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Solutions to a $(p(x), q(x))$-biharmonic elliptic problem on a bounded domain

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Abstract

Using variational methods and critical point results, we prove the existence and multiplicity of weak solutions of a $(p(x), q(x))$-biharmonic elliptic equation along with a singular term under Navier boundary conditions.

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1 Introduction

One of the important applied problems in analytical methods is solving the singular boundary value problems for differential equations. Singular problems have been intensively studied in the last decades. They arise naturally and repeatedly in physical models, often because of the coordinate system. These kinds of problems also appear in glacial advance, in transport of coal slurries down conveyor belts and in some other geophysical and industrial contents [8, 13, 21].

In the present paper, we consider the following $(p(x), q(x))$-biharmonic problem

$$\begin{cases} \Delta^2_{p(x)} u + \Delta^2_{q(x)} u + \theta(x) \frac{|u|^{s-2}u}{|x|^{2s}} = \lambda f(x, u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 2)$ is a bounded domain with boundary of class C^1; $p, q \in C_+(\overline{\Omega})$ satisfying the following inequalities

$$\max\{2, N/2\} < q^- \le q(x) \le q^+ < p^- \le p(x) \le p^+ < +\infty.$$

And,

$$\Delta^2_{r(x)} u := \Delta\left(|\Delta u|^{r(x)-2} \Delta u\right)$$

denotes $r(x)$-biharmonic operator for $r \in \{p, q\}$; $\theta \in L^\infty(\Omega)$ is a real function with $\operatorname{ess\,inf}_{x\in\overline{\Omega}} \theta(x) > 0$; s is a constant such that $1 < s < N/2$; $\lambda > 0$ is a real parameter, and

$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which holds the following growth condition:

$$\left|f(x,s)\right| \le a_1 + a_2|s|^{\gamma(x)-1} \tag{1.2}$$

for $(x,s) \in \Omega \times \mathbb{R}$, where a_1 and a_2 are positive constants and $\gamma \in C(\Omega)$ such that

$$1 < \gamma(x) \le p(x) \quad \text{a.e. } x \in \Omega.$$

In 2014, the existence of multiple weak solutions for the following nonlinear elliptic problem with the Navier boundary value involving the p-biharmonic operator was studied [4]

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 = \Delta u & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ with a smooth enough boundary $\partial\Omega$, λ is a positive parameter, and f is a suitable continuous function defined on the set $\overline{\Omega} \times \mathbb{R}$.

The existence of the solutions to the following weighted $(p(x), q(x))$-Laplacian problem consisting of a singular term

$$\begin{cases} -a(x)\Delta_{p(x)}u - b(x)\Delta_{q(x)}u + \frac{u|u|^{s-2}}{|x|^s} = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has been proved [17], where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $a, b \in L^\infty(\Omega)$ are positive functions with $a(x) \ge 1$ a.e. on Ω; $\lambda > 0$ is a real parameter, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following growth condition

$$\left|f(x,t)\right| \le \alpha + \beta|t|^{h(x)-1}$$

for all $(x,t) \in \Omega \times \mathbb{R}$. See also [2, 3] and the references therein. The existence of at least one positive radial solution of the p-biharmonic problem

$$\begin{aligned} &\Delta_{\mathbb{H}^n}\left(w(\xi)|\Delta_{\mathbb{H}^n}u|^{p-2}\Delta_{\mathbb{H}^n}u\right) + R(\xi)w(\xi)|u|^{p-2}u \\ &\quad = \sum_{i=1}^{m} a_i\left(|\xi|_{\mathbb{H}^n}\right)|u|^{q_i-2}u - \sum_{j=1}^{k} b_j\left(|\xi|_{\mathbb{H}^n}\right)|u|^{r_j-2}u, \end{aligned}$$

with the Navier boundary condition on a Korányi ball was proved [25] via a variational principle, where $w \in A_s$ is a Muckenhoupt weight function, and $\Delta^2_{\mathbb{H}^n,p}$ is the Heisenberg p-biharmonic operator.

The purpose of this paper is to prove the existence and multiplicity of weak solutions to the problem (1.1).

The structure of the paper is as follows: in Sect. 2, we recall some basic facts, which will be used later, and we also introduce our main tools. In Sect. 3, the existence of one weak solution for the problem (1.1) is proved; in Sect. 4, the existence of multiple weak solutions for the problem (1.1) is verified.

2 Basic definitions and preliminary results

Through the paper, we assume that $\Omega \subset \mathbb{R}^N$ $(N > 2)$ is a bounded domain with boundary of class C^1; $p, q \in C(\overline{\Omega})$, which hold the following inequalities

$$\max\{N/2, 2\} < q^- \leq q(x) \leq q^+ < p^- \leq p(x) \leq p^+ < \infty, \tag{2.1}$$

where

$$r^- := \inf_{x\in\Omega} r(x) \quad \text{and} \quad r^+ := \sup_{x\in\Omega} r(x)$$

for $r \in \{p, q\}$. We denote the variable exponent Lebesgue space by $L^{p(x)}(\Omega)$, i.e.,

$$L^{p(x)}(\Omega) = \left\{\Omega \longrightarrow \mathbb{R} : u \text{ is measurable and } \int_\Omega \left|u(x)\right|^{p(x)} dx < \infty\right\},$$

with the Luxemburg norm [10]

$$|u|_{p(x)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, where $L^{p'(x)}(\Omega)$ is the conjugate space of $L^{p(x)}(\Omega)$, the Hölder type inequality

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(x)}|v|_{p'(x)}$$

holds true.

Following the authors of [22], for any $\kappa > 0$, we put

$$\kappa^{\check{r}} := \begin{cases} \kappa^{r^+} & \kappa < 1, \\ \kappa^{r^-} & \kappa \geq 1, \end{cases} \tag{2.2}$$

and,

$$\kappa^{\hat{r}} := \begin{cases} \kappa^{r^-} & \kappa < 1, \\ \kappa^{r^+} & \kappa \geq 1 \end{cases} \tag{2.3}$$

for $r \in C_+(\Omega)$. The following proposition is well-known in Lebesgue spaces with variational exponent (for instance, see [15, Proposition 2.7]).

Proposition 2.1 *For each $u \in L^{p(x)}(\Omega)$, we have*

$$|u|_{p(x)}^{\check{p}} \leq \int_\Omega \left|u(x)\right|^{p(x)} dx \leq |u|_{p(x)}^{\hat{p}}.$$

Proposition 2.2 ([12]) *Let $p, q \in C_+(\overline{\Omega})$. If $q(x) \leq p(x)$, a.e. on Ω, then $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$; moreover, there is a constant k_q such that*

$$|u|_{q(x)} \leq k_q|u|_{p(x)}.$$

We denote the variable exponent Sobolev space $W^{k,p(x)}(\Omega)$ for $k = 1, 2$, by

$$W^{k,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\right\},$$

that in which $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_N} x_N}$ where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a multi-index with $|\alpha| = \Sigma_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$ with the norm

$$\|u\|_{k,p(x)} = \Sigma_{|\alpha| \leq k} \left|D^\alpha u\right|_{p(x)}$$

is a Banach separable and reflexive space. We assume that $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, which has the norm $\|u\|_{1,p(x)} = |Du|_{p(x)}$. In what follows, we set

$$X := W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega),$$

endowed with the norm

$$\|u\| := \inf\left\{\mu > 0 \int_\Omega \left|\frac{\Delta u}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

Remark 2.1 As a consequence of Proposition 2.2, if $q(x) \leq p(x)$ a.e on Ω, one has

$$W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,q(x)}(\Omega) \quad \text{and} \quad W^{2,p(x)}(\Omega) \hookrightarrow W^{2,q(x)}(\Omega).$$

In a special case,

$$X \hookrightarrow W_0^{1,p^-}(\Omega) \cap W^{2,p^-}(\Omega)$$

On the other hand, because $p^- > N/2$, so

$$W_0^{1,p^-}(\Omega) \cap W^{2,p^-}(\Omega) \hookrightarrow C^0(\overline{\Omega}).$$

Thus, the embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact; moreover, there exists constant $L > 0$ such that

$$|u|_\infty \leq L\|u\|, \tag{2.4}$$

where $|u|_\infty = \sup_{x\in\Omega} u(x)$.

Here, we recall the classical Hardy-Rellich inequality mentioned in [9].

Lemma 2.1 *Let* $1 < s < N/2$. *Then for* $u \in W_0^{1,s}(\Omega) \cap W^{2,s}(\Omega)$, *one has*

$$\int_\Omega \frac{|u(x)|^s}{|x|^{2s}} dx \leq \frac{1}{\mathcal{H}_s} \int_\Omega \left|\Delta u(x)\right|^s dx,$$

where $\mathcal{H}_s := \left(\frac{N(s-1)(N-2s)}{s^2}\right)^s$.

We mean by weak solution of the problem (1.1) is as follows.

Definition 2.1 We say that function $u \in X$ is a weak solution of Problem (1.1) if $u = \Delta u = 0$ on $\partial\Omega$ and

$$\begin{aligned}&\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v\,dx + \int_\Omega |\Delta u|^{q(x)-2}\Delta u\Delta v\,dx\\&\quad + \int_\Omega \theta(x)\frac{|u|^{s-2}}{|x|^{2s}}uv\,dx - \lambda\int_\Omega f(x,u)v\,dx = 0\end{aligned}$$

for every $v \in X$.

We continue by introducing the main tools of this paper. Do to this, we need the following definition.

Definition 2.2 Let Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I := \Phi - \Psi$ is said to verify the Palais-Smale condition cut of upper at r (in short $(PS)^{[r]}$) if any sequence $\{u_n\}_{n\in\mathbb{N}} \in X$ such that

- $I(u_n)$ is bounded;
- $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$;
- $\Phi(u_n) < r$ for each $n \in \mathbb{N}$;

has a convergent subsequence.

If $r = \infty$, we say that the functional $I := \Phi - \Psi$ verify the Palais-Smale condition.

The following is one of the main tools of the next section established in [6].

Theorem 2.1 *Let X be a real Banach space, and let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that*

$$\inf_{x\in X}\Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there exists positive constant $r \in \mathbb{R}$ and $\overline{x} \in X$ with $0 < \Phi(\overline{x}) < r$ such that

$$\frac{\sup_{x\in\Phi^{-1}(]-\infty,r[)}\Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}, \tag{2.5}$$

and for each

$$\lambda \in \Lambda := \left]\frac{\Phi(x)}{\Psi(x)}, \frac{r}{\sup_{x\in\Phi^{-1}(]-\infty,r[)}\Psi(x)}\right[$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition, then for each $\lambda \in \Lambda$, there is $x_\lambda \in \Phi^{-1}(]0,r[)$ such that $I_\lambda(x_\lambda) \le I_\lambda(x)$ for all $x \in \Phi^{-1}(]0,r[)$ and $I'_\lambda(u_\lambda) = 0$.

The other tool is the following abstract result proved in [5].

Theorem 2.2 *Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$.*

Fix $r > 0$ and assume that for each

$$\lambda \in \left]0, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the Palais-Smale condition, and it is unbounded from below. Then, for each

$$\lambda \in \left]0, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$

the functional I_λ admits two distinct critical points.

The other tool is the following theorem from [7].

Theorem 2.3 *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable whose Gâteaux derivative is compact such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that

(*i*) $\frac{\sup_{\Phi(x)<r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$;

(*ii*) *for each $\lambda \in \Lambda_r :=]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)<r} \Psi(x)}[$, the functional $I_\lambda := \Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

In the sequel, we put

$$\delta(x) = \sup\{\delta > 0 : B(x,\delta) \subseteq \Omega\} \quad \text{and} \quad R := \sup_{x \in \Omega} \delta(x).$$

Obviously, there exists $x^0 = (x_1^0, \dots, x_N^0) \in \Omega$ such that

$$B(x^0, R) \subseteq \Omega.$$

3 Existence result

Let $\Phi : X \to \mathbb{R}$ be a functional defined by

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx + \int_\Omega \frac{1}{q(x)} |\Delta u|^{q(x)}\, dx + \frac{1}{s} \int_\Omega \theta(x) \frac{|u(x)|^s}{|x|^{2s}}\, dx,$$

where $1 < s < N/2$, and the inequalities (2.1) hold.

Remark 3.1 Under the above assumptions, we gain

$$\frac{1}{p^+} \|u\|^{\check{p}} \le \Phi(u) \le K\left(\|u\|^{\hat{p}} + \|u\|^s\right),$$

where $K = \max\{\frac{2}{s}, \frac{2|\theta|_\infty}{s\mathcal{H}_s}\}$.

Proof Because $1 < s < N/2 < q^- \le q^+ < p^- \le p^+$, we have

$$\begin{aligned}\frac{1}{p^+}\|u\|^{\check{p}} &\le \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx \\ &\le \Phi(u) \\ &\le \frac{1}{s}\int_\Omega |\Delta u|^{p(x)}\,dx + \frac{1}{s}\int_\Omega |\Delta u|^{q(x)}\,dx + \frac{1}{s}\int_\Omega \theta(x)\frac{|u(x)|^s}{|x|^{2s}}\,dx.\end{aligned}$$

By applying the Hardy's inequality, we gain

$$\frac{1}{p^+}\|u\|^{\check{p}} \le \Phi(u) \le K\big(\|u\|^{\hat{p}} + \|u\|^s\big),$$

where $K = \max\{\frac{2}{s}, \frac{2|\theta|_\infty}{s\mathcal{H}_s}\}$, and then the proof is completed. □

It is known that Φ is a continuously Gâteaux differentiable functional; moreover,

$$\langle \Phi'(u), v\rangle = \int_\Omega \left(|\Delta u|^{p(x)-2}\Delta u\Delta v + |\Delta u|^{q(x)-2}\Delta u\Delta v + \theta(x)\frac{|u(x)|^{s-2}uv}{|x|^{2s}}\right)dx$$

for $u, v \in X$ (see [18]). Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with the growth condition (1.2) and define

$$F(x,t) := \int_0^t f(x,s)\,ds. \tag{3.1}$$

Then the functional $\Psi : X \to \mathbb{R}$ with

$$\Psi(u) := \int_\Omega F\big(x,u(x)\big)\,dx$$

for every $u \in X$ is continuously Gâteaux differentiable with the following compact derivative

$$\langle \Psi'(u), v\rangle := \int_\Omega f\big(x,u(x)\big)v(x)\,dx$$

for every u, v in X (see [1]). Now, define

$$I_\lambda = \Phi - \lambda\Psi.$$

If $I'_\lambda(u) = 0$, we have

$$\begin{aligned}\int_\Omega \bigg(&|\Delta u|^{p(x)-2}\Delta u\Delta v + |\Delta u|^{q(x)-2}\Delta u\Delta v \\ &+ \theta(x)\frac{|u|^{s-2}uv}{|x|^{2s}}\bigg)\,dx - \lambda\int_\Omega f(x,u)v\,dx = 0\end{aligned}$$

for every $u, v \in X$, then the critical points of I_λ are the weak solutions of Problem (1.1).

Lemma 3.1 *The functional I_λ verifies the Palais-Smale condition for every $\lambda > 0$.*

Proof Let $\{u_n\} \subseteq X$ be a Palais-Smale sequence, that is

$$\sup_n I_\lambda(u_n) < +\infty \quad \text{and} \quad \left\| I'_\lambda(u_n) \right\|_{X^*} \longrightarrow 0; \tag{3.2}$$

We prove that $\{u_n\} \subseteq X$ contains a convergent subsequence. By using (1.2), the Hölder inequality, and (2.4), we have

$$\begin{aligned} \Psi(u) &= \int_\Omega F\bigl(x, u(x)\bigr)\,dx \\ &\leq a_1 \int_\Omega \bigl|u(x)\bigr|\,dx + \frac{a_2}{\gamma^-} \int_\Omega \bigl|u(x)\bigr|^{\gamma(x)}\,dx \\ &\leq a_1 \bigl|u(x)\bigr|_\infty \int_\Omega 1\,dx + \frac{a_2}{\gamma^-} \bigl|u(x)\bigr|_\infty^{\hat{\gamma}} \int_\Omega 1\,dx \\ &\leq |\Omega| \biggl(a_1 L\|u\| + \frac{a_2}{\gamma^-} L^{\hat{\gamma}} \|u\|^{\hat{\gamma}} \biggr), \end{aligned} \tag{3.3}$$

where $|\Omega|$ is the Lebesgue measure of Ω. So, for n large enough, from Remark 3.1 and (3.3), one has

$$\begin{aligned} \bigl\langle I'_\lambda(u_n), u_n \bigr\rangle &= \bigl\langle \Phi'_\lambda(u_n), u_n \bigr\rangle - \lambda \bigl\langle \Psi'_\lambda(u_n), u_n \bigr\rangle \\ &\geq \|u_n\|^{\check{p}} - \lambda |\Omega| \biggl(a_1 L\|u_n\| + \frac{a_2}{\gamma^-} L^{\hat{\gamma}} \|u_n\|^{\hat{\gamma}} \biggr). \end{aligned}$$

Then, by applying (3.2), we have

$$\|u_n\|^{\check{p}} \leq \lambda |\Omega| \biggl(a_1 L\|u\| + \frac{a_2}{\gamma^-} L^{\hat{\gamma}} \|u\|^{\hat{\gamma}} \biggr);$$

since $\gamma(x) \leq p(x)$, it follows that $\{u_n\}$ is bounded. By the Eberlian-Smulyan theorem, passing to a subsequence if necessary, we can assume that $u_n \rightharpoonup u$. Then $\Psi'(u_n) \longrightarrow \Psi'(u)$ because of compactness. Since $I'_\lambda(u_n) = \Phi'(u_n) - \lambda\Psi'(u_n) \longrightarrow 0$, then $\Phi'(u_n) \longrightarrow \lambda\Psi'(u_n)$. By [11, Theorem 3.1], Φ' is a homeomorphism, then $u_n \longrightarrow u$, and so I_λ satisfies the Palais-Smale compactness condition. □

The next is one of the main results of this paper.

Theorem 3.1 *Let $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be Carathéodory function satisfying (1.2). Assume that there exist $r > 0$ and $\delta > 0$ such that*

$$\hat{K}\biggl(\biggl(\frac{2\delta N}{R^2 - (\frac{R}{2})^2} \biggr)^{\hat{p}} + \biggl(\frac{2\delta N}{R^2 - (\frac{R}{2})^2} \biggr)^{s} \biggr) m \biggl(R^N - \biggl(\frac{R}{2} \biggr)^N \biggr) < r, \tag{3.4}$$

where $m := \frac{\pi^{N/2}}{N/2\Gamma(N/2)}$ is the measure of unit ball of $\mathbb{R}^N$, and Γ is the Gamma function. Then, for each $\lambda \in]A_{r,\delta}, B_r[$, where

$$A_{r,\delta} := \frac{K((\frac{2\delta N}{R^2-(\frac{R}{2})^2})^{\hat{p}} + (\frac{2\delta N}{R^2-(\frac{R}{2})^2})^{s})(2^N - 1)}{\inf_{x\in\Omega} F(x,\delta)},$$

and

$$B_r := \frac{r}{|\Omega|(a_1 L(p^+ r)^{\frac{1}{\check{p}}} + \frac{a_2}{\gamma^-} L^{\hat{\gamma}} (p^+ r)^{\frac{\hat{\gamma}}{\check{p}}})},$$

Problem (1.1) *admits at least one non-trivial weak solution.*

Proof Using Theorem 2.1, one can prove the theorem. Thus, we need to show that the hypotheses of Theorem 2.1 are hold.

First of all, for the given $\lambda > 0$, from Lemma 3.1, the functional I_λ satisfies the $(PS)^{[r]}$ condition. Let $r > 0$ and $\delta > 0$ be as in (3.4) and the function $w \in X$ be defined by

$$w(x) := \begin{cases} 0 & x \in \Omega \setminus B(x^0, R), \\ \delta & x \in B(x^0, \frac{R}{2}), \\ \frac{\delta}{R^2 - (\frac{R}{2})^2}(R^2 - \sum_{i=1}^N (x_i - x_i^0)^2) & x \in B(x^0, R) \setminus B(x^0, \frac{R}{2}), \end{cases} \tag{3.5}$$

where $x = (x_1, \dots, x_N) \in \Omega$. Then,

$$\sum_{i=1}^N \frac{\partial^2 w}{\partial x_i^2}(x) = \begin{cases} 0 & x \in (\Omega \setminus B(x^0, R)) \cup B(x^0, \frac{R}{2}), \\ -\frac{2\delta N}{R^2 - (\frac{R}{2})^2} & x \in B(x_0, R) \setminus B(x^0, \frac{R}{2}). \end{cases}$$

So, by applying Remark 3.1, one has

$$\begin{aligned} &\frac{1}{p^+}\left(\frac{2\delta N}{R^2 - (\frac{R}{2})^2}\right)^{\check{p}} m\left(R^N - \left(\frac{R}{2}\right)^N\right) \\ &\quad < \Phi(w) \\ &\quad \le K\left(\left(\frac{2\delta N}{R^2 - (\frac{R}{2})^2}\right)^{\hat{p}} + \left(\frac{2\delta N}{R^2 - (\frac{R}{2})^2}\right)^{s}\right) m\left(R^N - \left(\frac{R}{2}\right)^N\right), \end{aligned}$$

then, we gain $\Phi(w) < r$. Plus that, one has

$$\Psi(w) \ge \int_{B(x^0, \frac{R}{2})} F(x, \delta)\, dx \ge \inf_{x \in \Omega} F(x, \delta) m\left(\frac{R}{2}\right)^N.$$

Then, we deduce that

$$\frac{\Psi(w)}{\Phi(w)} > \frac{\inf_{x \in \Omega} F(x, \delta)}{K((\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^{\hat{p}} + (\frac{2\delta N}{R^2 - (\frac{R}{2})^2})^{s})(2^N - 1)}.$$

On the other side, using Remark 3.1, for each $u \in \Phi^{-1}((-\infty, 1[)$, we have

$$\|u\| \le \left[p^+ \Phi(u)\right]^{\frac{1}{\check{p}}} \le \left(p^+ r\right)^{\frac{1}{\check{p}}}. \tag{3.6}$$

Hence, from (3.6) and (3.3), we deduce

$$\sup_{\Phi(u) < r} \Psi(u) \le |\Omega| \left(a_1 L\left(p^+ r\right)^{\frac{1}{\check{p}}} + \frac{a_2}{\gamma^-} L^{\hat{\gamma}} \left(p^+ r\right)^{\frac{\hat{\gamma}}{\check{p}}}\right).$$

Therefore, the conditions of Theorem 2.1 are verified. So, for each

$$\lambda \in]A_{r,\delta}, B_r[\subseteq \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}\right[,$$

the functional I_λ has at least one non-zero critical point, which is the weak solution of Problem (1.1). □

4 Multiplicity of weak solutions

Theorem 4.1 *Let $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be Carathéodory function satisfying Condition* (1.2), *and there exist constants $\mu > p^+$, $D > 0$ such that*

$$0 < \mu F(x,t) \leq tf(x,t) \tag{4.1}$$

for all $x \in \Omega$ and $|t| > D$. Then, for each $\lambda \in]0, B_r[$, where

$$B_r := \frac{r}{|\Omega|(a_1 L(p^+r)^{\frac{1}{\hat{p}}} + \frac{a_2}{\gamma^-}L^{\hat{\gamma}}(p^+r)^{\frac{\hat{\gamma}}{\hat{p}}})},$$

the problem (1.1) *admits at least two distinct weak solution.*

Proof By hypothesis (4.1) and simple computations, there exists $\alpha, \beta > 0$ such that

$$F(x,t) \geq \alpha|t|^\mu - \beta$$

for all $x \in \Omega$ and $|t| > D$. We show that I_λ is unbounded from below for $r > 1$

$$\begin{aligned} I_\lambda(ru) &= (\Phi - \lambda\Psi)(ru) \\ &\leq \frac{1}{p^+}K\left(r^{p^+}\|u\|^{\hat{p}} + r^s\|u\|^s\right) - \lambda\int_\Omega F(x,ru)\,dx \\ &\leq \frac{r^{p^+}}{p^+}K\left(\|u\|^{\hat{p}} + \|u\|^s\right) - \lambda r^\mu\int_\Omega |u|^\mu\,dx + \lambda\beta|\Omega|; \end{aligned}$$

since $\mu > p^+ > s$, so, I_λ is unbounded from below, and from Lemma 3.1, the functional I_λ verifies the Palais-Smale condition, so all hypotheses of Theorem 2.2 are verified. Then, for each $\lambda \in]0, B_r[$, I_λ admits at least two distinct critical points that are weak solutions of Problem (1.1). □

Theorem 4.2 *Let $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function satisfying* (1.2). *Then, for each $\lambda \in]A_{r,\delta}, B_r[$, where $A_{r,\delta}$ and B_r are given as in Theorem* 3.1, *those are*

$$A_{r,\delta} := \frac{K((\frac{2\delta N}{R^2-(\frac{R}{2})^2})^{\hat{p}} + (\frac{2\delta N}{R^2-(\frac{R}{2})^2})^s)(2^N - 1)}{\inf_{x\in\Omega}F(x,\delta)},$$

and

$$B_r := \frac{r}{|\Omega|(a_1 L(p^+r)^{\frac{1}{\hat{p}}} + \frac{a_2}{\gamma^-}L^{\hat{\gamma}}(p^+r)^{\frac{\hat{\gamma}}{\hat{p}}})},$$

the problem (1.1) *has at least three weak solutions.*

Proof The functionals Φ and Ψ defined in previous section satisfy all regularity assumptions requested in Theorem 2.3. So, our aim is to verify (i) and (ii) of Theorem 2.3. Put

$$\frac{1}{p^+}\left(\frac{2\delta N}{R^2-(\frac{R}{2})^2}\right)^{\check{p}} m\left(R^N-\left(\frac{R}{2}\right)^N\right)=r$$

and consider $w \in X$ as above, that is

$$w(x) := \begin{cases} 0 & x \in \Omega \setminus B(x^0,R), \\ \delta & x \in B(x^0,\frac{R}{2}), \\ \frac{\delta}{R^2-(\frac{R}{2})^2}(R^2-\sum_{i=1}^N (x_i-x_i^0)^2) & x \in B(x^0,R)\setminus B(x^0,\frac{R}{2}), \end{cases} \tag{4.2}$$

So, by applying Remark 3.1, we gain

$$\begin{aligned} \Phi(w) &= \int_\Omega \frac{1}{p(x)}|\Delta w|^{p(x)}\,dx + \int_\Omega \frac{1}{q(x)}|\Delta w|^{q(x)}\,dx + \frac{1}{s}\int_\Omega \theta(x)\frac{|w(x)|^s}{|x|^{2s}}\,dx \\ &> \frac{1}{p^+}\left(\frac{2\delta N}{R^2-(\frac{R}{2})^2}\right)^{\check{p}} m\left(R^N-\left(\frac{R}{2}\right)^N\right) \\ &= r. \end{aligned}$$

Therefore, the assumption (i) of Theorem 2.3 is satisfied. Now, we prove that the functional I_λ is coercive for all $\lambda > 0$.

From (3.3), we have

$$\Psi(u) \le |\Omega|\left(a_1 L\|u\| + \frac{a_2}{\gamma^-}L^{\hat{\gamma}}\|u\|^{\hat{\gamma}}\right);$$

and, from Remark 3.1, $\frac{1}{p^+}\|u\|^{\check{p}} \le \Phi(u)$. So, we gain that

$$I_\lambda(u) \ge \frac{1}{p^+}\|u\|^{\check{p}} - \lambda|\Omega|\left(a_1 L\|u\| + \frac{a_2}{\gamma^-}L^{\hat{\gamma}}\|u\|^{\hat{\gamma}}\right);$$

since $\check{p} > \hat{\gamma} > 1$, the functional I_λ is coercive. Then condition (ii) holds. So, all hypotheses of Theorem 2.3 are verified. Then, for each $\lambda \in]A_{r,\delta}, B_r[$, the functional I_λ admits at least three distinct critical points that are weak solutions of Problem (1.1). □

Remark 4.1 An interesting problem is to probe the existence and multiplicity of solutions of this equation under the Steklov-type boundary conditions [16, 19, 20, 24, 26] or in the Heisenberg Sobolev spaces and Orlicz-Sobolev spaces. Interested readers can see details of these spaces in [14, 22, 23, 25] and references therein.

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