Li *Boundary Value Problems* (2023) 2023:60
https://doi.org/10.1186/s13661-023-01749-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Uniform stability of a strong time-delayed viscoelastic system with Balakrishnan–Taylor damping

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Abstract

This paper studies a Balakrishnan–Taylor viscoelastic wave equation with strong time-dependent delay. Under suitable assumptions on the coefficients of the delay term, we establish a generalized stability result, which improve some earlier results in the literature.

MSC: 35B35; 35B40; 93D15

Keywords: Stability; Memory; Balakrishnan–Taylor damping; Time delay

1 Introduction

This paper studies a Balakrishnan–Taylor wave equation with memory and a strong time-dependent delay

$$\begin{cases} v_{tt} - (a + b\|\nabla v\|^2 + \sigma(\nabla v, \nabla v_t))\Delta v + \int_0^t k(t-s)\Delta v(s)\,ds \\ \quad - \mu_1 \Delta v_t(t) - \mu_2 \Delta v_t(t-\tau(t)) = 0, \quad x \in \Omega, t > 0, & (1.1) \\ v = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, & (1.2) \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad x \in \Omega, & (1.3) \\ v_t(x,t) = f_0(x,t), \quad x \in \Omega, t \in [-\tau(0), 0), & (1.4) \end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$ ($n \geq 1$) is a bounded domain with smooth boundary $\partial\Omega$. The term $(a + b\|\nabla v\|^2 + \sigma(\nabla v, \nabla v_t))$ represents the nonlinear stiffness of the membrane; μ_1, μ_2 are two constants. The function $k(t)$ is often called the kernel or relaxation function. $\tau(t) > 0$, which is dependent on time t, is the time delay. System (1.1)–(1.4) is related to the panel flutter equation with memory term and time delay control from the physical point of view.

Balakrishnan and Taylor [4] first introduced Balakrishnan–Taylor damping $\sigma(\nabla v, \nabla v_t)$; see also Bass and Zes [5]. If $\mu_2 = \sigma = 0$, the system, which is called Kirchhoff-type equation, has been well studied. Generally speaking, the wave equation with Balakrishnan–Taylor damping is given by

$$v_{tt} - \left(a + b\|\nabla v\|^2 + \sigma(\nabla v, \nabla v_t)\right)\Delta v + g(v_t) = f(v). \tag{1.5}$$

In the absence of damping term and $f(v) = |v|^p v$, the global existence and polynomial decay of energy were obtained by Zarai and Tatar, see [35]. In [36], an exponential decay and the blow up of solutions were established. If $f(u) = 0$, Park [34] obtained a general decay rate of solutions. Ha [14] studied (1.5) with a memory term, and a general decay result of energy was proved, which did not impose any restrictive growth assumption on the damping term. We can find more results concerning wave equation with Balakrishnan–Taylor damping in Clark [7, 11, 12, 15, 18, 37, 38, 41, 42], and so on.

The delay effects can be regarded as a source of instability. There are so many results on wave equation with weak time delay effects; see, for example, Datko et al. [9], Nicaise et al. [30–33], Xu et al. [39], and so on. For the wave equation with a memory term and weak time delay,

$$v_{tt} - \Delta v + h * \Delta v + \mu_1 g_1\left(v_t(t)\right) + \mu_2 g_2\left(v_t(t-\tau)\right) = 0, \tag{1.6}$$

if g_1 and g_2 are linear, the stability was established in [8, 19, 22, 23], etc. Benaissa, Benguessoum, and Messaoudi [6] considered (1.6) to prove a general decay of energy by assuming g_2 is linear-like. Regarding a wave equation with Balakrishnan–Taylor damping and weak time delay,

$$\begin{aligned} &v_{tt} - \left(a + b\|\nabla v\|^2 + \sigma(\nabla v, \nabla v_t)\right)\Delta v + h * \Delta v \\ &\quad + \mu_1 g_1\left(v_t(t)\right) + \mu_2 g_2\left(v_t(t-\tau)\right) = 0, \end{aligned} \tag{1.7}$$

when g_1 and g_2 are linear, one can find some stability results in Lee et al. [20, 21] and Liu et al. [24], and so on. Kang et al. [17] studied the general equation (1.7) and obtained a general decay result following some properties of convex functions introduced in [1–3]. Gheraibia and Boumaza [13] established a general decay rate by assuming $h'(t) \le -\xi(t)h(t)$ for the case of $g_1(s) = g_2(s) = |s|^{m-2}s$.

Concerning the wave equation with a strong time delay, in [29], Messaoudi et al. first introduced a wave equation with strong time delay of the form

$$v_{tt} - \Delta v - \mu_1 \Delta v_t - \mu_2 \Delta v_t(t-\tau) = 0, \tag{1.8}$$

studied the stability of the problem. Feng [10] considered (1.8) with viscoelastic damping. The author obtained a general decay rate of solution. With respect to viscoelastic delayed wave equation with Balakrishnan–Taylor damping, Hao and Wei [16] studied the case of the system with a weak time delay, and energy decay was established by assuming the relaxation function k such that $k'(t) \le -\zeta(t)k(t)$. Using the same assumption on the relaxation function, Yoon et al. [40] proved the general decay of a viscoelastic Kirchhoff Balakrishnan–Taylor equation with nonlinear delay and acoustic boundary conditions. Our goal in this paper is to study the energy decay of a wave equation with Balakrishnan–Taylor damping and strong time delay, i.e., problem (1.1)–(1.4) by considering a more assumption on relaxation function k:

$$k'(t) \le -\zeta(t)k^q(t), \quad 1 \le q < 3/2,$$

which is more general than considered in earlier papers. Hence our result improves and generalizes earlier results in the literature.

In Sect. 2, we give some preliminaries. The general decay result is established in Sect. 3.

2 Preliminaries

First we state some assumptions used in this paper.

We assume $k : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing differentiable function satisfying

$$k(0) > 0, \quad a - \int_0^\infty k(s)\,ds = l > 0, \tag{2.1}$$

and there exists a nonincreasing differentiable function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that for $t \ge 0$,

$$\zeta(t) > 0, \quad k'(t) \le -\zeta(t)k^p(t), \quad 1 \le p < \frac{3}{2}, \tag{2.2}$$

and

$$\int_0^\infty \zeta(t)\,dt = \infty.$$

For the delay $\tau(t)$, there exist constants $\tau_0 > 0$ and $\tau_1 > 0$ such that

$$\tau_0 \le \tau(t) \le \tau_1, \quad \forall t > 0. \tag{2.3}$$

In addition,

$$\tau(t) \in W^{2,\infty}(0,T) \quad \text{and} \quad \tau'(t) \le d < 1, \quad \forall T, t > 0. \tag{2.4}$$

The existence and uniqueness of problem (1.1)–(1.4), which can be proved by using the Faedo–Galerkin method, are given in the theorem below, see, for example, [35, 41].

Theorem 2.1 *Let* (2.1)–(2.4) *hold. Let* $\mu_2 \le \mu_1$, *and* $(v_0, v_1) \in (H^1(\Omega) \times L^2(\Omega))$, $f_0 \in H_0^1(\Omega \times (-\tau(0), 0))$. *Then system* (1.1)–(1.4) *admits only one weak solution* $(v, v_t) \in C(0,T; H_0^1(\Omega) \times L^2(\Omega))$ *such that for any* $T > 0$,

$$v \in L^\infty\big(0,T; H_0^1(\Omega)\big), \qquad v_t \in L^\infty\big(0,T; L^2(\Omega)\big).$$

The total energy is defined by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\|v_t(t)\|^2 + \frac{b}{4}\|\nabla v(t)\|^4 + \frac{1}{2}\left(a - \int_0^t k(s)\,ds\right)\|\nabla v(t)\|^2 + \frac{1}{2}(k \circ \nabla v)(t) \\ & + \frac{\xi}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\|\nabla v_t(s)\|^2\,ds, \end{aligned} \tag{2.5}$$

where ξ is a positive constant and the constant $\lambda > 0$ satisfies, see [32],

$$\lambda < \frac{1}{\tau_1}\left|\log\frac{|\mu_2|}{\sqrt{1-d}}\right|$$

and

$$(k \circ v)(t) = \int_0^t k(t-s)\|v(t)-v(s)\|^2\,ds.$$

Lemma 2.1 *Let $|\mu_2| < \sqrt{1-d}\mu_1$. For any $t \geq 0$, we have*

$$\begin{aligned} E'(t) \leq{}& \frac{1}{2}(k' \circ \nabla v)(t) - \frac{1}{2}k(t)\|\nabla v(t)\|^2 + \left(\frac{\xi}{2} - \mu_1 + \frac{|\mu_2|}{2\sqrt{1-d}}\right)\|\nabla v_t(t)\|^2 \\ &+ \left[\frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\xi}{2}(1-d)e^{-\lambda\tau_1}\right]\|\nabla v_t(t-\tau(t))\|^2 \\ &- \sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|^2\right)^2 \\ &- \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\|\nabla v_t(s)\|^2\,ds \\ \leq{}& 0. \end{aligned} \tag{2.6}$$

Proof First,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[(k \circ \nabla v)(t) - \left(\int_0^t k(s)\,ds\right)\|\nabla v(t)\|^2\right] \\ &\quad= \frac{1}{2}(k' \circ \nabla v)(t) - \frac{1}{2}k(t)\|\nabla v(t)\|^2 - \int_\Omega \nabla v_t(t)\int_0^t k(t-s)\nabla v(s)\,ds\,dx. \end{aligned} \tag{2.7}$$

Differentiating (2.5), we have

$$\begin{aligned} E'(t) ={}& \int_\Omega v_t(t)v_{tt}(t)\,dx - \frac{1}{2}k(t)\|\nabla v(t)\|^2 + \left(a - \int_0^t k(s)\,ds\right)\int_\Omega \nabla v(t)\cdot\nabla v_t(t)\,dx \\ &+ b\|\nabla v(t)\|^2\int_\Omega \nabla v(t)\nabla v_t(t)\,dx + \frac{1}{2}\frac{d}{dt}(k \circ \nabla v)(t) \\ &+ \frac{\xi}{2}\|\nabla v_t(t)\|^2 - \frac{\xi}{2}e^{-\lambda\tau(t)}\left(1-\tau'(t)\right)\|\nabla v_t(t-\tau(t))\|^2 \\ &- \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\|\nabla v_t(s)\|^2\,ds. \end{aligned}$$

Then it is obtained by using (1.1) that

$$\begin{aligned} E'(t) ={}& \frac{1}{2}(k' \circ \nabla v)(t) - \frac{1}{2}k(t)\|\nabla v(t)\|^2 - \sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|^2\right)^2 - \mu_1\|\nabla v_t(t)\|^2 \\ &+ \frac{\xi}{2}\|\nabla v_t(t)\|^2 - \mu_2\int_\Omega \nabla v_t(t)\cdot\nabla v_t(t-\tau(t))\,dx \\ &- \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\|\nabla v_t(s)\|^2\,ds - \frac{\xi}{2}e^{-\lambda\tau(t)}\left(1-\tau'(t)\right)\|\nabla v_t(t-\tau(t))\|^2, \end{aligned}$$

which, together with (2.7) and (2.3)–(2.4), implies

$$\begin{aligned} E'(t) \le{}& \frac{1}{2}\big(k' \circ \nabla v\big)(t) - \frac{1}{2}k(t)\big\|\nabla v(t)\big\|^2 - \sigma\left(\frac{1}{2}\frac{d}{dt}\big\|\nabla v(t)\big\|^2\right)^2 - \mu_1\big\|\nabla v_t(t)\big\|^2 \\ &+ \frac{\xi}{2}\big\|\nabla v_t(t)\big\|^2 - \mu_2\int_\Omega \nabla v_t(t)\cdot\nabla v_t\big(t-\tau(t)\big)\,dx \\ &- \frac{\xi}{2}(1-d)e^{-\lambda\tau_1}\big\|\nabla v_t\big(t-\tau(t)\big)\big\|^2 \\ &- \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\big\|\nabla v_t(s)\big\|^2\,ds. \end{aligned} \tag{2.8}$$

In view of Young's inequality,

$$\begin{aligned} &-\mu_2\int_\Omega \nabla v_t(t)\cdot\nabla v_t\big(t-\tau(t)\big)\,dx \\ &\quad\le \frac{|\mu_2|}{2\sqrt{1-d}}\big\|\nabla v_t(t)\big\|^2 + \frac{|\mu_2|}{2}\sqrt{1-d}\big\|\nabla v_t\big(t-\tau(t)\big)\big\|^2. \end{aligned} \tag{2.9}$$

Then (2.6) follows from (2.8)–(2.9).

Clearly, $e^{\lambda\tau_1} \to 1$ as $\lambda \to 0$. Recalling that the set of real numbers is continuous, we can choose $\lambda > 0$, small such that

$$\frac{e^{\lambda\tau_1}|\mu_2|}{\sqrt{1-d}} < \xi < \mu_1, \tag{2.10}$$

where ξ is a positive constant. Then it follows that

$$\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2} < 0 \quad \text{and} \quad \frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\xi}{2e^{\lambda\tau_1}}(1-d) < 0. \tag{2.11}$$

Combining (2.10) and (2.6), we get that the energy in (2.5) is nonincreasing. □

Lemma 2.2 *Let* (2.1) *and* (2.2) *hold. We have*

$$(k \circ \nabla v)(t) \le c\left(\int_0^t k^{\frac{1}{2}}(s)\,ds\right)^{\frac{2q-2}{2q-1}}\big(k^q \circ \nabla u\big)^{\frac{1}{2q-1}}(t). \tag{2.12}$$

Proof See, Messaoudi [25]. □

Lemma 2.3 *Let* (2.1) *and* (2.2) *hold. Then for any* $t \ge 0$,

$$\zeta(t)(k \circ \nabla v)(t) \le c\big(-E'(t)\big)^{\frac{1}{2q-1}}. \tag{2.13}$$

Proof First, we claim that for any $0 < \alpha < 2 - q$,

$$\int_0^\infty \zeta(t)k^{1-\alpha}(t)\,dt < +\infty. \tag{2.14}$$

Indeed, in view of (2.2) and $0 < \alpha < 2 - q$, we get

$$\begin{aligned}\int_0^\infty \zeta(t)k^{1-\alpha}(t)\,dt &= \int_0^\infty \zeta(t)k^q(t)k^{1-q-\alpha}(t)\,dt\\ &\le -\int_0^\infty k'(t)k^{1-q-\alpha}(t)\,dt\\ &= -\frac{k^{2-q-\alpha}}{2-q-\alpha}\bigg|_0^\infty < +\infty.\end{aligned}$$

Multiplying (2.12) by $\zeta(t)$, and using (2.6) and (2.14), we obtain

$$\begin{aligned}\zeta(t)(k\circ\nabla v)(t) &\le c\zeta^{\frac{2q-2}{2q-1}}(t)\left(\int_0^t k^{\frac{1}{2}}(s)\,ds\right)^{\frac{2q-2}{2q-1}}\left(\zeta k^p\circ\nabla v\right)^{\frac{1}{2q-1}}(t)\\ &\le c\left(\int_0^t \zeta(s)k^{\frac{1}{2}}(s)\,ds\right)^{\frac{2q-2}{2q-1}}\left(-k'\circ\nabla v\right)^{\frac{1}{2q-1}}(t)\\ &\le c\left(-E'(t)\right)^{\frac{1}{2q-1}}.\end{aligned}$$

□

Now we give the stability result.

Theorem 2.2 *Let* (2.1)–(2.4) *hold. Suppose* $|\mu_2| < \sqrt{1-d}\mu_1$. *Let* $(v_0, v_1) \in (H^1(\Omega) \times L^2(\Omega))$, $f_0 \in H^1(\Omega \times (-\tau(0), 0))$. *Then for any* $t_1 > 0$, $E(t)$ *satisfies for all* $t \ge t_1$,

$$E(t) \le \begin{cases} c\exp(-\eta\int_{t_1}^t \zeta(s)\,ds), & \text{if } q = 1,\\ c(1+\int_{t_1}^t \zeta^{2q-1}(s)\,ds)^{-\frac{1}{2q-2}}, & \text{if } 1 < q < \frac{3}{2},\end{cases} \tag{2.15}$$

where η *is a positive constant. In addition, if, for* $1 < q < \frac{3}{2}$,

$$\int_{t_1}^\infty \left(1 + \int_{t_1}^t \zeta^{2q-1}(s)\,ds\right)^{-\frac{1}{2q-2}} dt < +\infty, \tag{2.16}$$

then

$$E(t) \le c\left(1 + \int_{t_1}^t \zeta^q(s)\,ds\right)^{-\frac{1}{q-1}}, \quad 1 < q < \frac{3}{2}. \tag{2.17}$$

The examples are given to verify some decay rates of energy, see [26–28].

Example 1 Taking $\zeta(t) = a$, it is obtained by (2.15) that

$$E(t) \le \beta e^{-\gamma a t}.$$

Example 2 Taking $\zeta(t) = \frac{a}{1+t}$, it is inferred by (2.15) that

$$E(t) \le \frac{\beta}{(1+t)^{\gamma a}}.$$

Example 3 Taking $k(t) = ae^{-b(1+t)^{\alpha}}$ for $a, b > 0$ and $0 < \alpha \leq 1$, we can pick $\zeta(t) = b\alpha(1+t)^{\alpha-1}$. It is concluded from (2.15) that

$$E(t) \leq \beta \exp\bigl(-b\gamma(1+t)^{\alpha}\bigr).$$

Example 4 Consider $k(t) = \frac{a}{(1+t)^b}$ $(b > 2)$. We take $a > 0$ satisfying (2.1). If here we denote $\zeta(t) = \frac{b}{1+t}$, then we have

$$k'(t) = -\frac{ab}{(1+t)^{b+1}} = -\zeta(t)k(t). \tag{2.18}$$

Then it is obtained from $(2.15)_1$ that

$$E(t) \leq c \exp\left(-\eta \int_{t_1}^{t} \zeta(s)\, ds\right) = \frac{c}{(1+t)^{b\eta}}.$$

On the other hand, by denoting $\zeta(t) = \rho = \frac{b}{a^{\frac{1}{b}}}$, we rewrite (2.18) as

$$k'(t) = -\rho\left(\frac{a}{(1+t)^b}\right)^{\frac{b+1}{b}} = -\zeta(t)k^p(t), \tag{2.19}$$

with $p = \frac{b+1}{b} < \frac{3}{2}$. Then we get for any $t_1 > 0$,

$$\int_{t_1}^{\infty} \left(1 + \int_{t_1}^{t} \zeta^{2p-1}(s)\, ds\right)^{-\frac{1}{2p-2}} dt = \int_{t_1}^{\infty} \bigl[1 + c(t-t_1)\bigr]^{-\frac{1}{2p-2}} dt < +\infty.$$

Then from (2.17), we get

$$E(t) \leq c\left(1 + \int_{t_1}^{t} \zeta^p(s)\, ds\right)^{-\frac{1}{p-1}} = \frac{c}{(1+t)^b}.$$

3 Uniform decay

We first define two functionals,

$$\phi(t) = \int_{\Omega} v(t)v_t(t)\, dx + \frac{\sigma}{4}\bigl\|\nabla v(t)\bigr\|^4$$

and

$$\psi(t) = -\int_{\Omega} v_t(t) \cdot \int_0^t k(t-s)\bigl(v(t)-v(s)\bigr)\, ds\, dx.$$

Lemma 3.1 *For any $\delta_1 > 0$, we have*

$$\begin{aligned} \phi'(t) \leq{}& \bigl\|v_t(t)\bigr\|^2 - \bigl[(a-l) - 3\delta_1\bigr]\bigl\|\nabla v(t)\bigr\|^2 - b\bigl\|\nabla v(t)\bigr\|^4 \\ &+ \frac{\mu_1^2}{4\delta_1}\bigl\|\nabla v_t(t)\bigr\|^2 + \frac{\mu_2^2}{4\delta_1}\bigl\|\nabla v_t\bigl(t-\tau(t)\bigr)\bigr\|^2 + \frac{a-l}{4\delta_1}(k \circ \nabla v)(t). \end{aligned} \tag{3.1}$$

Proof By (1.1),

$$
\begin{aligned}
\phi'(t) &= \int_\Omega v_{tt}(t)v(t)\,dx + \|v_t(t)\|^2 + \sigma\|\nabla v(t)\|^2\big(\nabla v(t),\nabla v_t(t)\big) \\
&= \|v_t(t)\|^2 + \sigma\|\nabla v(t)\|^2\big(\nabla v(t),\nabla v_t(t)\big) \\
&\quad + \int_\Omega v(t)\cdot\bigg[\Big(a+b\|\nabla v(t)\|^2+\sigma\big(\nabla v(t),\nabla v_t(t)\big)\Big)\Delta v(t) \\
&\quad - \int_0^t k(t-s)\Delta v(s)\,ds + \mu_1\Delta v_t(t) + \mu_2\Delta v_t\big(t-\tau(t)\big)\bigg]dx \\
&= \|v_t(t)\|^2 - \big(a+b\|\nabla v(t)\|^2\big)\|\nabla v(t)\|^2 \\
&\quad + \int_\Omega \nabla v(t)\cdot\int_0^t k(t-s)\big(\nabla v(s)-\nabla v(t)\big)\,ds\,dx \\
&\quad + \int_0^t k(s)\,ds\cdot\|\nabla v(t)\|^2 - \mu_1\int_\Omega \nabla v(t)\cdot\nabla v_t(t)\,dx \\
&\quad - \mu_2\int_\Omega \nabla v(t)\cdot\nabla v_t\big(t-\tau(t)\big)\,dx.
\end{aligned}
\tag{3.2}
$$

By the Cauchy–Schwarz inequality, for any $\delta_1 > 0$, we obtain

$$
\begin{aligned}
&\int_\Omega \nabla v(t)\cdot\int_0^t k(t-s)\big(\nabla v(s)-\nabla v(t)\big)\,ds\,dx \\
&\quad\le \delta_1\|\nabla v(t)\|^2 + \frac{1}{4\delta_1}\int_\Omega\bigg(\int_0^t k(t-s)\big(\nabla v(s)-\nabla v(t)\big)\,ds\bigg)^2dx \\
&\quad\le \delta_1\|\nabla v(t)\|^2 + \frac{1}{4\delta_1}\int_0^t k(s)\,ds(k\circ\nabla v)(t) \\
&\quad\le \delta_1\|\nabla v(t)\|^2 + \frac{a-l}{4\delta_1}(k\circ\nabla v)(t),
\end{aligned}
\tag{3.3}
$$

$$
-\mu_1\int_\Omega \nabla v(t)\cdot\nabla v_t(t)\,dx \le \delta_1\|\nabla v(t)\|^2 + \frac{\mu_1^2}{4\delta_1}\|\nabla v_t(t)\|^2,
\tag{3.4}
$$

$$
-\mu_2\int_\Omega \nabla v(t)\cdot\nabla v_t\big(t-\tau(t)\big)\,dx \le \delta_1\|\nabla v(t)\|^2 + \frac{\mu_2^2}{4\delta_1}\big\|\nabla v_t\big(t-\tau(t)\big)\big\|^2.
\tag{3.5}
$$

Replacing (3.3)–(3.5) in (3.2), (3.1) follows. □

Lemma 3.2 *For any $\delta_2 > 0$, we have*

$$
\begin{aligned}
\psi'(t) &\le \sigma^2E(0)\bigg(\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|^2\bigg)^2 - \bigg(\int_0^t k(s)\,ds-\delta_2\bigg)\|v_t(t)\|^2 + \delta_2\|\nabla v_t(t)\|^2 \\
&\quad + c_1(k\circ\nabla v)(t) - c_2\big(k'\circ\nabla v\big)(t) + \delta_2\big\|\nabla v_t\big(t-\tau(t)\big)\big\|^2 + \delta_2\|\nabla v(t)\|^2,
\end{aligned}
\tag{3.6}
$$

where c_1 and c_2 are positive constants depending on δ_2.

Proof It is obtained by (1.1) that

$$\begin{aligned}
\psi'(t) &= -\int_\Omega v_{tt}(t)\cdot\int_0^t k(t-s)\big(v(t)-v(s)\big)\,ds\,dx \\
&\quad -\int_\Omega v_t(t)\bigg[v_t(t)\int_0^t k(t-s)\,ds+\int_0^t k'(t-s)\big(v(t)-v(s)\big)\,ds\bigg]\,dx \\
&= -\int_\Omega\bigg[\Big(a+b\|\nabla v(t)\|^2+\sigma\big(\nabla v(t),\nabla v_t(t)\big)\Big)\Delta v(t)-\int_0^t k(t-s)\Delta v(s)\,ds \\
&\quad +\mu_1\Delta v_t(t)+\mu_2\Delta v_t\big(t-\tau(t)\big)\bigg]\cdot\int_0^t k(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\quad -\int_\Omega v_t(t)\int_0^t k'(t-s)\big(v(t)-v(s)\big)\,ds\,dx-\int_0^t k(s)\,ds\|v_t(t)\|^2.
\end{aligned}$$

Then

$$\begin{aligned}
\psi'(t) &= \bigg(a-\int_0^t k(s)\,ds+b\|\nabla v(t)\|^2\bigg)\int_\Omega\nabla v(t)\cdot\int_0^t k(t-s)\big(\nabla v(t)-\nabla v(s)\big)\,ds\,dx \\
&\quad +\int_\Omega\bigg(\int_0^t k(t-s)\big(\nabla v(s)-\nabla v(t)\big)\,ds\bigg)^2\,dx-\int_0^t k(s)\,ds\|v_t(t)\|^2 \\
&\quad -\int_\Omega v_t(t)\int_0^t k'(t-s)\big(v(t)-v(s)\big)\,ds\,dx \\
&\quad +\mu_1\int_\Omega\nabla v_t(t)\cdot\int_0^t k(t-s)\big(\nabla v(t)-\nabla v(s)\big)\,ds\,dx \\
&\quad +\sigma\big(\nabla v(t),\nabla v_t(t)\big)\int_\Omega\nabla v(t)\int_0^t k(t-s)\big(\nabla v(t)-\nabla v(s)\big)\,ds\,dx \\
&\quad +\mu_2\int_\Omega\nabla v_t\big(t-\tau(t)\big)\cdot\int_0^t k(t-s)\big(\nabla v(t)-\nabla v(s)\big)\,ds\,dx.
\end{aligned} \tag{3.7}$$

Noting that $E'(t)\le 0$, then

$$\bigg(a-\int_0^t k(s)\,ds\bigg)\|\nabla v(t)\|^2\le 2E(t)\le 2E(0),$$

and using (2.1), we have

$$\|\nabla v(t)\|^2\le 2l^{-1}E(0). \tag{3.8}$$

We combine Hölder's and Cauchy–Schwarz inequalities and (3.8) to obtain, for any $\delta_2>0$,

$$\begin{aligned}
&\bigg(a-\int_0^t k(s)\,ds+b\|\nabla v(t)\|^2\bigg)\int_\Omega\nabla v(t)\cdot\int_0^t k(t-s)\big(\nabla v(t)-\nabla v(s)\big)\,ds\,dx \\
&\quad\le\bigg(a+\frac{2b}{l}E(0)\bigg)\bigg|\int_\Omega\nabla v(t)\cdot\int_0^t k(t-s)\big(\nabla v(t)-\nabla v(s)\big)\,ds\,dx\bigg| \\
&\quad\le\delta_2\|\nabla v(t)\|^2+(a-l)(4\delta_2)^{-1}\big(a+2bl^{-1}E(0)\big)^2(k\circ\nabla v)(t),
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\sigma\big(\nabla v(t),\nabla v_t(t)\big)\int_\Omega \nabla v(t)\int_0^t k(t-s)\big(\nabla v(t)-\nabla v(s)\big)\,ds\,dx\\
&\quad\le \sigma^2\big(\nabla v(t),\nabla v_t(t)\big)^2\frac{l}{2}\big\|\nabla v(t)\big\|^2\\
&\qquad+\frac{1}{2l}\int_\Omega\left(\int_0^t k(t-s)\big(\nabla v(t)-\nabla v(s)\big)\,ds\right)^2dx\\
&\quad\le \sigma^2E(0)\left(\frac{1}{2}\frac{d}{dt}\big\|\nabla v(t)\big\|^2\right)^2+(a-l)(2l)^{-1}(k\circ\nabla v)(t),
\end{aligned}\tag{3.10}$$

$$\begin{aligned}
&\mu_1\int_\Omega \nabla v_t(t)\int_0^t k(t-s)\big(\nabla v(t)-\nabla v(s)\big)\,ds\,dx\\
&\quad\le \delta_2\big\|\nabla v_t(t)\big\|^2+\mu_1^2(a-l)(4\delta_2)^{-1}(k\circ\nabla v)(t),
\end{aligned}\tag{3.11}$$

$$\begin{aligned}
&\mu_2\int_\Omega \nabla v_t\big(t-\tau(t)\big)\cdot\int_0^t k(t-s)\big(\nabla v(t)-\nabla v(s)\big)\,ds\,dx\\
&\quad\le \delta_2\big\|\nabla v_t\big(t-\tau(t)\big)\big\|^2+\mu_2^2(a-l)(4\delta_2)^{-1}(k\circ\nabla v)(t),
\end{aligned}\tag{3.12}$$

$$\begin{aligned}
&-\int_\Omega v_t(t)\int_0^t k'(t-s)\big(v(t)-v(s)\big)\,ds\,dx\\
&\quad\le \delta_2\big\|v_t(t)\big\|^2+\frac{1}{4\delta_2}\left(\int_0^t\big(-k'(t-s)\big)\big\|v(t)-v(s)\big\|\,ds\right)^2dx\\
&\quad\le \delta_2\big\|v_t(t)\big\|^2-k(0)(4\delta_2\lambda_1)^{-1}\big(k'\circ\nabla v\big)(t),
\end{aligned}$$

and

$$\int_\Omega\left(\int_0^t k(t-s)\big(\nabla v(s)-\nabla v(t)\big)\,ds\right)^2dx\le(a-l)(k\circ\nabla v)(t).$$

It is inferred by combining the above inequalities with (3.7) that (3.6) holds with

$$c_1=\frac{a-l}{4\delta_2}\big(a+2bl^{-1}E(0)\big)^2+(a-l)(2l)^{-1}+\mu_1^2(a-l)(4\delta_2)^{-1}+\mu_2^2(a-l)(4\delta_2)^{-1}+(a-l)$$

and

$$c_2=k(0)(4\delta_2\lambda_1)^{-1}.$$ □

We define $\tilde{E}(t)$ by

$$\tilde{E}(t):=E(t)+\varepsilon_1\phi(t)+\varepsilon_2\psi(t),$$

for $\varepsilon_1>0$ and $\varepsilon_2>0$.

Lemma 3.3 *For $\varepsilon_1>0$ and $\varepsilon_2>0$ small, it holds that*

$$\frac{1}{2}E(t)\le\tilde{E}(t)\le\frac{3}{2}E(t).\tag{3.13}$$

Proof It is concluded by Young's and Poincaré's inequalities that

$$\begin{aligned}\left|\tilde{E}(t)-E(t)\right| &= \varepsilon_1\phi(t)+\varepsilon_2\psi(t)\\ &\le \varepsilon_1\int_\Omega \left|v(t)\right|\left|v_t(t)\right|dx+\frac{\varepsilon_1\sigma}{4}\left\|\nabla v(t)\right\|^4\\ &\quad +\varepsilon_2\int_\Omega\left|v_t(t)\right|\int_0^t k(t-s)\left|v(t)-v(s)\right|ds\,dx\\ &\le \frac{\varepsilon_1}{2}\left\|v_t(t)\right\|^2+\frac{\varepsilon_1}{2\lambda_1}\left\|\nabla v(t)\right\|^2+\frac{\varepsilon_1\sigma}{4}\left\|\nabla v(t)\right\|^4\\ &\quad +\frac{\varepsilon_2}{2}\left\|v_t(t)\right\|^2+\frac{\varepsilon_2}{2\lambda_1}(a-l)(k\circ\nabla v)(t).\end{aligned}$$

Then there is a constant $\varepsilon>0$ such that

$$\left|\tilde{E}(t)-E(t)\right|\le\varepsilon E(t),$$

which implies (3.13) by taking $\varepsilon_1>0$ and $\varepsilon_2>0$ sufficiently small. □

Proof of Theorem 2.2 First, for any $t_1>0$, it is obtained that for any $t\ge t_1$,

$$\int_0^t k(s)\,ds\ge\int_0^{t_1}k(s)\,ds:=k_0.$$

We infer from (2.6), (3.1), and (3.6) that for any $t\ge t_1$,

$$\begin{aligned}\tilde{E}'(t)\le&-\left(\int_0^t k(s)\,ds-\delta_2\right)\varepsilon_2\left\|v_t(t)\right\|^2-\left[\left[(a-l)-3\delta_1\right]\varepsilon_1-\delta_2\varepsilon_2\right]\left\|\nabla v(t)\right\|^2\\ &+\left(\frac{\xi}{2}-\mu_1+\frac{|\mu_2|}{2\sqrt{1-d}}+\frac{\mu_1^2}{4\delta_1}\varepsilon_1+\delta_2\varepsilon_2\right)\left\|\nabla v_t(t)\right\|^2\\ &+\left[\frac{|\mu_2|}{2}\sqrt{1-d}-\frac{\xi}{2}(1-d)e^{-\lambda\tau_1}+\frac{\mu_2^2}{4\delta_1}\varepsilon_1+\delta_2\varepsilon_2\right]\left\|\nabla v_t\big(t-\tau(t)\big)\right\|^2\\ &+\left(\frac12-c_2\varepsilon_2\right)\left(k'\circ\nabla v\right)(t)+\left(\frac{a-l}{4\delta_1}\varepsilon_1+c_1\varepsilon_2\right)(k\circ\nabla v)(t)\\ &-\frac{\lambda\xi}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\left\|\nabla v_t(s)\right\|^2ds-\left(\sigma-\sigma^2E(0)\varepsilon_2\right)\left(\frac12\frac{d}{dt}\left\|\nabla v(t)\right\|^2\right)^2.\end{aligned}\tag{3.14}$$

First, we take $\delta_1>0$ satisfying

$$(a-l)-3\delta_1>\frac{a-l}{2}.$$

For any fixed $\delta_1>0$, we choose $\delta_2>0$ satisfying for $t\ge t_1$,

$$\int_0^t k(s)\,ds-\delta_2\ge\frac12 k_0.$$

At this point, for any fixed $\delta_1,\delta_2>0$, we take $\varepsilon_1>0$ small enough such that (3.13) holds,

$$\varepsilon_1<\min\left\{\frac{2\delta_1}{\mu_1}-\frac{\xi\delta_1}{\mu_1^2}-\frac{|\mu_2|\delta_1}{\mu_1^2\sqrt{1-d}},\frac{\xi\delta_1}{\mu_2^2}(1-d)e^{-\lambda\tau_1}-\frac{\delta_1}{|\mu_2|}\sqrt{1-d}\right\},$$

which gives us

$$\frac{\xi}{2}-\mu_1+\frac{|\mu_2|}{2\sqrt{1-d}}+\frac{\mu_1^2}{4\delta_1}\varepsilon_1<\frac{\xi}{4}-\frac{\mu_1}{2}+\frac{|\mu_2|}{4\sqrt{1-d}}$$

and

$$\frac{|\mu_2|}{2}\sqrt{1-d}-\frac{\xi}{2}(1-d)e^{-\lambda\tau_1}+\frac{\mu_2^2}{4\delta_1}\varepsilon_1<\frac{|\mu_2|}{4}\sqrt{1-d}-\frac{\xi}{4}(1-d)e^{-\lambda\tau_1}.$$

At last for any fixed $\delta_1,\delta_2>0$ and $\varepsilon_1>0$, we pick $\varepsilon_1>0$ so small that (3.13) holds, and further

$$\begin{aligned}\varepsilon_2<\min\biggl\{&\frac{1}{4c_2},\frac{1}{2\sigma E(0)},\frac{a-l}{2\delta_2}\varepsilon_1,\frac{\mu_1}{4\delta_2}-\frac{\xi}{8\delta_2}-\frac{|\mu_2|}{8\delta_2\sqrt{1-d}},\\&\frac{\xi}{8\delta_2}(1-d)e^{-\lambda\tau_1}-\frac{|\mu_2|}{8\delta_2}\sqrt{1-d}\biggr\},\end{aligned}$$

which gives us

$$\begin{aligned}&\frac{1}{2}-c_2\varepsilon_2>\frac{1}{4},\qquad\sigma-\sigma^2E(0)\varepsilon_2>\frac{\sigma}{2},\qquad\frac{a-l}{2}\varepsilon_2-\delta_2\varepsilon_2>0,\\&\frac{\xi}{4}-\frac{\mu_1}{2}+\frac{|\mu_2|}{4\sqrt{1-d}}+\delta_2\varepsilon_2<\frac{\xi}{8}-\frac{\mu_1}{4}+\frac{|\mu_2|}{8\sqrt{1-d}},\end{aligned}$$

and

$$\frac{|\mu_2|}{4}\sqrt{1-d}-\frac{\xi}{4}(1-d)e^{-\lambda\tau_1}+\delta_2\varepsilon_2<\frac{|\mu_2|}{8}\sqrt{1-d}-\frac{\xi}{8}(1-d)e^{-\lambda\tau_1}.$$

Then we can conclude that for any $t\geq t_1$,

$$\tilde{E}'(t)\leq-\gamma_1E(t)+\gamma_2(k\circ\nabla v)(t),\tag{3.15}$$

for $\gamma_1>0$ and $\gamma_2>0$.

Case 1. $q=1$

We multiply (3.15) by $\zeta(t)$ and use (2.2) to obtain

$$\begin{aligned}\zeta(t)\tilde{E}'(t)&\leq-\gamma_1\zeta(t)E(t)+\gamma_2\zeta(t)(k\circ\nabla v)(t)\\&\leq-\gamma_1\zeta(t)E(t)-\gamma_2\bigl(k'\circ\nabla v\bigr)(t)\\&\leq-\gamma_1\zeta(t)E(t)-\gamma_3E'(t),\quad\forall t\geq t_1,\end{aligned}\tag{3.16}$$

where $\gamma_3>0$.

Denoting

$$H(t)=\zeta(t)\tilde{E}(t)+\gamma_3E(t),$$

and recalling (3.13), we know that $H(t)\sim E(t)$, i.e.,

$$\beta_1E(t)\leq H(t)\leq\beta_2E(t),\tag{3.17}$$

where β_1 and β_2 are two constants. Noting that $\zeta(t)$ is nonincreasing, we can derive from (3.16)–(3.17) that

$$H'(t) \le -\frac{\gamma_1}{\beta_2}\zeta(t)H(t), \quad \forall t \ge t_1.$$

It is inferred that for any $t \ge t_1$,

$$H(t) \le H(t_0)\exp\left(-\frac{\gamma_1}{\beta_2}\int_{t_1}^{t}\zeta(s)\,ds\right), \tag{3.18}$$

which gives us $(2.15)_1$.

Case 2. It is concluded by multiplying (3.15) by $\zeta(t)$ that

$$\begin{aligned} \zeta(t)\tilde{E}'(t) &\le -\gamma_1\zeta(t)E(t) + \gamma_2\zeta(t)(k\circ\nabla v)(t) \\ &\le -\gamma_1\zeta(t)E(t) + c\big(-E'(t)\big)^{\frac{1}{2q-1}}. \end{aligned} \tag{3.19}$$

Multiplying (3.19) by $\zeta^{2q-2}(t)E^{2q-2}(t)$ implies

$$\zeta^{2q-1}(t)E^{2q-2}(t)\tilde{E}'(t) \le -\gamma_1(\zeta E)^{2q-1}(t) + c(\zeta E)^{2q-2}(t)\big(-E'(t)\big)^{\frac{1}{2q-1}}. \tag{3.20}$$

Using Young's inequality in (3.20) gives for any $\varepsilon_0 > 0$,

$$\zeta^{2q-1}(t)E^{2q-2}(t)\tilde{E}'(t) \le -(\gamma_1 - c\varepsilon_0)(\zeta E)^{2q-1}(t) - \frac{c}{\varepsilon_0}E'(t). \tag{3.21}$$

Since $\zeta(t)$ and $E(t)$ are nonincreasing, we can take ε_0 small enough such that $\gamma_1 - c\varepsilon_0 > 0$ to get that there exists $\gamma_3' > 0$ such that

$$\big(\zeta^{2q-1}E^{2q-2}\tilde{E}\big)'(t) \le \zeta^{2q-1}(t)E^{2q-2}(t)\tilde{E}'(t) \le -\gamma_3'(\zeta E)^{2q-1}(t) - cE'(t). \tag{3.22}$$

Define

$$J(t) = \zeta^{2q-1}E^{2q-2}\tilde{E}.$$

Clearly, $J(t) \sim E(t)$. Then for some $\gamma_4 > 0$,

$$J'(t) \le -\gamma_3(\zeta E)^{2q-1}(t) \le -\gamma_4\zeta^{2q-1}(t)J^{2q-1}(t). \tag{3.23}$$

Integrating (3.23) over (t_1, t) yields

$$E(t) \le c\left(1 + \int_{t_1}^{t}\zeta^{2q-1}(s)\,ds\right)^{-\frac{1}{2q-2}}.$$

To prove (2.17), we first observe from $(2.15)_2$ and (2.16) that

$$\int_0^\infty E(t)\,dt < +\infty.$$

Define

$$\theta(t) = \int_0^t \left\| \nabla v(t) - \nabla v(t-s) \right\|^2 ds.$$

We have

$$\begin{aligned} \theta(t) &\le 2\int_0^t \left(\left\| \nabla v(t) \right\|^2 + \left\| \nabla v(t-s) \right\|^2 \right) ds \\ &\le 4l^{-1} \int_0^t \left(E(t) + E(t-s) \right) ds \\ &\le 8l^{-1} \int_0^t E(t-s)\, ds = 8l^{-1} \int_0^t E(s)\, ds \\ &\le 8l^{-1} \int_0^\infty E(s)\, ds < +\infty. \end{aligned} \tag{3.24}$$

Multiplying (3.15) by $\zeta(t)$ and using (3.24) together with Jensen's inequality, we obtain

$$\begin{aligned} \zeta(t)\tilde{E}'(t) &\le -\gamma_1\zeta(t)E(t) + \gamma_2\zeta(t)(k \circ \nabla v)(t) \\ &\le -\gamma_1\zeta(t)E(t) + c\frac{\theta(t)}{\theta(t)} \int_0^t \left(\zeta^q k^q \right)^{\frac{1}{q}}(s) \left\| \nabla v(t) - \nabla v(t-s) \right\|^2 ds \\ &\le -\gamma_1\zeta(t)E(t) + c\theta(t) \left(\frac{1}{\theta(t)} \int_0^t \zeta^q(s)k^q(s) \left\| \nabla v(t) - \nabla v(t-s) \right\|^2 ds \right)^{\frac{1}{q}}, \end{aligned} \tag{3.25}$$

where we assume that $\theta(t)$ is positive, otherwise, for any $t \ge t_1$, $E(t) \le ce^{-kt}$, $k > 0$. Hence it is inferred from (3.25) that

$$\begin{aligned} \zeta(t)\tilde{E}'(t) &\le -\gamma_1\zeta(t)E(t) + c\theta^{\frac{q-1}{q}}(t) \left(\zeta^{q-1}(0) \int_0^t \zeta(s)k^q(s) \left\| \nabla v(t) - \nabla v(t-s) \right\|^2 ds \right)^{\frac{1}{q}} \\ &\le -\gamma_1\zeta(t)E(t) + c\left(-k' \circ \nabla v \right)^{\frac{1}{q}}(t) \\ &\le -\gamma_1\zeta(t)E(t) + c\left(-E'(t) \right)^{\frac{1}{q}}. \end{aligned} \tag{3.26}$$

Consequently, it is concluded by multiplying (3.26) by $\zeta^{q-1}(t)E^{q-1}(t)$ and repeating the above steps that

$$E(t) \le c\left(1 + \int_{t_1}^t \zeta^q(s)\, ds \right)^{-\frac{1}{q-1}}, \quad 1 < q < \frac{3}{2}.$$

This ends the proof of Theorem 2.2. □

4 Conclusion

This paper studies a Balakrishnan–Taylor viscoelastic strongly delayed wave equation. Under suitable assumptions on μ_1, μ_2 and the relaxation function, a more general energy decay result is proved by using Lyapunov functionals. The decay rate we established is more general than earlier results, hence our result improves and generalizes some previous works. Several rates of energy decay are illustrated by provided examples.

Acknowledgements
The author is grateful to the referees for their valuable suggestions.

Funding
This work was supported by the Natural Science Foundation of Ningxia (2020AAC03233) and by the General Project of North Minzu University (2022XYZSX02).

Availability of data and materials
No data were generated or analyzed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
Haiyan Li wrote the main manuscript text and reviewed the manuscript independently .

Received: 5 April 2023 Accepted: 24 May 2023 Published online: 31 May 2023

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