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A relation-theoretic set-valued version of Prešić-Ćirić theorem and applications

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Abstract
In this paper, we establish a relation-theoretic set-valued version of the fixed point result of Ćirić and Prešić (Acta Math. Univ. Comen. LXXVI(2):143–147, 2007) on metric spaces endowed with an arbitrary binary relation. The results of this paper, generalize and unify the fixed point results of Ćirić and Prešić (Acta Math. Univ. Comen. LXXVI(2):143–147, 2007), Shukla and López (Quaest. Math. 45(3):1–16, 2019), and Shukla and Radenović (An. Ştiinţ. Univ. 'Al.I. Cuza' Iaşi, Mat. 63(2):339–350, 2017) in product spaces. Some examples are provided that justify and establish the importance of our results. As applications of our main result, we have established the existence of solutions to differential inclusion problems and the weak asymptotical stability and a global attractivity of the equilibrium point of a difference inclusion problem. The use of arbitrary binary relations in our results permits us to apply the results to the differential inclusion problems and difference inclusion problems with weaker assumptions than those used in the papers mentioned above.

Keywords: Binary relation; Prešić-Ćirić operator; Fixed point; Differential inclusion; Difference inclusion

1 Introduction

Throughout the paper, $\aleph$ and $\Re$ will stand for the set of natural numbers and real numbers, respectively, while k will stand for a fixed natural number. For a nonempty set ß, by $\mathcal{P}(ß)$ we will denote the collection of all nonempty subsets of ß.

Let $(ß, d)$ be a metric space and $\Upsilon : ß^k \to ß$ be a mapping. A point $\flat \in ß$ is said to be a fixed point of Υ if $\Upsilon(\flat, \flat, \dots, \flat) = \flat$. Consider the following nonlinear difference equation of k^{th} order:

$$\flat_{n+k} = \Upsilon(\flat_n, \flat_{n+1}, \dots, \flat_{n+k-1}), \quad \text{where } \flat_n \in ß \text{ for all } n \in \aleph. \tag{1}$$

A point $\flat \in ß$ is called an equilibrium point of (1) if it is a fixed point of Υ (see, [3]).

In 1965, when considering the problem of convergence of the sequence $\{\flat_n\}$ defined by (1), Prešić [4, 5] generalized the famous Banach contraction principle on product spaces by proving a fixed point result for the mapping $\Upsilon : ß^k \to ß$ and proved the convergence of the sequence $\{\flat_n\}$. He established that if the mapping Υ satisfies some particular conditions and ß is complete, then such a sequence is convergent and its limit is a fixed point of Υ.

The mapping Υ is called a Prešić operator if there exist constants $\varsigma_i \in [0,1)$ such that $\sum_{i=1}^{k} \varsigma_i < 1$ and

$$d\big(\Upsilon(\flat_1,\flat_2,\dots,\flat_k),\Upsilon(\flat_2,\flat_3,\dots,\flat_{k+1})\big) \le \sum_{i=1}^{k} \varsigma_i d(\flat_i,\flat_{i+1}) \tag{2}$$

for all $\flat_i \in ß$, $i = 1,2,\dots,k+1$. If ß is complete, then Υ has a unique fixed point which is an equilibrium point of difference equation (1) (see, [4, 5]).

The mapping Υ is called a Prešić-Ćirić operator if there exists a constant $\varrho \in [0,1)$ such that $\varrho < 1$ and

$$d\big(\Upsilon(\flat_1,\flat_2,\dots,\flat_k),\Upsilon(\flat_2,\flat_3,\dots,\flat_{k+1})\big) \le \varrho \max\big\{d(\flat_i,\flat_{i+1}) \colon i = 1,2,\dots,k\big\} \tag{3}$$

for all $\flat_i \in ß$, $i = 1,2,\dots,k+1$. Ćirić and Prešić [1] showed that every Prešić-Ćirić operator on a complete metric space has at least one fixed point. The result of Prešić is a generalization of famous Banach contraction principle in product spaces and has theoretical importance as well as several applications (see, e.g., [3, 6–9] and the references therein).

On the other hand, Nadler [10] considered the set-valued mappings and gave a set-valued version of Banach contraction principle. Shukla et al. [11] generalized and unified the results of Prešić and Nadler and proved some common fixed point results. The study of fixed point theorems on metric spaces equipped with binary relations was introduced by Alam and Imdad [12]. They established a relation-theoretic version of Banach contraction principle. Shukla and López [2] extended the result of Alam and Imdad for set-valued mappings and proved a relation-theoretic version of the result of Nadler [10].

Initial and boundary value problems are an essential part of the study of physical systems, e.g., in the study of solid mechanics, thermoelasticity, exponential decay, theory of elastic bodies etc. (see, e.g., [13–18] and the references therein). On the other hand, differential inclusion problems are considered as an interpretation of discontinuous initial and boundary value problems which occurs in the study of mechanical systems, power electronics, and in the theory of heat conduction in metals, etc. (see, e.g., [19, 20] and the references therein). The fixed point results in the spaces equipped with binary relations are very useful in solving a verity of initial value and boundary value problems. While, the relation-theoretic version of result of Nadler may be useful in solving the differential inclusion problems and not investigated yet in the product spaces, which is the motivation for the work presented here. In addition, the relation-theoretic version of result of Nadler makes us able to find the equilibrium point of a difference inclusion as well.

The purpose of this work is to prove a relation-theoretic set-valued version of the result of Ćirić and Prešić [1], which generalizes and extends the results of Ćirić and Prešić [1], Shukla and López [2], Shukla and Radenović [3], Nadler [10], Shukla et al. [11], Alam and Imdad [12] and several other fixed point results in metric spaces equipped with an arbitrary binary relation. The existence of solution of a differential inclusion problem is considered and with relation theoretic approach the existence of solution is established. The new approach provides freedom to use Shukla and López's method with a weaker contractive condition. Another application of the main result to the existence of equilibrium point of difference inclusions has been established and the nature of the equilibrium point has been determined. Specifically, the results proved here can be used in the problems where the solutions are in form of the fixed point of a mapping, which satisfies the contractive condition governed by the relational structure associated with a suitable space. The contractive condition we have used is different and weaker than that used by Shukla and Radenović [3]. We point out that the results of [2] are applicable in establishing the existence of differential inclusion problems, while the results of [3] establish the equilibrium point and its globally asymptotically stability of a difference equation, and the results proved here can be used for both such problems with weaker conditions. In particular, our results are a unification of both these results. For illustration and justification of new results and concepts some examples are presented.

2 Preliminaries

We first state some known definitions and results, which will be used throughout the paper.

Let $(ß, d)$ be a metric space and $CB(ß)$ denotes the set of all nonempty closed and bounded subsets of ß. The distance between a point $\flat \in ß$ and a set $P \subseteq ß$ is defined by $d(\flat, P) = \inf\{d(\flat, p) : p \in P\}$. A function $H : CB(ß) \times CB(ß) \to [0, \infty)$ defined by $H(P, Q) = \max\{D(P, Q), D(Q, P)\}$, where $D(P, Q) = \sup_{p\in P} d(p, Q)$, $D(Q, P) = \sup_{q\in Q} d(q, P)$ for all $P, Q \in CB(ß)$ is called the Pompeiu-Hausdorff metric on $CB(ß)$. Note that H is indeed a metric on $CB(ß)$.

Remark 1 (Ali and Kamran [21]) Let $P, Q \in CB(ß)$ and $h \in (1, \infty)$ be given. Then for $p \in P$ there exists $q \in Q$ such that $d(p, q) \le hH(P, Q)$.

Definition 2.1 (See, [22]) Let ß be a nonempty set. A subset $\mathcal{R}$ of $ß \times ß$ is called a binary relation on ß. Notice that, for each pair $\flat_1, \flat_2 \in ß$, one of the following conditions holds:

(i) $(\flat_1, \flat_2) \in \mathcal{R}$, which amounts to saying that "$\flat_1$ is $\mathcal{R}$-related to $\flat_2$" or "$\flat_1$ relates to $\flat_2$ under $\mathcal{R}$". Sometimes we write $\flat_1 \mathcal{R} \flat_2$ instead of $(\flat_1, \flat_2) \in \mathcal{R}$.

(ii) $(\flat_1, \flat_2) \notin \mathcal{R}$, which means that "$\flat_1$ is not $\mathcal{R}$-related to $\flat_2$" or "$\flat_1$ does not relate to $\flat_2$ under $\mathcal{R}$".

Trivially, $ß \times ß$ and $\emptyset$, being subsets of $ß \times ß$, are binary relations on ß, which are respectively called the universal relation (or full relation) and the empty relation. Another important relation of this kind is the relation $\triangle_ß = \{(\flat, \flat) : \flat \in ß\}$ called the identity relation or the diagonal relation on ß.

Throughout this paper, $\mathcal{R}$ stands for a nonempty binary relation, but for the sake of simplicity we write only "binary relation" instead of "nonempty binary relation."

Definition 2.2 Let ß be a nonempty set and $\mathcal{R}$ be a binary relation on ß. A sequence $\{\flat_n\}$ in ß is called termwise related ($\mathcal{R}$-preserving) if $(\flat_n, \flat_{n+1}) \in \mathcal{R}$ for all $n \in \aleph$.

Definition 2.3 (Alam and Imdad [12]) Let ß be a nonempty set, $\mathcal{R}$ be a binary relation on ß, $l \in \aleph$ and $\flat_1, \flat_2 \in ß$. We say that there is a path $\{\nu^i\}_{i=1}^{l+1}$ of length l from $\flat_1$ to $\flat_2$ if there exist $\nu^i \in ß$, $i = 1, 2, \dots, l + 1$ such that $\nu^1 = \flat_1$, $\nu^{l+1} = \flat_2$ and $(\nu^i, \nu^{i+1}) \in \mathcal{R}$, $i = 1, 2, \dots, l$. The element $\flat_2 \in ß$ is called l-connected to $\flat_1 \in ß$, if there exists a path of length l from $\flat_1$ to $\flat_2$. Denote by $P(\flat_1, l)$ the set of all $\flat_2 \in ß$ such that there exists a path of length l from $\flat_1$ to $\flat_2$, i.e.

$$P(\flat_1, l) = \{\flat_2 \in ß : \text{there exists a path of length } l \text{ from } \flat_1 \text{ to } \flat_2\}.$$

We now discuss our main results.

3 Main results

We first establish concepts and some definitions and results, which will be needed in further discussion.

Definition 3.1 Let ß be a nonempty set and $\Upsilon\colon ß^k \to \mathcal{P}(ß)$ be a mapping. A sequence $\{\flat_n\}$ in ß is called a trajectory of Υ with initial points $\flat_i$, $i = 1,2,\dots,k$, if $\flat_{n+k} \in \Upsilon(\flat_n,\flat_{n+1},\dots,\flat_{n+k-1})$ for all $n \in \aleph$.

Definition 3.2 Consider a metric space $(ß,d)$, a mapping $\Upsilon\colon ß^k \to \mathcal{P}(ß)$ and a binary relation $\mathcal{R}$ on ß. The set of all Cauchy sequences in ß is denoted by ℓ_C and the set of all termwise related sequences in ß is denoted by $\ell_{\mathcal{R}}$, while the set of all trajectories of Υ with initial points in ß is denoted by ℓ_Υ. Then, ß is complete if every sequence in ℓ_C is convergent to some point in ß. We say that ß is $\mathcal{R}$-complete if every sequence in $\ell_C \cap \ell_{\mathcal{R}}$ is convergent to some point in ß. While, ß is said to be $\mathcal{R}$-Υ-complete if every sequence in $\ell_C \cap \ell_{\mathcal{R}} \cap \ell_\Upsilon$ is convergent to some point in ß.

We denote the set of all fixed points of Υ by $\mathrm{Fix}(\Upsilon)$, i.e., $\mathrm{Fix}(\Upsilon) = \{\flat \in ß\colon \flat \in \Upsilon(\flat,\flat,\dots,\flat)\}$.

Definition 3.3 Let $\mathcal{R}$ be a binary relation on a nonempty set ß, d be a metric on ß and $\Upsilon\colon ß^k \to CB(ß)$ be a set-valued mapping then the mapping Υ is called a set-valued relation-theoretic Prešić-Ćirić operator (set-valued $\mathcal{R}PC$-operator) if for every path $\{\flat_i\}_{i=1}^{k+1}$ in $\mathcal{R}$ the following conditions hold:

($\mathcal{R}PC1$) there exists $\varrho \in [0,1)$ such that

$$H\bigl(\Upsilon(\flat_1,\flat_2,\dots,\flat_k),\Upsilon(\flat_2,\flat_3,\dots,\flat_{k+1})\bigr) \le \varrho \max_{1\le i\le k}\bigl\{d(\flat_i,\flat_{i+1})\bigr\};$$

($\mathcal{R}PC2$) if $\flat_{k+1} \in \Upsilon(\flat_1,\flat_2,\dots,\flat_k)$ and $\flat_{k+2} \in \Upsilon(\flat_2,\flat_3,\dots,\flat_{k+1})$ are such that $d(\flat_{k+1},\flat_{k+2}) < \max_{1\le i\le k}\{d(\flat_i,\flat_{i+1})\}$, then $(\flat_{k+1},\flat_{k+2}) \in \mathcal{R}$.

The constant ϱ is called the contractive constant of Υ.

Theorem 3.1 *Consider a metric space $(ß,d)$ equipped with a binary relation $\mathcal{R}$ and a mapping $\Upsilon : ß^k \to CB(ß)$ such that $(ß,d)$ is $\mathcal{R}$-Υ-complete. If Υ is an $\mathcal{R}PC$-operator with contractive constant ϱ and the following assertions hold*:

(A) *there exist $\flat_i \in ß$, $i = 1,2,\dots,k+1$ such that $(\flat_i,\flat_{i+1}) \in \mathcal{R}$ for all $i = 1,2,\dots,k$ and $\flat_{k+1} \in \Upsilon(\flat_1,\flat_2,\dots,\flat_k)$;*

(B) *if a termwise related trajectory $\{\flat_n\} \subseteq ß$ of Υ converges to $\flat \in ß$, then $(\flat_n,\flat) \in \mathcal{R}$ for all $n \in \aleph$.*

Then, there exists a termwise related trajectory $\{\flat_n\} \subseteq ß$ of Υ such that $\lim_{n\to\infty} \flat_n = \flat^* \in ß$ and $\flat^* \in \Upsilon(\flat^*,\flat^*,\dots,\flat^*)$.

Proof Suppose, $\flat_i \in ß$, $i = 1,2,\dots,k+1$ such that $(\flat_i,\flat_{i+1}) \in \mathcal{R}$ for all $i = 1,2,\dots,k$ and $\flat_{k+1} \in \Upsilon(\flat_1,\flat_2,\dots,\flat_k)$. As, $\Upsilon\colon ß^k \to CB(ß)$, we have $\Upsilon(\flat_2,\flat_3,\dots,\flat_{k+1}) \in CB(ß)$ and by Remark 1, for $h = \frac{1}{\sqrt{\varrho}} > 1$ there exists $\flat_{k+2} \in \Upsilon(\flat_2,\flat_3,\dots,\flat_{k+1})$ such that

$$d(\flat_{k+1},\flat_{k+2}) \le \frac{1}{\sqrt{\varrho}} H\bigl(\Upsilon(\flat_1,\flat_2,\dots,\flat_k),\Upsilon(\flat_2,\flat_3,\dots,\flat_{k+1})\bigr).$$

As $(\flat_i, \flat_{i+1}) \in \mathcal{R}$ for $i = 1, 2, \ldots, k$ and Υ is a set-valued $\mathcal{R}PC$-operator with contractive constant ϱ, it follows from $(\mathcal{R}PC1)$ and the above inequality that

$$d(\flat_{k+1}, \flat_{k+2}) \leq \sqrt{\varrho} \max_{1 \leq i \leq k} \{d(\flat_i, \flat_{i+1})\}.$$

As $\varrho < 1$, it follows from the above inequality that

$$d(\flat_{k+1}, \flat_{k+2}) < \max_{1 \leq i \leq k} \{d(\flat_i, \flat_{i+1})\}.$$

By $(\mathcal{R}PC2)$ and the above inequality we have $(\flat_{k+1}, \flat_{k+2}) \in \mathcal{R}$. Thus, $(\flat_i, \flat_{i+1}) \in \mathcal{R}$ for all $i = 1, 2, \ldots, k, k+1$. Again, since $\Upsilon(\flat_3, \flat_4, \ldots, \flat_{k+2}) \in CB(ß)$ by Remark 1, for $h = \frac{1}{\sqrt{\varrho}} > 1$ there exists $\flat_{k+3} \in \Upsilon(\flat_3, \flat_4, \ldots, \flat_{k+2})$ such that

$$d(\flat_{k+2}, \flat_{k+3}) \leq \frac{1}{\sqrt{\varrho}} H\big(\Upsilon(\flat_2, \flat_3, \ldots, \flat_{k+1}), \Upsilon(\flat_3, \flat_4, \ldots, \flat_{k+2})\big).$$

As $(\flat_i, \flat_{i+1}) \in \mathcal{R}$ for all $i = 1, 2, \ldots, k, k+1$ and Υ is a set-valued $\mathcal{R}PC$-operator with contractive constant ϱ, it follows from $(\mathcal{R}PC1)$ and the above inequality that

$$d(\flat_{k+2}, \flat_{k+3}) \leq \sqrt{\varrho} \max_{2 \leq i \leq k+1} \{d(\flat_i, \flat_{i+1})\}.$$

As $\varrho < 1$, it follows from the above inequality that

$$d(\flat_{k+2}, \flat_{k+3}) < \max_{2 \leq i \leq k+1} \{d(\flat_i, \flat_{i+1})\}.$$

By $(\mathcal{R}PC2)$ and the above inequality we have $(\flat_{k+2}, \flat_{k+3}) \in \mathcal{R}$.

The above process yields a termwise related trajectory $\{\flat_n\}$ of Υ with initial points $\flat_1, \flat_2, \ldots, \flat_k \in ß$ such that

$$d(\flat_{n+k}, \flat_{n+k+1}) \leq \sqrt{\varrho} \max_{n \leq i \leq k+n-1} \{d(\flat_i, \flat_{i+1})\} \quad \text{for all } n \in \aleph. \tag{4}$$

We now prove that the trajectory $\{\flat_n\}$ of Υ with initial points $\flat_1, \flat_2, \ldots, \flat_k \in ß$, is a Cauchy sequence.

We define a number $\wp$ as follows:

$$\wp = \max\left\{\frac{d(\flat_i, \flat_{i+1})}{\jmath^i} : i = 1, 2, \ldots, k\right\},$$

where $\jmath = \varrho^{1/2k}$. By using the mathematical induction we prove the following inequality:

$$d(\flat_n, \flat_{n+1}) \leq \wp \jmath^n \quad \text{for all } n \in \aleph. \tag{5}$$

Then, by the definition of $\wp$ it is clear that the inequality (5) is true for $n = 1, 2, \ldots, k$. Let the k inequalities $d(\flat_n, \flat_{n+1}) \leq \wp \jmath^n$, $d(\flat_{n+1}, \flat_{n+2}) \leq \wp \jmath^{n+1}, \ldots, d(\flat_{n+k-1}, \flat_{n+k}) \leq \wp \jmath^{n+k-1}$ be the induction hypothesis. For every $n \in \aleph$, it follows from (4) that

$$d(\flat_{n+k}, \flat_{n+k+1}) \leq \sqrt{\varrho} \max_{n \leq i \leq k+n-1} \{d(\flat_i, \flat_{i+1})\}$$

$$
\begin{aligned}
&\le \sqrt{\varrho}\max\left\{\wp\jmath^{i+n-1} : i = 1,2,\dots,k\right\} \\
&= \sqrt{\varrho}\wp\jmath^{n} \\
&= \wp\jmath^{n+k} \quad \left(\text{as } \jmath = \varrho^{1/2k} < 1\right).
\end{aligned}
$$

Hence, the mathematical induction is complete. Now, for $n, m \in \aleph$, $m > n$, using (5), we obtain

$$
\begin{aligned}
d(\flat_n, \flat_m) &\le d(\flat_n, \flat_{n+1}) + d(\flat_{n+1}, \flat_{n+2}) + \cdots + d(\flat_{m-1}, \flat_m) \\
&\le \wp\jmath^{n} + \wp\jmath^{n+1} + \cdots + \wp\jmath^{m-1} \\
&\le \wp\jmath^{n}\left(1 + \jmath + \jmath^2 + \cdots\right) \\
&= \frac{\wp\jmath^{n}}{1-\jmath}.
\end{aligned}
$$

As $\jmath = \varrho^{1/2k} < 1$, the above inequality yields that

$$\lim_{n,m\to\infty} d(\flat_n, \flat_m) = 0.$$

Therefore, the trajectory $\{\flat_n\}$ of Υ is termwise related Cauchy sequence in ß. Thus, $\{\flat_n\} \in \ell_C \cap \ell_{\mathcal{R}} \cap \ell_{\Upsilon}$.

By $\mathcal{R}$-Υ-completeness of ß, there exists $\flat^* \in$ ß such that

$$\lim_{n\to\infty} d\left(\flat_n, \flat^*\right) = 0.$$

Thus, we have obtain the termwise related trajectory $\{\flat_n\} \subseteq$ ß of Υ such that $\lim_{n\to\infty} \flat_n = \flat^* \in$ ß. We shall show that $\flat^* \in \Upsilon(\flat^*, \flat^*, \dots, \flat^*)$, i.e. $\flat^*$ is a fixed point of Υ.

Since (B) holds, we have $(\flat_n, \flat^*) \in \mathcal{R}$ for all $j \in \aleph$. Because, for each $n \in \aleph$ we have $(\flat_n, \flat_{n+1}) \in \mathcal{R}$ and $\flat_{n+k} \in \Upsilon(\flat_n, \flat_{n+1}, \dots, \flat_{n+k-1})$, hence

$$
\begin{aligned}
d\left(\flat^*, T\left(\flat^*, \dots, \flat^*\right)\right) &\le d\left(\flat^*, \flat_{n+k}\right) + d\left(\flat_{n+k}, \Upsilon\left(\flat^*, \dots, \flat^*\right)\right) \\
&\le d\left(\flat^*, \flat_{n+k}\right) + H\left(\Upsilon(\flat_n, \flat_{n+1}, \dots, \flat_{n+k-1}), \Upsilon\left(\flat^*, \dots, \flat^*\right)\right) \\
&\le d\left(\flat^*, \flat_{n+k}\right) + H\left(\Upsilon(\flat_n, \flat_{n+1}, \dots, \flat_{n+k-1}), \Upsilon\left(\flat_{n+1}, \dots, \flat_{n+k-1}, \flat^*\right)\right) \\
&\quad + H\left(\Upsilon\left(\flat_{n+1}, \dots, \flat_{n+k-1}, \flat^*\right), \Upsilon\left(\flat_{n+2}, \dots, \flat_{n+k-1}, \flat^*, \flat^*\right)\right) \\
&\quad + \cdots + H\left(\Upsilon\left(\flat_{n+k-1}, \flat^*, \dots, \flat^*\right), \Upsilon\left(\flat^*, \dots, \flat^*\right)\right) \\
&\le d\left(\flat^*, \flat_{n+k}\right) + \varrho\max\left\{d(\flat_n, \flat_{n+1}), \dots, d(\flat_{n+k-2}, \flat_{n+k-1}),\right. \\
&\quad \left. d\left(\flat_{n+k-1}, \flat^*\right)\right\} + \varrho\max\left\{d(\flat_{n+1}, \flat_{n+2}), \dots, d(\flat_{n+k-2}, \flat_{n+k-1}),\right. \\
&\quad \left. d\left(\flat_{n+k-1}, \flat^*\right)\right\} + \cdots + \varrho d\left(\flat_{n+k-1}, \flat^*\right).
\end{aligned}
$$

Letting $n \to \infty$ in the above inequality and using the fact that $\lim_{n\to\infty} \flat_n = \flat^*$, we obtain $d(\flat^*, \Upsilon(\flat^*, \dots, \flat^*)) = 0$, that is, $\flat^* \in \Upsilon(\flat^*, \dots, \flat^*)$. Thus, $\flat^*$ is a fixed point of Υ. □

Remark 2 In the case $k = 1$, condition (B) of Theorem 3.1 can be replaced by the following weaker condition:

(B′) if a termwise related trajectory $\{\flat_n\} \subseteq ß$ of Υ converges to $\flat \in ß$, then there exist a subsequence $\{\flat_{n_j}\}$ and $j_0 \in \aleph$ such that $(\flat_{n_j}, \flat) \in \mathcal{R}$ for all $j > j_0$.

Example 1 Consider $ß = [0, \infty)$ with the usual metric $d(\flat_1, \flat_2) = |\flat_1 - \flat_2|$ for all $\flat_1, \flat_2 \in ß$. Denote by $\mathbb{Q}^+$ the set of all nonnegative rational numbers and $\mathfrak{L}_0^{\mathbb{Q}^+}$ the class of all nonincreasing sequences of nonnegative rational numbers converging to zero, i.e.,

$$\mathfrak{L}_0^{\mathbb{Q}^+} = \bigl\{\{\flat_n\} \subseteq \mathbb{Q}^+ : \flat_{n+1} \leq \flat_n \text{ for all } n \in \aleph, \flat_n \to 0 \text{ as } n \to \infty\bigr\}.$$

Obviously $\mathfrak{L}_0^{\mathbb{Q}^+} \neq \emptyset$. Let $\mathcal{R}$ be the binary relation defined on ß by

$$\mathcal{R} = \bigl\{(\flat_n, \flat_{n+1}), (0, \flat_n) : \{\flat_n\} \in \mathfrak{L}_0^{\mathbb{Q}^+}\bigr\}.$$

Then, it is clear that $\mathcal{R}$ is nonempty. For a fixed $a > 1$ define a mapping $\Upsilon : ß^2 \to CB(ß)$ by

$$\Upsilon(\flat_1, \flat_2) = \begin{cases} \{0, \frac{\max\{\flat_1, \flat_2\}}{a^2}\}, & \text{if } \flat_1, \flat_2 \in \mathbb{Q}^+; \\ \{0, \frac{\max\{\flat_1, \flat_2\}}{|\max\{\flat_1, \flat_2\}^2 - 1|}\}, & \text{otherwise.} \end{cases}$$

Then, we observe that for $k = 2$ all the conditions of Theorem 3.1 are satisfied. We have the following:

(I) ($\mathcal{R}PC1$) *is satisfied with* $\frac{1}{a^2} \leq \varrho < 1$. Let $\flat_1, \flat_2, \flat_3 \in ß$ are such that $(\flat_1, \flat_2), (\flat_2, \flat_3) \in \mathcal{R}$. Then, the following cases are possible:

Case (i): Suppose $\flat_1, \flat_2, \flat_3 \in \mathbb{Q}^+$ and $\flat_1 \geq \flat_2 \geq \flat_3 \geq 0$, then:

$$\begin{aligned} H\bigl(\Upsilon(\flat_1, \flat_2), \Upsilon(\flat_2, \flat_3)\bigr) &= H\left(\left\{0, \frac{\max\{\flat_1, \flat_2\}}{a^2}\right\}, \left\{0, \frac{\max\{\flat_2, \flat_3\}}{a^2}\right\}\right) \\ &= H\left(\left\{0, \frac{\flat_1}{a^2}\right\}, \left\{0, \frac{\flat_2}{a^2}\right\}\right) \\ &= \max\left\{\sup\left\{\min\left\{0, \frac{\flat_2}{a^2}\right\}, \min\left\{\frac{\flat_1}{a^2}, \frac{1}{a^2}|\flat_1 - \flat_2|\right\}\right\},\right. \\ &\quad \left.\sup\left\{\min\left\{0, \frac{\flat_1}{a^2}\right\}, \min\left\{\frac{\flat_2}{a^2}, \frac{1}{a^2}|\flat_1 - \flat_2|\right\}\right\}\right\} \\ &= \frac{1}{a^2}\max\bigl\{\sup\bigl\{0, \min\{\flat_1, \flat_1 - \flat_2\}\bigr\}, \\ &\quad \sup\bigl\{0, \min\{\flat_2, \flat_1 - \flat_2\}\bigr\}\bigr\} \\ &= \frac{1}{a^2}\max\bigl\{\flat_1 - \flat_2, \min\{\flat_2, \flat_1 - \flat_2\}\bigr\} \\ &= \frac{1}{a^2}(\flat_1 - \flat_2) \\ &= \frac{1}{a^2}d(\flat_1, \flat_2) \\ &\leq \varrho \max\bigl\{d(\flat_1, \flat_2), d(\flat_2, \flat_3)\bigr\}, \end{aligned}$$

where $\frac{1}{a^2} \leq \varrho < 1$.

Case (ii): If $\flat_1, \flat_2, \flat_3 \in \mathbb{Q}^+$ and $\flat_2 = 0$, then:

$$H\bigl(\Upsilon(\flat_1, \flat_2), \Upsilon(\flat_2, \flat_3)\bigr) = H\bigl(\Upsilon(\flat_1, 0), \Upsilon(0, \flat_3)\bigr)$$

$$\begin{aligned}
&= H\left(\left\{0, \frac{\max\{\flat_1, 0\}}{a^2}\right\}, \left\{0, \frac{\max\{0, \flat_3\}}{a^2}\right\}\right) \\
&= H\left(\left\{0, \frac{\flat_1}{a^2}\right\}, \left\{0, \frac{\flat_3}{a^2}\right\}\right) \\
&= \max\left\{\sup\left\{\min\left\{0, \frac{\flat_3}{a^2}\right\}, \min\left\{\frac{\flat_1}{a^2}, \frac{1}{a^2}|\flat_1 - \flat_3|\right\}\right\},\right. \\
&\qquad \left.\sup\left\{\min\left\{0, \frac{\flat_1}{a^2}\right\}, \min\left\{\frac{\flat_3}{a^2}, \frac{1}{a^2}|\flat_1 - \flat_3|\right\}\right\}\right\} \\
&= \frac{1}{a^2}\max\bigl\{\sup\bigl\{0, \min\{\flat_1, |\flat_1 - \flat_3|\}\bigr\}, \\
&\qquad \sup\bigl\{0, \min\{\flat_3, |\flat_1 - \flat_3|\}\bigr\}\bigr\} \\
&= \frac{1}{a^2}\max\bigl\{\min\{\flat_1, |\flat_1 - \flat_3|\}, \min\{\flat_3, |\flat_1 - \flat_3|\}\bigr\} \\
&\leq \varrho\max\{\flat_1, \flat_3\} \\
&= \varrho\max\bigl\{d(\flat_1, \flat_2), d(\flat_2, \flat_3)\bigr\},
\end{aligned}$$

where $\frac{1}{a^2} \leq \varrho < 1$.

Case (iii): If $\flat_1, \flat_2, \flat_3 \in \mathbb{Q}^+$ and $\flat_2 \neq 0$, then two subcases are possible: (a) $\flat_1 = 0, \flat_2 \geq b_3 > 0$; (b) $\flat_1 = 0, \flat_2 \geq b_3 = 0$. In both subcases, the left-hand side of desired inequality becomes zero, hence the inequalities hold trivially.

Hence, $(\mathcal{R}PC1)$ is satisfied.

(II) $(\mathcal{R}PC2)$ *is satisfied.* Suppose that $(\flat_1, \flat_2), (\flat_2, \flat_3) \in \mathcal{R}$, $\flat_3 \in \Upsilon(\flat_1, \flat_2) = \{0, \frac{\max\{\flat_1, \flat_2\}}{a^2}\}$ and $\flat_4 \in \Upsilon(\flat_2, \flat_3) = \{0, \frac{\max\{\flat_2, \flat_3\}}{a^2}\}$. Then, we consider the following cases:

(a) if $\flat_3 = \flat_4 = 0$; or $\flat_3 = \frac{\max\{\flat_1, \flat_2\}}{a^2}$, $\flat_4 = 0$, then, since for every nonincreasing sequence $\{\flat_n\}$ which is eventually constant and has limit 0, we have $(\flat_n, \flat_{n+1}) \in \mathcal{R}$ for all $n \in \aleph$, hence we have $(\flat_3, \flat_4) \in \mathcal{R}$.

(b) If $\flat_3 = 0$, $\flat_4 = \frac{\max\{\flat_2, \flat_3\}}{a^2}$, then by definition of $\mathcal{R}$ we have $(\flat_3, \flat_4) \in \mathcal{R}$.

(c) If $\flat_3 = \frac{\max\{\flat_1, \flat_2\}}{a^2}$, $\flat_4 = \frac{\max\{\flat_2, \flat_3\}}{a^2}$, then if $\flat_2 = 0$ we must have $\flat_1 \geq \flat_3$, while if $\flat_2 \neq 0$, then by the definition of $\mathcal{R}$ we must have $\flat_1 \geq \flat_2 \geq \flat_3 > 0$. In both the cases there exists a nonincreasing sequence of nonnegative rational numbers which converges to zero and has two successive terms $\flat_3$, $\flat_4$, and so, $(\flat_3, \flat_4) \in \mathcal{R}$.

Hence, Υ is a set-valued $\mathcal{R}PC$-operator with contractive constant ϱ such that $\frac{1}{a^2} \leq \varrho < 1$.

(III) *Condition (A) of Theorem* 3.1 *is satisfied.* For every $\flat \in \mathbb{Q}^+$, we have $\{\flat_n\} \in \mathfrak{L}_0^{\mathbb{Q}^+}$, where $\flat_1 = \flat_2 = \flat$ and $\flat_{n+2} = \frac{\max\{\flat_n, \flat_{n+1}\}}{a^2}$ for all $n \in \aleph$. Hence, we have $\flat_{n+2} \in \Upsilon(\flat_n, \flat_{n+1})$ and $(\flat_n, \flat_{n+1}) \in \mathcal{R}$ for all $n \in \aleph$.

(IV) *Condition (B) of Remark* 2 *is satisfied.* Suppose that $\{\flat_n\}$ is a termwise related trajectory of Υ converging to $\flat \in ß$. As, $\{\flat_n\}$ is a termwise related trajectory of Υ we must have $\flat_3 \in \Upsilon(\flat_1, \flat_2)$, and so, the sequence $\{\flat_n\}$ is either an eventually constant sequence converging to 0, or it is given by: $\flat_{2n+1} = \max\{\frac{\flat_1}{a^{2n}}, \frac{\flat_2}{a^{2n}}\}$ and $\flat_{2n+2} = \max\{\frac{\flat_1}{a^{2n+2}}, \frac{\flat_2}{a^{2n}}\}$ for all $n \in \aleph$. In each case, the sequence $\{\flat_n\}$ converges to 0 and $(\flat_n, 0) \in \mathcal{R}$ for all $n \in \aleph$.

Thus, all the conditions of Theorem 3.1 are satisfied, hence Υ must possess a fixed point. Indeed, the set of all fixed points of Υ, i.e., $\mathrm{Fix}(\Upsilon) = \{0, \sqrt{2}\}$.

We next consider a generalization of the closedness of set-valued mappings in product spaces and a weaker version of this generalized closedness, which provides an alternate of the condition (B) of Theorem 3.1.

Definition 3.4 Let $(ß,d)$ be a metric space equipped with a binary relation $\mathcal{R}$ and $\Upsilon: ß^k \to \mathcal{P}(ß)$ be a set-valued mapping. Then the set $\mathfrak{G}(\Upsilon)$ is called the graph of Υ and

$$\mathfrak{G}(\Upsilon) = \{(\flat_0, \flat_1, \ldots, \flat_k) \in ß^{k+1} : \flat_k \in \Upsilon(\flat_0, \flat_1, \ldots, \flat_{k-1})\}.$$

We say that Υ is a closed mapping if for every collection of convergent sequences $\{\flat_n^i\}$, $i = 0, 1 \ldots, k$ in ß such that $(\flat_n^0, \flat_n^1, \ldots, \flat_n^k) \in \mathfrak{G}(\Upsilon)$ for all $n \in \aleph$ we have $(\flat^0, \flat^1, \ldots, \flat^k) \in \mathfrak{G}(\Upsilon)$, where $\flat^i \in ß$, $i = 0, 1, \ldots, k$ are the limits of the sequences $\{\flat_n^i\}$, $i = 0, 1, \ldots, k$ respectively. It is obvious that Υ is closed if and only if $\mathfrak{G}(\Upsilon)$ is a closed subset of $ß^{k+1}$. We say that Υ is $\mathcal{R}$-closed if for every convergent sequence $\{\flat_n\} \in \ell_{\mathcal{R}}$ such that $(\flat_n, \flat_{n+1}, \ldots, \flat_{n+k}) \in \mathfrak{G}(\Upsilon)$ for all $n \in \aleph$ we have $(\flat, \flat, \ldots, \flat) \in \mathfrak{G}(\Upsilon)$, where $\flat \in ß$ is the limit of the sequence $\{\flat_n\}$.

Remark 3 Suppose that $\{\flat_n\} \in \ell_{\mathcal{R}}$ is a convergent sequence with limit $\flat$ in ß such that $(\flat_n, \flat_{n+1}, \ldots, \flat_{n+k}) \in \mathfrak{G}(\Upsilon)$ for all $n \in \aleph$. Define the sequences $\{\flat_n^i\}$, $i = 0, 1, \ldots, k$ by $\{\flat_n^i\} = \{\flat_{n+i}\}$, $i = 0, 1, \ldots, k$, then each $\{\flat_n^i\}$ is convergent to $\flat$ and we have $(\flat_n^1, \flat_n^2, \ldots, \flat_n^{k+1}) \in \mathfrak{G}(\Upsilon)$ for all $n \in \aleph$, hence the implication "Υ is closed $\Longrightarrow$ Υ is $\mathcal{R}$-closed" holds. The reverse implication of the above does not hold (see the example below).

Example 2 Let $ß = \Re$, d is the usual metric on ß and $\mathfrak{L}_0^{\mathbb{Q}^+}$ be the class of all nonincreasing sequences of nonnegative rational numbers converging to 0. Consider the relation $\mathcal{R}$ on ß defined by $\mathcal{R} = \{(\flat_n, \flat_{n+1}) \in ß^2 : \{\flat_n\} \in \mathfrak{L}_0^{\mathbb{Q}^+}\}$ and a mapping $\Upsilon: ß^2 \to \mathcal{P}(ß)$ defined by

$$\Upsilon(\flat_1, \flat_2) = \begin{cases} \{0, \frac{\flat_1 + \flat_2}{2}\}, & \text{if } \flat_1, \flat_2 \text{ are nonnegative rationals;} \\ \{0, \flat_1 + \flat_2 + 1\}, & \text{otherwise.} \end{cases}$$

Note that if $\{\flat_n\}$ is a sequence of irrational numbers converging to 0, then $(\flat_n, \flat_{n+1}, \flat_n^1) \in \mathfrak{G}(\Upsilon)$ for all $n \in \aleph$, where $\flat_n^1 = \flat_n + \flat_{n+1} + 1$ for all $n \in \aleph$ and $(\flat_n, \flat_{n+1}, \flat_n^1) \to (0, 0, 1)$ as $n \to \infty$. While $1 \notin \Upsilon(0,0)(= \{0\})$, and so Υ is not a closed mapping. On the other hand, if $\{\flat_n\} \in \ell_{\mathcal{R}}$ such that $(\flat_n, \flat_{n+1}, \flat_{n+2}) \in \mathfrak{G}(\Upsilon)$ for all $n \in \aleph$, then we must have $\{\flat_n\} \in \mathfrak{L}_0^{\mathbb{Q}^+}$, and so, if the sequence $\{\flat_n\}$ converges, it must converge to 0. As $0 \in \Upsilon(0,0)$, i.e., $(0,0,0) \in \mathfrak{G}(\Upsilon)$, hence Υ is $\mathcal{R}$-closed.

Proposition 1 *If $(ß,d)$ is a metric space, $\mathcal{R}$ is a binary relation on ß and $\Upsilon: ß^k \to CB(ß)$ is an $\mathcal{R}$-closed mapping. If there exists a termwise related convergent trajectory of Υ, then Υ has a fixed point.*

Proof Let $\{\flat_n\}$ be a termwise related convergent trajectory of Υ and $\flat_n \to \flat$ as $n \to \infty$. Then, by definition, we have $\{\flat_n\} \in \ell_{\mathcal{R}}$, $(\flat_n, \flat_{n+1}, \ldots, \flat_{n+k}) \in \mathfrak{G}(\Upsilon)$ for all $n \in \aleph$. Since $\flat_n \to \flat$ as $n \to \infty$ by $\mathcal{R}$-closedness of Υ we have $(\flat, \flat, \ldots, \flat) \in \mathfrak{G}(\Upsilon)$, i.e., $\flat \in \Upsilon(\flat, \flat, \ldots, \flat)$. Hence, Υ has a fixed point $\flat$. □

In the next theorem, the condition (B) of Theorem 3.1 is replaced by the $\mathcal{R}$-closedness of Υ.

Theorem 3.2 *Consider a metric space $(ß,d)$ equipped with a binary relation $\mathcal{R}$ and a mapping $\Upsilon: ß^k \to CB(ß)$ such that $(ß,d)$ is $\mathcal{R}$-Υ-complete. If Υ is an $\mathcal{RPC}$-operator with contractive constant ϱ and the following assertions hold*:

(A) *there exist* $\flat_i \in ß$, $i = 1, 2, \dots, k+1$ *such that* $(\flat_i, \flat_{i+1}) \in \mathcal{R}$ *for all* $i = 1, 2, \dots, k$ *and* $\flat_{k+1} \in \Upsilon(\flat_1, \flat_2, \dots, \flat_k)$;

(B) Υ *is* $\mathcal{R}$*-closed.*

Then, there exists a termwise related trajectory $\{\flat_n\} \subseteq ß$ *of* Υ *such that* $\lim_{n\to\infty} \flat_n = \flat^* \in ß$ *and* $\flat^* \in \Upsilon(\flat^*, \flat^*, \dots, \flat^*)$.

Proof Similar to the proof of Theorem 3.1, we obtain a termwise related trajectory $\{\flat_n\}$ of Υ with initial values $\flat_1, \flat_2, \dots, \flat_k$ such that the sequence $\{\flat_n\}$ converges to some $\flat^* \in ß$. Now, using Proposition 1, we obtain that $\flat^*$ is a fixed point of Υ. □

Corollary 3.3 *Consider a metric space* $(ß, d)$ *equipped with a binary relation* $\mathcal{R}$ *and a mapping* $\Upsilon : ß^k \to CB(ß)$ *such that* $(ß, d)$ *is* $\mathcal{R}$-Υ*-complete. If* Υ *is an* $\mathcal{R}PC$*-operator with contractive constant* ϱ *and the following assertions hold*:

(A) *there exist* $\flat_i \in ß$, $i = 1, 2, \dots, k+1$ *such that* $(\flat_i, \flat_{i+1}) \in \mathcal{R}$ *for all* $i = 1, 2, \dots, k$ *and* $\flat_{k+1} \in \Upsilon(\flat_1, \flat_2, \dots, \flat_k)$;

(B) Υ *is closed.*

Then, there exists a termwise related trajectory $\{\flat_n\} \subseteq ß$ *of* Υ *such that* $\lim_{n\to\infty} \flat_n = \flat^* \in ß$ *and* $\flat^* \in \Upsilon(\flat^*, \flat^*, \dots, \flat^*)$.

Proof The proof follows by using Remark 3 and Theorem 3.2. □

In the next section, we present several consequences of our main results.

4 Consequences

We derive several fixed point results as corollaries of Theorem 3.1 and Theorem 3.2.

Let $(ß, \sqsubseteq)$ be a partially ordered set. A pair $(\flat_1, \flat_2)$ of elements of ß is called comparable if $\flat_1 \sqsubseteq \flat_2$ or $\flat_2 \sqsubseteq \flat_1$. Define a binary relation $\mathcal{R}_\sqsubseteq$ by:

$$\mathcal{R}_\sqsubseteq = \left\{(\flat_1, \flat_2) \in ß^2 : \flat_1 \sqsubseteq \flat_2\right\}.$$

Replacing the relation $\mathcal{R}$ by $\mathcal{R}_\sqsubseteq$ in Theorems 3.1 and 3.2, we obtain the following generalized and unified set-valued version of results of Ran and Reurings [23] and Nieto and López [24] in product spaces.

Corollary 4.1 *Let* $(ß, \sqsubseteq)$ *be a partially ordered set,* d *a metric on* $ß$ *and* $\Upsilon : ß^k \to CB(ß)$ *be such that* $(ß, d)$ *is* $\mathcal{R}_\sqsubseteq$-Υ*-complete. Suppose that* Υ *is an* $\mathcal{R}_\sqsubseteq PC$*-operator with contractive constant* ϱ *and the following assertions hold*:

(i) *there exist* $\flat_i \in ß$, $i = 1, 2, \dots, k+1$ *such that* $(\flat_i, \flat_{i+1}) \in \mathcal{R}_\sqsubseteq$ *for all* $i = 1, 2, \dots, k$ *and* $\flat_{k+1} \in \Upsilon(\flat_1, \flat_2, \dots, \flat_k)$;

(ii) *at least one of the following conditions is satisfied*:

  (a) Υ *is* $\mathcal{R}_\sqsubseteq$*-closed;*

  (b) *if a termwise related trajectory* $\{\flat_n\} \subseteq ß$ *of* Υ *converges to* $\flat \in ß$*, then* $\flat_n \sqsubseteq \flat$ *for all* $n \in \aleph$.

Then, there exists a trajectory $\{\flat_n\} \subseteq ß$ *of* Υ *such that* $\flat_n \sqsubseteq \flat_{n+1}$ *for all* $n \in \aleph$ *and* $\lim_{n\to\infty} \flat_n = \flat^* \in ß$ *and* $\flat^* \in \Upsilon(\flat^*, \flat^*, \dots, \flat^*)$.

Remark 4 Define:

$$\mathcal{R}_{\sqsupseteq} = \{(\flat_1, \flat_2) \in ß^2 : \flat_2 \sqsubseteq \flat_1\}; \quad \text{and} \quad \mathcal{R}_{\asymp} = \{(\flat_1, \flat_2) \in ß^2 : (\flat_1, \flat_2) \text{ is comparable}\}.$$

Then, similar to the above corollary, we can deduce the corresponding fixed point results for $\mathcal{R}_{\sqsupseteq}PC$-operators and $\mathcal{R}_{\asymp}PC$-operators.

The following corollary is a set-valued version of the result of Ćirić and Prešić [1].

Corollary 4.2 *Let* $(ß, d)$ *be a complete metric space and* $\Upsilon : ß^k \to CB(ß)$ *be a set-valued Prešić-Ćirić type contraction, i.e., the following condition holds*:

$$H\big(\Upsilon(\flat_1, \flat_2, \dots, \flat_k), \Upsilon(\flat_2, \flat_3, \dots, \flat_{k+1})\big) \le \varrho \max_{1 \le i \le k}\{d(\flat_i, \flat_{i+1})\}$$

for all $\flat_1, \flat_2, \dots, \flat_{k+1} \in ß$, *where* $0 \le \varrho < 1$. *Then,* Υ *has a fixed point in* $ß$. *Moreover, for arbitrary* $\flat_1, \flat_2, \dots, \flat_k \in ß$, *there exists a sequence* $\{\flat_n\}$ *in* $ß$ *such that* $\flat_{n+k} \in \Upsilon(\flat_n, \flat_{n+1}, \dots, \flat_{n+k-1})$ *for all* $n \in \aleph$ *and* $\{\flat_n\}$ *converges to a fixed point of* Υ.

Proof Consider the universal relation $\mathcal{R} = ß^2$. Then, it is easy to see that all the conditions of Theorem 3.1 are satisfied, hence the conclusion follows from Theorem 3.1. □

A mapping $\Upsilon : ß^k \to ß$ is called a relation theoretic Prešić-Ćirić operator ($\mathcal{R}PC$-operator) if for every path $\{\flat_i\}_{i=1}^{k+1}$ in $\mathcal{R}$ the following hold:

($\mathcal{R}PC1'$) there exist $\varrho \in [0, 1)$ such that

$$d\big(\Upsilon(\flat_1, \flat_2, \dots, \flat_k), \Upsilon(\flat_2, \flat_3, \dots, \flat_{k+1})\big) \le \varrho \max_{1 \le i \le k}\{d(\flat_i, \flat_{i+1})\};$$

($\mathcal{R}PC2'$) if $\flat_{k+1} = \Upsilon(\flat_1, \flat_2, \dots, \flat_k)$ and $\flat_{k+2} = \Upsilon(\flat_2, \flat_3, \dots, \flat_{k+1})$ are such that $d(\flat_{k+1}, \flat_{k+2}) < \max_{1 \le i \le k}\{d(\flat_i, \flat_{i+1})\}$, then $(\flat_{k+1}, \flat_{k+2}) \in \mathcal{R}$.

The constant ϱ is called the contractive constant of Υ. A sequence $\{\flat_n\}$ in ß is called a *PP*-sequence of Υ with initial values $\flat_1, \flat_2, \dots, \flat_k \in ß$ if $\flat_{n+k} = \Upsilon(\flat_n, \flat_{n+1}, \dots, \flat_{n+k-1})$ for all $n \in \aleph$ (see, [25]). The mapping Υ is called *P*-$\mathcal{R}$-continuous at $\flat \in ß$ if for every *PP*-sequence $\{\flat_n\}$ in $\ell_{\mathcal{R}}$ such that $\flat_n \to \flat$ as $n \to \infty$ we have $\Upsilon(\flat_n, \flat_{n+1}, \dots, \flat_{n+k-1}) \to \Upsilon(\flat, \flat, \dots, \flat)$ as $n \to \infty$. The mapping Υ is called *P*-$\mathcal{R}$-continuous on ß if it is *P*-$\mathcal{R}$-continuous at every point of ß.

Remark 5 For a given single-valued mapping $\Upsilon : ß^k \to ß$ we define its corresponding set-valued mapping $\Upsilon_s : ß^k \to \mathcal{P}(ß)$ by $\Upsilon_s(\flat_1, \flat_2, \dots, \flat_k) = \{\Upsilon(\flat_1, \flat_2, \dots, \flat_k)\}$ for all $\flat_1, \flat_2, \dots, \flat_k \in ß$. Since $H(\{\flat_1\}, \{\flat_2\}) = d(\flat_1, \flat_2)$, hence we observe the following:

(a) If Υ is an $\mathcal{R}PC$-operator with contractive constant ϱ, then Υ_s is a set-valued $\mathcal{R}PC$-operator with contractive constant ϱ and vice-versa.
(b) A termwise related *PP*-sequence of Υ is a termwise related trajectory of Υ_s and vice-versa.
(c) If Υ is *P*-$\mathcal{R}$-continuous, then Υ_s is $\mathcal{R}$-closed and vice-versa.

The following corollary is a relation theoretic version of the main result of Ćirić and Prešić [1].

Corollary 4.3 *Let (β, d) be a complete metric space, $\mathcal{R}$ a binary relation on β and $\Upsilon : \beta^k \to \beta$ be an $\mathcal{R}PC$-operator with contractive constant ϱ. Suppose that the following assertions hold*:

(i) *there exist $\flat_i \in \beta$, $i = 1, 2, \dots, k+1$ such that $(\flat_i, \flat_{i+1}) \in \mathcal{R}$ for all $i = 1, 2, \dots, k$ and $\flat_{k+1} = \Upsilon(\flat_1, \flat_2, \dots, \flat_k)$*;

(ii) *at least one of the following conditions is satisfied*:

  (a′) *for any PP-sequence $\{\flat_n\} \subseteq \beta$ of Υ which converges to $\flat \in \beta$, then $(\flat_n, \flat) \in \mathcal{R}$ for all $n \in \aleph$*;

  (b′) *Υ is a P-$\mathcal{R}$-continuous mapping.*

Then, there exists a PP-sequence $\{\flat_n\} \subset \beta$ of Υ such that $\lim_{n\to\infty} \flat_n = \flat^* \in \beta$ and $\flat^* = \Upsilon(\flat^*, \flat^*, \dots, \flat^*)$.

Proof We consider the corresponding set-valued mapping Υ_s of Υ. Then, by using Remark 5, one can easily verify that: (i) Υ_s is a set-valued $\mathcal{R}PC$-operator with contractive constant ϱ; (ii) condition (i) implies condition (A) of Theorem 3.1; (iii) condition (a′) implies condition (B) of Theorem 3.1 (iv) condition (b′) implies condition (B) of Theorem 3.2. Hence, the result follows from Theorem 3.1 and Theorem 3.2. □

Let (β, d) be a metric space and $\Delta = \{(\flat, \flat) : \flat \in \beta\}$. Let Ξ be a directed graph (see [9] and the reference therein) with $V(\Xi) = \beta$ and $E(\Xi) \subseteq \beta^2$ and Ξ has no parallel edges. Then, we say that β is endowed with the graph Ξ. For $\flat^1, \flat^2 \in \beta$, a path in Ξ from $\flat^1$ to $\flat^2$ of length $N \in \aleph \cup \{0\}$ is a finite sequence $\{\flat_i\}_{i=0}^N$ of $N+1$ points (vertices) such that $\flat_0 = \flat^1$, $\flat_N = \flat^2$ and $(\flat_{i-1}, \flat_i) \in E(\Xi)$ for $i = 1, \dots, N$. A sequence $\{\flat_n\}$ in β is called a termwise connected sequence if $(\flat_n, \flat_{n+1}) \in E(\Xi)$ for all $n \in \aleph$.

The following corollary is the main result of Shahzad and Shukla [9].

Corollary 4.4 *Let (β, d) be a complete metric space endowed with a graph Ξ and $\Upsilon : \beta^k \to CB(\beta)$ be a set-valued Ξ-Prešić operator, i.e., for every path $\{\flat_i\}_{i=1}^{k+1}$ of $k+1$ vertices in Ξ the following conditions are satisfied*:

(ΞP1) *there exist nonnegative numbers ς_i's such that $0 \le \sum_{i=1}^k \varsigma_i < 1$ and*

$$H\big(\Upsilon(\flat_1, \flat_2, \dots, \flat_k), \Upsilon(\flat_2, \flat_3, \dots, \flat_{k+1})\big) \le \sum_{i=1}^{k} \varsigma_i d(\flat_i, \flat_{i+1});$$

(ΞP2) *if $\flat_{k+1} \in \Upsilon(\flat_1, \flat_2, \dots, \flat_k)$ and $\flat_{k+2} \in \Upsilon(\flat_2, \flat_3, \dots, \flat_{k+1})$ are such that $d(\flat_{k+1}, \flat_{k+2}) < \max_{1\le i\le k}\{d(\flat_i, \flat_{i+1})\}$, then $(\flat_{k+1}, \flat_{k+2}) \in E(\Xi)$.*

Suppose, the following conditions hold:

(a″) *there exists a path $\{\flat_i\}_{i=1}^{k+1}$ of $k+1$ vertices in Ξ, such that $\flat_{k+1} \in \Upsilon(\flat_1, \flat_2, \dots, \flat_k)$*;

(b″) *for any termwise connected sequence $\{\flat_n\}$ converges to $\flat \in \beta$, and $\flat_{n+k} \in \Upsilon(\flat_n, \flat_{n+1}, \dots, \flat_{n+k-1})$ for all $n \in \aleph$, then $(\flat_n, \flat) \in E(\Xi)$ for all $n \in \aleph$.*

Then, Υ has a fixed point in β. Moreover, there exists a termwise connected sequence $\{\flat_n\}$ in β such that $\flat_{n+k} \in \Upsilon(\flat_n, \flat_{n+1}, \dots, \flat_{n+k-1})$ for all $n \in \aleph$ and $\{\flat_n\}$ converges to a fixed point of Υ.

Proof Consider the binary relation $\mathcal{R}_\Xi$ on β defined by $\mathcal{R}_\Xi = \{(\flat_1, \flat_2) \in \beta^2 : (\flat_1, \flat_2) \in E(\Xi)\}$. First, note that $\sum_{i=1}^k \varsigma_i d(\flat_i, \flat_{i+1}) \le [\sum_{i=1}^k \varsigma_i] \max_{1\le i\le k}\{d(\flat_i, \flat_{i+1})\}$, and so condition

(ΞP1) implies condition ($\mathcal{R}_{\Xi}PC1$). Also, it is easy to see that condition (ΞP2) implies condition ($\mathcal{R}_{\Xi}PC2$). By definition of $\mathcal{R}_{\Xi,}$ the conditions (a″) and (b″) imply the conditions (A) and (B) of Theorem 3.1. Hence, the existence of sequence $\{\flat_n\}$ with desired properties follows from Theorem 3.1. □

Remark 6 The existence results for set-valued Prešić type operators in ε-chainable spaces (see, [26] and [9]) and the existence results for a single-valued cyclic-Prešić operator (see, [25] and the references therein) can be derived in a similar way, here we are omitting the detailed proof.

5 Applications

In this section, as applications of our main results, we establish existence of solution of a particular differential inclusion problem and prove the existence and stability of equilibrium point of a difference inclusion.

5.1 Application to the solution of a differential inclusion

We first consider a differential inclusion problem for a solution under some suitable conditions.

Suppose that $\varpi > 0$ and $I = [0, \varpi]$. By $C(I, \Re)$ we denote the space of all continuous real-valued functions on I under the supremum norm $\|\cdot\|_\infty$, while $AC(I, \Re)$ denotes the space of all absolutely continuous real-valued functions on I with same norm. For $r > 0$ and $\flat \in C(I, \Re)$ we define a subset $B[\flat, r]$ of $C(I, \Re)$ by: $B[\flat, r] = \{\flat_1 \in C(I, \Re) : \|\flat - \flat_1\|_\infty \le r\}$.

Consider the time-dependent differential inclusion problem of the following form:

$$\begin{aligned} &\flat'(t) \in \Lambda\bigl(t, \flat(t)\bigr) + g(t) \quad \text{for a.e. } t \in I, \\ &\flat(0) = \alpha \in \Re, \end{aligned} \tag{6}$$

in which $\Lambda : I \times \Re \to \mathcal{P}(\Re)$ is a set-valued mapping considered with some suitable conditions, and $g : I \to \Re$ is a continuous function.

Suppose that $\Omega^1_\Lambda(\flat)$ represents the set of all Lebesgue integrable selections of $\Lambda(\cdot, \flat(\cdot))$, i.e.,

$$\Omega^1_\Lambda(\flat) = \bigl\{\nu \in L^1(I, \Re) : \nu(t) \in \Lambda\bigl(t, \flat(t)\bigr) \text{ for a.e. } t \in I\bigr\}.$$

We assume that $\Omega^1_\Lambda(\flat)$ is nonempty and closed.

Definition 5.1 A function $\flat \in AC(I, \Re)$ is called a solution to (6) if there is $\nu \in \Omega^1_\Lambda(\flat)$ such that $\flat'(t) = \nu(t) + g(t)$ for a.e. $t \in I$ and $\flat(0) = \alpha$. On the other hand:

(i) a function $\flat_L \in AC(I, \Re)$ is said to be a lower solution of (6) if there is a $\nu_L \in \Omega^1_\Lambda(\flat_L)$ such that $\flat'_L(t) \le \nu_L(t) + g(t)$ for a.e. $t \in I$ and $\flat_L(0) \le \alpha$;

(ii) a function $\flat_U \in AC(I, \Re)$ is said to be an upper solution of (6) if there is a $\nu_U \in \Omega^1_\Lambda(\flat_U)$ such that $\flat'_U(t) \ge \nu_U(t) + g(t)$ for a.e. $t \in I$ and $\flat_U(0) \ge \alpha$.

Theorem 5.1 *Suppose that the following conditions are satisfied*:

(a) $\Omega^1_\Lambda(\flat) \ne \emptyset$ *for all* $\flat \in C(I, \Re)$;

(b) *if* $\flat \in C(I,\Re)$ *is fixed and* $\{\nu_n\}$ *is a sequence in* $\Omega^1_\Lambda(\flat)$*, then there exists a subsequence* $\{\nu_{n_i}\}$ *of* $\{\nu_n\}$*, which converges at almost every point* $t \in I$ *to a* $\nu \in L^1(I,\Re)$ *such that* $\int_0^t \nu_{n_i}(s)\,ds \to \int_0^t \nu(s)\,ds$*, as* $i \to \infty$ *for every* $t \in I$;

(c) $\Lambda(t,\flat)$ *is closed for all* $(t,\flat) \in I \times \Re$ *and for each fixed* $\flat \in C(I,\Re)$ *we have* $\Lambda(\cdot,\flat(\cdot))$ *is bounded on* I;

(d) *there exist* $0 \le \mu < 1$ *and* $r > 0$ *such that, if* $\flat_1 \in C(I,\Re)$, $\flat_2 \in B[\flat_1, r]$ *with* $\flat_1(t) \le \flat_2(t)$ *for all* $t \in I$*, then*

$$\left|\nu_{\flat_1}(t) - \nu_{\flat_2}(t)\right| \le \frac{\mu}{\varpi}\left[\flat_2(t) - \flat_1(t)\right], \quad \textit{for a.e. } t \in I \tag{7}$$

for all $\nu_{\flat_1} \in \Omega^1_\Lambda(\flat_1)$, $\nu_{\flat_2} \in \Omega^1_\Lambda(\flat_2)$;

(e) *if* $\flat_1 \in C(I,\Re)$, $\flat_2 \in B[\flat_1, r]$ *with* $\flat_1(t) \le \flat_2(t)$ *for all* $t \in I$*, then*

$$\nu_{\flat_2}(t) - \nu_{\flat_1}(t) \ge 0, \quad \textit{for a.e. } t \in I, \tag{8}$$

for all $\nu_{\flat_1} \in \Omega^1_\Lambda(\flat_1)$, $\nu_{\flat_2} \in \Omega^1_\Lambda(\flat_2)$ *with* $d(\int_0^t \nu_{\flat_1}(s)\,ds, \int_0^t \nu_{\flat_2}(s)\,ds) \le d(\flat_1,\flat_2)$.

If there exists a lower solution (or an upper solution) $\flat_L$ *(or* $\flat_U$*) of the differential inclusion* (6) *such that* $d(\flat_L, u) \le r$ *whenever* $u = \alpha + \int_0^t [\nu(s) + g(s)]\,ds$, $t \in I$, $\nu \in \Omega^1_\Lambda(\flat_L)$ *(or* $d(\flat_U, u) \le r$ *whenever* $u = \alpha + \int_0^t [\nu(s) + g(s)]\,ds$, $t \in I$, $\nu \in \Omega^1_\Lambda(\flat_U)$*), then there exists a solution of* (6).

Proof Let ß $= C(I,\Re)$ (complete metric space with the supremum norm) and define a set3-valued operator Υ on ß by

$$\Upsilon(\flat) = \left\{u \in ß\colon u(t) = \alpha + \int_0^t \left[\nu(s) + g(s)\right] ds, t \in I, \nu \in \Omega^1_\Lambda(\flat)\right\} \quad \text{for all } \flat \in ß.$$

Then, Υ is well-defined, since $\Omega^1_\Lambda(\flat) \ne \emptyset$ for all $\flat \in$ ß. Now, it is obvious that the differential inclusion (6) is equivalent to the following inclusion:

$$\flat(t) \in \Upsilon\big(\flat(t)\big), \quad t \in I,$$

i.e., $\flat \in \Upsilon(\flat)$. Therefore, the solutions to the differential inclusion (6) are the fixed points of the operator Υ. Define a binary relation $\mathcal{R}$ on ß by:

$$\mathcal{R} = \left\{(\flat_1,\flat_2) \in ß^2\colon \flat_2 \in B[\flat_1, r], \flat_1 \le \flat_2\right\},$$

where $\flat_1 \le \flat_2$ means $\flat_1(t) \le \flat_2(t)$ for all $t \in I$.

We observe that ß, Υ and $\mathcal{R}$ meet all the conditions of Theorem 3.1 with $k = 1$. In fact:

(I) $\Upsilon\colon ß \to CB(ß)$.

Note that, $\Omega^1_\Lambda(\flat) \ne \emptyset$ for all $\flat \in$ ß, this shows that $\Upsilon(\flat) \ne \emptyset$ for all $\flat \in$ ß. We show that $\Upsilon(\flat)$ is closed and bounded in ß for each $\flat \in$ ß. Suppose that $\flat \in$ ß is fixed $\{u_n\}$ be a sequence in $\Upsilon(\flat)$ and $u_n \to u \in$ ß. We show that $u \in \Upsilon(\flat)$. Then, by definition, there is a sequence $\{\nu_n\}$ in $\Omega^1_\Lambda(\flat)$ with

$$u_n(t) = \alpha + \int_0^t \left[\nu_n(s) + g(s)\right] ds, \quad t \in I.$$

By (b) there is a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that $\{v_{n_i}\}$ converges at almost every point $t \in I$ towards $v \in L^1(I, \Re)$ as $i \to \infty$ and $\int_0^t [v_{n_i}(s) + g(s)]\,ds \to \int_0^t [v(s) + g(s)]\,ds$, as $i \to \infty$ for every $t \in I$.

Since each v_{n_i} is in $\Lambda(t, \flat(t))$ which is closed for all $t \in I$, hence $v(t) \in \Lambda(t, \flat(t))$ for a.e. $t \in I$ which with the fact $v \in L^1(I, \Re)$ shows that $v \in \Omega^1_\Lambda(\flat)$. Moreover,

$$u(t) = \lim_{i\to\infty} u_{n_i}(t) = \alpha + \int_0^t \bigl[v(s) + g(s)\bigr]\,ds, \quad \text{for all } t \in I.$$

This shows that $u \in \Upsilon(\flat)$ and so $\Upsilon(\flat)$ is closed in ß.

Further, since $\Lambda(\cdot, \flat(\cdot))$ is bounded on I for each fixed $\flat \in$ ß, there exists a number $M > 0$ such that $|v(t)| \le M$ for a.e. $t \in I$. Since $g\colon I \to \Re$ is continuous, hence for all $u \in \Upsilon(\flat)$ we have

$$\begin{aligned}
\sup_{t\in I}\bigl|u(t)\bigr| &\le |\alpha| + \sup_{t\in I}\int_0^t \bigl|v(s) + g(s)\bigr|\,ds \\
&\le |\alpha| + \int_0^t [M + g_{\sup}]\,ds \\
&\le |\alpha| + [M + g_{\sup}]\varpi,
\end{aligned}$$

where $g_{\sup} = \sup_{t\in I} g(t)$. Therefore, Υ has bounded values in ß.

(II) *Υ is a set-valued $\mathcal{R}PC$-operator.*

Let $\flat_1, \flat_2 \in$ ß with $(\flat_1, \flat_2) \in \mathcal{R}$, then we have $\flat_2 \in B[\flat_1, r]$ and $\flat_1(t) \le \flat_2(t)$ for all $t \in I$. Then, we have

$$\begin{aligned}
D\bigl(\Upsilon(\flat_1), \Upsilon(\flat_2)\bigr) &= \sup_{u_{\flat_1}\in\Upsilon(\flat_1)} d\bigl(u_{\flat_1}, \Upsilon(\flat_2)\bigr) \\
&= \sup_{u_{\flat_1}\in\Upsilon(\flat_1)} \inf_{u_{\flat_2}\in\Upsilon(\flat_2)} d(u_{\flat_1}, u_{\flat_2}) \\
&= \sup_{u_{\flat_1}\in\Upsilon(\flat_1)} \inf_{u_{\flat_2}\in\Upsilon(\flat_2)} \sup_{t\in I}\bigl|u_{\flat_1}(t) - u_{\flat_2}(t)\bigr| \\
&\le \sup_{u_{\flat_1}\in\Upsilon(\flat_1)} \inf_{u_{\flat_2}\in\Upsilon(\flat_2)} \sup_{t\in I}\int_0^t \bigl|v_{\flat_1}(s) - v_{\flat_2}(s)\bigr|\,ds,
\end{aligned}$$

for some $v_{\flat_1} \in \Omega^1_\Lambda(\flat_1)$, $v_{\flat_2} \in \Omega^1_\Lambda(\flat_2)$. By use of (7) the above inequality yields:

$$\begin{aligned}
D\bigl(\Upsilon(\flat_1), \Upsilon(\flat_2)\bigr) &\le \frac{\mu}{\varpi} \sup_{u_{\flat_1}\in\Upsilon(\flat_1)} \inf_{u_{\flat_2}\in\Upsilon(\flat_2)} \sup_{t\in I}\int_0^t \bigl|\flat_1(s) - \flat_2(s)\bigr|\,ds \\
&\le \frac{\mu}{\varpi} d(\flat_1, \flat_2) \sup_{u_{\flat_1}\in\Upsilon(\flat_1)} \inf_{u_{\flat_2}\in\Upsilon(\flat_2)} \sup_{t\in I}\int_0^t ds \\
&= \mu d(\flat_1, \flat_2).
\end{aligned}$$

By a similar calculation, one can obtain $D(\Upsilon(\flat_2), \Upsilon(\flat_1)) \le \mu d(\flat_1, \flat_2)$. Hence,

$$\begin{aligned}
H\bigl(\Upsilon(\flat_1), \Upsilon(\flat_2)\bigr) &= \max\bigl\{D\bigl(\Upsilon(\flat_1), \Upsilon(\flat_2)\bigr), D\bigl(\Upsilon(\flat_2), \Upsilon(\flat_1)\bigr)\bigr\} \\
&\le \mu d(\flat_1, \flat_2).
\end{aligned}$$

Hence, $(\mathcal{R}PC1)$ is satisfied. Now suppose that $(\flat_1,\flat_2)\in\mathcal{R}$, $\flat_2\in\Upsilon(\flat_1)$ and $\flat_3\in\Upsilon(\flat_2)$. Then, we have $\flat_2\in B[\flat_1,r]$ and $\flat_1(t)\leq\flat_2(t)$ for all $t\in I$. Hence, by definition, there exist $\nu_{\flat_1}\in\Omega^1_\Lambda(\flat_1)$ and $\nu_{\flat_2}\in\Omega^1_\Lambda(\flat_2)$ such that

$$\flat_2(t)=\alpha+\int_0^t\left[\nu_{\flat_1}(s)+g(s)\right]ds \quad\text{and}\quad \flat_3(t)=\alpha+\int_0^t\left[\nu_{\flat_2}(s)+g(s)\right]ds,\quad t\in I.$$

Therefore, using (d), we obtain:

$$\begin{aligned}
d(\flat_2,\flat_3) &= d\left(\alpha+\int_0^t\left[\nu_{\flat_1}(s)+g(s)\right]ds,\alpha+\int_0^t\left[\nu_{\flat_2}(s)+g(s)\right]ds\right)\\
&= d\left(\int_0^t\nu_{\flat_1}(s)\,ds,\int_0^t\nu_{\flat_2}(s)\,ds\right)\\
&\leq\sup_{t\in I}\int_0^t\left|\nu_{\flat_2}(s)-\nu_{\flat_1}(s)\right|ds\\
&\leq\frac{\mu}{\varpi}\sup_{t\in I}\int_0^t\left[\flat_2(s)-\flat_1(s)\right]ds\\
&\leq\frac{\mu}{\varpi}d(\flat_1,\flat_2)\int_0^t ds\\
&\leq\mu d(\flat_1,\flat_2)\\
&< d(\flat_1,\flat_2)\leq r.
\end{aligned}$$

Hence, we have $\flat_3\in B[\flat_2,r]$ and with the help of property (e) we obtain:

$$\flat_3(t)-\flat_2(t)=\int_0^t\left[\nu_{\flat_2}(s)-\nu_{\flat_1}(s)\right]ds\geq 0,\quad t\in I.$$

Hence, $(\flat_2,\flat_3)\in\mathcal{R}$, and so, $(\mathcal{R}PC2)$ is satisfied.

(III) *Condition (A) of Theorem 3.1 is satisfied.* By hypothesis, there exists a lower solution $\flat_{\mathrm{L}}$ of the differential inclusion (6) such that $d(\flat_{\mathrm{L}},u)\leq r$ for all $u\in\Upsilon(\flat_{\mathrm{L}})$. Hence, there exists $\nu_{\mathrm{L}}\in\Omega^1_\Lambda(\flat_{\mathrm{L}})$ such that $\flat'_{\mathrm{L}}(t)\leq\nu_{\mathrm{L}}(t)+g(t)$ for a.e. $t\in I$, $\flat_{\mathrm{L}}(0)\leq\alpha$. This shows that

$$\begin{aligned}
\flat_{\mathrm{L}}(t) &\leq \flat_{\mathrm{L}}(0)+\int_0^t\left[\nu_{\mathrm{L}}(s)+g(s)\right]ds\\
&\leq\alpha+\int_0^t\left[\nu_{\mathrm{L}}(s)+g(s)\right]ds=:u_{\mathrm{L}}(t),\quad t\in I,
\end{aligned}$$

and $u_{\mathrm{L}}\in\Upsilon(\flat_{\mathrm{L}})$. Further, since $u_{\mathrm{L}}\in\Upsilon(\flat_{\mathrm{L}})$ by hypothesis we have $d(\flat_{\mathrm{L}},u_{\mathrm{L}})\leq r$, i.e., $u_{\mathrm{L}}\in B[\flat_{\mathrm{L}},r]$. Hence, $(\flat_{\mathrm{L}},u_{\mathrm{L}})\in\mathcal{R}$ and $u_{\mathrm{L}}\in\Upsilon(\flat_{\mathrm{L}})$. So, the condition (A) of Theorem 3.1 is satisfied.

(IV) *Condition (B') of Remark 2 is satisfied.* Suppose that $\{\flat_n\}$ is a termwise related trajectory of Υ that converges to $\flat\in ß$, then $\flat_{n+1}\in\Upsilon(\flat_n)$ and $\flat_n\leq\flat_{n+1}$ for all $n\in\aleph$. By definition of $\Upsilon(\flat_n)$, there exists a sequence $\{\nu_n\}$ in $\Omega^1_\Lambda(\flat_n)$ such that

$$\flat_{n+1}=\alpha+\int_0^t\left[\nu_n(s)+g(s)\right]ds$$

and by condition (b) there exists a subsequence $\{\flat_{n_i}\}$ that converges to some $\nu \in L^1(I,\Re)$ with $\int_0^t \nu_{n_i}(s)\,ds \to \int_0^t \nu(s)\,ds$, as $i \to \infty$ for every $t \in I$. Therefore, $\flat = \alpha + \int_0^t [\nu(s) + g(s)]\,ds$ and we can find $i_0 \in \aleph$ such that $d(\nu_{n_i}, \nu) \leq \frac{r}{\varpi}$ for all $i > i_0$. Also, since $\flat_n \leq \flat_{n+1}$ for all $n \in \aleph$, we must have $\flat_{\nu_i} \leq \flat$ for all $i \in \aleph$. Finally, for all $i > i_0$ we have:

$$\begin{aligned} d(\flat_{n_i}, \flat) &= \sup_{t\in I} \left|\flat_{n_i}(t) - \flat(t)\right| \\ &\leq \sup_{t\in I} \int_0^t \left|\nu_{n_i}(t) - \nu(t)\right| ds \\ &\leq d(\nu_{n_i}, \nu) \sup_{t\in I} \int_0^t ds \\ &\leq r \end{aligned}$$

Hence, $(\flat_{n_i}, \flat) \in \mathcal{R}$ for all $i > i_0$, and so, the condition (B') of Remark 2 is also satisfied for $k = 1$.

Therefore, the existence of a fixed point of Υ, i.e., a solution to the differential inclusion (6) follows by Theorem 3.1 and Remark 2.

On the other hand, it is easy to see that a similar conclusion can be drawn in case of existence of upper solution of (6) by choosing the relation $\mathcal{R}$ accordingly. □

The next example validate and illustrate the above theorem.

Example 3 Consider the following differential inclusion problem:

$$\begin{aligned} &\flat'(t) \in \Lambda\big(t, \flat(t)\big) + \sin t \quad \text{for a.e. } t \in I = [0,1], \\ &\flat(0) = 1, \end{aligned} \tag{9}$$

where $\Lambda : I \times \Re \to \mathcal{P}(\Re)$ is given by

$$\Lambda\big(t, \flat(t)\big) = \begin{cases} \{x\flat, y\flat\}, & \text{if } \flat \in \mathbb{Q}; \\ \{x\flat\}, & \text{otherwise} \end{cases}$$

and $0 < x < 1$, $y \in \Re$ are fixed real numbers. Then, all the conditions of Theorem 5.1 are satisfied with $\varpi = 1$, $g(t) = \sin t$, $t \in I$, $r = e^x$ and $\mu = x$. In fact, we have:

(a) If $\nu(t) = x\flat$ for all $t \in I$, then ν is a Lebesgue integrable selection of $\Lambda(\cdot, \flat(\cdot))$. Therefore, $\Omega^1_\Lambda(\flat) \neq \emptyset$ for all $\flat \in C(I,\Re)$.

(b) For any fixed $\flat \in C(I,\Re)$ if $\{\nu_n\}$ is a sequence in $\Omega^1_\Lambda(\flat)$, then by definition of $\Omega^1_\Lambda(\flat)$ there exists a subsequence $\{\nu_{n_i}\} = \{x\flat\}$ of $\{\nu_n\}$ which converges at almost every point $t \in I$ to $\nu = x\flat \in L^1(I,\Re)$ such that $\int_0^t \nu_{n_i}(s)\,ds \to \int_0^t \nu(s)\,ds$, as $i \to \infty$ for every $t \in I$.

(c) By definition, $\Lambda(t, \flat)$ is closed for all $(t, \flat) \in I \times \Re$ and for each fixed $\flat \in C(I,\Re)$ we have $\Lambda(\cdot, \flat(\cdot))$ is bounded on I;

(d) Here, $\varpi = 1$. If $\flat_1, \flat_2 \in C(I,\Re)$, $\flat_2 \in B[\flat_1, e^x]$ with $\flat_1(t) \leq \flat_2(t)$ for all $t \in I$ and $\nu_{\flat_1} \in \Omega^1_\Lambda(\flat_1)$, $\nu_{\flat_2} \in \Omega^1_\Lambda(\flat_2)$, then for $0 < \mu = x < 1$ and $r = e^x > 0$ we have

$$\left|\nu_{\flat_1}(t) - \nu_{\flat_2}(t)\right| \leq \frac{\mu}{\varpi}\big[\flat_2(t) - \flat_1(t)\big], \quad \text{for a.e. } t \in I.$$

(e) If $\flat_1 \in C(I,\Re)$, $\flat_2 \in B[\flat_1, e^x]$ with $\flat_1(t) \leq \flat_2(t)$ for all $t \in I$, then

$$\nu_{\flat_2}(t) - \nu_{\flat_1}(t) = x\flat_2 - x\flat_1 \geq 0, \quad \text{for a.e. } t \in I,$$

for all $\nu_{\flat_1} \in \Omega^1_\Lambda(\flat_1)$, $\nu_{\flat_2} \in \Omega^1_\Lambda(\flat_2)$ with $d(\int_0^t \nu_{\flat_1}(s)\,ds, \int_0^t \nu_{\flat_2}(s)\,ds) \leq d(\flat_1,\flat_2)$.

Observe that $\flat_L(t) = e^{xt}$ for all $t \in [0,1]$ is a lower solution of inclusion problem (9), as $\flat_L'(t) = xe^{xt} \leq e^{xt} + \sin t$ for a.e. $t \in [0,1]$ and $\flat_L(0) = 1$. Also $d(\flat_L, u) \leq r = e^x$ whenever $u = 1 + \int_0^t [\nu(s) + \sin s]\,ds$, $t \in I$, $\nu \in \Omega^1_\Lambda(\flat_{\mathrm{L}})$. Hence, by Theorem 5.1, there exists a solution of (9).

5.2 Application to equilibrium point of a difference inclusion

We now consider the problem of obtaining an equilibrium point of k-th order nonlinear difference inclusions and its weak stability and global attractivity.

Suppose that Θ is a subset of a real Banach space ß and $\|\cdot\|$ denotes the norm on ß. Let $\Upsilon: \Theta^k \to \mathcal{P}(\Theta)$ be a mapping, with nonempty values. Let $\flat_1, \flat_2, \dots, \flat_k \in \Theta$ and consider the k-th order nonlinear difference inclusion on Θ:

$$\flat_{n+k} \in \Upsilon(\flat_n, \flat_{n+1}, \dots, \flat_{n+k-1}), \quad n = 1, 2, \dots. \tag{10}$$

A function $\tau: \aleph \to \Theta$ is called a solution of (10) (see, [9]) if for every $n \in \aleph$, $\tau(n+k) = \tau_{n+k} \in \Upsilon(\tau_n, \tau_{n+1}, \dots, \tau_{n+k-1})$. We say that $\flat \in \Theta$ is an equilibrium point of (10) if, $\flat \in \Upsilon(\flat, \flat, \dots, \flat)$. Obviously, an equilibrium point of (10) is a fixed point of Υ and vice-versa. An equilibrium point $\flat \in \Theta$ of (10) is said to be weakly stable if, given $\varepsilon > 0$, there exists $\delta > 0$ such that for at least one solution of (10) with initial values $\flat_1, \flat_2, \dots, \flat_k$ and $\|\flat_1 - \flat\| + \|\flat_2 - \flat\| + \cdots + \|\flat_k - \flat\| < \delta$ implies $\|\flat_n - \flat\| < \varepsilon$ for all $n \in \aleph$. While the equilibrium point $\flat$ is said to be weakly asymptotically stable if it is weakly stable and $\lim_{n\to\infty} \flat_n = \flat$, and $\flat$ is said to be global attractor if for every $\flat_1, \flat_2, \dots, \flat_k \in \Theta$ we have $\lim_{n\to\infty} \flat_n = \flat$.

Theorem 5.2 *If Θ is a closed subset of ß and $\Upsilon: \Theta^k \to CB(\Theta)$ satisfies the following condition*:

$$H\big(\Upsilon(\flat_1, \flat_2, \dots, \flat_k), \Upsilon(\flat_2, \flat_3, \dots, \flat_{k+1})\big) \leq \varrho \max_{1\leq i\leq k} \big\{d(\flat_i, \flat_{i+1})\big\}$$

for all $\flat_1, \flat_2, \dots, \flat_{k+1} \in$ ß, where $0 \leq \varrho < 1$. Then for every set of initial conditions $\flat_1, \flat_2, \dots, \flat_k \in \Theta$ the difference inclusion (10) has an equilibrium point $\flat \in \Theta$. Furthermore, the equilibrium point $\flat$ is weakly asymptotically stable and a global attractor.

Proof By Corollary 4.2, the set-valued mapping Υ has a fixed point in Θ, which is an equilibrium point of (10). Also, by the method used in the proof of Corollary 4.2, we observe that for all $\flat_1, \flat_2, \dots, \flat_k \in \Theta$ the trajectory $\{\flat_n\}$ converges to $\flat$, therefore, $\flat$ is weakly asymptotically stable and a global attractor. □

Example 4 Consider the real Banach space $\Re$ with usual norm, and the following nonlinear difference inclusion of order 2:

$$\flat_{n+2} \in \left[0, \frac{\sigma + \flat_{n+1}^2}{\varsigma + \flat_n^2}\right], \tag{11}$$

where $0 \leq \sigma \leq \varsigma$ and $\varsigma > 2$. Then the difference inclusion (11) has an equilibrium point in $\flat \in [0, 1/2]$. Furthermore, the equilibrium point $\flat$ is weakly asymptotically stable and a global attractor.

Proof Let $\text{ß} = \Re$ with the usual distance and $\Theta = I = [0, 1/2]$ and define a set-valued mapping $\Upsilon : I^2 \to CB(I)$ by:

$$\Upsilon(\flat_1, \flat_2) = \left[0, \frac{\sigma + \flat_2^2}{\varsigma + \flat_1^2}\right] \quad \text{for all } \flat_1, \flat_2 \in I.$$

We show that Υ satisfies the contractive condition of Theorem 5.2. Then, for all $\flat_1, \flat_2, \flat_3 \in I$ we have:

$$\begin{aligned} H\big(\Upsilon(\flat_1, \flat_2), \Upsilon(\flat_2, \flat_3)\big) &= \left|\frac{\sigma + \flat_2^2}{\varsigma + \flat_1^2} - \frac{\sigma + \flat_3^2}{\varsigma + \flat_2^2}\right| \\ &= \left|\frac{(\sigma + \flat_2^2)(\flat_2^2 - \flat_1^2) + (\varsigma + \flat_1^2)(\flat_2^2 - \flat_3^2)}{(\varsigma + \flat_1^2)(\varsigma + \flat_2^2)}\right| \\ &\leq \frac{\flat_1 + \flat_2}{\varsigma + \flat_1^2}|\flat_1 - \flat_2| + \frac{\flat_2 + \flat_3}{\varsigma + \flat_2^2}|\flat_2 - \flat_3| \\ &\leq \frac{2}{\varsigma}\max\big\{|\flat_1 - \flat_2|, |\flat_2 - \flat_3|\big\} \\ &\leq \varrho \max\big\{|\flat_1 - \flat_2|, |\flat_2 - \flat_3|\big\}, \end{aligned}$$

where $\varrho \leq \frac{2}{\varsigma} < 1$. Hence, Υ satisfies the contractive condition of Theorem 5.2. Therefore, by Theorem 5.2, the difference inclusion (11) has an equilibrium point $\flat_{\sigma,\varsigma} \in I$ which is weakly asymptotically stable and a global attractor. Indeed, the global attractor $\flat_{\sigma,\varsigma}$ will depend on σ and ς and $\inf_{\sigma\in[0,\varsigma]} \flat_{\sigma,\varsigma} = 0$. □

6 Conclusion

In this paper, we have considered a relation-theoretic set-valued generalization of the result of Prešić and established the existence of fixed point of mappings under suitable assumptions. The main result of this paper is a unification and generalization of the results of [2, 3, 10, 12]. More importantly, this unification permits us to use the techniques of [2] and [3] with weaker contractive conditions. The results have been successfully applied to the differential inclusion problems and to obtain the equilibrium points of difference inclusions. The result proved here can be further generalized and extended, e.g., more weaker types of contractive conditions can be used, more than one mapping can be involved in the contractive conditions, so that the scope and applicability of the results can be broaden. We point out that the contractive condition and the method of proof used here may not be compatible with the structure of a graphical metric spaces (see, [27]). It is worth investigating under which assumption(s) the result of this paper can be further generalized in the generalized setting of graphical metric spaces.

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