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Dynamics properties for a viscoelastic Kirchhoff-type equation with nonlinear boundary damping and source terms

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Abstract

This work is devoted to studying a viscoelastic Kirchhoff-type equation with nonlinear boundary damping-source interactions in a bounded domain. Under suitable assumptions on the kernel function g, density function, and M, the global existence and general decay rate of solution are established. Moreover, we prove the finite time blow-up result of solution with negative initial energy.

Keywords: Viscoelastic wave equation; Kirchhoff-type equation; Density function; Global existence; General decay; Blow-up

1 Introduction

Damping describes transformation of the mechanical energy of a structure that is subjected to an oscillatory deformation to a thermal energy and its dissipation per cycle of motion. Passive damping is used to reduce vibrations and noise resulting from a failure of one of the components of the material which has led many authors to study these kinds of problems.

In this paper, we consider the following viscoelastic Kirchhoff-type equation with velocity-dependent density and nonlinear boundary damping-source interaction:

$$\begin{cases} |u_t|^\rho u_{tt} - (a + b\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)\,ds - \alpha\Delta u_{tt} = 0, & \text{in } \Omega \times (0,+\infty), \\ u = 0, & \text{in } \Gamma_0 \times (0,+\infty), \\ (a + b\|\nabla u\|_2^2)\frac{\partial u}{\partial \nu} + \alpha\frac{\partial u_{tt}}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u(s)}{\partial \nu}\,ds + |u_t|^{m-2}u_t & \\ \quad = |u|^{p-2}u, & \text{in } \Gamma_1 \times (0,+\infty), \\ u(0) = u_0(x), \qquad u_t(0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of R^n $(n \geq 1)$ with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$, ρ, a, b, and $\alpha > 0$ are fixed positive constants, and we denote by ν and $\frac{\partial}{\partial \nu}$ the outward normal and the unit outer normal derivative to Γ respectively. $m \geq 2$, $p > 2$, and g is a positive nonincreasing kernel function.

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Problem (1) with $b = 0$, without nonlinear boundary damping and source, has been extensively studied, and results concerning existence, asymptotic behavior, and blow-up have been established. Cavalcanti et al. [8] considered the following equation:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + a(x)u_t + b|u|^\gamma u = 0, \quad \text{in } \Omega \times (0,\infty), \tag{2}$$

where $\lambda > 0$. By supposing the relaxation function $g(t)$ decays exponentially, they established an exponential decay result of solution energy. Berrimi and Messaoudi [2] studied (2) with $a(x) \equiv 0$, established a local existence result, and showed that the local solution is global and decays uniformly if the initial data are small enough. Later, Messaoudi [31] studied (2) with $a(x) \equiv 0$ and $b = 0$, and they established a general decay result that is not necessarily of exponential or polynomial type.

Park et al. [36] considered the following equation:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + h(u_t) = |u|^{p-2}u \tag{3}$$

and proved the blow-up result of solution with positive initial energy as well as nonpositive initial energy under a weaker assumption on the damping term. Messaoudi [28] studied (3) with $h = u_t|u_t|^{m-2}$ and proved the blow-up result of solutions with negative initial energy and $p > m > 2$. Messaoudi [30] studied (3) with $h = 0$ and established a local existence result, showing that the local solution is global and decays uniformly if the initial data are small enough. Song and Zhong [39] studied (3) with $(h(u_t) = \Delta u_t)$ and established the blow-up result of solutions with positive initial energy.

Cavalcanti et al. [6] considered the following nonlinear viscoelastic equation:

$$|u_t|^\rho u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds - \gamma\Delta u_t - \Delta u_{tt} = 0 \tag{4}$$

and established the global existence of weak solution and uniform decay rates of the energy. Messaoudi and Tatar [33] investigated the behavior of solutions to the nonlinear viscoelastic equation given by [6] with $\gamma = 0$ and Dirichlet boundary condition. In addition, they considered a nonlinear source term that is dependent on the solution u. By introducing a new functional and using the potential well method, they showed that the viscoelastic term is enough to ensure the global existence and uniform decay of solutions provided that the initial data are in the same stable set. Later, Wu [43] studied (4) with $\gamma = 0$, nonlinear source, and weak damping terms. He discussed the general uniform decay estimate of energy solution under suitable conditions on the relaxation function g and the initial data.

In 1883, Kirchhoff introduced a model given in [20] as a generalization of the well known d'Alembert's wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial t}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{5}$$

for free vibrations of elastic strings. The parameters in the above equation have physical significant meanings as follows: L is the length of the string, h is the area of the cross section, E is Young's modulus of the material, ρ is the mass density, and P_0 is the initial tension. This type of problem has been considered by many authors during the past decades, and many results have been obtained, we refer the interested readers to [9, 14, 17, 27, 34, 35, 40, 46, 55] and the references therein. For the viscoelastic Kirchhoff-type equation, the following equation

$$u_{tt} - M\big(\|\nabla u\|_2^2\big)\Delta u + \int_0^t h(t-s)\Delta u(s)\,ds + h(u_t) = f(u), \tag{6}$$

has been considered by many authors. Wu and Tsai [47] investigated the global existence, asymptotic behavior, and blow-up properties for (6). Yang and Gong [51] studied (6) with $M(s) = 1 + bs^\gamma$ $(b \geq 0, \gamma > 0, s \geq 0)$, $h(u_t) = u_t$, and $f(u) = |u|^{p-1}u$. Under certain assumptions on the kernel g and the initial data, they established a new blow-up result for arbitrary positive initial energy. Guesmia et al. [15] studied (6) with $h = g \equiv 0$ and investigated the well-posedness and the optimal decay rate estimate of energy. Recently, Draifia [12] considered the following nonlinear viscoelastic equation with the Kirchhoff-type damping:

$$|u_t|^\rho u_{tt} - \big(\xi_0 + \xi_1\|\nabla u\|_2^2\big)\Delta u + \int_0^t g(t-s)\Delta u(s)\,ds = 0, \tag{7}$$

where $\rho \geq 0$, $\xi_0, \xi_1 > 0$ are positive constants. He studied the intrinsic decay rates for the energy of relaxation kernels described by the inequality $g'(t) \leq H(g(t))$, $t \geq 0$, we also refer to other works [4, 5, 13, 16, 18, 19, 37, 38, 42].

In recent years, the viscoelastic wave equation with boundary damping and source terms has been studied by many authors. In the case that ($g = 0$), Vitillaro [41] studied the following initial boundary value problem:

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \Gamma_0 \times (0,\infty), \\ u_\nu = -|u_t|^{m-2}u_t + |u|^{p-2}u, & \text{on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{8}$$

He showed that the superlinear damping term $-|u_t|^{m-2}u_t$, when $2 \leq m \leq p$, implies the existence of global solutions for arbitrary initial data, in contrast with the nonexistence phenomenon that occurs when $m = 2 < p$. Zhang and Hu [54] proved the asymptotic behavior of the solutions to problem (8) when the initial data are inside a stable set, the nonexistence of the solution when $p > m$, and the initial data are inside an unstable set.

In the presence of the viscoelastic term ($g \neq 0$), Cavalcanti et al. [7] considered the following problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds = 0, & \Omega \times (0,+\infty), \\ u = 0, & \Gamma_0 \times (0,+\infty), \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}\,ds + h(u_t) = 0, & \Gamma_1 \times (0,+\infty), \\ u(0) = u_0(x), \qquad u_t(0) = u_0(x), & \Omega. \end{cases} \tag{9}$$

They proved the existence of strong and weak solutions by using the Faedo–Galerkin method, and assuming that the kernel function g is small enough, they proved uniform decay. Messaoudi and Mustafa in [32] established the general decay rate of solution of (9) without strong conditions on damping term. Wu [44] considered the following initial boundary value problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds = a|u|^{p-1}u, & \Omega \times (0,+\infty), \\ u(x,t) = 0, & \Gamma_0 \times (0,+\infty), \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}\,ds + h(u_t) = b|u|^{k-1}u, & \Gamma_1 \times (0,+\infty), \\ u(0) = u_0(x), u_t(0) = u_1(x), & \Omega. \end{cases} \tag{10}$$

Under appropriate assumptions imposed on the source and damping terms, he established both the existence of solutions and uniform decay rate of the solution energy. He also exhibited the finite time blow-up phenomenon of the solution for certain initial data in the unstable set. Liu and Yu [25] studied (10) with $a = 0$, $b = 1$, and $h(u_t) = |u_t|^{m-2}u_t$. They obtained a general decay result for the global solution under suitable assumptions on the relaxation function g in two cases: $m = 2$ and $m > 2$. Furthermore, they proved two blow-up results: one is for certain solutions with nonpositive initial energy as well as positive initial energy in the case $m = 2$, and the other is for certain solutions with arbitrary positive initial energy in the case $m \geq 2$.

Di et al. [11] considered the following initial boundary value problem for a viscoelastic wave equation with nonlinear boundary source term:

$$\begin{cases} |u_t|^{\rho-1}u_{tt} - \Delta u + \int_0^t g(t-s)\,ds = 0, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \Gamma_0 \times (0,\infty), \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}\,ds = f(u), & \text{on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{11}$$

They obtained the global existence of a weak solution under some assumptions on g and f. They supposed that $I(u_0) \geq 0$ and $E(0) = d$, and when $I(u_0) < 0$ and $E(0) < \beta\delta$, they established the blow-up in finite time. Later, Di and Shang [10] studied (11) with $f(u) \equiv 0$, nonlinear boundary damping, and internal source terms. First, they proved the existence of global weak solutions by using a combination of Galerkin approximation, potential well, and monotonicity-compactness methods. They also established decay rates and finite time blow-up of solutions under some assumptions on g and the initial data.

For the viscoelastic Kirchhoff-type wave equation with nonlinear boundary damping, Wu [45] considered the following viscoelastic equation with Balakrishnan–Taylor damping term and nonlinear boundary/interior sources:

$$\begin{cases} u_{tt} - M(t)\Delta u + \int_0^t g(t-s)\Delta u(s)\,ds = |u|^{p-1}u, & \text{in } \Omega \times (0,+\infty), \\ u = 0, & \text{on } \Gamma_0 \times (0,+\infty), \\ M((t)\frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}\,ds + h(u_t) = |u|^{k-1}u, & \text{on } \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{12}$$

where $M(t) = a + b\|\nabla u\|_2^2 + \sigma \int_\Omega \nabla u.\nabla u_t\,dx$. This model was introduced by Balakrishnan and Taylor in [1] to study the inherent damping problem in flutter structures. In the problem at hand, Wu discussed the uniform decay rates by imposing usual assumptions on the kernel function, damping, and source term. Zarai et al. [53] studied (12) with $h = \alpha u_t$ and without a source term ($|u|^{p-1}u$). They proved the global existence of solutions and a general decay result for the energy by using the multiplier technique.

Li and Xi [22] considered the following nonlinear viscoelastic Kirchhoff-type equation with acoustic control boundary conditions:

$$\begin{cases} u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t h(t-s)\Delta u(s)\,ds + a|u_t|^{m-2}u & \\ \quad = |u|^{p-2}u, & \text{in } \Omega \times (0,+\infty), \\ u = 0, & \text{on } \Gamma_1 \times (0,+\infty), \\ M(\|\nabla u\|_2^2)\frac{\partial u}{\partial \nu} - \int_0^t h(t-s)\frac{\partial u}{\partial \nu}\,ds = y_t, & \text{on } \Gamma_0 \times (0,+\infty), \\ u_t + \alpha(x)y_t + \beta y = 0, & \text{on } \Gamma_0 \times (0,+\infty), \\ u(0) = u_0(x), \qquad u_t(0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{13}$$

where $a \geq 0$, $m \geq 2$, $p > 2$. By using multiplier techniques and under certain conditions on M, h, α, β, and on the initial data, they demonstrated that the rate at which the energy of the solution decreases over time as $t \longrightarrow +\infty$ is determined by the characteristics of the convolution kernel h at infinity. Later, Li et al. [23] studied (13), proved the finite time blow-up of solutions, and gave an upper bound of the blow-up time T^*.

Motivated by the previous works, our objective in this work is to examine the global existence, general decay, and the finite time blow-up of solutions. So, to achieve this goal, we organized our paper as follows: In Sect. 2, we give and recall some preliminaries and lemmas and put the necessary assumptions. In Sect. 3 we obtain global existence of the solution. In Sect. 4, we establish the decay rates of solution. In Sect. 5, we prove the finite time blow-up of solutions.

2 Material and assumptions

In this section we give some notation for function spaces and preliminary lemmas. We denote by $\|u\|_p$ and $\|u\|_{p,\Gamma_1}$ to the usual $L^p(\Omega)$ and $L^p(\Gamma_1)$ norms, respectively. For Sobolev space $H_0^1(\Omega)$ norm, we use the notation

$$\|u\|_{H_0^1} = \|\nabla u\|_2.$$

Let

$$H_{\Gamma_0}^1(\Omega) = \{u \in H^1(\Omega) | u_{|\Gamma_0} = 0\},$$

and c_*, c_p be the Poincaré-type constants defined as the smallest positive constants such that

$$\|u\|_p \leq c_p\|\nabla u\|_2, \quad \forall u \in H^1(\Omega), \tag{14}$$

and

$$\|u\|_{p,\Gamma_1} \leq c_*\|\nabla u\|_2, \quad \forall u \in H_{\Gamma_0}^1(\Omega). \tag{15}$$

To state and prove our results, we need the following assumptions:

(G_1): The kernel function g is a decreasing C^1-function satisfying for $s > 0$

$$g(s) \geq 0, \qquad g'(s) \leq 0, \qquad a - \int_0^{+\infty} g(s)\,ds = l \geq 0.$$

(G_2): There exists a positive differentiable function ξ such that

$$g'(s) \leq -\xi(s)g(s) \quad \forall s > 0.$$

(G_3): The constant p satisfies

$$4 < m < p, \quad \text{if } n = 1, 2, \quad \text{and} \quad 4 < m < p < \frac{2(n-1)}{n-2} \quad \text{if } n \geq 3.$$

Assume further that g satisfies

$$\int_0^{+\infty} g(s)\,ds < \frac{a(\zeta/2 - 1)}{\zeta/2 - 1 + 1/2\zeta}. \tag{16}$$

Now, we define the energy associated with problem (1) by

$$\begin{aligned} E(t) = {}& \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(a - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 + \frac{1}{2}(g \circ \nabla u)(t) \\ & + \frac{\alpha}{2}\|\nabla u_t\|_2^2 - \frac{1}{p}\|u\|_{p.\Gamma_1}^p. \end{aligned} \tag{17}$$

Lemma 2.1 *Let u be a solution of problem* (1). *Then*

$$E'(t) \leq -\|u_t\|_{m,\Gamma_1}^m - \frac{1}{2}g(t)\|\nabla u\|_2^2 + \frac{1}{2}\big(g' \circ \nabla u\big)(t) \leq 0. \tag{18}$$

Proof Multiplying the first equation in (1) by u_t and integrating it over Ω, we get (18). □

Next, we define the following functionals:

$$I(t) = I\big(u(t)\big) = \left(a - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^4 + (g \circ \nabla u)(t) - \|u\|_{p.\Gamma_1}^p \tag{19}$$

and

$$J(t) = J\big(u(t)\big) = \frac{1}{2}\left(a - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u\|_{p.\Gamma_1}^p. \tag{20}$$

Then, by (19) and (20), it is obvious that

$$E(t) = \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{\alpha}{2}\|\nabla u_t\|_2^2 + J(t). \tag{21}$$

Lemma 2.2 ([29]) *Suppose that $p \leq 2\frac{n-1}{n-2}$ holds. Then there exists a positive constant $C > 1$ depending only on Ω such that*

$$\|u\|_p^s \leq C\big(\|\nabla u\|_2^2 + \|u\|_p^p\big)$$

for any $u \in H_0^1(\Omega)$, $2 \leq s \leq p$.

As in [29], we can prove the following lemma.

Lemma 2.3 *Suppose that $p \leq 2\frac{(n-1)}{n-2}$ holds, then there exists a positive constant $C > 1$ depending only on Γ_1 such that*

$$\|u\|_{p,\Gamma_1}^s \leq C_*\big(\|\nabla u\|_2^2 + \|u\|_{p,\Gamma_1}^p\big)$$

for any $u \in H_{\Gamma_0}^1(\Omega)$, $2 \leq s \leq p$.

Now, concerning the study of local existence, we will just state the theorem below and the proof can be found in [3, 21, 24, 26, 48–50, 52].

Theorem 2.4 *Assume that $(G_1) - (G_3)$ hold. Then, for any $u_0 \in H_{\Gamma_0}^1(\Omega)$ and $u_1 \in H_{\Gamma_0}^1(\Omega) \cap L^m(\Gamma_1)$ be given. Then there exists a unique local solution u of problem (1) such that*

$$u \in L^\infty\big([0,T]; H_{\Gamma_0}^1(\Omega)\big), \qquad u_t \in L^\infty\big([0,T]; H_{\Gamma_0}^1(\Omega) \cap L^m(\Gamma_1)\big)$$

for some $T > 0$.

3 Global existence

In this section, we prove that the solution established in problem (1) is global in time.

Lemma 3.1 *Assuming that (G_1)–(G_3) hold, and for any $(u_0, u_1) \in H_{\Gamma_0}^1(\Omega) \times L^2(\Omega)$ satisfy*

$$I(0) > 0, \qquad \vartheta = \frac{B_*^p}{l}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{22}$$

then

$$I(t) > 0, \quad \forall t > 0. \tag{23}$$

Proof Since $I(0) > 0$, then by the continuity of $u(t)$, there exists a time $T_* < T$ such that

$$I(t) > 0, \quad \forall t \in [0, T_*). \tag{24}$$

Let t_0 be such that

$$\big\{I(t_0) = 0, \text{ and } I(t) > 0, \forall 0 < t_0 < T_*\big\}. \tag{25}$$

This implies that, for all $t \in [0, T_*)$,

$$\begin{aligned} J(t) &= \frac{p-2}{2p}\left[\left(a - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^4 + (g \circ \nabla u)(t)\right] + \frac{1}{p}I(t) \\ &\geq \frac{p-2}{2p}\left[\left(a - \int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^4 + (g \circ \nabla u)(t)\right]. \end{aligned} \tag{26}$$

Hence, from (G_1), (26), (21), and Lemma 2.1, we obtain

$$\begin{aligned} l\|\nabla u\|_2^2 &\le \left(a-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 \\ &\le \frac{2p}{p-2}J(t) \le \frac{2p}{p-2}E(t) \le \frac{2p}{p-2}E(0), \quad \forall t\in[0,T_*). \end{aligned} \tag{27}$$

By exploiting (15), (27), (22), and (G_1), we obtain

$$\begin{aligned} \|u\|_{p,\Gamma_1}^p &\le c_*^p\|\nabla u\|_2^p \le \frac{c_*^p}{l}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{p-2}{2}} l\|\nabla u\|_2^2 \\ &= \vartheta l\|\nabla u\|_2^2 < \left(a-\int_0^t g(s)\,ds\right)\big\|\nabla u(t_0)\big\|_2^2, \quad \forall t\in[0,T_*). \end{aligned} \tag{28}$$

Hence, we can get

$$I(t_0) > 0.$$

which contradicts (25). Thus, $I(t) > 0$ on $(0,T_*)$.

Repeating this procedure and using the fact that

$$\lim_{t\to T^*}\frac{c_*^p}{l}\left(\frac{2p}{l(p-2)}E\big(u(t),u_t(t)\big)\right)^{\frac{p-2}{2}} \le \nu < 1,$$

T_* is extended to T. □

Theorem 3.2 *Assuming that the conditions of Lemma 3.1 hold, solution (1) is global and bounded.*

Proof It suffices to show that $\|\nabla u\|_2^2 + \|\nabla u_t\|_2^2$ is bounded independently of t. It follows from (18), (21), and (26) that

$$\begin{aligned} E(0) \ge E(t) &= \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{\alpha}{2}\|\nabla u_t\|_2^2 + J(t) \\ &\ge \frac{p-2}{2p}\big(l\|\nabla u\|_2^2\big) + \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{\alpha}{2}\|\nabla u_t\|_2^2. \end{aligned} \tag{29}$$

Therefore,

$$\|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 \le KE(0), \tag{30}$$

where K is a positive constant. The proof is complete. □

4 Decay of solution

This section is devoted to the study of the stability of the solution of problem (1). So, to prove our main results, we put the following functionals:

$$\phi(t) = \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u\,dx + \int_\Omega \alpha\nabla u_t\nabla u\,dx \tag{31}$$

and

$$\psi(t) = -\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\ -\int_\Omega \alpha\nabla u_t \int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx. \tag{32}$$

Next, we define the following functional:

$$L(t) = NE(t) + \epsilon\phi(t) + \psi(t). \tag{33}$$

Then, we have the following lemmas.

Lemma 4.1 *For ϵ small enough and choosing N large enough, we have*

$$\beta_1 E(t) \le L(t) \le \beta_2 E(t) \tag{34}$$

holds for two positive constants β_1 and β_2.

Proof By using Hölder's, Young's inequalities, and (14), we get

$$\begin{aligned} \big|L(t)-NE(t)\big| &\le \epsilon c_\rho \|u_t\|_{\rho+2}^{\rho+1}\|\nabla u\|_2 + \epsilon c_\alpha\big(\|\nabla u\|_2^2 + \|\nabla u_t\|_2^2\big) \\ &\quad + c_\rho\int_0^t g(t-s)\big\|u(t)-u(s)\big\|_2\,ds\|u_t\|_{\rho+2}^{\rho+1} \\ &\quad + c_\alpha\int_0^t g(t-s)\big\|\nabla u(t)-\nabla u(s)\big\|_2\,ds\|\nabla u_t\|_2 \\ &\le \epsilon c\big(\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 + \|\nabla u_t\|_2^2\big) \\ &\quad + c\big(\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 + (g\circ\nabla u)(t)\big) \\ &\le c(\epsilon)E(t), \end{aligned} \tag{35}$$

where c is a positive constant dependent on ρ and $E(0)$. If we take ϵ to be sufficiently small, then (34) follows from (35). □

Lemma 4.2 *The functional $\phi(t)$ defined in (31) satisfies*

$$\phi'(t) \le \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \alpha\|\nabla u_t\|_2^2 - k_0\|\nabla u\|_2^2 - b\|\nabla u\|_2^4 \\ + \frac{a-l}{4\eta}(g\circ\nabla u)(t) + k_1\|u\|_{p,\Gamma_1}^p + \eta\|u_t\|_{m,\Gamma_1}^m, \tag{36}$$

where k_0 and k_1 are positive constants dependent on η.

Proof Differentiating (31) with respect to t and using equation (1), we get

$$\phi'(t) = \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \alpha\|\nabla u_t\|_2^2 - a\|\nabla u\|_2^2 - b\|\nabla u\|_4^4 + \|u\|_{\Gamma_1,p}^p \\ + \int_\Omega \nabla u\int_0^t g(t-s)\nabla u(s)\,ds\,dx - \int_{\Gamma_1}|u_t|^{m-2}u_t u\,d\Gamma. \tag{37}$$

Employing Holder's and Young's inequalities and Lemma 2.2, we estimate the third and fourth terms on the right-hand side of (37) as follows:

$$\int_\Omega \nabla u \int_0^t g(t-s)\nabla u(s)\,ds\,dx \le (\eta + a - l)\|\nabla u\|_2^2 + \frac{a-l}{4\eta}(g\circ\nabla u)(t) \tag{38}$$

and

$$\begin{aligned}\int_{\Gamma_1} |u_t|^{m-2}u_t u\,d\Gamma &\le c(\eta)\|u\|_{m,\Gamma_1}^m + \eta\|u_t\|_{m,\Gamma_1}^m \\ &\le c(\eta)C\big(\|\nabla u\|_2^2 + \|u\|_{p,\Gamma_1}^p\big) + \eta\|u_t\|_{m,\Gamma_1}^m.\end{aligned} \tag{39}$$

A substitution of (38)–(39) into (37) yields (36). □

Lemma 4.3 *The functional $\psi(t)$ defined in* (32) *satisfies*

$$\begin{aligned}\psi'(t) \le{}& -\frac{1}{\rho+1}\left[\int_0^t g(s)\,ds - \lambda k_5\right]\|u_t\|_{\rho+2}^{\rho+2} - \alpha\left[\int_0^t g(s)\,ds - \lambda\right]\|\nabla u_t\|_2^2 \\ &+ \lambda\big(a + 2(a-l)^2\big)\|\nabla u\|_2^2 + \lambda b\|\nabla u\|_2^4 + k_\lambda(g\circ\nabla u)(t) \\ &- \frac{k_{\alpha,\rho}}{4\lambda}\big(g'\circ\nabla u\big)(t) + c(\lambda)\big(\|u\|_{p,\Gamma_1}^p + \|u_t\|_{m,\Gamma_1}^m\big).\end{aligned} \tag{40}$$

Proof Differentiating (32) with respect to t and using equation (1), we get

$$\begin{aligned}\psi'(t) ={}& -\int_\Omega |u_t|^\rho u_{tt}\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\ &- \frac{1}{\rho+1}\left(\int_0^t g(s)\,ds\right)\int_\Omega |u_t|^{\rho+2}\,dx \\ &- \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\ &- \int_\Omega \alpha\nabla u_t\left(\int_0^t g(t-s)\nabla u_t(t)\,ds\right)dx \\ &- \int_\Omega \alpha\nabla u_t\int_0^t g'(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx \\ &- \int_\Omega \alpha\nabla u_{tt}\left(\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right)dx \\ ={}& \int_\Omega \big(a - b\|\nabla u\|_2^2\big)\nabla u\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx \\ &- \int_\Omega\left(\int_0^t g(t-s)\nabla u(s)\,ds\right)\left(\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right)dx \\ &- \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\ &- \int_\Omega \alpha\nabla u_t\int_0^t g'(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx \\ &+ \int_{\Gamma_1} |u_t|^{m-2}u_t\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,d\Gamma\end{aligned} \tag{41}$$

$$
\begin{aligned}
&- \int_{\Gamma_1} |u^{p-2}u \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,d\Gamma \\
&- \left(\frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \alpha\|\nabla u_t\|_2^2\right)\int_0^t g(s)\,ds \\
&= I_1 + \cdots I_6 - \left(\frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \alpha\|\nabla u_t\|_2^2\right)\int_0^t g(s)\,ds.
\end{aligned}
$$

Applying Young's and Holder's inequalities, we obtain for $\lambda > 0$

$$
\begin{aligned}
I_1 &= \big(a + b\|\nabla u\|_2^2\big)\int_\Omega \nabla u \int_0^t g(t-s)\big(\nabla u(t) - \nabla u(s)\big)\,ds\,dx \\
&\le a\lambda\|\nabla u\|_2^2 + b\lambda\|\nabla u\|_2^4 + \frac{k_2(a-l)}{4\lambda}(g\circ\nabla u)(t)
\end{aligned}
\tag{42}
$$

and

$$
\begin{aligned}
I_2 &= \int_\Omega \left(\int_0^t g(t-s)\nabla u(s)\,ds\right)\left(\int_0^t g(t-s)\big(\nabla u(t) - \nabla u(s)\big)\,ds\right)dx \\
&\le 2\lambda(a-l)^2\|\nabla u\|_2^2 + \left(2\lambda + \frac{1}{4\lambda}\right)(a-l)(g\circ\nabla u)(t).
\end{aligned}
\tag{43}
$$

By using Young's, Holder's inequalities, (G_1), (15), and Lemma 2.1, we obtain the following estimates:

$$
\begin{aligned}
I_3 &= \int_{\Gamma_1} |u_t|^{m-2}u_t \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,d\Gamma \\
&\le c(\lambda)\|u_t\|_{m,\Gamma_1}^m + \lambda c_*^m (a-l)^{m-1}\int_0^t g(t-s)\big\|\nabla u(s) - \nabla u(t)\big\|_2^m\,ds \\
&\le c(\lambda)\|u_t\|_{m,\Gamma_1}^m + \lambda k_3(g\circ\nabla u)(t)
\end{aligned}
\tag{44}
$$

and

$$
I_4 = \int_{\Gamma_1} |u|^{p-2}u \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,d\Gamma \le c(\lambda)\|u\|_{p,\Gamma_1}^p + \lambda k_4(g\circ\nabla u)(t),
\tag{45}
$$

where k_3, k_4 are positive constants, which depend only on $E(0)$, m, and p.

Since $0 < -\int_0^t g'(s)\,ds \le g(0)$, we have

$$
\begin{aligned}
I_5 &\le \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\le \frac{\lambda k_5}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} - \frac{g(0)c_p^2}{4\lambda(\rho+1)}\big(g'\circ\nabla u\big)(t)
\end{aligned}
\tag{46}
$$

and

$$
I_6 = \alpha\int_\Omega \nabla u_t \int_0^t g'(t-s)\big(\nabla u(t) - \nabla u(s)\big)\,ds\,dx \le \alpha\lambda\|\nabla u_t\|_2^2 - \alpha\frac{g(0)}{4\lambda}\big(g'\circ\nabla u\big)(t),
\tag{47}
$$

where k_5 is a positive constant, which depends only on $E(0)$ and ρ.

A substitution of (42)–(47) into (41) yields

$$\begin{aligned}\psi'(t) \le &-\frac{1}{\rho+1}\left[\int_0^t g(s)\,ds - \lambda k_5\right]\|u_t\|_{\rho+2}^{\rho+2} - \alpha\left[\int_0^t g(s)\,ds - \lambda\right]\|\nabla u_t\|_2^2 \\ &+ \lambda\big(a + 2(a-l)^2\big)\|\nabla u\|_2^2 + \lambda b\|\nabla u\|_2^4 + k_\lambda(g\circ\nabla u)(t) \\ &- \frac{k_{\alpha,\rho}}{4\lambda}\big(g'\circ\nabla u\big)(t) + c(\lambda)\big(\|u\|_{p,\Gamma_1}^p + \|u_t\|_{m,\Gamma_1}^m\big),\end{aligned} \tag{48}$$

where

$$k_\lambda = \frac{k_2(a-l)}{4\lambda} + \left(2\lambda + \frac{1}{4\lambda}\right)(a-l) + \lambda(k_3+k_4) \quad \text{and} \quad k_{\alpha,\rho} = g(0)\left(\frac{c_p^2}{\rho+1} + \alpha\right). \qquad \square$$

Lemma 4.4 *Assume that* (G_1)–(G_3) *hold. Let* $u_0 \in H^1_{\Gamma_0}(\Omega)$ *and* $u_1 \in H^1_{\Gamma_0}(\Omega)\cap L^m(\Gamma_1)$ *be given and satisfy* (22), *then, for any* $t_0 > 0$, *the functional* $L(t)$ *verifies*

$$L'(t) \le -\kappa_1 E(t) + \kappa_2(g\circ\nabla u)(t) \tag{49}$$

for some $\kappa_i > 0$, $(i = 1, 2)$.

Proof From Lemmas 2.1, 4.2, and 4.3, we have

$$\begin{aligned}L'(t) \le &-\frac{1}{\rho+1}(g_0 - \lambda k_5 - \epsilon)\|u_t\|_{\rho+2}^{\rho+2} \\ &- \big\{\epsilon k_0 - \lambda\big(a + 2(a-l)^2\big)\big\}\|\nabla u\|_2^2 - b(\epsilon-\lambda)\|\nabla u\|_2^4 \\ &+ \big(\epsilon k_1 + c(\lambda)\big)\|u\|_{p,\Gamma}^p - \alpha(g_0 - \lambda - \epsilon)\|\nabla u_t\|_2^2 + \left(k_\lambda + \epsilon\frac{(a-l)}{4\eta}\right)(g\circ\nabla u)(t) \\ &+ \left(\frac{N}{2} - \frac{k_{\alpha,\rho}}{4\lambda}\right)\big(g'\circ\nabla u\big)(t) - \big(N - C(\lambda) - \epsilon\eta\big)\|u_t\|_{m,\Gamma_1}^m,\end{aligned} \tag{50}$$

where $g_0 = \int_0^{t_0} g(s)\,ds$. First, we choose λ to satisfy

$$g_0 - \lambda k_5 > 0, \qquad g_0 - \lambda > 0.$$

When λ is fixed, we pick N to be sufficiently large such that (34) remains valid and

$$N - C(\delta) > 0, \qquad \frac{k_{\alpha,\rho}}{4\lambda} > 0.$$

Once λ and N are fixed, we select ϵ such that

$$\begin{aligned}&g_0 - \lambda k_5 - \epsilon > 0, \qquad \epsilon - \lambda > 0, \qquad \epsilon k_0 - \lambda\big(a + 2(a-l)^2\big) > 0, \\ &g_0 - \delta - \epsilon > 0, \qquad N - C(\lambda) - \epsilon\eta > 0,\end{aligned}$$

which yields for $\kappa_i > 0$, $i = 1, 2$,

$$L'(t) \le -\kappa_1 E(t) + \kappa_2(g\circ\nabla u)(t). \tag{51}$$

$\square$

Theorem 4.5 *Assume that the conditions of Lemma 4.4 hold. Then there exist two positives constants k, ω such that, for each $t_0 > 0$, the energy of the solution to problem (1) satisfies*

$$E(t) \leq k \exp\left(-\omega \int_{t_0}^{t} \xi(s)\,ds\right). \tag{52}$$

Proof Multiplying (51) by $\xi(t)$, we get

$$\begin{aligned} \xi(t)L'(t) &\leq -\kappa_1 \xi(t)E(t) - \kappa_2\bigl(g' \circ \nabla u\bigr)(t) \\ &\leq -\kappa_1 \xi(t)E(t) - 2\kappa_2 E'(t), \end{aligned} \tag{53}$$

which implies

$$\xi(t)L'(t) + 2\kappa_2 E'(t) \leq -\kappa_1 \xi(t)E(t). \tag{54}$$

We define the Lyapunov functional as follows:

$$F(t) = \xi(t)L(t) + 2\kappa_2 E(t).$$

It is easy to show that $F(t)$ is equivalent to $E(t)$ because of (34). Using the fact that $\xi'(t) \leq 0$, we obtain

$$F'(t) \leq -\frac{\alpha_1}{\beta_2}\xi(t)F(t). \tag{55}$$

Then, by performing a simple integration of Eq. (55) over (t_0, t), we get

$$F(t) \leq F(t_0) \exp\left(\int_{t_0}^{t} \xi(s)\,ds\right).$$

Therefore, (52) is obtained. □

5 Blow-up of solution

Theorem 5.1 *Suppose that $(G_1) - (G_3)$, (16), $\rho + 2 < p$, and $E(0) < 0$ hold. Let $u_0 \in H^1_{\Gamma_0}(\Omega)$ and $u_1 \in H^1_{\Gamma_0}(\Omega) \cap L^m(\Gamma_1)$, then the solution of problem (1) blows up in finite time.*

Proof Let

$$H(t) = -E(t), \tag{56}$$

then (18), (17), and (56) give

$$H'(t) \geq \|u_t\|^m_{m.\Gamma_1} \tag{57}$$

and

$$H(0) \leq H(t) \leq \frac{1}{p}\|u\|^p_{p,\Gamma_1}. \tag{58}$$

Next, we define

$$\Gamma(t) = H^{1-\sigma}(t) + \varepsilon \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u\,dx + \varepsilon\alpha\int_\Omega \nabla u \nabla u_t\,dx, \tag{59}$$

where σ is a small constant and will be chosen later, and

$$0 < \sigma \le \min\left\{\frac{p-m}{p(m-1)}, \frac{1}{\rho+2} - \frac{1}{p}\right\}. \tag{60}$$

Taking a derivative of (59) and using (1), we obtain

$$\begin{aligned}\Gamma'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \frac{\varepsilon}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \varepsilon\alpha\|\nabla u_t\|_2^2 - \varepsilon b\|\nabla u\|_2^4 - \varepsilon a\|\nabla u\|_2^2 \\ &\quad + \varepsilon\|u\|_{p,\Gamma_1}^p + \varepsilon\int_\Omega \nabla u\int_0^t g(t-s)\nabla u(s)\,ds\,dx - \varepsilon\int_{\Gamma_1}|u_t|^{m-2}u_t u\,d\Gamma.\end{aligned} \tag{61}$$

Applying Young's inequality, for $\eta, \delta > 0$, we obtain

$$\int_\Omega \nabla u\int_0^t g(t-s)\nabla u(s)\,ds\,dx \ge \left(1-\frac{1}{4\eta}\right)\left(\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 - \eta(g\circ\nabla u)(t)$$

and

$$\begin{aligned}\int_{\Gamma_1}|u_t|^{m-2}u_t u\,d\Gamma &\le \frac{\delta^m}{m}\|u\|_{m,\Gamma_1}^m + \frac{m-1}{m}\delta^{-m/m-1}\|u_t\|_{m,\Gamma_1}^m \\ &\le \frac{\delta^m}{m}\|u\|_{m,\Gamma_1}^m + \frac{m-1}{m}\delta^{-m/m-1}H'(t).\end{aligned}$$

Then inequality (61) becomes

$$\begin{aligned}\Gamma'(t) &\ge \left[(1-\sigma)H^{-\sigma}(t) - \varepsilon\frac{m-1}{m}\delta^{-m/m-1}\right]H'(t) + \frac{\varepsilon}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \varepsilon\alpha\|\nabla u_t\|_2^2 \\ &\quad - \varepsilon b\|\nabla u\|_2^4 - \varepsilon\left[a - \left(1-\frac{1}{4\eta}\right)\left(\int_0^t g(s)\,ds\right)\right]\|\nabla u\|_2^2 - \eta(g\circ\nabla u)(t) \\ &\quad + \varepsilon\|u\|_{p,\Gamma_1}^p - \frac{\delta^m}{m}\|u\|_{m,\Gamma_1}^m.\end{aligned} \tag{62}$$

It follows from (17) and (56), for constant $\zeta > 0$, that

$$\begin{aligned}\Gamma'(t) &\ge \left\{(1-\sigma)H^{-\sigma}(t) - \varepsilon\frac{m-1}{m}\delta^{-m/m-1}\right\}H'(t) + \varepsilon\left(\frac{1}{\rho+1} + \frac{\zeta}{\rho+2}\right)\|u_t\|_{\rho+2}^{\rho+2} \\ &\quad + \varepsilon\alpha\left(\frac{\zeta}{2}+1\right)\|\nabla u_t\|_2^2 \\ &\quad + \varepsilon\left[a\left(\frac{\zeta}{2}-1\right) - \left(\frac{\zeta}{2} - 1 + \frac{1}{4\eta}\right)\left(\int_0^t g(s)\,ds\right)\right]\|\nabla u\|_2^2 \\ &\quad + \varepsilon b\left(\frac{\zeta}{4}-1\right)\|\nabla u\|_2^4 + \left(\frac{\zeta}{2}-\eta\right)(g\circ\nabla u)(t) + \varepsilon\left(1-\frac{\zeta}{p}\right)\|u\|_{p,\Gamma_1}^p \\ &\quad + \varepsilon\zeta H(t) - \frac{\delta^m}{m}\|u\|_{m,\Gamma_1}^m.\end{aligned} \tag{63}$$

Using (63), we find that, for some number $0 < \eta < \frac{\zeta}{2}$,

$$\begin{aligned}\Gamma'(t) \geq & \left\{(1-\sigma)H^{-\sigma}(t) - \varepsilon\frac{m-1}{m}\delta^{-m/m-1}\right\}H'(t) + \varepsilon c_1\|u_t\|_{\rho+2}^{\rho+2} + \varepsilon c_2\|\nabla u_t\|_2^2 \\ & + \varepsilon c_3\|\nabla u\|_2^2 + \varepsilon c_4\|\nabla u\|_2^4 + \varepsilon c_5(g\circ\nabla u)(t) \\ & + \varepsilon c_6\|u\|_{p,\Gamma_1}^p + \varepsilon\zeta H(t) - \varepsilon\frac{\delta^m}{m}\|u\|_{m,\Gamma_1}^m,\end{aligned} \tag{64}$$

where $4 < \zeta < p$ and

$$\begin{aligned}& c_1 = \frac{1}{\rho+1} + \frac{\zeta}{\rho+2}, \qquad c_2 = \frac{\zeta}{2} > +1, \\ & c_3 = a\left(\frac{\zeta}{2}-1\right) - \left(\frac{\zeta}{2} - 1 + \frac{1}{4\eta}\right)\left(\int_0^t g(s)\,ds\right) > 0. \\ & c_4 = \frac{\zeta}{4} - 1 > 0, \qquad c_5 = \frac{\zeta}{2} - \eta > 0, \qquad c_6 = 1 - \frac{\zeta}{p} > 0.\end{aligned}$$

Therefore, by taking $\delta = (\frac{mk}{m-1}H(t)^{-\sigma})^{-\frac{m-1}{m}}$, where k is a positive constant to be specified later, we can obtain

$$\begin{aligned}\Gamma'(t) \geq & \left\{(1-\sigma) - \varepsilon k\right\}H^{-\sigma}(t)H'(t) + \varepsilon c_1\|u_t\|_{\rho+2}^{\rho+2} + \varepsilon c_2\|\nabla u_t\|_2^2 + \varepsilon c_3\|\nabla u\|_2^2 \\ & + \varepsilon c_4\|\nabla u\|_2^4 + \varepsilon c_5(g\circ\nabla u)(t) + \varepsilon c_6\|u\|_{p,\Gamma_1}^p \\ & + \varepsilon\zeta H(t) - \varepsilon k^{1-m}c_7H^{\sigma(m-1)}(t)\|u\|_{m,\Gamma_1}^m,\end{aligned} \tag{65}$$

where $c_7 = (m/m-1)^{1-m} > 0$.

Since $m < p$, and from (58), (60), and Lemma 2.3, we deduce

$$\begin{aligned}H^{\sigma(m-1)}(t)\|u\|_{m,\Gamma_1}^m \leq c_mH^{\sigma(m-1)}(t)\|u\|_{p,\Gamma_1}^m & \leq \frac{c_m}{p^{\sigma(m-1)}}\|u\|_{p,\Gamma_1}^{m+\sigma p(m-1)} \\ & \leq \frac{c_mC_*}{p^{\sigma(m-1)}}\left(\|\nabla u\|_2^2 + \|u\|_{p,\Gamma_1}^p\right)\end{aligned} \tag{66}$$

for $s = m + \sigma p(m-1) \leq p$. Combining (66) with (65), we get

$$\begin{aligned}\Gamma'(t) \geq & \left\{(1-\sigma) - \varepsilon k\right\}H^{-\sigma}(t)H'(t) + \varepsilon c_1\|u_t\|_{\rho+2}^{\rho+2} + \varepsilon c_2\|\nabla u_t\|_2^2 + \varepsilon c_4\|\nabla u\|_2^4 \\ & + \varepsilon c_5(g\circ\nabla u)(t) + \varepsilon\left(c_3 - c_8k^{1-m}\right)\|\nabla u\|_2^2 \\ & + \varepsilon\left(c_6 - c_8k^{1-m}\right)\|u\|_{p,\Gamma_1}^p + \varepsilon\zeta H(t),\end{aligned} \tag{67}$$

where $c_8 = c_7\frac{c_mC_*}{p^{\sigma(m-1)}}$. First, we choose $k > 0$ large enough such that

$$c_3 - c_8k^{1-m} > 0, \qquad c_6 - c_8k^{1-m} > 0.$$

Once k is fixed, we select ε small enough such that

$$(1-\sigma) - \varepsilon k > 0$$

and

$$\Gamma(0) = H^{1-\sigma}(0) + \varepsilon \frac{1}{\rho+1} \int_\Omega |u_1|^\rho u_1 u_0 \, dx + \varepsilon\alpha \int_\Omega \nabla u_0 \nabla u_1 \, dx.$$

Thus, we obtain

$$\Gamma'(t) \geq \lambda \bigl(\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 + (g \circ \nabla u)(t) + \|\nabla u\|_2^4 + \|\nabla u_t\|_2^2 + H(t) + \|u\|_{p,\Gamma_1}^p \bigr), \tag{68}$$

where λ is a positive constant.

On the other hand, we have

$$\Gamma^{\frac{1}{1-\sigma}}(t) \leq C_1 \biggl[H(t) + \biggl(\int_\Omega |u_t|^\rho u_t u \, dx \biggr)^{\frac{1}{1-\delta}} + \biggl(\int_\Omega \nabla u \nabla u_t \, dx \biggr)^{\frac{1}{1-\sigma}} \biggr]. \tag{69}$$

Using Holder's and Young's inequalities, we have

$$\begin{aligned} \biggl(\int_\Omega |u_t|^\rho u_t u \, dx \biggr)^{\frac{1}{1-\sigma}} &\leq \|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}} \|u\|_{\rho+2}^{\frac{1}{1-\sigma}} \leq c_1 \|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}} \|u\|_p^{\frac{1}{1-\sigma}} \\ &\leq c_2 \bigl(\|u_t\|_{\rho+2}^{\frac{q(\rho+1)}{1-\sigma}} + \|u\|_p^{\frac{q_*}{1-\sigma}} \bigr) \end{aligned} \tag{70}$$

for $\frac{1}{q} + \frac{1}{q_*} = 1$. Taking $q = \frac{(1-\sigma)(\rho+2)}{\rho+1} > 1$, then by (60) we have $\frac{q_*}{(1-\sigma)} = \frac{\rho+2}{(1-\sigma)(\rho+2)-(\rho+1)} < p$. Applying Lemma 2.2 and (30), we get

$$\begin{aligned} \biggl(\int_\Omega |u_t|^\rho u_t u \, dx \biggr)^{\frac{1}{1-\sigma}} &\leq c_3 \bigl(\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 + \|u\|_p^p \bigr) \\ &\leq C_3 \bigl(\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 + \|\nabla u\|_2^p \bigr) \\ &\leq c_4 \bigl(\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 \bigr). \end{aligned} \tag{71}$$

Similar to (70), we have

$$\begin{aligned} \biggl(\int_\Omega \nabla u \nabla u_t \, dx \biggr)^{\frac{1}{1-\sigma}} &\leq c_5 \bigl(\|\nabla u_t\|_2^2 + \|\nabla u\|_2^{\frac{1}{1-2\sigma}} \bigr) \\ &\leq c_5 \biggl(\|\nabla u_t\|_2^2 + \bigl(KE(0) \bigr)^{\frac{1}{1-2\sigma}} \frac{H(t)}{H(0)} \biggr) \\ &\leq c_6 \bigl(\|\nabla u_t\|_2^2 + H(t) \bigr). \end{aligned} \tag{72}$$

Therefore, from (71) and (72), we get

$$\Gamma^{\frac{1}{1-\sigma}}(t) \leq \kappa \bigl(H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 \bigr). \tag{73}$$

Combining (68) and (73), we find that

$$\Gamma'(t) \geq \xi \Gamma^{\frac{1}{1-\sigma}}(t), \tag{74}$$

where ξ is a positive constant. A simple integration of (74) over $(0,t)$ yields

$$\Gamma^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Gamma^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\xi\sigma t}{1-\sigma}}, \tag{75}$$

which allows us to deduce that $\Gamma(t)$ blows up in finite time T^*, and

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Gamma^{\frac{\sigma}{1-\sigma}}(0)}. \tag{76}$$

$\square$

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