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Existence and H-U stability of a tripled system of sequential fractional differential equations with multipoint boundary conditions

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Abstract

In this paper, we introduce a new coupled system of sequential fractional differential equations with coupled boundary conditions. We establish existence and uniqueness results using the Leray–Schauder alternative and Banach contraction principle. We examine the stability of the solutions involved in the Hyers–Ulam type. As an application, we present a few examples to illustrate the main results.

Keywords: Sequential fractional differential equations; Existence; Uniqueness; Stability; Fixed point theorems

1 Introduction

Over the last few decades, fractional calculus (FC) has evolved as an interesting subject of research. The FC methods greatly improved the study of integer-order mathematical models associated with real-world problems in a variety of scientific and technological disciplines, including finance, control theory [1], ecology [2], signal and image processing [3], blood flow phenomena [4], biophysics [5], and chaotic synchronization [6]. Fractional differential equations (FDEs) are more effective than classic integer-order differential equations (DEs) at representing real-world phenomena such as the knowledge and heredity properties of various materials. As a result, numerous scholars have examined FDEs in the mathematical modeling of a wide range of physical and technical processes [7–13]. Along with Riemann–Liouville, these operators are referred to in the literature as Grünwald–Letnikov, Caputo, Hilfer, and Hadamard.

Coupled systems with fractional differential equations are very important to study since they appear to have a wide range of problems in a variety of real-world scenarios. Scholars have also done numerous investigations of coupled systems of FDEs. Consider the following example: some of the most current results on the problem are contained in a series of papers [11, 14–17] and the references given in [18–22].

Stability analysis is another field of research that has received much attention to fractional differential equations in the last few decades. Various kinds of stability have been in-

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vestigated in the literature, including Mittag-Leffler, Lyapunov, and others. To our knowledge, the Ulam–Hyers stability of a coupled system of fractional differential equations has been studied very rarely.

Ulam and Hyers discovered a novel type of stability called the Hyers–Ulam stability. This type of research can aid in understanding biochemical processes and fluid motion, as well as semiconductors, population dynamics, heat conduction, and elasticity. This paper summarizes research on integral and nonlocal boundary value problems for coupled FDEs. The papers [16, 23–29] provide more insight into the theoretical approaches to the topic.

Zada, Yar, and Li [26] studied the nonlinear sequential coupled system of Caputo fractional differential equations

$$\begin{cases} (\mathcal{D}^\eta + \varphi \mathcal{D}^{\eta-1})\mathbf{p}(\varsigma) = \hat{\mathcal{F}}_1(\varsigma, \mathbf{p}(\varsigma), \mathbf{q}(\varsigma)), & 2 < \eta \leq 3, \\ (\mathcal{D}^\xi + \varphi \mathcal{D}^{\xi-1})\mathbf{q}(\varsigma) = \hat{\mathcal{F}}_2(\varsigma, \mathbf{p}(\varsigma), \mathbf{q}(\varsigma)), & 2 < \xi \leq 3, \\ \mathbf{p}(0) = 0, \quad \mathbf{p}(T) = \sum_{j=1}^k \eta_j \mathcal{I}^{\rho_j} \mathbf{q}(\varrho_j), \\ \mathbf{q}(0) = 0, \quad \mathbf{q}(T) = \sum_{j=1}^k \beta_j \mathcal{I}^{\gamma_j} \mathbf{q}(\varpi_j), \end{cases}$$

where ${}^C\mathcal{D}^\eta$ and ${}^C\mathcal{D}^\xi$ denote the Caputo fractional derivatives of orders η and ξ , \mathcal{I}^{ρ_j} and \mathcal{I}^{γ_j} are the Riemann–Liouville fractional integrals of orders $\rho_j, \gamma_j > 0$, $\beta_j, \eta_j \in (0, T)$, $k \in (\mathbb{R})^+$, $\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2 : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^2$, and $\rho_j, \gamma_j \in \mathbb{R}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, are real constants. The existence of solutions is established by the Banach contraction principle, and the uniqueness of solutions is established by the Leray–Schauder alternative. The Hyers–Ulam stability was also considered.

In [27] the authors studied a new kind of coupled system of three fractional differential equations with coupled boundary conditions:

$$\begin{cases} {}^C\mathcal{D}_{a+}^\eta u(\varsigma) = \rho(\varsigma, u(\varsigma), \hat{\mathcal{G}}_1(\varsigma), \hat{\mathcal{G}}_2(\varsigma)), & 1 < \eta \leq 2, \varsigma \in [a, b], \\ {}^C\mathcal{D}_{a+}^\xi \hat{\mathcal{G}}_1(\varsigma) = \varphi(\varsigma, u(\varsigma), \hat{\mathcal{G}}_1(\varsigma), \hat{\mathcal{G}}_2(\varsigma)), & 1 < \xi \leq 2, \varsigma \in [a, b], \\ {}^C\mathcal{D}_{a+}^\zeta \hat{\mathcal{G}}_2(\varsigma) = \psi(\varsigma, u(\varsigma), \hat{\mathcal{G}}_1(\varsigma), \hat{\mathcal{G}}_2(\varsigma)), & 2 < \zeta \leq 3, \varsigma \in [a, b], \\ u(a) = u_0, \quad u(b) = \sum_{i=1}^m p_i \hat{\mathcal{G}}_1(\alpha_i), \\ \hat{\mathcal{G}}_1(a) = 0, \quad \hat{\mathcal{G}}_1(b) = \sum_{j=1}^n q_j \hat{\mathcal{G}}_2(\beta_j), \\ \hat{\mathcal{G}}_2(\xi_1) = 0, \quad \hat{\mathcal{G}}_2(\xi_2) = 0, \quad \hat{\mathcal{G}}_2(b) = \sum_{k=1}^l r_k u(\gamma_k), \\ a < \xi_1 < \xi_2 < \alpha_1 < \dots < \alpha_m < \beta_1 < \dots < \beta_n < \gamma_1 < \dots < \gamma_l < b, \end{cases}$$

where ${}^C\mathcal{D}^\chi$ is the Caputo fractional derivative of order $\chi \in \{\eta, \xi, \zeta\}$, $\rho, \varphi, \psi : [a, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given functions, and $p_i, q_j, r_k \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, n$, $k = 1, \dots, l$. The existence is proved via the Leray–Schauder alternative, whereas the existence of a unique solution is established via the Banach contraction mapping principle. We suggest the reader a series of publications on FDE-coupled systems [27, 28, 30–34]. In the last two decades, the fractional-order differential equations appeared and began to study the predator–prey models in the fractional-order form. In [35] the authors studied the

predator–prey model of Holling-type II with harvesting and predator in disease

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{k}) - \frac{ayx}{m+x} - azx - h_1x, \\ \frac{dy}{dt} = bxy + \alpha yz + \frac{\gamma yx}{m+x} - h_2y, \\ \frac{dz}{dt} = bzx - \alpha yz - dz, \end{cases}$$

where x , y , and z are the prey, infected predator, and susceptible predator, respectively, and r , k , a , b , γ , α , h_1 , h_2 , d are assumed to be positive constants. They have studied the existence of a positive biological equilibrium and the uniform boundedness of the system. Local stability conditions are also defined based on Routh–Hurwitz. In [36] the authors discussed the fractional-order model of a two-prey-one-predator system

$$\begin{cases} {}^cD_*^\alpha x_1(t) = f_1(x_1, x_2, x_3) = \alpha x_1(t)(1 - x_1(t)) - x_1(t)x_3(t) + x_1(t)x_2(t)x_3(t), & t \in [0, T], \\ {}^cD_*^\alpha x_2(t) = f_2(x_1, x_2, x_3) = bx_2(t)(1 - x_2(t)) - x_2(t)x_3(t) + x_1(t)x_2(t)x_3(t), & t \in [0, T], \\ {}^cD_*^\alpha x_3(t) = f_3(x_1, x_2, x_3) = -cx_3^2(t) + dx_1(t)x_3(t) + ex_2(t)x_3(t), & t \in [0, T], \end{cases}$$

where c is the death rate of the predator, $0 \leq \alpha \leq 1$, $x_1(t) \geq 0$, $x_2(t) \geq 0$, $x_3(t) \geq 0$, and a , b , c , d , and e are all positive constants. They have studied the local asymptotic stability of the equilibrium solutions of the proposed model. One of the most important disciplines in the study of fractional-order differential equations is the theory of existence, uniqueness, and stability of solutions. In the present paper, inspired by the above-mentioned works, we introduce and investigate the existence and stability of solutions for the following coupled system of sequential fractional differential equations with nonlocal multipoint coupled boundary conditions:

$$\begin{cases} ({}^cD^\eta + \varphi {}^cD^{\eta-1})p(\varsigma) = \hat{F}_1(\varsigma, p(\varsigma), q(\varsigma), r(\varsigma)), & 1 < \eta \leq 2, \\ ({}^cD^\xi + \varphi {}^cD^{\xi-1})q(\varsigma) = \hat{F}_2(\varsigma, p(\varsigma), q(\varsigma), r(\varsigma)), & 1 < \xi \leq 2, \\ ({}^cD^\zeta + \varphi {}^cD^{\zeta-1})r(\varsigma) = \hat{F}_3(\varsigma, p(\varsigma), q(\varsigma), r(\varsigma)), & 2 < \zeta \leq 3, \\ p(0) = 0, \quad p(1) = \beta_1 \sum_{j=1}^{k-2} w_j q(\varrho_j), \\ q(0) = 0, \quad q(1) = \beta_2 \sum_{j=1}^{k-2} v_j r(\varpi_j), \\ r(0) = 0, \quad r'(0) = 0, \quad r(1) = \beta_3 \sum_{j=1}^{k-2} \vartheta_j p(\rho_j), \\ 0 < \varrho_1 < \varpi_1 < \rho_1 < \varrho_2 < \varpi_2 < \rho_2 \dots < \varrho_{k-2} < \varpi_{k-2} < \rho_{k-2} < 1, \end{cases} \quad (1)$$

where ${}^cD^\chi$ is the Caputo fractional derivative (CFD) of order $\chi \in \{\eta, \xi, \zeta\}$, $f, g, h : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given functions, φ is a positive real number, and $w_j, v_j, \vartheta_j \in \mathbb{R}$, $j = 1, \dots, k-2$, β_1 , β_2 , and β_3 are real constants.

The CFD ${}^cD^\chi$ of order χ is defined by

$${}^cD^\chi v(\varsigma) = \frac{1}{\Gamma(n-\chi)} \int_0^\varsigma (\varsigma - \varsigma)^{n-\chi-1} \left(\frac{d}{d\varsigma} \right)^n v(\varsigma) d\varsigma, \quad n-1 < \chi < n,$$

$$n = [\chi] + 1,$$

and the Riemann–Liouville integral of fractional order χ is defined by

$${}^{RL}\mathcal{I}^\chi v(\varsigma) = \frac{1}{\Gamma(\chi)} \int_0^\varsigma (\varsigma - \varsigma)^{\chi-1} v(\varsigma) d\varsigma, \quad \chi > 0,$$

This investigation is unique in that it also investigates a coupled system of three sequential fractional differential equations (SFDEs) of various orders in an arbitrary domain with multipoint boundary conditions. The multipoint boundary conditions, as we can see, are cyclic in nature and occur in a variety of nonlocal areas. As a result, our findings are more general and have a considerable impact on current research. Existence and uniqueness results can be obtained using fixed point theory. The Hyers–Ulam stability study is also performed.

The rest of the paper is organized as follows. In Sect. 2, we discuss several fundamental definitions and lemmas of fractional calculus. Additionally, we prove an auxiliary lemma involving a linear function of (1), which is necessary for obtaining the main results.

Section 3 summarizes the main results. We obtain the existence of a solution to the problem at hand using the Leray–Schauder alternative and also verify the existence of a unique solution using Banach's contraction mapping principle.

In Sect. 4, we prove that the proposed problem (1) is Ulam–Hyers stable under certain conditions.

In Sect. 5, we provide examples to illustrate the theoretical results.

2 Preliminaries

Here we recall some notations and definitions of fractional calculus [7–10, 37, 38].

Definition 1 The fractional integral of order α with the lower limit zero for a function f is defined as

$$I^\alpha f(\varsigma) = \frac{1}{\Gamma(\alpha)} \int_0^\varsigma \frac{f(s)}{(\varsigma - s)^{1-\alpha}} ds, \quad \varsigma > 0, \alpha > 0, \quad (2)$$

provided that the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function defined by $\Gamma(\alpha) = \int_0^\infty \varsigma^{\alpha-1} e^{-\varsigma} d\varsigma$.

Definition 2 The Riemann–Liouville fractional derivative of order $\alpha > 0$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$, is defined as

$$D_{0+}^\alpha f(\varsigma) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{d\varsigma} \right)^n \int_0^\varsigma (\varsigma - s)^{n-\alpha-1} f(s) ds, \quad \varsigma > 0, \quad (3)$$

where the function f has absolutely continuous derivatives up to order $(n - 1)$.

Definition 3 The Caputo derivative of order $r \in [n - 1, n)$ for a function $f : [0, \infty) \rightarrow \mathbb{R}$ can be written as

$${}^C D_{0+}^r f(\varsigma) = D_{0+}^r \left(f(\varsigma) - \sum_{k=0}^{n-1} \frac{\varsigma^k}{k!} f^{(k)}(0) \right), \quad \varsigma > 0, n - 1 < r < n. \quad (4)$$

Note that the Caputo fractional derivative of order $r \in [n - 1, n)$ exists almost everywhere on $[0, \infty)$ if $f \in AC^n([0, \infty), (\mathbb{R}))$.

Remark 1 If $f \in C^n[0, \infty)$, then

$${}^C D_{0+}^r \hat{f}(\varsigma) = \frac{1}{\Gamma(n-r)} \int_0^\varsigma \frac{\hat{f}^{(n)}(s)}{(\varsigma - s)^{r+1-n}} ds = I^{n-r} \hat{f}^{(n)}(\varsigma), \quad \varsigma > 0, n - 1 < r < n.$$

Now we are ready to present an essential solution we obtained for (1).

Lemma 1 Let $\hat{\mathcal{G}}_1, \hat{\mathcal{G}}_2, \hat{\mathcal{G}}_3 \in \mathcal{C}[0, 1]$ and $\Upsilon \neq 0$. Then the unique solution of the system

$$\begin{cases} (^c\mathcal{D}^\eta + \varphi ^c\mathcal{D}^{\eta-1})\mathfrak{p}(\varsigma) = \hat{\mathcal{G}}_1(\varsigma), & 1 < \eta \leq 2, \varsigma \in [0, 1], \\ (^c\mathcal{D}^\xi + \varphi ^c\mathcal{D}^{\xi-1})\mathfrak{q}(\varsigma) = \hat{\mathcal{G}}_2(\varsigma), & 1 < \xi \leq 2, \varsigma \in [0, 1], \\ (^c\mathcal{D}^\zeta + \varphi ^c\mathcal{D}^{\zeta-1})\mathfrak{r}(\varsigma) = \hat{\mathcal{G}}_3(\varsigma), & 2 < \zeta \leq 3, \varsigma \in [0, 1], \\ \mathfrak{p}(0) = 0, \quad \mathfrak{p}(1) = \beta_1 \sum_{j=1}^{k-2} w_j \mathfrak{q}(\varrho_j), \\ \mathfrak{q}(0) = 0, \quad \mathfrak{q}(1) = \beta_2 \sum_{j=1}^{k-2} v_j \mathfrak{r}(\varpi_j), \\ \mathfrak{r}(0) = 0, \quad \mathfrak{r}'(0) = 0, \quad \mathfrak{r}(1) = \beta_3 \sum_{j=1}^{k-2} \vartheta_j \mathfrak{p}(\rho_j), \\ 0 < \varrho_1 < \varpi_1 < \rho_1 < \varrho_2 < \varpi_2 < \rho_2 \dots < \varrho_{k-2} < \varpi_{k-2} < \rho_{k-2} < 1, \end{cases} \quad (5)$$

is given by

$$\mathfrak{p}(\varsigma) = \left(\frac{1 - e^{-\varphi \varsigma}}{\varphi \Upsilon} \right) \quad (6)$$

$$\begin{aligned} & \times \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds \right. \right. \\ & - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds \Bigg) \\ & - \hat{\mathfrak{A}}_5 \hat{\mathfrak{A}}_2 \left(\beta_2 \sum_{j=1}^{k-2} v_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds \right. \\ & - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds \Bigg) \\ & + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds \right. \\ & - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds \Bigg) \\ & \left. \left. + \int_0^\varsigma e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds, \right\} \right) \quad (7) \end{aligned}$$

$$\begin{aligned} \mathfrak{q}(\varsigma) = & \left(\frac{1 - e^{-\varphi \varsigma}}{\varphi \hat{\mathfrak{A}}_2} \right) \left\{ \beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds \right. \\ & - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds \\ & - \frac{1}{\Upsilon} \left[\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds \right. \right. \\ & - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds \Bigg) \end{aligned}$$

$$\begin{aligned}
& - \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_5 \mathfrak{A}_2 \left(\beta_2 \sum_{j=1}^{k-2} v_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds \right. \\
& \quad \left. - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds \right) \\
& + \hat{\mathfrak{A}}_1 \mathfrak{A}_2 \hat{\mathfrak{A}}_3 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds \right. \\
& \quad \left. - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds \right] \Bigg) \\
& + \int_0^\varsigma e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds,
\end{aligned} \tag{8}$$

$$\begin{aligned}
\mathfrak{r}(\varsigma) &= \frac{(\varphi \varsigma - 1 + e^{-\varphi \varsigma})}{\Upsilon \varphi^2} \\
&\times \left\{ \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds \right. \right. \\
&\quad \left. \left. - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds \right) \right. \\
&\quad \left. - \left[\hat{\mathfrak{A}}_6 \hat{\mathfrak{A}}_4 \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds \right. \right. \right. \\
&\quad \left. \left. \left. - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds \right) \right] \right. \\
&\quad \left. - \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_6 \left(\beta_2 \sum_{j=1}^{k-2} v_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds \right. \right. \\
&\quad \left. \left. - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds \right) \right] \right\} \\
& + \int_0^\varsigma e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds,
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
\hat{\mathfrak{A}}_1 &= \frac{(1 - e^{-\varphi})}{\varphi}, \quad \mathfrak{A}_2 = -\beta_1 \sum_{j=1}^{k-2} w_j \frac{(1 - e^{-\varphi \varrho_j})}{\varphi}, \\
\hat{\mathfrak{A}}_3 &= -\beta_2 \sum_{j=1}^{k-2} v_j \frac{(\varphi \varpi_j - 1 + e^{-\varphi \varpi_j})}{\varphi}, \quad \hat{\mathfrak{A}}_4 = \frac{(1 - e^{-\varphi})}{\varphi}, \\
\hat{\mathfrak{A}}_5 &= \frac{(\varphi - 1 + e^{-\varphi})}{\varphi^2}, \quad \hat{\mathfrak{A}}_6 = -\beta_3 \sum_{j=1}^{k-2} \vartheta_j \frac{(1 - e^{-\varphi \rho_j})}{\varphi},
\end{aligned} \tag{10}$$

$$\Upsilon = (\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \hat{\mathfrak{A}}_6),$$

$$\begin{aligned}
\mathcal{I}_1 &= \beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds \\
&\quad - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds, \\
\mathcal{I}_2 &= \beta_2 \sum_{j=1}^{k-2} v_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds \\
&\quad - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds, \\
\mathcal{I}_3 &= \beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds \\
&\quad - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds.
\end{aligned} \tag{11}$$

Proof As argued in [9], the solution of FDEs (5) can be written as

$$\mathfrak{p}(\varsigma) = c_0 e^{-\varphi \varsigma} + \frac{c_1}{\varphi} (1 - e^{-\varphi \varsigma}) + \int_0^\varsigma e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds, \tag{12}$$

$$\mathfrak{q}(\varsigma) = d_0 e^{-\varphi \varsigma} + \frac{d_1}{\varphi} (1 - e^{-\varphi \varsigma}) + \int_0^\varsigma e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds, \tag{13}$$

$$\begin{aligned}
\mathfrak{r}(\varsigma) &= b_0 e^{-\varphi \varsigma} + \frac{b_1}{\varphi} (1 - e^{-\varphi \varsigma}) + \frac{b_2}{\varphi^2} (\varphi \varsigma - 1 + e^{-\varphi \varsigma}) \\
&\quad + \int_0^\varsigma e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds.
\end{aligned} \tag{14}$$

Using the condition $\mathfrak{p}(0) = 0$ in (12), we get $c_0 = 0$, and the condition $\mathfrak{q}(0) = 0$ in (13) yields $d_0 = 0$, whereas the conditions $\mathfrak{r}(0) = 0$ and $r'0 = 0$ in (14) yield $b_0 = 0$ and $b_1 = 0$. Consequently, we have

$$\begin{aligned}
\mathfrak{p}(\varsigma) &= \frac{c_1}{\varphi} (1 - e^{-\varphi \varsigma}) + \int_0^\varsigma e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds, \\
\mathfrak{q}(\varsigma) &= \frac{d_1}{\varphi} (1 - e^{-\varphi \varsigma}) + \int_0^\varsigma e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds, \\
\mathfrak{r}(\varsigma) &= \frac{b_2}{\varphi^2} (\varphi \varsigma - 1 + e^{-\varphi \varsigma}) + \int_0^\varsigma e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds.
\end{aligned} \tag{15}$$

Using the conditions $\mathfrak{p}(1) = \beta_1 \sum_{j=1}^{k-2} w_j \mathfrak{q}(\varrho_j)$, $\mathfrak{q}(1) = \beta_2 \sum_{j=1}^{k-2} v_j \mathfrak{r}(\varpi_j)$, and $\mathfrak{r}(1) = \beta_3 \times \sum_{j=1}^{k-2} \vartheta_j \mathfrak{p}(\rho_j)$ in (15), we find that

$$\begin{aligned}
c_1 &= \frac{1}{\Upsilon} \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds \right. \right. \\
&\quad \left. \left. - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& - \hat{\mathfrak{A}}_5 \hat{\mathfrak{A}}_2 \left(\beta_2 \sum_{j=1}^{k-2} v_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds \right. \\
& \quad \left. - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds \right) \\
& \quad + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds \right. \\
& \quad \left. - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds \right) \Big\}, \tag{16}
\end{aligned}$$

$$\begin{aligned}
d_1 = & \frac{1}{\hat{\mathfrak{A}}_2} \left\{ \beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds \right. \\
& - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds \\
& - \frac{1}{\Upsilon} \left[\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds \right. \right. \\
& \quad \left. \left. - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds \right) \right] \Big\}, \tag{17}
\end{aligned}$$

$$\begin{aligned}
& - \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_5 \hat{\mathfrak{A}}_2 \left(\beta_2 \sum_{j=1}^{k-2} v_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds \right. \\
& \quad \left. - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds \right) \\
& \quad + \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds \right. \\
& \quad \left. - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds \right) \Big\}, \\
b_2 = & \frac{1}{\Upsilon} \left\{ \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds \right. \right. \\
& - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds \\
& - \left[\hat{\mathfrak{A}}_6 \hat{\mathfrak{A}}_4 \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds \right. \right. \\
& \quad \left. \left. - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{G}}_1(u) du \right) ds \right) \right] \\
& - \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_6 \left(\beta_2 \sum_{j=1}^{k-2} v_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{G}}_3(u) du \right) ds \right) \Big\}, \tag{18}
\end{aligned}$$

$$-\int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{G}}_2(u) du \right) ds \Bigg] \Bigg\},$$

from which by substituting into (15), we get the solutions (6)–(8)–(9). The converse follows by direct computation. This completes the proof. \square

3 Main results

Let $\mathcal{J} = \mathcal{C}([0, 1], \mathbb{R})$ be space equipped with the norm $\|q\| = \sup\{|q(\varsigma)|, \varsigma \in [0, 1]\}$. Obviously, $(\mathcal{J}, \|\cdot\|)$ is a Banach space, and, consequently, $(\mathcal{J} \times \mathcal{J} \times \mathcal{J}, \|(\mathfrak{p}, q, r)\|_{\mathcal{J}})$ is also a Banach space equipped with the norm $\|(\mathfrak{p}, q, r)\|_{\mathcal{J}} = \|\mathfrak{p}\| + \|q\| + \|r\|, \mathfrak{p}, q, r \in \mathcal{J}$.

In view of Lemma 1, we define the operator $\mathcal{S} : \mathcal{J} \times \mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J} \times \mathcal{J} \times \mathcal{J}$ by $\mathcal{S}(\mathfrak{p}(\varsigma), q(\varsigma), r(\varsigma)) = (\mathcal{S}_1(\mathfrak{p}(\varsigma), q(\varsigma), r(\varsigma)), \mathcal{S}_2(\mathfrak{p}(\varsigma), q(\varsigma), r(\varsigma)), \mathcal{S}_3(\mathfrak{p}(\varsigma), q(\varsigma), r(\varsigma)))$, where

$$\begin{aligned} & \mathcal{S}_1(\mathfrak{p}(\varsigma), q(\varsigma), r(\varsigma)) \\ &= \left(\frac{1 - e^{-\varphi\varsigma}}{\varphi\Upsilon} \right) \\ & \quad \times \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), q(u), r(u)) du \right) ds \right. \right. \\ & \quad - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{F}}_1(u, \mathfrak{p}(u), q(u), r(u)) du \right) ds \Big) \\ & \quad - \hat{\mathfrak{A}}_5 \hat{\mathfrak{A}}_2 \left(\beta_2 \sum_{j=1}^{k-2} v_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{F}}_3(u, \mathfrak{p}(u), q(u), r(u)) du \right) ds \right. \\ & \quad - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), q(u), r(u)) du \right) ds \Big) \\ & \quad + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{F}}_1(u, \mathfrak{p}(u), q(u), r(u)) du \right) ds \right. \\ & \quad - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{F}}_3(u, \mathfrak{p}(u), q(u), r(u)) du \right) ds \Big) \Big) \\ & \quad \left. + \int_0^{\varsigma} e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{F}}_1(u, \mathfrak{p}(u), q(u), r(u)) du \right) ds \right\} \\ & \mathcal{S}_2(\mathfrak{p}(\varsigma), q(\varsigma), r(\varsigma)) \\ &= \left(\frac{1 - e^{-\varphi\varsigma}}{\varphi\mathfrak{A}_2} \right) \\ & \quad \times \left\{ \beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), q(u), r(u)) du \right) ds \right. \\ & \quad - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{F}}_1(u, \mathfrak{p}(u), q(u), r(u)) du \right) ds \\ & \quad - \frac{1}{\Upsilon} \left[\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \right. \end{aligned}$$

$$\begin{aligned}
& \times \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right. \\
& - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \Big) \\
& - \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_5 \mathfrak{A}_2 \left(\beta_2 \sum_{j=1}^{k-2} v_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right. \\
& - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \Big) \\
& + \hat{\mathfrak{A}}_1 \mathfrak{A}_2 \hat{\mathfrak{A}}_3 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right. \\
& - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \Big) \Big] \Big\} \\
& + \int_0^\varsigma e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds, \\
& \mathcal{S}_3(\mathfrak{p}(\varsigma), \mathfrak{q}(\varsigma), \mathfrak{r}(\varsigma)) \\
& = \frac{(\varphi\varsigma - 1 + e^{-\varphi\varsigma})}{\Upsilon\varphi^2} \\
& \times \left\{ \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right. \right. \\
& - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \Big) \\
& - \left[\hat{\mathfrak{A}}_6 \hat{\mathfrak{A}}_4 \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right. \right. \\
& - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \Big) \\
& - \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_6 \left(\beta_2 \sum_{j=1}^{k-2} v_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right. \\
& - \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \Big) \Big] \Big\} \\
& + \int_0^\varsigma e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds.
\end{aligned}$$

We further use the following notations:

$$\mathcal{W}_1 = \left(\frac{1-e^{-\varphi\varsigma}}{\varphi\Upsilon} \right) \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left[\frac{(1-e^{-\varphi})}{\varphi\Gamma(\eta)} \right] + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left[\beta_3 \sum_{j=1}^{k-2} |\vartheta_j| \rho_j^{\eta-1} \frac{(1-e^{-\varphi\rho_j})}{\varphi\Gamma(\eta)} \right] \right\}$$

$$\begin{aligned}
& + \frac{(1 - e^{-\varphi})}{\varphi \Gamma(\eta)}, \\
\mathcal{V}_1 &= \left(\frac{1 - e^{-\varphi \zeta}}{\varphi \Upsilon} \right) \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \varrho_j^{\xi-1} \frac{(1 - e^{-\varphi \varrho_j})}{\varphi \Gamma(\xi)} \right) + \left(\hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_5 \left[\frac{(1 - e^{-\varphi})}{\varphi \Gamma(\xi)} \right] \right) \right\}, \\
\mathcal{U}_1 &= \left(\frac{1 - e^{-\varphi \zeta}}{\varphi \Upsilon} \right) \left\{ \hat{\mathfrak{A}}_3 \hat{\mathfrak{A}}_2 \left[\frac{(1 - e^{-\varphi})}{\varphi \Gamma(\zeta)} \right] + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_5 \left(\beta_2 \sum_{j=1}^{k-2} |\nu_j| \varpi_j^{\zeta-1} \frac{(1 - e^{-\varphi \varpi_j})}{\varphi \Gamma(\zeta)} \right) \right\}, \\
\mathcal{W}_2 &= \left(\frac{1 - e^{-\varphi \zeta}}{\varphi \mathfrak{A}_2} \right) \left\{ \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3}{\Upsilon} \left(\beta_3 \sum_{j=1}^{k-2} |\vartheta_j| \rho_j^{\eta-1} \frac{(1 - e^{-\varphi \rho_j})}{\varphi \Gamma(\eta)} \right) + \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5}{\Upsilon} \left(\frac{(1 - e^{-\varphi})}{\varphi \Gamma(\eta)} \right) \right. \\
& \quad \left. + \frac{(1 - e^{-\varphi})}{\varphi \Gamma(\eta)} \right\}, \\
\mathcal{V}_2 &= \left(\frac{1 - e^{-\varphi \zeta}}{\varphi \mathfrak{A}_2} \right) \left\{ \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \varrho_j^{\xi-1} \frac{(1 - e^{-\varphi \varrho_j})}{\varphi \Gamma(\xi)} \right) \right. \\
& \quad \left. + \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5}{\Upsilon} \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \varrho_j^{\xi-1} \frac{(1 - e^{-\varphi \varrho_j})}{\varphi \Gamma(\xi)} \right) \right. \\
& \quad \left. + \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5}{\Upsilon} \left[\frac{(1 - e^{-\varphi})}{\varphi \Gamma(\xi)} \right] \right\} + \left[\frac{(1 - e^{-\varphi})}{\varphi \Gamma(\xi)} \right], \\
\mathcal{U}_2 &= \left(\frac{1 - e^{-\varphi \zeta}}{\varphi \mathfrak{A}_2} \right) \left\{ \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3}{\Upsilon} \left(\frac{(1 - e^{-\varphi})}{\varphi \Gamma(\zeta)} \right) + \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5}{\Upsilon} \left(\beta_2 \sum_{j=1}^{k-2} |\nu_j| \varpi_j^{\zeta-1} \frac{(1 - e^{-\varphi \varpi_j})}{\varphi \Gamma(\zeta)} \right) \right\}, \\
\mathcal{W}_3 &= \left(\frac{(\varphi \zeta - 1 + e^{-\varphi \zeta})}{\varphi^2 \Upsilon} \right) \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_1 \left(\beta_3 \sum_{j=1}^{k-2} |\vartheta_j| \rho_j^{\eta-1} \frac{(1 - e^{-\varphi \rho_j})}{\varphi \Gamma(\eta)} \right) + \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_6 \left(\frac{(1 - e^{-\varphi})}{\varphi \Gamma(\eta)} \right) \right\}, \\
\mathcal{V}_3 &= \left(\frac{(\varphi \zeta - 1 + e^{-\varphi \zeta})}{\varphi^2 \Upsilon} \right) \left\{ \hat{\mathfrak{A}}_6 \hat{\mathfrak{A}}_4 \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \varrho_j^{\xi-1} \frac{(1 - e^{-\varphi \varrho_j})}{\varphi \Gamma(\xi)} \right) + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_6 \left(\frac{(1 - e^{-\varphi})}{\varphi \Gamma(\xi)} \right) \right\}, \\
\mathcal{U}_3 &= \left(\frac{(\varphi \zeta - 1 + e^{-\varphi \zeta})}{\varphi^2 \Upsilon} \right) \left\{ \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_6 \left(\beta_2 \sum_{j=1}^{k-2} |\nu_j| \varpi_j^{\zeta-1} \frac{(1 - e^{-\varphi \varpi_j})}{\varphi \Gamma(\zeta)} \right) + \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \left(\frac{(1 - e^{-\varphi})}{\varphi \Gamma(\zeta)} \right) \right\} \\
& \quad + \frac{(1 - e^{-\varphi})}{\varphi \Gamma(\zeta)}. \tag{19}
\end{aligned}$$

Now we provide our first finding, a proof of the existence of a solution to problem (1) using the Leray–Schauder alternative [39].

Lemma 2 Let $\mathfrak{E}: \mathfrak{I} \rightarrow \mathfrak{I}$ be a completely continuous (c.c.) operator. Let $\mathcal{Y}(\mathfrak{E}) = \{q \in \mathfrak{I} : q = \eta \mathfrak{E}(q) \text{ for some } 0 < \eta < 1\}$.

Then either the set $\mathcal{Y}(\mathfrak{E})$ is unbounded, or \mathfrak{E} has at least one fixed point (Leray–Schauder alternative) [39].

Theorem 1 Let $\Upsilon \neq 0$, where Υ is defined by (10).

Assume that $(M_2): \hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2, \hat{\mathcal{F}}_3: [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions and there exist constants $\kappa_i, \lambda_i, \varepsilon_i \geq 0$ ($i = 1, 2, 3$) and $\kappa_0 > 0, \lambda_0 > 0, \varepsilon_0 > 0$ such that for all

$\mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in \mathbb{R}$ and $\varsigma \in [0, 1]$,

$$\begin{aligned} |\hat{\mathcal{F}}_1(\varsigma, \mathfrak{p}, \mathfrak{q}, \mathfrak{r})| &\leq \kappa_0 + \kappa_1 |\mathfrak{p}| + \kappa_2 |\mathfrak{q}| + \kappa_3 |\mathfrak{r}|, \\ |\hat{\mathcal{F}}_2(\varsigma, \mathfrak{p}, \mathfrak{q}, \mathfrak{r})| &\leq \lambda_0 + \lambda_1 |\mathfrak{p}| + \lambda_2 |\mathfrak{q}| + \lambda_3 |\mathfrak{r}|, \\ |\hat{\mathcal{F}}_3(\varsigma, \mathfrak{p}, \mathfrak{q}, \mathfrak{r})| &\leq \varepsilon_0 + \varepsilon_1 |\mathfrak{p}| + \varepsilon_2 |\mathfrak{q}| + \varepsilon_3 |\mathfrak{r}|. \end{aligned}$$

Then problem (1) has at least one solution on $[0, 1]$, provided that

$$\begin{aligned} (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\lambda_1 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\varepsilon_1 &< 1, \\ (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_2 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\lambda_2 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\varepsilon_2 &< 1, \\ (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_3 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\lambda_3 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\varepsilon_3 &< 1, \end{aligned} \quad (20)$$

where $\mathcal{W}_i, \mathcal{V}_i, \mathcal{U}_i, i = 1, 2, 3$, are given in (19).

Proof The operator $\mathcal{S} : \mathcal{J} \times \mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J} \times \mathcal{J} \times \mathcal{J}$ is completely continuous since the functions $\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2$, and $\hat{\mathcal{F}}_3$ are completely continuous. Next, let $\hat{\Omega}_1 \subset \mathcal{J} \times \mathcal{J} \times \mathcal{J}$ be a bounded set to show the uniform boundedness. The operator \mathcal{S} is also continuous such that

$$\begin{aligned} |\hat{\mathcal{F}}_1(\varsigma, \mathfrak{p}(\varsigma), \mathfrak{q}(\varsigma), \mathfrak{r}(\varsigma))| &\leq \wp_1, \\ |\hat{\mathcal{F}}_2(\varsigma, \mathfrak{p}(\varsigma), \mathfrak{q}(\varsigma), \mathfrak{r}(\varsigma))| &\leq \wp_2, \\ |\hat{\mathcal{F}}_3(\varsigma, \mathfrak{p}(\varsigma), \mathfrak{q}(\varsigma), \mathfrak{r}(\varsigma))| &\leq \wp_3, \quad (\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) \in \hat{\Omega}_1, \end{aligned}$$

for nonnegative constants \wp_1, \wp_2 , and \wp_3 . Then, for any $(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) \in \hat{\Omega}_1$,

$$\begin{aligned} &|\mathcal{S}_1(\mathfrak{p}(\varsigma), \mathfrak{q}(\varsigma), \mathfrak{r}(\varsigma))| \\ &\leq \left(\frac{1 - e^{-\varphi\varsigma}}{\varphi\Upsilon} \right) \\ &\times \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} |\hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \right. \right. \\ &+ \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} |\hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \Bigg) \\ &+ \hat{\mathfrak{A}}_5 \hat{\mathfrak{A}}_2 \left(\beta_2 \sum_{j=1}^{k-2} v_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} |\hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \right. \\ &+ \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} |\hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \Bigg) \\ &+ \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} |\hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \right. \Bigg) \\ &+ \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} |\hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \Bigg) \\ &+ \int_0^\varsigma e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} |\hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{1-e^{-\varphi\zeta}}{\varphi\Upsilon} \right) \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left[\frac{(1-e^{-\varphi})}{\varphi\Gamma(\eta)} \right] + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left[\beta_3 \sum_{j=1}^{k-2} |\vartheta_j| \rho_j^{\eta-1} \frac{(1-e^{-\varphi\rho_j})}{\varphi\Gamma(\eta)} \right] \right\} \\
&\quad + \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \varrho_j^{\xi-1} \frac{(1-e^{-\varphi\varrho_j})}{\varphi\Gamma(\xi)} \right) + \left(\hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_5 \left[\frac{(1-e^{-\varphi})}{\varphi\Gamma(\xi)} \right] \right) \right\} \\
&\quad + \left\{ \hat{\mathfrak{A}}_3 \hat{\mathfrak{A}}_2 \left[\frac{(1-e^{-\varphi})}{\varphi\Gamma(\zeta)} \right] + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_5 \left(\beta_2 \sum_{j=1}^{k-2} |\nu_j| \varpi_j^{\zeta-1} \frac{(1-e^{-\varphi\varpi_j})}{\varphi\Gamma(\zeta)} \right) \right\} \\
&\quad + \frac{(1-e^{-\varphi})}{\varphi\Gamma(\eta)} \\
&\leq \left(\frac{1-e^{-\varphi\zeta}}{\varphi\Upsilon} \right) (\mathcal{W}_1 \wp_1 + \mathcal{V}_1 \wp_2 + \mathcal{U}_1 \wp_3),
\end{aligned}$$

which implies that

$$\|\mathcal{S}_1(\mathfrak{p}(\zeta), \mathfrak{q}(\zeta), \mathfrak{r}(\zeta))\|_{\mathcal{J}} \leq \left(\frac{1-e^{-\varphi\zeta}}{\varphi\Upsilon} \right) (\mathcal{W}_1 \wp_1 + \mathcal{V}_1 \wp_2 + \mathcal{U}_1 \wp_3).$$

Similarly, we can conclude that

$$\|\mathcal{S}_2(\mathfrak{p}(\zeta), \mathfrak{q}(\zeta), \mathfrak{r}(\zeta))\|_{\mathcal{J}} \leq \left(\frac{1-e^{-\varphi\zeta}}{\mathfrak{A}_2\varphi} \right) (\mathcal{W}_2 \wp_1 + \mathcal{V}_2 \wp_2 + \mathcal{U}_2 \wp_3)$$

and

$$\begin{aligned}
&|\mathcal{S}_3(\mathfrak{p}(\zeta), \mathfrak{q}(\zeta), \mathfrak{r}(\zeta))| \\
&\leq \sup_{\zeta \in [0,1]} \left| \frac{(\varphi\zeta - 1 + e^{-\varphi\zeta})}{\varphi^2\Upsilon} \right| \\
&\quad \times \left\{ \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} |\hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \right. \right. \\
&\quad + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} |\hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \Bigg) \\
&\quad + \left[\hat{\mathfrak{A}}_6 \hat{\mathfrak{A}}_4 \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} |\hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \right. \right. \\
&\quad + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} |\hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \Bigg) \\
&\quad + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_6 \left(\beta_2 \sum_{j=1}^{k-2} \nu_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} |\hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \right) \\
&\quad + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} |\hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \Bigg] \Bigg) \\
&\quad + \int_0^{\zeta} e^{-\varphi(\zeta-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} |\hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{(\varphi\varsigma - 1 + e^{-\varphi\varsigma})}{\varphi^2\Upsilon} \right) \left[\left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_1 \left(\beta_3 \sum_{j=1}^{k-2} |\vartheta_j| \rho_j^{\eta-1} \frac{(1 - e^{-\varphi\rho_j})}{\varphi\Gamma(\eta)} \right) + \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_6 \left(\frac{(1 - e^{-\varphi})}{\varphi\Gamma(\eta)} \right) \right\} \right. \\
&\quad + \left\{ \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \varrho_j^{\xi-1} \frac{(1 - e^{-\varphi\varrho_j})}{\varphi\Gamma(\xi)} \right) + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_6 \left(\frac{(1 - e^{-\varphi})}{\varphi\Gamma(\xi)} \right) \right\} \\
&\quad \left. + \left\{ \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_6 \left(\beta_2 \sum_{j=1}^{k-2} |\nu_j| \varpi_j^{\zeta-1} \frac{(1 - e^{-\varphi\varpi_j})}{\varphi\Gamma(\zeta)} \right) + \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \left(\frac{(1 - e^{-\varphi})}{\varphi\Gamma(\zeta)} \right) \right\} \right] + \frac{(1 - e^{-\varphi})}{\varphi\Gamma(\zeta)} \\
&\leq \left[\frac{(\varphi\varsigma - 1 + e^{-\varphi\varsigma})}{\varphi^2\Upsilon} \right] (\mathcal{W}_3\wp_1 + \mathcal{V}_3\wp_2 + \mathcal{U}_3\wp_3),
\end{aligned}$$

which accumulates to

$$\|\mathcal{S}_3(\mathfrak{p}(\varsigma), \mathfrak{q}(\varsigma), \mathfrak{r}(\varsigma))\|_{\mathcal{J}} \leq \left[\frac{(\varphi\varsigma - 1 + e^{-\varphi\varsigma})}{\varphi^2\Upsilon} \right] (\mathcal{W}_3\wp_1 + \mathcal{V}_3\wp_2 + \mathcal{U}_3\wp_3).$$

As a result, the operator \mathcal{S} is uniformly bounded, that is,

$$\begin{aligned}
&\|\mathcal{S}(\mathfrak{p}(\varsigma), \mathfrak{q}(\varsigma), \mathfrak{r}(\varsigma))\|_{\mathcal{J}} \\
&\leq \left[\left(\frac{1 - e^{-\varphi\varsigma}}{\Upsilon\varphi} \right) + \left(\frac{1 - e^{-\varphi\varsigma}}{\mathfrak{A}_2\varphi} \right) + \frac{(\varphi\varsigma - 1 + e^{-\varphi\varsigma})}{\varphi^2\Upsilon} \right] \\
&\quad + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\wp_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\wp_2 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\wp_3.
\end{aligned}$$

Next, we show that \mathcal{S} is equicontinuous.

Let $\varsigma_1, \varsigma_2 \in [0, 1]$ with $\varsigma_1 < \varsigma_2$. Then we have

$$\begin{aligned}
&|\mathcal{S}_1(\mathfrak{p}(\varsigma_2), \mathfrak{q}(\varsigma_2), \mathfrak{r}(\varsigma_2)) - \mathcal{S}_1(\mathfrak{p}(\varsigma_1), \mathfrak{q}(\varsigma_1), \mathfrak{r}(\varsigma_1))| \\
&\leq \frac{(e^{-\varphi\varsigma_2} - e^{-\varphi\varsigma_1})}{\Upsilon\varphi} \\
&\quad \times \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right. \right. \\
&\quad + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \Big) \\
&\quad + \hat{\mathfrak{A}}_5 \hat{\mathfrak{A}}_2 \left(\beta_2 \sum_{j=1}^{k-2} \nu_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right. \\
&\quad + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \Big) \\
&\quad + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right. \\
&\quad + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \Big) \Big\} \\
&\quad + \left| \int_0^{\varsigma_1} (e^{-\varphi(\varsigma_2-s)} - e^{-\varphi(\varsigma_1-s)}) \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right|
\end{aligned}$$

$$\begin{aligned}
& + \left| \int_{\varsigma_2}^{\varsigma_1} e^{-\varphi(\varsigma_2-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right| \\
& \leq \frac{(e^{-\varphi\varsigma_2} - e^{-\varphi\varsigma_1})}{\Upsilon\varphi} \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left[\frac{(1-e^{-\varphi})}{\varphi\Gamma(\eta)} \right] + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left[\beta_3 \sum_{j=1}^{k-2} |\vartheta_j| \rho_j^{\eta-1} \frac{(1-e^{-\varphi\rho_j})}{\varphi\Gamma(\eta)} \right] \right\} \wp_1 \\
& \quad + \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \rho_j^{\xi-1} \frac{(1-e^{-\varphi\rho_j})}{\varphi\Gamma(\xi)} \right) + \left(\hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_5 \left[\frac{(1-e^{-\varphi})}{\varphi\Gamma(\xi)} \right] \right) \right\} \wp_2 \\
& \quad + \left\{ \hat{\mathfrak{A}}_3 \hat{\mathfrak{A}}_2 \left[\frac{(1-e^{-\varphi})}{\varphi\Gamma(\zeta)} \right] + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_5 \left(\beta_2 \sum_{j=1}^{k-2} |\nu_j| \varpi_j^{\zeta-1} \frac{(1-e^{-\varphi\varpi_j})}{\varphi\Gamma(\zeta)} \right) \right\} \wp_3 \\
& \quad + \left| \int_0^{\varsigma_1} (e^{-\varphi(\varsigma_2-s)} - e^{-\varphi(\varsigma_1-s)}) \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} du \right) ds \right. \\
& \quad \left. + \int_{\varsigma_2}^{\varsigma_1} e^{-\varphi(\varsigma_2-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} du \right) ds \right| \wp_1.
\end{aligned}$$

In a similar way,

$$\begin{aligned}
& |\mathcal{S}_2(\mathfrak{p}(\varsigma_2), \mathfrak{q}(\varsigma_2), \mathfrak{r}(\varsigma_2)) - \mathcal{S}_2(\mathfrak{p}(\varsigma_1), \mathfrak{q}(\varsigma_1), \mathfrak{r}(\varsigma_1))| \\
& \leq \frac{(e^{-\varphi\varsigma_2} - e^{-\varphi\varsigma_1})}{\varphi\Upsilon} \\
& \quad \times \left\{ \beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right. \\
& \quad + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \\
& \quad + \frac{1}{\Upsilon} \left[\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \right. \\
& \quad \times \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right. \\
& \quad + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \left. \right] \\
& \quad + \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_5 \hat{\mathfrak{A}}_2 \left(\beta_2 \sum_{j=1}^{k-2} \nu_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right) \\
& \quad + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \left. \right] \\
& \quad + \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right) \\
& \quad + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \left. \right] \\
& \quad + \int_0^{\varsigma} e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds,
\end{aligned}$$

$$\begin{aligned}
& + \left| \int_0^{\varsigma_1} (e^{-\varphi(\varsigma_2-s)} - e^{-\varphi(\varsigma_1-s)}) \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right. \\
& \quad \left. + \int_{\varsigma_1}^{\varsigma_2} (e^{-\varphi(\varsigma_2-s)}) \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u)) du \right) ds \right| \\
& \leq \left(\frac{1-e^{-\varphi\xi}}{\varphi \mathfrak{A}_2} \right) \\
& \quad \times \left[\left\{ \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3}{\Upsilon} \left(\beta_3 \sum_{j=1}^{k-2} |\vartheta_j| \rho_j^{\eta-1} \frac{(1-e^{-\varphi\rho_j})}{\varphi\Gamma(\eta)} \right) \right. \right. \\
& \quad + \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5}{\Upsilon} \left(\frac{(1-e^{-\varphi})}{\varphi\Gamma(\eta)} \right) + \frac{(1-e^{-\varphi})}{\varphi\Gamma(\eta)} \Big\} \mathfrak{B}_1 \\
& \quad + \left\{ \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \varrho_j^{\xi-1} \frac{(1-e^{-\varphi\rho_j})}{\varphi\Gamma(\xi)} \right) + \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5}{\Upsilon} \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \varrho_j^{\xi-1} \frac{(1-e^{-\varphi\rho_j})}{\varphi\Gamma(\xi)} \right) \right. \\
& \quad + \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5}{\Upsilon} \left[\frac{(1-e^{-\varphi})}{\varphi\Gamma(\xi)} \right] \Big\} \mathfrak{B}_2 + \left\{ \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3}{\Upsilon} \left(\frac{(1-e^{-\varphi})}{\varphi\Gamma(\zeta)} \right) \right. \\
& \quad + \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5}{\Upsilon} \left(\beta_2 \sum_{j=1}^{k-2} |\nu_j| \varpi_j^{\zeta-1} \frac{(1-e^{-\varphi\varpi_j})}{\varphi\Gamma(\zeta)} \right) \Big\} \mathfrak{B}_3 \Big] \\
& \quad + \left| \int_0^{\varsigma_1} (e^{-\varphi(\varsigma_2-s)} - e^{-\varphi(\varsigma_1-s)}) \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} du \right) ds \right. \\
& \quad \left. + \int_{\varsigma_1}^{\varsigma_2} (e^{-\varphi(\varsigma_2-s)}) \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} du \right) ds \right| \mathfrak{B}_2,
\end{aligned}$$

and

$$\begin{aligned}
& |\mathcal{S}_3(\mathfrak{p}(\varsigma_2), \mathfrak{q}(\varsigma_2), \mathfrak{r}(\varsigma_2)) - \mathcal{S}_3(\mathfrak{p}(\varsigma_1), \mathfrak{q}(\varsigma_1), \mathfrak{r}(\varsigma_1))| \\
& \leq \left| \frac{(\varphi(\varsigma_2 - \varsigma_1) + e^{-\varphi\varsigma_2} - e^{-\varphi\varsigma_1})}{\varphi^2 \Upsilon} \right| \\
& \quad \times \left\{ \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} |\hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \right. \right. \\
& \quad + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} |\hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \Big) \\
& \quad + \left[\hat{\mathfrak{A}}_6 \hat{\mathfrak{A}}_4 \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} |\hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \right. \right. \\
& \quad + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} |\hat{\mathcal{F}}_1(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \Big) \\
& \quad + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_6 \left(\beta_2 \sum_{j=1}^{k-2} \nu_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} |\hat{\mathcal{F}}_3(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \right) \\
& \quad \left. \left. + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} |\hat{\mathcal{F}}_2(u, \mathfrak{p}(u), \mathfrak{q}(u), \mathfrak{r}(u))| du \right) ds \right) \right] \Big\}
\end{aligned}$$

$$\begin{aligned}
& + \left| \int_0^{\varsigma_1} (e^{-\varphi(\varsigma_2-s)} - e^{-\varphi(\varsigma_1-s)}) \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} |\hat{\mathcal{F}}_3(u, p(u), q(u), r(u))| du \right) ds \right. \\
& \quad \left. + \int_{\varsigma_1}^{\varsigma_2} e^{-\varphi(\varsigma_2-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} |\hat{\mathcal{F}}_3(u, p(u), q(u), r(u))| du \right) ds \right| \\
& \leq \left| \frac{(\varphi(\varsigma_2 - \varsigma_1) + e^{-\varphi\varsigma_2} - e^{-\varphi\varsigma_1})}{\varphi^2 \Upsilon} \right| \\
& \quad \times \left[\left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_1 \left(\beta_3 \sum_{j=1}^{k-2} |\vartheta_j| \rho_j^{\eta-1} \frac{(1-e^{-\varphi\rho_j})}{\varphi\Gamma(\eta)} \right) + \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_6 \left(\frac{(1-e^{-\varphi})}{\varphi\Gamma(\eta)} \right) \right\} \wp_1 \right. \\
& \quad + \left\{ \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \varpi_j^{\xi-1} \frac{(1-e^{-\varphi\varpi_j})}{\varphi\Gamma(\xi)} \right) + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_6 \left(\frac{(1-e^{-\varphi})}{\varphi\Gamma(\xi)} \right) \right\} \wp_2 \\
& \quad \left. + \left\{ \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_6 \left(\beta_2 \sum_{j=1}^{k-2} |\nu_j| \varpi_j^{\zeta-1} \frac{(1-e^{-\varphi\nu_j})}{\varphi\Gamma(\zeta)} \right) + \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \left(\frac{(1-e^{-\varphi})}{\varphi\Gamma(\zeta)} \right) \right\} \wp_3 \right] \\
& \quad + \left| \int_0^{\varsigma_1} (e^{-\varphi(\varsigma_2-s)} - e^{-\varphi(\varsigma_1-s)}) \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} du \right) ds \right. \\
& \quad \left. + \int_{\varsigma_1}^{\varsigma_2} e^{-\varphi(\varsigma_2-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} du \right) ds \right| \wp_3.
\end{aligned}$$

As $\varsigma_1 \rightarrow \varsigma_2$ is independent of p, q, r with respect to the boundedness of $\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2$, and $\hat{\mathcal{F}}_3$, the operator $\mathcal{S}(p, q, r)$ is equicontinuous. Thus the operator $\mathcal{S}(p, q, r)$ is completely continuous.

Finally, we show that the set $\mathcal{P} = \{(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) \in \mathcal{J} \times \mathcal{J} \times \mathcal{J} : (\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) = \nu \mathcal{S}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}), 0 \leq \nu \leq 1\}$ is bounded. Let $(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) \in \mathcal{P}$ with $(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) = \nu \mathcal{S}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})$. For any $\varsigma \in [0, 1]$, we have

$$\mathfrak{p}(\varsigma) = \nu \mathcal{S}_1(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(\varsigma),$$

$$\mathfrak{q}(\varsigma) = \nu \mathcal{S}_2(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(\varsigma),$$

$$\mathfrak{r}(\varsigma) = \nu \mathcal{S}_3(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(\varsigma).$$

Then by (\mathcal{M}_2)

$$\begin{aligned}
|\mathfrak{p}(\varsigma)| & \leq \left(\frac{1-e^{-\varphi\varsigma}}{\Upsilon\varphi} \right) + \mathcal{W}_1 (\kappa_0 + \kappa_1 |\mathfrak{p}| + \kappa_2 |\mathfrak{q}| + \kappa_3 |\mathfrak{r}|) \\
& \quad + \mathcal{V}_1 (\lambda_0 + \lambda_1 |\mathfrak{p}| + \lambda_2 |\mathfrak{q}| + \lambda_3 |\mathfrak{r}|) + \mathcal{U}_1 (\varepsilon_0 + \varepsilon_1 |\mathfrak{p}| + \varepsilon_2 |\mathfrak{q}| + \varepsilon_3 |\mathfrak{r}|) \\
& \leq \left(\frac{1-e^{-\varphi\varsigma}}{\Upsilon\varphi} \right) + \mathcal{W}_1 \kappa_0 + \mathcal{V}_1 \lambda_0 + \mathcal{U}_1 \varepsilon_0 \\
& \quad + (\mathcal{W}_1 \kappa_1 + \mathcal{V}_1 \lambda_1 + \mathcal{U}_1 \varepsilon_1) |\mathfrak{p}| \\
& \quad + (\mathcal{W}_1 \kappa_2 + \mathcal{V}_1 \lambda_2 + \mathcal{U}_1 \varepsilon_2) |\mathfrak{q}| \\
& \quad + (\mathcal{W}_1 \kappa_3 + \mathcal{V}_1 \lambda_3 + \mathcal{U}_1 \varepsilon_3) |\mathfrak{r}|, \\
|\mathfrak{q}(\varsigma)| & \leq \left(\frac{1-e^{-\varphi\varsigma}}{\varphi\mathfrak{A}_2} \right) + \mathcal{W}_2 \kappa_0 + \mathcal{V}_2 \lambda_0 + \mathcal{U}_2 \varepsilon_0 \\
& \quad + (\mathcal{W}_2 \kappa_1 + \mathcal{V}_2 \lambda_1 + \mathcal{U}_2 \varepsilon_1) |\mathfrak{p}| \\
& \quad + (\mathcal{W}_2 \kappa_2 + \mathcal{V}_2 \lambda_2 + \mathcal{U}_2 \varepsilon_2) |\mathfrak{q}|
\end{aligned}$$

$$+ (\mathcal{W}_2\kappa_3 + \mathcal{V}_2\lambda_3 + \mathcal{U}_2\varepsilon_3)|\mathfrak{r}|,$$

and

$$\begin{aligned} |\mathfrak{r}(\varsigma)| &\leq \left[\frac{(\varphi\varsigma - 1 + e^{-\varphi\varsigma})}{\varphi^2\Upsilon} \right] + \mathcal{W}_3\kappa_0 + \mathcal{V}_3\lambda_0 + \mathcal{U}_3\varepsilon_0 \\ &+ (\mathcal{W}_3\kappa_1 + \mathcal{V}_3\lambda_1 + \mathcal{U}_3\varepsilon_1)|\mathfrak{p}| \\ &+ (\mathcal{W}_3\kappa_2 + \mathcal{V}_3\lambda_2 + \mathcal{U}_3\varepsilon_2)|\mathfrak{q}| \\ &+ (\mathcal{W}_3\kappa_3 + \mathcal{V}_3\lambda_3 + \mathcal{U}_3\varepsilon_3)|\mathfrak{r}|. \end{aligned}$$

As a result, we can conclude that

$$\begin{aligned} \|\mathfrak{p}(\varsigma)\| &\leq \left(\frac{1 - e^{-\varphi\varsigma}}{\Upsilon\varphi} \right) + \mathcal{W}_1\kappa_0 + \mathcal{V}_1\lambda_0 + \mathcal{U}_1\varepsilon_0 + (\mathcal{W}_1\kappa_1 + \mathcal{V}_1\lambda_1 + \mathcal{U}_1\varepsilon_1)\|\mathfrak{p}\| \\ &+ (\mathcal{W}_1\kappa_2 + \mathcal{V}_1\lambda_2 + \mathcal{U}_1\varepsilon_2)\|\mathfrak{q}\| + (\mathcal{W}_1\kappa_3 + \mathcal{V}_1\lambda_3 + \mathcal{U}_1\varepsilon_3)\|\mathfrak{r}\|, \\ \|\mathfrak{q}(\varsigma)\| &\leq \left(\frac{1 - e^{-\varphi\varsigma}}{\varphi\mathfrak{A}_2} \right) + \mathcal{W}_2\kappa_0 + \mathcal{V}_2\lambda_0 + \mathcal{U}_2\varepsilon_0 + (\mathcal{W}_2\kappa_1 + \mathcal{V}_2\lambda_1 + \mathcal{U}_2\varepsilon_1)\|\mathfrak{p}\| \\ &+ (\mathcal{W}_2\kappa_2 + \mathcal{V}_2\lambda_2 + \mathcal{U}_2\varepsilon_2)\|\mathfrak{q}\| + (\mathcal{W}_2\kappa_3 + \mathcal{V}_2\lambda_3 + \mathcal{U}_2\varepsilon_3)\|\mathfrak{r}\|, \\ \|\mathfrak{r}(\varsigma)\| &\leq \left[\frac{(\varphi\varsigma - 1 + e^{-\varphi\varsigma})}{\varphi^2\Upsilon} \right] + \mathcal{W}_3\kappa_0 + \mathcal{V}_3\lambda_0 + \mathcal{U}_3\varepsilon_0 + (\mathcal{W}_3\kappa_1 + \mathcal{V}_3\lambda_1 + \mathcal{U}_3\varepsilon_1)\|\mathfrak{p}\| \\ &+ (\mathcal{W}_3\kappa_2 + \mathcal{V}_3\lambda_2 + \mathcal{U}_3\varepsilon_2)\|\mathfrak{q}\| + (\mathcal{W}_3\kappa_3 + \mathcal{V}_3\lambda_3 + \mathcal{U}_3\varepsilon_3)\|\mathfrak{r}\|. \end{aligned}$$

By the previous three inequalities we arrive at

$$\begin{aligned} \|\mathfrak{p}\| + \|\mathfrak{q}\| + \|\mathfrak{r}\| &\leq \left(\frac{1 - e^{-\varphi\varsigma}}{\Upsilon\varphi} \right) + \left(\frac{1 - e^{-\varphi\varsigma}}{\varphi\mathfrak{A}_2} \right) + \left[\frac{(\varphi\varsigma - 1 + e^{-\varphi\varsigma})}{\varphi^2\Upsilon} \right] \\ &+ (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_0 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\lambda_0 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\varepsilon_0 \\ &+ [(\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\lambda_1 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\varepsilon_1]\|\mathfrak{p}\| \\ &+ [(\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_2 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\lambda_2 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\varepsilon_2]\|\mathfrak{q}\| \\ &+ [(\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_3 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\lambda_3 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\varepsilon_3]\|\mathfrak{r}\|, \end{aligned}$$

implying that

$$\begin{aligned} \|(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})\|_{\mathcal{J}} &\leq \frac{1}{\Phi} \left[\left(\frac{1 - e^{-\varphi\varsigma}}{\Upsilon\varphi} \right) + \left(\frac{1 - e^{-\varphi\varsigma}}{\varphi\mathfrak{A}_2} \right) + \left[\frac{(\varphi\varsigma - 1 + e^{-\varphi\varsigma})}{\varphi^2\Upsilon} \right] \right. \\ &\quad \left. + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_0 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\lambda_0 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\varepsilon_0 \right], \end{aligned}$$

where $\Phi = \min\{1 - [(\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_i + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\lambda_i + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\varepsilon_i], i = 1, 2, 3\}$. which means that \mathcal{P} is bounded. Thus by the Leray–Schauder alternative [39] the operator \mathcal{S} has at least one fixed point, which implies that problem (1) has at least one solution on $[0, 1]$. This completes the proof. \square

Banach's principle of contraction mapping provides the basis for our next results on the existence and uniqueness.

Theorem 2 Let $\Upsilon \neq 0$, where Υ is defined by (10) and (11). In addition, we assume that

(\mathcal{T}_1) $\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2, \hat{\mathcal{F}}_3 : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, and there exist non-negative constants Θ_1, Θ_2 , and Θ_3 such that for all $\varsigma \in [0, 1]$ and $p_i, q_i, r_i \in \mathbb{R}, i = 1, 2, 3$, we have

$$\begin{aligned} |\hat{\mathcal{F}}_1(\varsigma, q_1, q_2, q_3) - \hat{\mathcal{F}}_1(\varsigma, r_1, r_2, r_3)| &\leq \Theta_1(|q_1 - r_1| + |q_2 - r_2| + |q_3 - r_3|), \\ |\hat{\mathcal{F}}_2(\varsigma, q_1, q_2, q_3) - \hat{\mathcal{F}}_2(\varsigma, r_1, r_2, r_3)| &\leq \Theta_2(|q_1 - r_1| + |q_2 - r_2| + |q_3 - r_3|), \\ |\hat{\mathcal{F}}_3(\varsigma, q_1, q_2, q_3) - \hat{\mathcal{F}}_3(\varsigma, r_1, r_2, r_3)| &\leq \Theta_3(|q_1 - r_1| + |q_2 - r_2| + |q_3 - r_3|) \end{aligned}$$

if

$$(\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\Theta_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\Theta_2 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\Theta_3 < 1, \quad (21)$$

where $\mathcal{W}_i, \mathcal{V}_i, \mathcal{U}_i$ are given in (19). Then system (1) has a unique solution on $[0, 1]$.

Proof Let $\sup_{\varsigma \in [0, 1]} \hat{\mathcal{F}}_1(\varsigma, 0, 0, 0) = Q_1 < \infty$, $\sup_{\varsigma \in [0, 1]} \hat{\mathcal{F}}_2(\varsigma, 0, 0, 0) = Q_2 < \infty$, and $\sup_{\varsigma \in [0, 1]} \hat{\mathcal{F}}_3(\varsigma, 0, 0, 0) = Q_3 < \infty$, and let $\Psi > 0$ be such that

$$\Psi > \frac{\left(\frac{1-e^{-\varphi\varsigma}}{\Upsilon\varphi}\right) + \left(\frac{1-e^{-\varphi\varsigma}}{\mathfrak{A}_2\varphi}\right) + \left[\frac{(\varphi\varsigma-1+e^{-\varphi\varsigma})}{\varphi^2\Upsilon}\right] + \mathcal{O}_1}{1 - (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\Theta_1 - (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\Theta_2 - (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\Theta_3},$$

where $\mathcal{O}_1 = (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)Q_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)Q_2 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)Q_3$.

We will show that $\mathcal{S}B_\Psi \subset B_\Psi$, where $B_\Psi = \{(\mathfrak{p}, q, r) \in X \times X \times X : \|(\mathfrak{p}, q, r)\| \leq \Psi\}$.

By assumption (\mathcal{M}_2), for $(\mathfrak{p}, q, r) \in B_\Psi$, $\varsigma \in [0, 1]$, we have

$$\begin{aligned} |\mathfrak{p}(\varsigma, p(\varsigma), q(\varsigma), r(\varsigma))| &\leq |\mathfrak{p}(\varsigma, p(\varsigma), q(\varsigma), q(\varsigma)) - \mathfrak{p}(\varsigma, 0, 0, 0)| \\ &\leq \Theta_1(|p(\varsigma)| + |q(\varsigma)| + |r(\varsigma)|) + Q_1 \\ &\leq \Theta_1(\|\mathfrak{p}\| + \|q\| + \|r\|) + Q_1 \leq \Theta_1\Psi + Q_1, \end{aligned} \quad (22)$$

$$\begin{aligned} |q(\varsigma, p(\varsigma), q(\varsigma), r(\varsigma))| &\leq |q(\varsigma, p(\varsigma), q(\varsigma), q(\varsigma)) - q(\varsigma, 0, 0, 0)| \\ &\leq \Theta_2(|p(\varsigma)| + |q(\varsigma)| + |r(\varsigma)|) + Q_2 \\ &\leq \Theta_2(\|\mathfrak{p}\| + \|q\| + \|r\|) + Q_2 \leq \Theta_2\Psi + Q_2, \end{aligned} \quad (23)$$

$$\begin{aligned} |r(\varsigma, p(\varsigma), q(\varsigma), r(\varsigma))| &\leq |r(\varsigma, p(\varsigma), q(\varsigma), q(\varsigma)) - r(\varsigma, 0, 0, 0)| \\ &\leq \Theta_3(|p(\varsigma)| + |q(\varsigma)| + |r(\varsigma)|) + Q_3 \\ &\leq \Theta_3(\|\mathfrak{p}\| + \|q\| + \|r\|) + Q_3 \leq \Theta_3\Psi + Q_3, \end{aligned} \quad (24)$$

using (22), (23), and (24), This leads to

$$\begin{aligned} |\mathcal{S}_1((\mathfrak{p}, q, r)(\varsigma))| \\ \leq \left(\frac{1 - e^{-\varphi\varsigma}}{\varphi\Upsilon} \right) \end{aligned}$$

$$\begin{aligned}
& \times \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left[\frac{(1-e^{-\varphi})}{\varphi \Gamma(\eta)} \right] + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left[\beta_3 \sum_{j=1}^{k-2} |\vartheta_j| \rho_j^{\eta-1} \frac{(1-e^{-\varphi \rho_j})}{\varphi \Gamma(\eta)} \right] \right\} \|\hat{\mathcal{F}}_1\| \\
& + \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \rho_j^{\xi-1} \frac{(1-e^{-\varphi \rho_j})}{\varphi \Gamma(\xi)} \right) + \left(\hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_5 \left[\frac{(1-e^{-\varphi})}{\varphi \Gamma(\xi)} \right] \right) \right\} \|\hat{\mathcal{F}}_2\| \\
& + \left\{ \hat{\mathfrak{A}}_3 \hat{\mathfrak{A}}_2 \left[\frac{(1-e^{-\varphi})}{\varphi \Gamma(\zeta)} \right] + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_5 \left(\beta_2 \sum_{j=1}^{k-2} |\nu_j| \varpi_j^{\zeta-1} \frac{(1-e^{-\varphi \nu_j})}{\varphi \Gamma(\zeta)} \right) \right\} \|\hat{\mathcal{F}}_3\| \\
& + \frac{(1-e^{-\varphi})}{\varphi \Gamma(\eta)} \\
& \leq \left(\frac{1-e^{-\varphi \varsigma}}{\Upsilon \varphi} \right) + \mathcal{W}_1(\Theta_1 \Psi + Q_1) + \mathcal{V}_1(\Theta_2 \Psi + Q_2) + \mathcal{U}_1(\Theta_3 \Psi + Q_3) \\
& \leq \left(\frac{1-e^{-\varphi \varsigma}}{\Upsilon \varphi} \right) + (\mathcal{W}_1 \Theta_1 + \mathcal{V}_1 \Theta_2 + \mathcal{U}_1 \Theta_3) \Psi + \mathcal{W}_1 Q_1 + \mathcal{V}_1 Q_2 + \mathcal{U}_1 Q_3,
\end{aligned}$$

which, on taking the norm on $\varsigma \in [0, 1]$, yields

$$\begin{aligned}
\|\mathcal{S}_1(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})\|_{\mathcal{J}} & \leq \left(\frac{1-e^{-\varphi \varsigma}}{\Upsilon \varphi} \right) + (\mathcal{W}_1 \Theta_1 + \mathcal{V}_1 \Theta_2 + \mathcal{U}_1 \Theta_3) \Psi \\
& + \mathcal{W}_1 Q_1 + \mathcal{V}_1 Q_2 + \mathcal{U}_1 Q_3.
\end{aligned}$$

Likewise, we can find that

$$\begin{aligned}
\|\mathcal{S}_2(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})\|_{\mathcal{J}} & \leq \left(\frac{1-e^{-\varphi \varsigma}}{\mathfrak{A}_2 \varphi} \right) + (\mathcal{W}_2 \Theta_1 + \mathcal{V}_2 \Theta_2 + \mathcal{U}_2 \Theta_3) \Psi \\
& + \mathcal{W}_2 Q_1 + \mathcal{V}_2 Q_2 + \mathcal{U}_2 Q_3
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{S}_3(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})\|_{\mathcal{J}} & \leq \left[\frac{(\varphi \varsigma - 1 + e^{-\varphi \varsigma})}{\varphi^2 \Upsilon} \right] + (\mathcal{W}_3 \Theta_1 + \mathcal{V}_3 \Theta_2 + \mathcal{U}_3 \Theta_3) \Psi \\
& + \mathcal{W}_3 Q_1 + \mathcal{V}_3 Q_2 + \mathcal{U}_3 Q_3.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|\mathcal{S}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})\|_{\mathcal{J}} & \leq \left(\frac{1-e^{-\varphi \varsigma}}{\Upsilon \varphi} \right) + \left(\frac{1-e^{-\varphi \varsigma}}{\mathfrak{A}_2 \varphi} \right) + \left[\frac{(\varphi \varsigma - 1 + e^{-\varphi \varsigma})}{\varphi^2 \Upsilon} \right] \\
& + [(\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3) \Theta_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3) \Theta_2 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3) \Theta_3] \Psi \\
& + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3) Q_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3) Q_2 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3) Q_3 \\
& \leq \Psi.
\end{aligned}$$

Now, for $(\mathfrak{p}_1, \mathfrak{q}_1, \mathfrak{r}_1), (\mathfrak{p}_2, \mathfrak{q}_2, \mathfrak{r}_2) \in \mathcal{J} \times \mathcal{J} \times \mathcal{J}$ and for any $\varsigma \in [0, 1]$, we get

$$\begin{aligned}
& |\mathcal{S}_1((\mathfrak{p}_2, \mathfrak{q}_2, \mathfrak{r}_2)(\varsigma)) - \mathcal{S}_1((\mathfrak{p}_1, \mathfrak{q}_1, \mathfrak{r}_1)(\varsigma))| \\
& \leq \left(\frac{1-e^{-\varphi \varsigma}}{\Upsilon \varphi} \right)
\end{aligned}$$

$$\begin{aligned}
& \times \left[\left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left[\frac{(1-e^{-\varphi})}{\varphi \Gamma(\eta)} \right] + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left[\beta_3 \sum_{j=1}^{k-2} |\vartheta_j| \rho_j^{\eta-1} \frac{(1-e^{-\varphi \rho_j})}{\varphi \Gamma(\eta)} \right] \right\} \right. \\
& \times \Theta_1 (\|\mathfrak{p}_2 - \mathfrak{p}_1\| + \|\mathfrak{q}_2 - \mathfrak{q}_1\| + \|\mathfrak{r}_2 - \mathfrak{r}_1\|) \\
& + \left. \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \varrho_j^{\xi-1} \frac{(1-e^{-\varphi \varrho_j})}{\varphi \Gamma(\xi)} \right) + \left(\hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_5 \left[\frac{(1-e^{-\varphi})}{\varphi \Gamma(\xi)} \right] \right) \right\} \right. \\
& \times \Theta_2 (\|\mathfrak{p}_2 - \mathfrak{p}_1\| + \|\mathfrak{q}_2 - \mathfrak{q}_1\| + \|\mathfrak{r}_2 - \mathfrak{r}_1\|) \\
& + \left. \left\{ \hat{\mathfrak{A}}_3 \hat{\mathfrak{A}}_2 \left[\frac{(1-e^{-\varphi})}{\varphi \Gamma(\zeta)} \right] + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_5 \left(\beta_2 \sum_{j=1}^{k-2} |\nu_j| \varpi_j^{\zeta-1} \frac{(1-e^{-\varphi \varpi_j})}{\varphi \Gamma(\zeta)} \right) \right\} \right] \\
& \times \Theta_3 (\|\mathfrak{p}_2 - \mathfrak{p}_1\| + \|\mathfrak{q}_2 - \mathfrak{q}_1\| + \|\mathfrak{r}_2 - \mathfrak{r}_1\|) + \frac{(1-e^{-\varphi})}{\varphi \Gamma(\eta)} \\
& \leq (\mathcal{W}_1 \Theta_1 + \mathcal{V}_1 \Theta_2 + \mathcal{U}_1 \Theta_3) (\|\mathfrak{p}_2 - \mathfrak{p}_1\| + \|\mathfrak{q}_2 - \mathfrak{q}_1\| + \|\mathfrak{r}_2 - \mathfrak{r}_1\|),
\end{aligned}$$

from which we obtain

$$\begin{aligned}
& \|\mathcal{S}_1((\mathfrak{p}_2, \mathfrak{q}_2, \mathfrak{r}_2)(\varsigma)) - \mathcal{S}_1((\mathfrak{p}_1, \mathfrak{q}_1, \mathfrak{r}_1)(\varsigma))\|_{\mathcal{J}} \\
& \leq (\mathcal{W}_1 \Theta_1 + \mathcal{V}_1 \Theta_2 + \mathcal{U}_1 \Theta_3) (\|\mathfrak{p}_2 - \mathfrak{p}_1\| + \|\mathfrak{q}_2 - \mathfrak{q}_1\| + \|\mathfrak{r}_2 - \mathfrak{r}_1\|).
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \|\mathcal{S}_2((\mathfrak{p}_2, \mathfrak{q}_2, \mathfrak{r}_2)(\varsigma)) - \mathcal{S}_2((\mathfrak{p}_1, \mathfrak{q}_1, \mathfrak{r}_1)(\varsigma))\|_{\mathcal{J}} \\
& \leq (\mathcal{W}_2 \Theta_1 + \mathcal{V}_2 \Theta_2 + \mathcal{U}_2 \Theta_3) (\|\mathfrak{p}_2 - \mathfrak{p}_1\| + \|\mathfrak{q}_2 - \mathfrak{q}_1\| + \|\mathfrak{r}_2 - \mathfrak{r}_1\|),
\end{aligned}$$

and

$$\begin{aligned}
& \|\mathcal{S}_3((\mathfrak{p}_2, \mathfrak{q}_2, \mathfrak{r}_2)(\varsigma)) - \mathcal{S}_3((\mathfrak{p}_1, \mathfrak{q}_1, \mathfrak{r}_1)(\varsigma))\|_{\mathcal{J}} \\
& \leq (\mathcal{W}_3 \Theta_1 + \mathcal{V}_3 \Theta_2 + \mathcal{U}_3 \Theta_3) (\|\mathfrak{p}_2 - \mathfrak{p}_1\| + \|\mathfrak{q}_2 - \mathfrak{q}_1\| + \|\mathfrak{r}_2 - \mathfrak{r}_1\|), \\
& \|\mathcal{S}(\mathfrak{p}_2, \mathfrak{q}_2, \mathfrak{r}_2) - \mathcal{S}(\mathfrak{p}_1, \mathfrak{q}_1, \mathfrak{r}_1)\|_{\mathcal{J}} \\
& \leq [(\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3) \Theta_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3) \Theta_2 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3) \Theta_3] \\
& \times (\|\mathfrak{p}_2 - \mathfrak{p}_1\| + \|\mathfrak{q}_2 - \mathfrak{q}_1\| + \|\mathfrak{r}_2 - \mathfrak{r}_1\|).
\end{aligned}$$

In view of this inequality and (21), \mathcal{S} is a contraction. As a result of Banach's fixed point theorem, there exists a unique fixed point for the operator \mathcal{S} , which corresponds to a unique solution to problem (1) on $[0, 1]$. The proof is complete. \square

4 Hyers–Ulam stability

Let us define the nonlinear operators $\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3 \in \mathcal{C}([0, 1], \mathbb{R}) \times \mathcal{C}([0, 1], \mathbb{R}) \times \mathcal{C}([0, 1], \mathbb{R}) \rightarrow \mathcal{C}([0, 1], \mathbb{R})$ by

$$\begin{cases} (^c\mathcal{D}^\eta + \varphi ^c\mathcal{D}^{\eta-1})\mathfrak{p}(\varsigma) - \hat{\mathcal{F}}_1(\varsigma, \mathfrak{p}(\varsigma), \mathfrak{q}(\varsigma), \mathfrak{r}(\varsigma)) = \mathfrak{J}_1(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(\varsigma), \\ (^c\mathcal{D}^\xi + \varphi ^c\mathcal{D}^{\xi-1})\mathfrak{q}(\varsigma) - \hat{\mathcal{F}}_2(\varsigma, \mathfrak{p}(\varsigma), \mathfrak{q}(\varsigma), \mathfrak{r}(\varsigma)) = \mathfrak{J}_2(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(\varsigma), \\ (^c\mathcal{D}^\zeta + \varphi ^c\mathcal{D}^{\zeta-1})\mathfrak{r}(\varsigma) - \hat{\mathcal{F}}_3(\varsigma, \mathfrak{p}(\varsigma), \mathfrak{q}(\varsigma), \mathfrak{r}(\varsigma)) = \mathfrak{J}_3(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(\varsigma) \end{cases} \quad (25)$$

for $\varsigma \in [0, 1]$. For some $\pi_1, \pi_2, \pi_3 > 0$, we consider the following inequalities:

$$\|\mathfrak{J}_1(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})\| \leq \pi_1, \quad \|\mathfrak{J}_2(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})\| \leq \pi_2, \quad \|\mathfrak{J}_3(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})\| \leq \pi_3. \quad (26)$$

Definition 4 The coupled system (1) is said to be stable in the Hyers–Ulam sense if there exist $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 > 0$ such that there is a unique solution $(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) \in \mathcal{C}([0, 1], \mathbb{R}) \times \mathcal{C}([0, 1], \mathbb{R}) \times \mathcal{C}([0, 1], \mathbb{R})$ of problem (1) with

$$\|(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) - (\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})\| \leq \mathcal{K}_1 \pi_1 + \mathcal{K}_2 \pi_2 + \mathcal{K}_3 \pi_3$$

for every solution $(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})$ belonging to $\mathcal{C}([0, 1], \mathbb{R}) \times \mathcal{C}([0, 1], \mathbb{R}) \times \mathcal{C}([0, 1], \mathbb{R})$ of inequality (26).

Theorem 3 Suppose that (\mathcal{M}_2) holds. Then the BVP (1) is Hyers–Ulam stable.

Proof Let $(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) \in \mathcal{C}([0, 1], \mathbb{R}) \times \mathcal{C}([0, 1], \mathbb{R}) \times \mathcal{C}([0, 1], \mathbb{R})$ be a solution of problem (1) that satisfies the main results. Let $(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})$ be any solution satisfying (26):

$$\begin{cases} (^c\mathcal{D}^\eta + \varphi ^c\mathcal{D}^{\eta-1})\mathfrak{p}(\varsigma) = \hat{\mathcal{F}}_1(\varsigma, \mathfrak{p}(\varsigma), \mathfrak{q}(\varsigma), \mathfrak{r}(\varsigma)) + \mathfrak{J}_1(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(\varsigma), \\ (^c\mathcal{D}^\xi + \varphi ^c\mathcal{D}^{\xi-1})\mathfrak{q}(\varsigma) = \hat{\mathcal{F}}_2(\varsigma, \mathfrak{p}(\varsigma), \mathfrak{q}(\varsigma), \mathfrak{r}(\varsigma)) + \mathfrak{J}_2(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(\varsigma), \\ (^c\mathcal{D}^\zeta + \varphi ^c\mathcal{D}^{\zeta-1})\mathfrak{r}(\varsigma) = \hat{\mathcal{F}}_3(\varsigma, \mathfrak{p}(\varsigma), \mathfrak{q}(\varsigma), \mathfrak{r}(\varsigma)) + \mathfrak{J}_3(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(\varsigma) \end{cases} \quad (27)$$

for $\varsigma \in [0, 1]$. Then

$$\begin{aligned} \widehat{\mathfrak{p}}(\varsigma) &= \mathcal{S}_1(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\varsigma) + \left(\frac{1 - e^{-\varphi \varsigma}}{\varphi \Upsilon} \right) \\ &\quad \times \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \mathfrak{J}_2(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) du \right) ds \right. \right. \\ &\quad + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \mathfrak{J}_1(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) du \right) ds \Big) \\ &\quad + \hat{\mathfrak{A}}_5 \hat{\mathfrak{A}}_2 \left(\beta_2 \sum_{j=1}^{k-2} v_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \mathfrak{J}_3(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) du \right) ds \right. \\ &\quad + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \mathfrak{J}_2(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) du \right) ds \Big) \\ &\quad + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \mathfrak{J}_1(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) du \right) ds \right. \\ &\quad + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \mathfrak{J}_3(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) du \right) ds \Big) \Big) \\ &\quad \left. + \int_0^{\varsigma} e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \mathfrak{J}_1(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) du \right) ds \right\} \\ |\widehat{\mathfrak{p}}(\varsigma) - \mathcal{S}_1(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\varsigma)| &\leq \left(\frac{1 - e^{-\varphi \varsigma}}{\varphi \Upsilon} \right) \end{aligned}$$

$$\begin{aligned}
& \times \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \pi_2 du \right) ds \right. \right. \\
& + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \pi_1 du \right) ds \Big) \\
& + \hat{\mathfrak{A}}_5 \mathfrak{A}_2 \left(\beta_2 \sum_{j=1}^{k-2} v_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \pi_3 du \right) ds \right. \\
& + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} \pi_2 du \right) ds \Big) \\
& + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \pi_1 du \right) ds \right. \\
& + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} \pi_3 du \right) ds \Big) \Big\} \\
& + \int_0^\varsigma e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} \pi_1 du \right) ds \\
& \leq \left(\frac{1-e^{-\varphi\varsigma}}{\Upsilon\varphi} \right) \\
& \times \left\{ \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left[\frac{(1-e^{-\varphi})}{\varphi\Gamma(\eta)} \right] + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left[\beta_3 \sum_{j=1}^{k-2} |\vartheta_j| \rho_j^{\eta-1} \frac{(1-e^{-\varphi\rho_j})}{\varphi\Gamma(\eta)} \right] \pi_1 \right\} \right. \\
& + \left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \varrho_j^{\xi-1} \frac{(1-e^{-\varphi\varrho_j})}{\varphi\Gamma(\xi)} \right) + \left(\mathfrak{A}_2 \hat{\mathfrak{A}}_5 \left[\frac{(1-e^{-\varphi})}{\varphi\Gamma(\xi)} \right] \right) \right\} \pi_2 \\
& + \left. \left\{ \hat{\mathfrak{A}}_3 \mathfrak{A}_2 \left[\frac{(1-e^{-\varphi})}{\varphi\Gamma(\zeta)} \right] + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_5 \left(\beta_2 \sum_{j=1}^{k-2} |v_j| \varpi_j^{\zeta-1} \frac{(1-e^{-\varphi\varpi_j})}{\varphi\Gamma(\zeta)} \right) \right\} \pi_3 \right] \\
& + \frac{(1-e^{-\varphi})}{\varphi\Gamma(\eta)} \\
& \leq (\mathcal{W}_1 \pi_1 + \mathcal{V}_1 \pi_2 + \mathcal{U}_1 \pi_3).
\end{aligned}$$

In the same way,

$$\begin{aligned}
\widehat{\mathfrak{q}}(\varsigma) &= \mathcal{S}_2(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\varsigma) + \left(\frac{1-e^{-\varphi\varsigma}}{\varphi\mathfrak{A}_2} \right) \\
&\times \left\{ \beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} G_2(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(u) du \right) ds \right. \\
&+ \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} G_1(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(u) du \right) ds \\
&+ \frac{1}{\Upsilon} \left[\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5 \left(\beta_1 \sum_{j=1}^{k-2} w_j \int_0^{\varrho_j} e^{-\varphi(\varrho_j-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} G_2(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(u) du \right) ds \right. \right. \\
&\left. \left. \right]
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} G_1(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(u) du \right) ds \\
& + \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_5 \hat{\mathfrak{A}}_2 \left(\beta_2 \sum_{j=1}^{k-2} v_j \int_0^{\varpi_j} e^{-\varphi(\varpi_j-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} G_3(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(u) du \right) ds \right. \\
& \quad \left. + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} G_2(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(u) du \right) ds \right) \\
& + \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3 \left(\beta_3 \sum_{j=1}^{k-2} \vartheta_j \int_0^{\rho_j} e^{-\varphi(\rho_j-s)} \left(\int_0^s \frac{(s-u)^{\eta-2}}{\Gamma(\eta-1)} G_1(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(u) du \right) ds \right. \\
& \quad \left. + \int_0^1 e^{-\varphi(1-s)} \left(\int_0^s \frac{(s-u)^{\zeta-2}}{\Gamma(\zeta-1)} G_3(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(u) du \right) ds \right) \Big] \Big] \Big\} \\
& + \int_0^\varsigma e^{-\varphi(\varsigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-2}}{\Gamma(\xi-1)} G_2(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(u) du \right) ds,
\end{aligned}$$

$$\begin{aligned}
& |\widehat{\mathfrak{q}}(\varsigma) - \mathcal{S}_2(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\varsigma)| \\
& \leq \left(\frac{1 - e^{-\varphi\varsigma}}{\varphi \hat{\mathfrak{A}}_2} \right) \\
& \times \left[\left\{ \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3}{\Upsilon} \left(\beta_3 \sum_{j=1}^{k-2} |\vartheta_j| \rho_j^{\eta-1} \frac{(1 - e^{-\varphi\rho_j})}{\varphi \Gamma(\eta)} \right) \right. \right. \\
& \quad \left. \left. + \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5}{\Upsilon} \left(\frac{(1 - e^{-\varphi})}{\varphi \Gamma(\eta)} \right) + \frac{(1 - e^{-\varphi})}{\varphi \Gamma(\eta)} \right\} \pi_1 \right. \\
& \quad \left. + \left\{ \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \varrho_j^{\xi-1} \frac{(1 - e^{-\varphi\varrho_j})}{\varphi \Gamma(\xi)} \right) + \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5}{\Upsilon} \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \varrho_j^{\xi-1} \frac{(1 - e^{-\varphi\varrho_j})}{\varphi \Gamma(\xi)} \right) \right. \right. \\
& \quad \left. \left. + \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5}{\Upsilon} \left[\frac{(1 - e^{-\varphi})}{\varphi \Gamma(\xi)} \right] \right\} \pi_2 + \left\{ \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_3}{\Upsilon} \left(\frac{(1 - e^{-\varphi})}{\varphi \Gamma(\zeta)} \right) \right. \right. \\
& \quad \left. \left. + \frac{\hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_5}{\Upsilon} \left(\beta_2 \sum_{j=1}^{k-2} |\vartheta_j| \varpi_j^{\zeta-1} \frac{(1 - e^{-\varphi\varpi_j})}{\varphi \Gamma(\zeta)} \right) \right\} \pi_3 \right] + \frac{(1 - e^{-\varphi})}{\varphi \Gamma(\xi)} \\
& \leq (\mathcal{W}_2 \pi_1 + \mathcal{V}_2 \pi_2 + \mathcal{U}_2 \pi_3).
\end{aligned}$$

Similarly,

$$\begin{aligned}
& |\widehat{\mathfrak{r}}(\varsigma) - \mathcal{S}_3(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\varsigma)| \\
& \leq \left(\frac{(\varphi\varsigma - 1 + e^{-\varphi\varsigma})}{\varphi^2 \Upsilon} \right) \\
& \times \left[\left\{ \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_1 \left(\beta_3 \sum_{j=1}^{k-2} |\vartheta_j| \rho_j^{\eta-1} \frac{(1 - e^{-\varphi\rho_j})}{\varphi \Gamma(\eta)} \right) + \hat{\mathfrak{A}}_4 \hat{\mathfrak{A}}_6 \left(\frac{(1 - e^{-\varphi})}{\varphi \Gamma(\eta)} \right) \right\} \pi_1 \right. \\
& \quad \left. + \left\{ \left(\beta_1 \sum_{j=1}^{k-2} |w_j| \varrho_j^{\xi-1} \frac{(1 - e^{-\varphi\varrho_j})}{\varphi \Gamma(\xi)} \right) + \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_6 \left(\frac{(1 - e^{-\varphi})}{\varphi \Gamma(\xi)} \right) \right\} \pi_2 \right]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ \hat{\mathfrak{A}}_2 \hat{\mathfrak{A}}_6 \left(\beta_2 \sum_{j=1}^{k-2} |\nu_j| \varpi_j^{\zeta-1} \frac{(1-e^{-\varphi \omega_j})}{\varphi \Gamma(\zeta)} \right) + \hat{\mathfrak{A}}_1 \hat{\mathfrak{A}}_4 \left(\frac{(1-e^{-\varphi})}{\varphi \Gamma(\zeta)} \right) \right\} \pi_3 \\
& + \frac{(1-e^{-\varphi})}{\varphi \Gamma(\zeta)} \\
& \leq (\mathcal{W}_3 \pi_1 + \mathcal{V}_3 \pi_2 + \mathcal{U}_3 \pi_3),
\end{aligned}$$

where $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{U}_1, \mathcal{U}_2$, and \mathcal{U}_3 are described in the main results. Therefore the operator \mathcal{S} defined in the main results can be excluded from the fixed point property as follows. We have

$$\begin{aligned}
|\mathfrak{p}(\zeta) - \widehat{\mathfrak{p}}(\zeta)| & = |\mathfrak{p}(\zeta) - \mathcal{S}_1(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\zeta) + \mathcal{S}_1(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\zeta) - \widehat{\mathfrak{p}}(\zeta)| \\
& \leq |\mathcal{S}_1(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(\zeta) - \mathcal{S}_1(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\zeta)| + |\mathcal{S}_1(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\zeta) - \widehat{\mathfrak{p}}(\zeta)| \\
& \leq (\mathcal{W}_1 k_1 + \mathcal{V}_1 \lambda_1 + \mathcal{U}_1 \varepsilon_1) + (\mathcal{W}_1 k_2 + \mathcal{V}_1 \lambda_2 + \mathcal{U}_1 \varepsilon_2) \\
& \quad + (\mathcal{W}_1 k_3 + \mathcal{V}_1 \lambda_3 + \mathcal{U}_1 \varepsilon_3) \|(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) - (\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})\| \\
& \quad + (\mathcal{W}_1 \pi_1 + \mathcal{V}_1 \pi_2 + \mathcal{U}_1 \pi_3),
\end{aligned} \tag{28}$$

so we obtain

$$\begin{aligned}
|\mathfrak{q}(\zeta) - \widehat{\mathfrak{q}}(\zeta)| & = |\mathfrak{q}(\zeta) - \mathcal{S}_2(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\zeta) + \mathcal{S}_2(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\zeta) - \widehat{\mathfrak{q}}(\zeta)| \\
& \leq |\mathcal{S}_2(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(\zeta) - \mathcal{S}_2(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\zeta)| + |\mathcal{S}_2(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\zeta) - \widehat{\mathfrak{q}}(\zeta)| \\
& \leq (\mathcal{W}_2 k_1 + \mathcal{V}_2 \lambda_1 + \mathcal{U}_2 \varepsilon_1) + (\mathcal{W}_2 k_2 + \mathcal{V}_2 \lambda_2 + \mathcal{U}_2 \varepsilon_2) \\
& \quad + (\mathcal{W}_2 k_3 + \mathcal{V}_2 \lambda_3 + \mathcal{U}_2 \varepsilon_3) \|(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) - (\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})\| \\
& \quad + (\mathcal{W}_2 \pi_1 + \mathcal{V}_2 \pi_2 + \mathcal{U}_2 \pi_3)
\end{aligned} \tag{29}$$

and, in the same way,

$$\begin{aligned}
|\mathfrak{r}(\zeta) - \widehat{\mathfrak{r}}(\zeta)| & = |\mathfrak{r}(\zeta) - \mathcal{S}_3(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\zeta) + \mathcal{S}_3(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\zeta) - \widehat{\mathfrak{r}}(\zeta)| \\
& \leq |\mathcal{S}_3(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})(\zeta) - \mathcal{S}_3(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\zeta)| + |\mathcal{S}_3(\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})(\zeta) - \widehat{\mathfrak{r}}(\zeta)| \\
& \leq (\mathcal{W}_3 k_1 + \mathcal{V}_3 \lambda_1 + \mathcal{U}_3 \varepsilon_1) + (\mathcal{W}_3 k_2 + \mathcal{V}_3 \lambda_2 + \mathcal{U}_3 \varepsilon_2) \\
& \quad + (\mathcal{W}_3 k_3 + \mathcal{V}_3 \lambda_3 + \mathcal{U}_3 \varepsilon_3) \|(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) - (\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})\| \\
& \quad + (\mathcal{W}_3 \pi_1 + \mathcal{V}_3 \pi_2 + \mathcal{U}_3 \pi_3).
\end{aligned} \tag{30}$$

From (28), (29), and (4) it follows that

$$\begin{aligned}
& \|(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) - (\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})\| \\
& \leq (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3) \pi_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3) \pi_2 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3) \pi_3 \\
& \quad + (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3) (\kappa_1 + \lambda_1 + \varepsilon_1) \\
& \quad + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3) (\kappa_2 + \lambda_2 + \varepsilon_2) \\
& \quad + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3) (\kappa_3 + \lambda_3 + \varepsilon_3) \|(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) - (\widehat{\mathfrak{p}}, \widehat{\mathfrak{q}}, \widehat{\mathfrak{r}})\|,
\end{aligned}$$

$$\begin{aligned} & \|(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) - (\hat{\mathfrak{p}}, \hat{\mathfrak{q}}, \hat{\mathfrak{r}})\| \\ & \leq \frac{(\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\pi_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\pi_2 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\pi_3}{1 - ((\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)(\kappa_1 + \lambda_1 + \varepsilon_1) + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)(\kappa_2 + \lambda_2 + \varepsilon_2) + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)(\kappa_3 + \beta_3 + \varepsilon_3))} \\ & \leq \mathcal{K}_1\pi_1 + \mathcal{K}_2\pi_2 + \mathcal{K}_3\pi_3 \end{aligned}$$

with

$$\mathcal{K}_1$$

$$= \frac{(\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)}{1 - ((\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)(\kappa_1 + \lambda_1 + \varepsilon_1) + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)(\kappa_2 + \lambda_2 + \varepsilon_2) + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)(\kappa_3 + \beta_3 + \varepsilon_3))},$$

$$\mathcal{K}_2$$

$$= \frac{(\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)}{1 - ((\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)(\kappa_1 + \lambda_1 + \varepsilon_1) + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)(\kappa_2 + \lambda_2 + \varepsilon_2) + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)(\kappa_3 + \beta_3 + \varepsilon_3))},$$

$$\mathcal{K}_3$$

$$= \frac{(\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)}{1 - ((\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)(\kappa_1 + \lambda_1 + \varepsilon_1) + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)(\kappa_2 + \lambda_2 + \varepsilon_2) + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)(\kappa_3 + \beta_3 + \varepsilon_3))}.$$

Therefore the BVPs (1) is H-U stable. \square

5 Example

Example 1 Consider the following coupled fractional differential system:

$$\begin{cases} ({^c}\mathcal{D}^{\frac{3}{2}} + \varphi {^c}\mathcal{D}^{\frac{3}{2}-1})\mathfrak{p}(\zeta) = \hat{\mathcal{F}}_1(\zeta, \mathfrak{p}(\zeta), \mathfrak{q}(\zeta), \mathfrak{r}(\zeta)), & 1 < \eta \leq 2, \\ ({^c}\mathcal{D}^{\frac{3}{2}} + \varphi {^c}\mathcal{D}^{\frac{3}{2}-1})\mathfrak{q}(\zeta) = \hat{\mathcal{F}}_2(\zeta, \mathfrak{p}(\zeta), \mathfrak{q}(\zeta), \mathfrak{r}(\zeta)), & 1 < \xi \leq 2, \\ ({^c}\mathcal{D}^{\frac{1}{4}} + \varphi {^c}\mathcal{D}^{\frac{1}{4}-1})\mathfrak{r}(\zeta) = \hat{\mathcal{F}}_3(\zeta, \mathfrak{p}(\zeta), \mathfrak{q}(\zeta), \mathfrak{r}(\zeta)), & 2 < \zeta \leq 3, \\ \mathfrak{p}(0) = 0, \quad \mathfrak{p}(1) = \beta_1 \sum_{j=1}^4 w_j \mathfrak{q}(\varrho_j), \\ \mathfrak{q}(0) = 0, \quad \mathfrak{q}(1) = \beta_2 \sum_{j=1}^4 v_j \mathfrak{r}(\varpi_j), \\ \mathfrak{r}(0) = 0, \quad \mathfrak{r}'(0) = 0, \quad \mathfrak{r}(1) = \beta_3 \sum_{j=1}^4 \vartheta_j \mathfrak{p}(\rho_j), \\ 0 < \varrho_1 < \lambda_1 < \vartheta_1 < \varrho_2 < \lambda_2 < \vartheta_2 \dots < \varrho_{k-2} < \lambda_{k-2} < \vartheta_{k-2} < 1, \end{cases} \quad (31)$$

Here $\eta = \frac{3}{2}$, $\xi = \frac{3}{2}$, $\zeta = \frac{1}{4}$, $\beta_1 = \frac{3}{2}$, $\beta_2 = \frac{6}{5}$, $\beta_3 = \frac{1}{3}$, $w_1 = \frac{1}{40}$, $w_2 = \frac{7}{200}$, $w_3 = \frac{9}{200}$, $w_4 = \frac{11}{200}$, $\varrho_1 = \frac{5}{4}$, $\varrho_2 = \frac{7}{5}$, $\varrho_3 = \frac{33}{20}$, $\varrho_4 = \frac{9}{5}$, $v_1 = \frac{1}{50}$, $v_2 = \frac{9}{200}$, $v_3 = \frac{3}{50}$, $v_4 = \frac{17}{200}$, $\varpi_1 = \frac{6}{5}$, $\varpi_2 = \frac{29}{20}$, $\varpi_3 = \frac{8}{5}$, $\varpi_4 = \frac{37}{20}$, $\vartheta_1 = \frac{1}{40}$, $\vartheta_2 = \frac{1}{25}$, $\vartheta_3 = \frac{13}{200}$, $\vartheta_4 = \frac{12}{25}$, $\rho_1 = \frac{21}{40}$, $\rho_2 = \frac{107}{200}$, $\rho_3 = \frac{109}{200}$, $\rho_4 = \frac{111}{200}$. With this data, we find that $\mathcal{W}_1 = 0.8316369829$, $\mathcal{W}_2 = 0.1631960492$, $\mathcal{W}_3 = 0.0482076794$, $\mathcal{V}_1 = 0.1048341679$, $\mathcal{V}_2 = 0.8246454202$, $\mathcal{V}_3 = 0.0008716269$, $\mathcal{U}_1 = 0.0186054209$, $\mathcal{U}_2 = 0.0070833630$, $\mathcal{U}_3 = 0.2236302088$.

(I) To illustrate Theorem 1, we take

$$\begin{aligned} \hat{\mathcal{F}}_1(\zeta, \mathfrak{p}, \mathfrak{q}, \mathfrak{r}) &= \frac{1}{30e} + \frac{7}{50}\mathfrak{p} \cos \mathfrak{q} + \frac{1}{40e}\mathfrak{q} \sin \mathfrak{r} + \frac{e^{-\zeta}}{2}\mathfrak{r} \cos \mathfrak{p}, \\ \hat{\mathcal{F}}_2(\zeta, \mathfrak{p}, \mathfrak{q}, \mathfrak{r}) &= \zeta \sqrt{\zeta + 3} + \frac{1}{189}\mathfrak{p} \tan^{-1} \mathfrak{q} + \frac{7}{\sqrt{48 + \zeta^2}}\mathfrak{q} + \frac{1}{4}\mathfrak{r} \sin \mathfrak{p}, \\ \hat{\mathcal{F}}_3(\zeta, \mathfrak{p}, \mathfrak{q}, \mathfrak{r}) &= \frac{e^{-\zeta}}{4} + \frac{e^{(-\zeta)}}{3}\mathfrak{p} + \frac{1}{\zeta + 8}\mathfrak{q} + \frac{e^{-\zeta}}{10}\mathfrak{r} \cos \mathfrak{q}. \end{aligned} \quad (32)$$

It is easy to check that condition (H_2) is satisfied with $\kappa_0 = \frac{1}{30e}$, $\kappa_1 = \frac{7}{50}$, $\kappa_2 = \frac{1}{40e}$, $\kappa_3 = \frac{e^{-\varsigma}}{2}$, $\lambda_0 = 2$, $\lambda_1 = \frac{1}{189}$, $\lambda_2 = \frac{1}{7}$, $\beta_3 = \frac{1}{4}$, $\varepsilon_0 = \frac{e^{-\varsigma}}{4}$, $\varepsilon_1 = \frac{e^{-\varsigma}}{3}$, $\varepsilon_2 = \frac{1}{9}$, $\varepsilon_3 = \frac{e^{-\varsigma}}{10}$. Furthermore,

$$(\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\lambda_1 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\varepsilon_1 \simeq 0.4292961592 < 1,$$

$$(\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_2 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\lambda_2 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\varepsilon_2 \simeq 0.1705329068 < 1,$$

$$(\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\kappa_3 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\lambda_3 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\varepsilon_3 \simeq 0.7699169503 < 1.$$

Clearly, the hypotheses of Theorem 1 are satisfied, and hence the conclusion of Theorem 1 applies to problem (31) with p, q, r given by (32).

Example 2 To illustrate Theorem 2, we take

$$\begin{aligned}\hat{\mathcal{F}}_1(\varsigma, p, q, r) &= \frac{e^{-\varsigma}}{\sqrt{99 + \varsigma^2}} \cos p + \cos \varsigma, \\ \hat{\mathcal{F}}_2(\varsigma, p, q, r) &= \frac{1}{9 + \varsigma^2} (\sin p + |q|) + e^{-\varsigma}, \\ \hat{\mathcal{F}}_3(\varsigma, p, q, r) &= \frac{e^{-\varsigma}}{9} \sin r + \tan^{-1} \varsigma.\end{aligned}\tag{33}$$

Then condition (M_2) is clearly satisfied with $\Theta_1 = \frac{1}{10e}$, $\Theta_2 = \frac{1}{10}$, and $\Theta_3 = \frac{1}{9e}$. Moreover,

$$(\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3)\Theta_1 + (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3)\Theta_2 + (\mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3)\Theta_3 \simeq 0.2080311651 < 1,$$

Thus the hypothesis of Theorem 2 holds, and consequently there exists a unique solution for problem (31) on $[0, 1]$ with p, q, r given by (33).

6 Conclusions

We examined the existence and stability of solutions to a linked system of Caputo sequential fractional differential equations with standard conditions using the Leray–Schauder alternative, Banach–Kranoselskii fixed-point theorem, and Hyer–Ulam stability. We obtain new results for the given system of three sequential fractional differential equations under the specified conditions when we apply the combined solution to all three case values ($w_j = 0, j = 1, \dots, k-2, v_j = 0, j = 1, \dots, k-2, \vartheta_j = 0, j = 1, \dots, k-2$) to the series of three sequential fractional differential equations. In a fragmented research field, it appears that the first multipoint boundary value problem to be stated in scientific research employs a triple system of sequential fractional differential equations. This paper discusses original research that contributes significantly to the body of expertise on the topic.

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Author contributions

M.M. and S.M. wrote the original version. J.A. checked the validation and confirm the correctness. T.N.G. search the sources and edit the original draft.

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