{2}\big|\omega_1(\mathrm{u}) - \omega_2(\mathrm{u})\big|,
\end{aligned}$$

then $\ell = \frac{1}{2} > 0$, and by taking $\Delta(\mu - n) = 1$, we have

$$\ell\left(\frac{(n+1-\mu)[\phi(\tau) - \phi(\iota)]^n}{\Delta(\mu - n)\Gamma(n+1)} + \frac{(\mu - n)[\phi(\tau) - \phi(\iota)]^{\mu}}{\Delta(\mu - n)\Gamma(\mu+1)}\right) = 0.376583 < 1. \tag{4.3}$$

Hence, all the conditions of Theorem 3.14 hold. Therefore, the solution of the system (4.1) and (4.2) exists and is unique.

*Example* 4.2 Consider the following initial fractional differential equation:

$${}^{ABC}\mathfrak{D}_0^{2.5,\phi}\omega(\mathrm{u}) = \mathrm{u}^2 - \omega(\mathrm{u}), \quad \mathrm{u} \in \mathrm{J} = [0, 1], \tag{4.4}$$

$$\omega(0) = 0, \qquad \omega'_{\phi}(0) = 0, \qquad \omega''_{\phi}(0) = 2, \tag{4.5}$$

where $\mu = 2.5 \in (2,3]$, $\phi(\mathrm{u}) = \mathrm{u}$, and has an exact solution $\omega(\mathrm{u}) = \mathrm{u}^2$.

Now, we will check the conditions of Theorem 3.14 as follows:

$$\left|\mathfrak{h}(\mathrm{u},\omega_1) - \mathfrak{h}(\mathrm{u},\omega_2)\right| = \left|\mathrm{u}^2 - \omega_1(\mathrm{u}) - \mathrm{u}^2 + \omega_2(\mathrm{u})\right| \leq |\omega_1 - \omega_2|,$$

then $\ell = 1 > 0$, and by taking $\Delta(\mu - n) = 1$, we have

$$\ell\left(\frac{(n+1-\mu)[\phi(\tau) - \phi(\iota)]^n}{\Delta(\mu - n)\Gamma(n+1)} + \frac{(\mu - n)[\phi(\tau) - \phi(\iota)]^\mu}{\Delta(\mu - n)\Gamma(\mu + 1)}\right) = 0.350676 < 1. \tag{4.6}$$

Hence, all the conditions of Theorem 3.14 hold. Therefore, the solution of the system (4.4) and (4.5) exists and is unique.

Next, we will compute the solution of the system (4.4) and (4.5) by Picard's iterative method as follows:

$$\omega_i(\mathrm{u}) = \omega_0(\mathrm{u}) + {}^{AB}\mathfrak{I}_0^{2.5,\phi}\left(\mathrm{u}^2 - \omega_{i-1}(\mathrm{u})\right), \qquad \omega_0(\mathrm{u}) = \mathrm{u}^2, \quad i \in \mathbb{N}. \tag{4.7}$$

Then,

$$\begin{aligned}
\omega_1(\mathrm{u}) &= \mathrm{u}^2 + {}^{AB}\mathfrak{I}_0^{2.5,\phi}\left(\mathrm{u}^2 - \omega_0(\mathrm{u})\right) = \mathrm{u}^2, \\
\omega_2(\mathrm{u}) &= \mathrm{u}^2 + {}^{AB}\mathfrak{I}_0^{2.5,\phi}\left(\mathrm{u}^2 - \omega_1(\mathrm{u})\right) = \mathrm{u}^2, \\
\omega_3(\mathrm{u}) &= \mathrm{u}^2, \\
&\vdots,
\end{aligned}$$

which matches the exact solution.

## 5 Conclusion

We conclude the following:

- We have extended $\phi - ABC$ *and* $\phi - ABR$ *fractional derivatives to higher arbitrary orders. The corresponding* $\phi - AB$ fractional integrals are presented as well.
- The *introduced* generalization in the Caputo sense agrees with and generalizes those presented by Abdeljawad in [2]. However, our approach in this work is different in the Reimann–Liouville case (ABR) for either the derivative or the corresponding integral. Applying the $n$th-order derivative with respect to another function outside in the ABR higher-order case and the Riemann–Liouville integral of order $n$ with respect to another function outside in the $AB$-integral higher case shows that the integral and differential $ABR$ operators are inverses of each other for all orders and not only for the order between 0 and 1, as was the case for the higher-order extension obtained in [2].
- Some properties and actions of the extended higher-order integrals and derivatives on each other have been given.
- Gronwall's inequality has been *established in the framework of a* $\phi - AB$ *fractional integral operator*.

- The existence and uniqueness theory for initial value problems in the sense of higher-order $ABC$ of a function with respect to another function has been studied briefly.
- Some examples to illustrate the new extensions have been given.
- This work is a key attraction to researchers to study this type of extended calculus. *Thus, in the future we will focus our attention to apply these extension operators on real-life dynamics systems along with investigating new properties and inequities related to such operators.*

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## Declarations

**Competing interests**
The authors declare no competing interests.

**Author contributions**
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