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Optimal partial regularity of discontinuous subelliptic systems with VMO coefficients related to Hörmander's vector fields

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Abstract

In this paper, we study discontinuous subelliptic systems with VMO coefficients related to Hörmander's vector fields. In the case of growth exponential $p \geq 2$, the regularity results of the partial Hölder continuity of weak solutions are established based on the $\mathcal{A}$-harmonic approximation method under the controllable growth condition.

Keywords: Hörmander's vector fields; Partial regularity; Discontinuous subelliptic systems; VMO coefficients; $\mathcal{A}$-harmonic approximation technique

1 Introduction

Let Ω be a bounded open domain in a homogeneous group of Hörmander's vector fields, $X_1, \ldots, X_k$ $(k < n)$. In this paper, we consider the partial regularity of weak solutions of nonlinear subelliptic systems with the following form:

$$-\sum_{i=1}^{k} X_i A_i^{\alpha}(x, u, Xu) = B^{\alpha}(x, u, Xu), \tag{1.1}$$

where $\alpha = 1, 2, \ldots, N$ and $Xu = (X_1 u, \ldots, X_k u)$ is the generalized gradient of u .

At present, the partial regularity of weak solutions of linear or nonlinear elliptic equations with smooth continuous coefficients has been studied extensively. From the exploratory work of De Giorgi [1], Campanto [2] generalized the results to lower dimensional spaces; in [3], the partial regularity of quasilinear elliptic systems with uniformly continuous coefficients was proved by using contradiction. Later, in the Euclidean space, Chen and Tan considered the regularity results under the natural growth condition or the controllable growth condition in [4, 5] respectively. In [6], Qiu obtained the partial regularity with Dini continuous coefficients. Then, Marco Bramanti pointed out that a Euclidean space is a special case of the nilpotent Lie group in [7]. Furthermore, subelliptic equations and systems on nilpotent Lie groups have received extensive attention. Wang and Liao et al. in [8, 9] studied the partial regularity of Dini continuous coefficients in the Heisenberg groups, and in [10, 11], they developed it in Carnot groups. For Hörmander's vector field,

Gao and Niu [12], using the inverse Hölder inequality, obtained the partial regularity results of degenerate subelliptic systems. Then, Wang [13] considered the internal Hölder continuity.

In recent years, scholars have been devoted to the study discontinuous coefficients. The most popular one is VMO function, which is between the space of continuous functions and the space of L^∞. Although it is discontinuous, it has similar properties of continuous functions. Based on the commutator theory of VMO for Calderón–Zygmund singular integral operator, Chiarenza, Frasca, and Longo [14] studied $W^{2,p}$ for strong solutions of nondivergence elliptic equations with VMO coefficients. Further, Fazio and Ragusa [15] received the regularity of linear elliptic equations with VMO in the Morrey space $L^{2,\lambda}$. Lieberman [16] obtained the overall estimate with VMO coefficient of linear elliptic equations in the Morrey space $L^{p,\lambda}$ based on the L^p theory from Chiarenza, Frasca, and Longo with some primary methods. Zheng [17] discussed the partial regularity of the divergence form quasilinear elliptic systems with VMO coefficients in the Morrey space.

Compared with the elliptic equations and quasilinear systems, the main new aspect of our paper is the fact that we are able to deal with the more general subelliptic nonlinear systems with VMO coefficients in divergence form, which is associated with Hörmander's vector fields.

In this paper, we use the $\mathcal{A}$-harmonic approximation method to prove our results. This method is derived from Simon [18] and is used to prove the regularity of harmonic functions. Later, Duzaar and Grotowski [19] developed an approximate lemma, established the corresponding harmonic form, and studied the partial regularity of weak solutions of nonlinear elliptic equations. More recently, Duzaar in [20] developed it into the parabolic form of $\mathcal{A}$-harmonic approximation lemma. This is to connect $\mathcal{A}$-harmonic functions with nonlinear partial differential equations by the $\mathcal{A}$-harmonic approximation lemma. By constructing a special function related to the weak solution u, and then using the $\mathcal{A}$-harmonic approximation lemma, we can find that there exists an $\mathcal{A}$-harmonic function sufficiently approaching the special function in the sense of L^2. The harmonic function has good properties, and the desired attenuation estimation is derived. Finally, partial regularity results can be obtained. Using this method, not only the process of proof is greatly simplified but, more importantly, the optimal partial regularity result is obtained.

Assume that $X_0, X_1, \dots, X_k$ are a family of smooth vector fields in $\Omega \subset \mathbb{R}^n$ $(n > k)$, i.e.,

$$X_i = \sum_{j=1}^{k} b_{ij}(\xi)\frac{\partial}{\partial \xi_j}, \quad b_{ij}(\xi) \in C^\infty(\Omega),$$

where $i = 0, 1, \dots, k$ and $j = 1, 2, \dots, k$. Let

$$L = \sum_{i=1}^{k} b_{ij} X_i^2 + X_0.$$

Hörmander in his important article [21] proved that if smooth vector fields $\{X_0, X_1, \dots, X_k\}$ satisfy the finite rank conditions (see Sect. 2), then the operator L is the hypoelliptic. An example is the most typical Heisenberg groups, Heisenberg type groups, and even the more general Carnot groups, whose Lie algebras have a family of smooth vector fields satisfying finite rank conditions, so the sum of squares (usually called sub-Laplace operators) composed of these vector fields are subelliptic.

Then can we establish $\Gamma^{0,\gamma}$ $(0 < \gamma < 1)$ continuity for weak solutions to more general nonlinear subelliptic systems arising from Hörmander's vector fields? Compared with the Euclidean space, our research faces three main challenges: first, the lack of commutativity between elements makes Taylor's expansion invalid; second, the vector fields lack homogeneity, so the scaling method from the indirect method in the Euclidean space cannot be used; third, Hörmander's vector fields have no specific form, so it is difficult to establish the Caccioppoli type inequality. For the first difficulty, we construct appropriate auxiliary functions and apply Poincaré's inequality related to Hörmander's vector fields proved by Jerison [22] instead of Taylor's expansion to establish suitable estimates. For the second one, we use the new method—$\mathcal{A}$ harmonic approximation method adopted in this paper, which allows us to solve this question. For the last challenge, we employ Lemma 2.2 for the vector's fields established by Xu and Zuily [23]. In this paper, we construct the decay function

$$\Phi(x_0, r, l) = \fint_{B_r(x_0)} \left[\frac{|Xu - Xl|^2}{(1 + |Xl|)^2} + \frac{|Xu - Xl|^p}{(1 + |Xl|)^p} \right] dx$$

such that the corresponding Caccioppoli type inequality is proved, where l is an affine function in Hörmander's vector fields that $l : \mathbb{R}^k \to \mathbb{R}^N$. Meanwhile, the weak solution of (1.1) is defined

$$\int_\Omega A_i^\alpha(x, u, Xu) X\varphi \, dx = \int_\Omega B(x, u, Xu)\varphi \, dx, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{1.2}$$

We now state the precise structure assumptions we are dealing with.

(H1) The coefficient $A_i^\alpha(x, u, P)$ satisfies the ellipticity conditions

$$\big(D_P A_i^\alpha(x, u, P)P_0, P_0\big) \geq \lambda\big(1 + |P|\big)^{p-2}|P_0|^2, \tag{1.3}$$

whenever $\forall x \in \Omega$, $u, u_0 \in \mathbb{R}^N$, $p \geq 2$, and $P, P_0 \in \mathbb{R}^{k\times N}$.

We can derive from (H1) that

$$\big(A_i^\alpha(x, u, P) - A_i^\alpha(x, u, P_0)\big)(P - P_0) \geq \lambda\big(\big(1 + |P_0|\big)^{p-2}|P - P_0|^2 + |P - P_0|^p\big), \tag{1.4}$$

where λ is a positive constant.

(H2) The coefficient $A_i^\alpha(x, u, P)$ satisfies the following growth conditions:

$$\big|A_i^\alpha(x, u, P)\big| + \big(1 + |P|\big)\big|D_P A_i^\alpha(x, u, P)\big| \leq L\big(1 + |P|\big)^{p-1}, \tag{1.5}$$

where $1 \leq L < \infty$.

(H3) The coefficient $A_i^\alpha(x, u, P)$ of the second variable u is continuous, and there exists a bounded, convex, nondecreasing continuous module $\omega : [0, \infty) \to [0, 1]$ with $\lim_{s\to 0}\omega(s) = 0 = \omega(0)$ such that

$$\big|A_i^\alpha(x, u, P) - A_i^\alpha(x, u_0, P)\big| \leq L\omega\big(|u - u_0|^2\big)\big(1 + |P|\big)^{p-1}, \tag{1.6}$$

whenever $x \in \Omega$, $u, u_0 \in \mathbb{R}^N$, $P, P_0 \in \mathbb{R}^{k\times N}$.

(H4) $D_P A_i^\alpha(x,u,P)$ about the third variable P is continuous, and there exists a bounded, convex, nondecreasing continuous module $\mu : [0,\infty) \to [0,1]$ with $\mu(s) \le s$, $\lim_{s\to 0}\mu(s) = 0 = \mu(0)$ such that

$$\left|D_P A_i^\alpha(x,u,P) - D_P A_i^\alpha(x,u,P_0)\right| \le L\mu\left(\frac{|P-P_0|}{1+|P|+|P_0|}\right)\left(1+|P|+|P_0|\right)^{p-2}. \tag{1.7}$$

(H5) Mapping $x \mapsto \frac{A_i^\alpha(x,u,P)}{(1+|P|)^{p-1}}$ on u, P satisfies the following VMO conditions:

$$\left|A_i^\alpha(x,u,P) - \left(A_i^\alpha(x,u,P)\right)_{x_0,r}\right| \le \nu_{x_0}(x,r)\left(1+|P|\right)^{p-1}, \quad \forall x \in B_r(x_0), \tag{1.8}$$

whenever $x_0 \in \Omega$, $r \in (0,\rho_0]$, $u \in \mathbb{R}^N$, $P \in \mathbb{R}^{k\times N}$, $\rho_0 > 0$, $\nu_{x_0} : \mathbb{R}^n \times [0,\rho_0] \to [0,2L]$ is a bounded function satisfying $\lim_{\rho\to 0} V(\rho) = 0$, where

$$V(\rho) = \sup_{x_0\in\Omega}\sup_{0<r\le\rho}\left(\fint_{B_r(x_0)\cap\Omega} \nu_{x_0}(x,r)\,dx\right),$$
$$\left(A_i^\alpha(x,u,P)\right)_{x_0,r} = \fint_{B_r(x_0)\cap\Omega} A_i^\alpha(x,u,P)\,dx.$$

(H6) (Controllable growth conditions) Let $r = \frac{pQ}{Q-p}$ if $p < Q$; or $r \in [p,\infty)$ if $p = Q$ such that

$$\left|B(x,u,P)\right| \le a|P|^{p(1-\frac{1}{r})} + b|u|^{r-1} + c, \tag{1.9}$$

where a, b, c are positive constants, Q is the dimension of a homogeneous group of Hörmander's vector fields.

Under this set of assumptions, we have the following partial regularity result.

Theorem 1 *Assume that coefficients A_i^α and B satisfy* (H1)–(H6), $\Omega \subset \mathbb{R}^n$, $u \in HW^{1,p}(\Omega, R^N)$ *is a weak solution to system* (1.1). *Then there exists an open subset $\Omega_0 \subset \Omega$ such that*

$$u \in \Gamma_{\mathrm{loc}}^{0,\gamma}\left(\Omega_0,\mathbb{R}^N\right), \quad \gamma \in (0,1).$$

Further,

$$\Omega \setminus \Omega_0 \subseteq \Sigma_1 \cup \Sigma_2, \qquad \mathrm{meas}(\Omega \setminus \Omega_0) = 0,$$

where

$$\Sigma_1 = \left\{x_0 \in \Omega : \lim_{r\to 0^+}\sup\left(|u_{x_0,r}| + \left|(Xu)_{x_0,r}\right|\right) = \infty\right\},$$
$$\Sigma_2 = \left\{x_0 \in \Omega : \lim_{r\to 0^+}\inf \fint_{B_r(x_0)} \left|Xu - (Xu)_{x_0,r}\right|^p dx > 0\right\}.$$

This paper is organized as follows: in Sect. 2, we introduce the knowledge of Hörmander's vector fields, function spaces in Hörmander's vector fields, affine functions, and some necessary lemmas. Then we show a Caccioppoli type inequality in Sect. 3, and Sect. 4 includes three lemmas and is devoted to the proof of Theorem 1.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open, and path-connected domain and $\{X_i\}_{i=1}^k$ $(k < n)$ be a family of C^∞ real-valued vector fields defined in a neighborhood of the closure $\bar{\Omega}$ of Ω. For a multi-index $\alpha = (i_1, i_2, \dots, i_k)$, we denote by X_α the commutator

$$\bigl[X_{i_1}, \bigl[X_{i_2}, \dots, [X_{i_{k-1}}, X_{i_k}]\bigr]\bigr]$$

of length $l = |\alpha|$. We say that the vector fields satisfy Hörmander's condition if there exists some positive integer s such that $\{X_\alpha\}_{|\alpha|\le s}$ span the tangent space of $\mathbb{R}^n$ at each point of Ω, that is, rank Lie $[X_1, \dots, X_k] \equiv n$. Throughout the paper, we assume that the vector fields are free. Let $\mathcal{G}(s,k)$ be the free Lie algebra of step s with k generators, that is, the quotient of the free Lie algebra with k generators by the ideal generated by commutators of length at least $s+1$. Then we say that $\{X_\alpha\}_{|\alpha|\le s}$ are free up to order s if and only if $n = \dim \mathcal{G}(s,k)$.

Now, we introduce a metric in the following way. An admissible path γ is a Lipschitz curve $\gamma : [0,b] \mapsto \Omega$ such that there exist $c_i(t)$, $0 \le t \le b$, $i = 1, \dots, k$, satisfying $\sum_{i=1}^k c_i^2(t) \le 1$ and $\gamma'(t) = \sum_{i=1}^k c_i(t) X_i(\gamma(t))$ for a.e. $t \in [0,b]$. Then a *Carnot-Carathéodory (C-C)* metric on Ω associated with $\{X_i\}_{i=1}^k$ is defined by

$$\varrho_{cc}(x,\eta) = \min\bigl\{b \ge 0 : \exists \gamma : [0,b] \mapsto \Omega, \text{s.t. } \gamma(0) = \xi, \gamma(b) = \eta\bigr\}.$$

So, we can define the C–C ball and the C–C sphere

$$B_r(x) = \bigl\{\eta : \varrho_{cc}(x,\eta) < r\bigr\}, \qquad \partial B_r(x) = \bigl\{\eta : \varrho_{cc}(x,\eta) = r\bigr\},$$

respectively. A fundamental doubling property of the Lebesgue measure with respect to the C–C metric balls was showed by Nagel, Stein, and Wainger in [24]; namely, given a bounded set $\Omega \subset \mathbb{R}^n$ for any $\Omega' \Subset \Omega$, there exist positive constants R_0 and C such that $|B_{2R}(x_0)| \le C|B_R(x_0)|$, only for $x_0 \in \Omega'$ and $0 < R < R_0$. Furthermore, it follows

$$\bigl|B_R(x)\bigr| = \omega_X R^Q$$

for free Hörmander's vector fields, where $\omega_X(x)$ is positively bounded and depends only on the center x.

The following results for free Hörmander's vector fields will be employed to establish the Caccioppoli type inequality. See [23] for more details.

Lemma 1 *Let $P = \sum_{i,m=1}^k X_i(a_{im}X_m)$ be a second order differential operator with a_{im} being bounded measurable functions satisfying the uniform ellipticity condition, and denote by A the matrix $A = (a_{im})_{1\le i,m\le k}$. For any $x_0 \in \Omega$, one can find coordinates in a neighborhood V of x_0 and a matrix $T(x) \in GL(k,\mathbb{R})$, which is C^∞ in V such that if we set $(Y_1, \dots, Y_k) = T(x)(X_1, \dots, X_k)$, then*

1.

$$Y_i = \frac{\partial}{\partial x_i} + \sum_{l=k+1}^n \frac{\partial}{\partial x_l}, \quad i = 1, \dots, k;$$

2. *The set $(Y_1, \dots, Y_k)$ satisfies Hörmander's condition of order s and is free up to the order s in V;*

3.

$$P = \sum_{i,m=1}^{k} Y_i(b_{im}Y_m),$$

where $(b_{im}) = C^T(x)AC(x)$, $C(x)$ *with* C^∞ *an invertible matrix with* C^∞ *entries in* V.

Next, we introduce some spaces with norm.

Let $\Omega \subset \mathbb{R}^n$ be an open set and denote the Sobolev space by

$$HW^{1,p}(\Omega) = \{u \in L^p(\Omega) | X_i u \in L^p(\Omega), i = 1, \dots, k\}.$$

The $HW^{1,p}(\Omega)$ is a Banach space under the norm

$$\|u\|_{HW^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{k} \|X_i u\|_{L^p(\Omega)}.$$

The local Sobolev space is defined by

$$HW^{1,p}_{\mathrm{loc}}(\Omega) = \{u | \eta u \in HW^{1,p}(\Omega), \eta \in C_0^\infty(\Omega)\}.$$

Jerison [22] showed a Poincaré type inequality related to Hörmander's vector fields:

$$\int_{B_\rho(x_0)} \left|u(x) - u_{x_0,\rho}\right|^p dx \le C_p \rho^p \int_{B_\rho(x_0)} |Xu|^p dx, \quad u \in HW^{1,p}\big(B_\rho(x_0), \mathbb{R}^N\big), \tag{2.1}$$

where $u_{x_0,\rho} = \fint_{B_\rho(x_0)} u(x)\,dx$, C_p only relies on p and Q. Without loss of generality, we may assume $C_p > 1$.

Let $\gamma \in (0,1)$ and denote

$$\Gamma^{0,\gamma}(\Omega) = \{u \in L^\infty(\Omega) | u \in \Gamma^\gamma(\Omega), i = 1, \dots, k\},$$

where $\Gamma^\gamma(\Omega) = \{u \in L^\infty(\Omega) | \sup_{x,x_0 \in \Omega, x \ne x_0} \frac{|u(x)-u(x_0)|}{\varrho_{cc}^\gamma(x,x_0)} < \infty\}$. We say that $\Gamma^{0,\gamma}(\Omega)$ is a Folland–Stein space with the norm

$$\|u\|_{\Gamma^{0,\gamma}(\Omega)} = \|u\|_{L^\infty(\Omega)} + \sup_{x,x_0 \in \Omega, x \ne x_0} \frac{|u(x) - u(x_0)|}{\varrho_{cc}^\gamma(x,x_0)}.$$

Capogna in [25] pointed out that any gauge ball $B_r(x) \Subset \Omega$ fits **A**– property and proved a Campanato type lemma for Hörmander's vector fields. First, we introduce a Campanato space and the norm as follows.

Let $\Omega \subset \mathbb{R}^n$, $1 \le p < \infty$, $\mu \ge 0$, and denote

$$\mathcal{L}^{p,\mu}(\Omega) = \left\{u \in L^p(\Omega) \middle| \sup_{x \in \Omega, 0 < \rho < \operatorname{diam}\Omega} \rho^{-\mu} \int_{B_\rho(x) \cap \Omega} \left|u(z) - u_{x,\rho}\right|^p dz < +\infty\right\}.$$

Then $\mathcal{L}^{p,\mu}(\Omega)$ is said to be a Campanato space with the norm

$$\|u\|_{\mathcal{L}^{p,\mu}(\Omega)} = \|u\|_{L^p(\Omega)} + \left\{u \in L^p(\Omega) \middle| \sup_{x \in \Omega, 0 < \rho < \operatorname{diam}\Omega} \rho^{-\mu} \int_{B_\rho(x) \cap \Omega} \left|u(z) - u_{x,\rho}\right|^p dz\right\}^{\frac{1}{p}}.$$

We recall that an open set $\Omega \subset \mathbb{R}^n$ has the $\mathbf{A}-$ property if and only if there exists $A > 0$ such that for every $x \in \Omega$ and $\rho > 0$,

$$\left|\Omega \cap B_\rho(x)\right| \geq A\left|B_\rho(x)\right|.$$

Based on the knowledge of affine function in Hörmander's vector fields, elementary calculations yield the following estimates (for more details, see [26]).

Let $u \in L^2(B_r(x_0), \mathbb{R}^N)$, $x_0 \in \mathbb{R}^n$ and consider the horizontal components

$$\bar{x} = \left(x^1, x^2, \ldots, x^k\right) \quad \text{and} \quad \bar{x}_0 = \left(x_0^1, x_0^2, \ldots, x_0^k\right).$$

If the function

$$l(\bar{x}) = l_{x_0,r}(\bar{x}_0) + Xl_{x_0,r}(\bar{x} - \bar{x}_0)$$

minimizes the functional

$$l \mapsto \fint_{B_r(x_0)} |u - l|^2 \, dx$$

among affine function $l : \mathbb{R}^k \to \mathbb{R}^N$, then

$$l_{x_0,r}(\bar{x}_0) = u_{x_0,r} = \fint_{B_r(x_0)} u \, dx.$$

Lemma 2 *Let $u \in L^2(B_r(x_0), \mathbb{R}^N)$, $\theta \in (0,1)$. We denote by $l_{x_0,r}$ and $l_{x_0,\theta^k r}$, the affine function defined as above for radii r and $\theta^k r$. Then we have*

$$|Xl_{x_0,\theta^k r} - Xl_{x_0,r}|^2 \leq C \fint_{B_{\theta^k r}(x_0)} |u - l_{x_0,r}|^2 \, dx, \tag{2.2}$$

and more generally

$$|Xl_{x_0,r} - Xl|^2 \leq C \fint_{B_r(x_0)} |u - l|^2 \, dx. \tag{2.3}$$

Lemma 3 *Let $p > 1$, $r > 0$, $Q < \mu \leq Q + p$, then*

$$\mathcal{L}^{p,\mu}\left(B_\rho(x)\right) \subset \Gamma^\gamma\left(B_\rho(x)\right), \quad \gamma = \frac{\mu - Q}{p}.$$

Wang [13] obtained the following result by [23].

Lemma 4 *For all $x_0 \in \Omega$, $R \in (0,1)$ such that $B(x_0, R) \subset \Omega$ and all $k \in \mathbb{N}$,*

$$R^k \|u\|_{HW^{k,2}(B(x_0), \frac{R}{2^k})} \leq C \|u\|_{L^2(B(x_0,R))},$$

where C is a constant independent of x_0, R, u.

So, we can conclude the following inequality:

$$r^Q \sup_{B_{\frac{r}{2}}(x_0)} |Xu|^2 \le C_0 \int_{B_r(x_0)} |Xu|^2 \, dx, \tag{2.4}$$

where $Q \ge 2$.

Last, we give the introduction with the precise statement of the $\mathcal{A}$-harmonic approximation lemma. For the proof about this lemma, see [27]. We say that if for any testing function $\varphi \in C_0^\infty(B_r(x_0), \mathbb{R}^N)$ one has

$$\fint_{B_r(x_0)} \mathcal{A}(Xh, X\varphi) \, dx = 0, \tag{2.5}$$

then a map $h \in C^\infty(B_r(x_0), \mathbb{R}^N)$ is $\mathcal{A}$-harmonic.

Lemma 5 *Let* λ, L *be a positive number, fixed* $n, N \in \mathbb{N}$, *and* $n \ge 2$. *If for any given* $\varepsilon > 0$ *there exists* $\delta = \delta(n, N, p, \lambda, L, \varepsilon) \in (0, 1]$, *then one has the following properties*:

(1) $\mathcal{A}$ *is a bilinear form on* $\mathbb{R}^{k\times N}$ *with the properties*

$$\mathcal{A}(v, v) \ge \lambda |v|^2, \qquad \mathcal{A}(v, \overline{v}) \ge L|v||\overline{v}|, \quad v, \overline{v} \in \mathbb{R}^{k\times N}; \tag{2.6}$$

(2) *Let* $w \in HW^{1,p}(B_r(x_0), \mathbb{R}^N)$ *be an approximately* $\mathcal{A}$-*harmonic map in the sense that there holds*

$$\left| \fint_{B_r(x_0)} \mathcal{A}(Xw, X\varphi) \, dx \right| \le \delta \sup_{B_r(x_0)} |X\varphi|, \quad \forall \varphi \in C_0^\infty\big(B_r(x_0), \mathbb{R}^N\big), \tag{2.7}$$

and that

$$\fint_{B_r(x_0)} |Xw|^2 \, dx \le 1. \tag{2.8}$$

Then there exists an $\mathcal{A}$-*harmonic map* $h \in C^\infty(B_r(x_0), \mathbb{R}^N)$ *that satisfies*

$$\fint_{B_r(x_0)} |Xh|^2 + |Xh|^p \, dx \le 2^{Q+1}, \tag{2.9}$$

and

$$\fint_{B_r(x_0)} \left| \frac{h - w}{r} \right|^2 + \left| \frac{h - w}{r} \right|^p dx \le \varepsilon. \tag{2.10}$$

3 Caccioppoli type inequality

The crucial point in the following Caccioppoli type inequality is based on the fact that the constant appearing on the right-hand side depends only upon the structural constants of the subelliptic system. Later on, we apply this inequality with the affine function.

At the first, we introduce some notations for convenience:

$$\Phi(x_0, r, l) = \fint_{B_r(x_0)} \left[\frac{|Xu - Xl|^2}{(1 + |Xl|)^2} + \frac{|Xu - Xl|^p}{(1 + |Xl|)^p} \right] dx,$$

$$\Psi(x_0,r,l)=\fint_{B_r(x_0)}\left[\frac{|u-l|^2}{r^2(1+|Xl|)^2}+\frac{|u-l|^p}{r^p(1+|Xl|)^p}\right]dx,$$
$$f(x_0,r)=r^2\fint_{B_r(x_0)}\left[|Xu|^p+|u|^r+1\right]^2dx,$$
$$\Psi_*(x_0,r,l)=\Psi(x_0,r,l)+\omega\left(\fint_{B_r(x_0)}\left|u-l(x_0)\right|^2dx\right)+V(r)+f(x_0,r).$$

Lemma 6 *Let $\Omega\subset\mathbb{R}^n$, $u\in HW^{1,p}(B_r(x_0),\mathbb{R}^N)$ be a weak solution to system (1.1) with (H1)–(H6). Then, for the minimizing affine function in Hörmander's vector fields $l\equiv l_{x_0,r}:\mathbb{R}^k\to\mathbb{R}^N$ introduced in Sect. 2 and for any $x_0\in B_r(x_0)\subseteq\Omega\subset\mathbb{R}^k\times\mathbb{R}^{n-k}$, $r:0<r<\min\{1,\operatorname{dist}(x_0,\partial\Omega)\}$, we have the estimate*

$$\begin{aligned}&\fint_{B_{\frac{r}{2}}(x_0)}\left[\frac{|Xu-Xl|^2}{(1+|Xl|)^2}+\frac{|Xu-Xl|^p}{(1+|Xl|)^p}\right]dx\\&\quad\le C(p,L,\lambda)\left\{\fint_{B_r(x_0)}\left[\frac{|u-l|^2}{r^2(1+|Xl|)^2}+\frac{|u-l|^p}{r^p(1+|Xl|)^p}\right]dx\right.\\&\qquad\left.+\omega\left(\fint_{B_r(x_0)}\left|u-l(x_0)\right|^2dx\right)+V(r)+r^2\fint_{B_r(x_0)}\left[|Xu|^p+|u|^r+1\right]^2dx\right\},\end{aligned}$$

i.e.,

$$\Phi\left(x_0,\frac{r}{2},l\right)\le C_c\Psi_*(x_0,r,l),\tag{3.1}$$

where $C_c=C(p,L,\lambda)$.

Proof Taking a testing function $\varphi=\phi^p(u-l)$, $l=l(x_0)-Xl(x-x_0)$, where $\phi\in C_0^\infty(B_r(x_0))$ is a cut-off function, satisfying $\phi\equiv1$ on $B_{\frac{r}{2}}(x_0)$, and $0\le\phi\le1$, $|X\phi|\le\frac{4}{r}$, $0\le\phi\le1$, $|X\phi|\le\frac{4}{r}$, we can substitute it into the weak solution systems (1.2) and take the average integral of both sides. Then

$$\begin{aligned}&\fint_{B_r(x_0)}A_i^\alpha(x,u,Xu)\phi^p(Xu-Xl)\,dx\\&\quad=-p\fint_{B_r(x_0)}A_i^\alpha(x,u,Xu)\phi^{p-1}(u-l)X\phi\,dx+\fint_{B_r(x_0)}B_i^\alpha(x,u,Xu)\phi^p(u-l)\,dx.\end{aligned}$$

In a similar way, we replace Xu with Xl in $\fint_{B_r(x_0)}A_i^\alpha(x,u,Xu)X\phi\,dx$, and we can get

$$\begin{aligned}&-\fint_{B_r(x_0)}A_i^\alpha(x,u,Xl)\phi^p(Xu-Xl)\,dx\\&\quad=p\fint_{B_r(x_0)}A_i^\alpha(x,u,Xl)\phi^{p-1}(u-l)X\phi\,dx-\fint_{B_r(x_0)}A_i^\alpha(x,u,Xl)X\varphi\,dx.\end{aligned}$$

Since $A_i^\alpha(x_0,l(x_0),Xl)$ is a constant, which $l(x_0)=u_{x_0,\theta^kr}$, so we have

$$\fint_{B_r(x_0)}A_i^\alpha\left(x_0,l(x_0),Xl\right)dx=\left(A_i^\alpha\left(x_0,l(x_0),Xl\right)\right)_{x_0,r}=0,$$

it infers from the integration by parts that

$$\fint_{B_r(x_0)} \big(A_i^\alpha\big(x_0, l(x_0), Xl\big)\big)_{x_0,r} X\varphi \, dx = 0.$$

Combined with the above equations, it follows

$$\begin{aligned}
&\fint_{B_r(x_0)} \big[A_i^\alpha(x,u,Xu) - A_i^\alpha(x,u,Xl)\big]\phi^p(Xu - Xl)\, dx \\
&\quad = -p \fint_{B_r(x_0)} \big[A_i^\alpha(x,u,Xu) - A_i^\alpha(x,u,Xl)\big]\phi^{p-1}(u-l)X\phi \, dx \\
&\qquad + \fint_{B_r(x_0)} \big[A_i^\alpha\big(x, l(x_0), Xl\big) - A_i^\alpha(x,u,Xl)\big]X\varphi \, dx \\
&\qquad + \fint_{B_r(x_0)} \big[\big(A_i^\alpha\big(x_0, l(x_0), Xl\big)\big)_{x_0,r} - A_i^\alpha\big(x, l(x_0), Xl\big)\big]X\varphi \, dx \\
&\qquad + \fint_{B_r(x_0)} B(x,u,Xu)\phi^p(u-l)\, dx \\
&\quad := I + II + III + IV.
\end{aligned} \tag{3.2}$$

The left-hand side can be estimated via version (1.4) of the ellipticity condition, which leads to

$$\begin{aligned}
&\fint_{B_r(x_0)} \big[A_i^\alpha(x,u,Xu) - A_i^\alpha(x,u,Xl)\big]\phi^p(Xu - Xl)\, dx \\
&\quad \geq \lambda \fint_{B_r(x_0)} \phi^p\big[\big(1 + |Xl|\big)^{p-2}|Xu - Xl|^2 + |Xu - Xl|^p\big]\, dx.
\end{aligned} \tag{3.3}$$

Now, we will treat the terms I, II, III, IV of the right-hand side.

For the first term I, according to (1.5) and Young's inequality, and $|X\phi| \leq \frac{4}{r}$, we gain the following estimates:

$$\begin{aligned}
I &\leq p \fint_{B_r(x_0)} \big|A_i^\alpha(x,u,Xu) - A_i^\alpha(x,u,Xl)\big||\phi|^{p-1}|u-l||X\phi| \, dx \\
&= p \fint_{B_r(x_0)} \left|\int_0^1 D_P A_i^\alpha\big(x,u,Xl + t(Xu - Xl)\big)(Xu - Xl)\, dt\right||\phi|^{p-1}|u-l||X\phi| \, dx \\
&\leq pL \fint_{B_r(x_0)} \big(1 + |Xl| + |Xu - Xl|\big)^{p-2}|Xu - Xl||\phi|^{p-1}|u-l||X\phi| \, dx \\
&\leq C(p,L) \fint_{B_r(x_0)} \big(1 + |Xl| + |Xu - Xl|\big)^{p-2}|Xu - Xl||\phi|^{p-1}\frac{|u-l|}{r} \, dx \\
&\leq C(p,L) \fint_{B_r(x_0)} \big(1 + |Xl|\big)^{\frac{p-2}{2}}|Xu - Xl|\big(1 + |Xl|\big)^{\frac{p-2}{2}}\frac{|u-l|}{r} \, dx \\
&\quad + C(p,L)|\phi|^{p-1} \fint_{B_r(x_0)} |Xu - Xl|^{p-1}\frac{|u-l|}{r} \, dx
\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&\le C(p,L)|\phi|^{p-1} \fint_{B_r(x_0)} \left[\left(1+|Xl|\right)^{p-2} |Xu - Xl|^2 + \left(1+|Xl|\right)^{p-2} \frac{|u-l|^2}{r^2} \right] dx \\
&\quad + C(p,L)|\phi|^{p-1} \fint_{B_r(x_0)} \left[|Xu - Xl|^p + \frac{|u-l|^p}{r^p} \right] dx.
\end{aligned}
$$

Next, based on (1.6) and Young's inequality,

$$
\begin{aligned}
II &\le C(p)L \fint_{B_r(x_0)} \omega\left(\left|l(x_0)-u\right|^2\right)\left(1+|Xl|\right)^{p-1} \left(\phi|Xu - Xl| + \frac{|u-l|}{r} \right) dx \\
&\le C(p)L \fint_{B_r(x_0)} \left[\phi^p |Xu - Xl|^p + \frac{|u-l|^p}{r^p} \right] dx \\
&\quad + C(p)L \fint_{B_r(x_0)} \omega^{\frac{p}{p-1}}\left(\left|l(x_0)-u\right|^2\right)\left(1+|Xl|\right)^p dx \\
&\le C(p,L) \fint_{B_r(x_0)} \left[\phi^p |Xu - Xl|^p + \frac{|u-l|^p}{r^p} \right] dx \\
&\quad + C(p,L)\left(1+|Xl|\right)^p \omega\left(\fint_{B_r(x_0)} \left|l(x_0)-u\right|^2 dx \right).
\end{aligned} \tag{3.5}
$$

Here we used $\frac{1}{p} + \frac{p-1}{p} = 1$ and Jensen's inequality in the last step.

Then, using (1.8) and Young's inequality, we derive the following bound for III:

$$
\begin{aligned}
III &\le \fint_{B_r(x_0)} \left| \left(A_i^\alpha\left(x_0, l(x_0), Xl\right)\right)_{x_0,r} - A_i^\alpha\left(x, l(x_0), Xl\right) \right| |X\varphi| \, dx \\
&\le \fint_{B_r(x_0)} \nu_{x_0}(x,r)\left(1+|Xl|\right)^{p-1} \left(\phi|Xu - Xl| + \frac{|u-l|}{r} \right) dx \\
&\le \fint_{B_r(x_0)} \nu_{x_0}^{\frac{p}{p-1}}(x,r)\left(1+|Xl|\right)^p dx + C(p) \fint_{B_r(x_0)} \left(\phi^p |Xu - Xl|^p + \frac{|u-l|^p}{r^p} \right) dx.
\end{aligned}
$$

Let us estimate $\fint_{B_r(x_0)} \nu_{x_0}^{\frac{p}{p-1}}(x,r)(1+|Xl|)^p \, dx$. Because of the quality of ν_{x_0} and $\lim_{\rho\to 0} V(\rho) = 0$, we get

$$
\begin{aligned}
&\left(1+|Xl|\right)^p \fint_{B_r(x_0)} \nu_{x_0}^{\frac{p}{p-1}}(x,r)\, dx \\
&\quad \le \left(1+|Xl|\right)^p \left(\fint_{B_r(x_0)} \nu_{x_0}(x,r)\, dx \right)^{\frac{p}{p-1}} \\
&\quad \le \left(1+|Xl|\right)^p V^{\frac{p}{p-1}}(r) \\
&\quad \le \left(1+|Xl|\right)^p V(r).
\end{aligned}
$$

Therefore,

$$
III \le C(p) \fint_{B_r(x_0)} \left(\phi^p |Xu - Xl|^p + \frac{|u-l|^p}{r^p} \right) dx + \left(1+|Xl|\right)^p V(r). \tag{3.6}
$$

Finally, the term IV can be estimated by (1.9) and Young's inequality, this yields

$$\begin{aligned} IV &\le \fint_{B_r(x_0)} \left[a|Xu|^{p(1-\frac{1}{r})} + b|u|^{r-1} + c\right]\phi^p(u-l)\,dx \\ &\le C \fint_{B_r(x_0)} \left||Xu|^{p(1-\frac{1}{r})} + |u|^{r-1} + 1\right||\phi|^p(u-l)\,dx \\ &\le Cr^2 \fint_{B_r(x_0)} \left(|Xu|^{p(1-\frac{1}{r})} + |u|^{r-1} + 1\right)^2 dx + C \fint_{B_r(x_0)} \frac{|u-l|^2}{r^2}\,dx \\ &\le Cr^2\left(1+|Xl|\right)^p \fint_{B_r(x_0)} \left(|Xu|^p + |u|^r + 1\right)^2 dx \\ &\quad + C\left(1+|Xl|\right)^{p-2} \fint_{B_r(x_0)} \frac{|u-l|^2}{r^2}\,dx. \end{aligned} \tag{3.7}$$

We note that in the last second line we used in turn $0 \le \phi \le 1$ and Hölder's inequality.

Plug (3.4)–(3.7) into (3.2) and combine with (3.3) estimate of the items on the left. At the same time, the left- and right-hand sides with division to $(1+|Xl|)^p$, and let $\lambda > 3C(p,L)$, we can absorb the first integral of the right-hand side into the left. So, in the end, we arrive at this estimate

$$\begin{aligned} &\fint_{B_{\frac{r}{2}}(x_0)} \left[\frac{|Xu-Xl|^2}{(1+|Xl|)^2} + \frac{|Xu-Xl|^p}{(1+|Xl|)^p}\right] dx \\ &\quad \le \fint_{B_r(x_0)} \left[\frac{|Xu-Xl|^2}{(1+|Xl|)^2} + \frac{|Xu-Xl|^p}{(1+|Xl|)^p}\right] dx \\ &\quad \le C(p,L,\lambda)\left\{\fint_{B_r(x_0)} \left[\frac{|u-l|^2}{r^2(1+|Xl|)^2} + \frac{|u-l|^p}{r^p(1+|Xl|)^p}\right] dx \right. \\ &\qquad \left. + \omega\left(\fint_{B_r(x_0)} \left|u-l(x_0)\right|^2 dx\right) + V(r) + r^2 \fint_{B_r(x_0)} \left[|Xu|^p + |u|^r + 1\right]^2 dx\right\}, \end{aligned} \tag{3.8}$$

where we also scale the left-hand side down to $B_{\frac{r}{2}}(x_0)$, so one can get $\phi \equiv 1$, and $C(p,L,\lambda) = \frac{C(p,L)}{\lambda - 3C(p,L)}$ in the right-hand side, i.e., (3.1) holds. □

4 Proof of Theorem 1

In this section, we first establish three lemmas and finally prove Theorem 1. The following lemma provides a linearization strategy for nonlinear elliptic systems as in (1.1). Later on, this will be the starting point for the application of the $\mathcal{A}$-harmonic approximation lemma.

Lemma 7 *Under the assumptions of Theorem 1, we consider a ball $B_{2r}(x_0) \subseteq \Omega$ with $r \le r_0$ and an arbitrary affine function in Hörmander's vector fields $l : \mathbb{R}^k \to \mathbb{R}^N$. We define*

$$\begin{aligned} \mathcal{A} &= \frac{D_P A_i^\alpha(x, l(x_0), Xl)}{(1+|Xl|)^{p-1}}, \\ w &= u - l. \end{aligned}$$

Then, for $\forall\varphi \in C_0^\infty(B_r(x_0);\mathbb{R}^N)$, *it follows*

$$\begin{aligned}
&\fint_{B_r(x_0)} \mathcal{A}(Xw, X\varphi)\,dx \\
&\quad \le C_1\big[\mu\big(\Psi_*^{\frac{1}{2}}(x_0,2r,l)\big)\Psi_*^{\frac{1}{2}}(x_0,2r,l) + \Psi_*(x_0,2r,l)\big] \sup_{B_r(x_0)} |X\varphi|.
\end{aligned} \tag{4.1}$$

Proof A direct calculation gives

$$\begin{aligned}
&\fint_{B_r(x_0)} \mathcal{A}(Xw, X\varphi)\,dx \\
&\quad = \frac{1}{(1+|Xl|)^{p-1}} \fint_{B_r(x_0)} D_P A_i^\alpha\big(x, l(x_0), Xl\big)\cdot Xw \cdot X\varphi\,dx \\
&\quad = \frac{1}{(1+|Xl|)^{p-1}} \\
&\qquad \times \fint_{B_r(x_0)} \int_0^1 \big[D_P A_i^\alpha\big(x, l(x_0), Xl\big) - D_P A_i^\alpha\big(x, l(x_0), Xl + tXw\big)\big]\,dt \cdot Xw \cdot X\varphi\,dx \\
&\qquad + \frac{1}{(1+|Xl|)^{p-1}} \fint_{B_r(x_0)} \int_0^1 D_P A_i^\alpha\big(x, l(x_0), Xl + tXw\big)\,dt \cdot Xw \cdot X\varphi\,dx \\
&\quad := \frac{1}{(1+|Xl|)^{p-1}}(I + II).
\end{aligned} \tag{4.2}$$

To estimate the first term, we apply (1.7) and $t \in [0,1]$ to get the pointwise bound

$$\begin{aligned}
&\int_0^1 \big|D_P A_i^\alpha\big(x, l(x_0), Xl\big) - D_P A_i^\alpha\big(x, l(x_0), Xl + tXw\big)\big|\,dt \\
&\quad \le \int_0^1 L\mu\left(\frac{|tXw|}{1+|Xl|+|Xl+tXw|}\right)\big(1+|Xl|+|Xl+tXw|\big)^{p-2}\,dt \\
&\quad \le \int_0^1 C(L,p)\mu\left(\frac{|Xu-Xl|}{1+|Xl|}\right)\big(1+2|Xl|+|tXw|\big)^{p-2}\,dt \\
&\quad \le C(L,p)\mu\left(\frac{|Xu-Xl|}{1+|Xl|}\right)\big(1+|Xl|+|Xl-Xu|\big)^{p-2}.
\end{aligned}$$

This pointwise estimate leads to the following bound for the first term:

$$\begin{aligned}
I &= \fint_{B_r(x_0)} \left[\int_0^1 D_P A_i^\alpha\big(x, l(x_0), Xl\big) - D_P A_i^\alpha\big(x, l(x_0), Xl + tXw\big)\,dt\right] \\
&\quad \cdot Xw \cdot \sup_{B_r(x_0)} |X\varphi|\,dx \\
&\le \fint_{B_r(x_0)} \left[\int_0^1 \big|D_P A_i^\alpha\big(x, l(x_0), Xl\big) - D_P A_i^\alpha\big(x, l(x_0), Xl + tXw\big)\big|\,dt\right] \\
&\quad \cdot |Xu - Xl| \sup_{B_r(x_0)} |X\varphi|\,dx \\
&\le C(L,p) \fint_{B_r(x_0)} \mu\left(\frac{|Xu-Xl|}{1+|Xl|}\right)\big[\big(1+|Xl|\big)^{p-2} + |Xl-Xu|^{p-2}\big] \\
&\quad \cdot |Xu - Xl| \sup_{B_r(x_0)} |X\varphi|\,dx
\end{aligned} \tag{4.3}$$

$$
\begin{aligned}
&\le C(L,p)\big(1+|Xl|\big)^{p-1} \fint_{B_r(x_0)} \mu\left(\frac{|Xu-Xl|}{1+|Xl|}\right)\frac{|Xu-Xl|}{1+|Xl|} \sup_{B_r(x_0)} |X\varphi|\,dx \\
&\quad + C(L,p)\big(1+|Xl|\big)^{p-1} \fint_{B_r(x_0)} \mu\left(\frac{|Xu-Xl|}{1+|Xl|}\right)\frac{|Xu-Xl|^{p-1}}{(1+|Xl|)^{p-1}} \sup_{B_r(x_0)} |X\varphi|\,dx.
\end{aligned}
$$

And then, by Hölder's inequality and Jensen's inequality, we find that

$$
\begin{aligned}
I &\le C(L,p)\big(1+|Xl|\big)^{p-1}\left(\fint_{B_r(x_0)} \mu^2\left(\frac{|Xu-Xl|}{1+|Xl|}\right)dx\right)^{\frac{1}{2}} \\
&\quad \cdot \left(\fint_{B_r(x_0)} \frac{|Xu-Xl|^2}{(1+|Xl|)^2}\,dx\right)^{\frac{1}{2}} \sup_{B_r(x_0)} |X\varphi| \\
&\quad + C(L,p)\big(1+|Xl|\big)^{p-1}\left(\fint_{B_r(x_0)} \mu^p\left(\frac{|Xu-Xl|^p}{1+|Xl|}\right)dx\right)^{\frac{1}{p}} \\
&\quad \cdot \left(\fint_{B_r(x_0)} \frac{|Xu-Xl|^p}{(1+|Xl|)^p}\,dx\right)^{\frac{p-1}{p}} \sup_{B_r(x_0)} |X\varphi| \\
&\le C(L,p)\big(1+|Xl|\big)^{p-1}\mu^{\frac{1}{2}}\left(\left[\fint_{B_r(x_0)} \frac{|Xu-Xl|^2}{(1+|Xl|)^2}\,dx\right]^{\frac{1}{2}}\right) \\
&\quad \cdot \left(\fint_{B_r(x_0)} \frac{|Xu-Xl|^2}{(1+|Xl|)^2}\,dx\right)^{\frac{1}{2}} \sup_{B_r(x_0)} |X\varphi| \\
&\quad + C(L,p)\big(1+|Xl|\big)^{p-1}\mu^{\frac{1}{p}}\left(\left[\fint_{B_r(x_0)} \frac{|Xu-Xl|^2}{(1+|Xl|)^2}\,dx\right]^{\frac{1}{p}}\right) \\
&\quad \cdot \left(\fint_{B_r(x_0)} \frac{|Xu-Xl|^p}{(1+|Xl|)^p}\,dx\right)^{\frac{p-1}{p}} \sup_{B_r(x_0)} |X\varphi| \\
&\le C(L,p)\big(1+|Xl|\big)^{p-1}\mu^{\frac{1}{2}}\big(\sqrt{\Phi(x_0,r,l)}\big)\sqrt{\Phi(x_0,r,l)} \sup_{B_r(x_0)} |X\varphi| \\
&\quad + C(L,p)\big(1+|Xl|\big)^{p-1}\mu^{\frac{1}{p}}\big[(\Phi^{\frac{1}{p}}(x_0,r,l)\big]\Phi^{\frac{p-1}{p}}(x_0,r,l) \sup_{B_r(x_0)} |X\varphi| \\
&\le C(L,p)\big(1+|Xl|\big)^{p-1}\big[\mu^{\frac{1}{p}}\big(\Phi^{\frac{1}{p}}(x_0,r,l)\big)\Phi^{\frac{1}{2}}(x_0,r,l)+\Phi(x_0,r,l)\big] \sup_{B_r(x_0)} |X\varphi|,
\end{aligned}
\tag{4.4}
$$

where we used the fact that $\mu^{\frac{1}{2}} < \mu^{\frac{1}{p}}$ as $\mu \le 1$ and $\Phi^{\frac{p-1}{p}}(x_0,r,l) \le \Phi^{\frac{1}{2}}(x_0,r,l)$ when $0 < \Phi(x_0,r,l) < 1$ in the last inequality.

Next let us estimate the last term II. Since $A_i^\alpha(x_0,l(x_0),Xl)$ is a constant, then

$$
\int_0^1 D_p A_i^\alpha\big(x_0,l(x_0),Xl\big)\,dt = 0.
$$

In view of these facts, we can rewrite the integral II, and similar to the treatment of I, it follows

$$
II \le \fint_{B_r(x_0)} \int_0^1 \big|D_p A_i^\alpha\big(x,l(x_0),Xl+tXw\big) - D_p A_i^\alpha\big(x_0,l(x_0),Xl\big)\big||Xu-Xl|\,dt
$$

$$
\begin{aligned}
&\cdot \sup_{B_r(x_0)} |X\varphi|\,dx \\
&\le C(L) \fint_{B_r(x_0)} \mu\left(\frac{|Xu - Xl|}{1 + |Xl|}\right)\big(1 + |Xl| + |Xu - Xl|\big)^{p-2}|Xu - Xl| \sup_{B_r(x_0)} |X\varphi|\,dx \\
&\le C(L,p)\big(1 + |Xl|\big)^{p-1}\big[\mu^{\frac{1}{p}}\big(\Phi^{\frac{1}{p}}(x_0,r,l)\big)\Phi^{\frac{1}{2}}(x_0,r,l) + \Phi(x_0,r,l)\big] \sup_{B_r(x_0)} |X\varphi|.
\end{aligned}
\tag{4.5}
$$

Replace (4.3), (4.4), and (4.5) in (4.2) to see

$$
\begin{aligned}
&\big(1 + |Xl|\big)^{p-1} \fint_{B_r(x_0)} \mathcal{A}(Xw, X\varphi)\,dx \\
&\quad \le C(L,p)\big(1 + |Xl|\big)^{p-1}\big[\mu^{\frac{1}{p}}\big(\Phi^{\frac{1}{p}}(x_0,r,l)\big)\Phi^{\frac{1}{2}}(x_0,r,l) + \Phi(x_0,r,l)\big] \sup_{B_r(x_0)} |X\varphi|.
\end{aligned}
$$

By Lemma 6, the following inequality is finally obtained:

$$
\fint_{B_r(x_0)} \mathcal{A}(Xw, X\varphi)\,dx \le C_1\big[\mu^{\frac{1}{p}}\big(\Psi_*^{\frac{1}{p}}(x_0,2r,l)\big)\Psi_*^{\frac{1}{2}}(x_0,2r,l) + \Psi_*(x_0,2r,l)\big] \sup_{B_r(x_0)} |X\varphi|,
$$

where $C_1 = C(L,p,C_c)$. □

Next, we are in the position to establish the excess improvement. The strategy of the proof is to approximate the given solution by $\mathcal{A}$-harmonic functions, for which suitable decay estimates are available from the classical theory.

Lemma 8 *Suppose that the assumptions of Theorem 1 are satisfied and the ball $B_r(x_0) \subseteq \Omega$ with $r \le r_0 \in (0,1]$. For constants $\theta \in (0, \frac{1}{8}]$, $\delta = \delta(Q,N,p,L,\theta) \in (0,1]$, we impose the following smallness conditions*:

(1) $\mu^{\frac{1}{p}}(\Psi_*^{\frac{1}{p}}(x_0,2r,l)) + \Psi_*^{\frac{1}{2}}(x_0,2r,l) \le 1$;

(2) $\gamma = (\frac{\delta}{2})^{-1}\sqrt{\Psi_*(x_0,2r,l)} \le 1$,

then there holds the excess improvement estimate

$$
\Psi(x_0,\theta r,l_{x_0,r}) \le C_6\theta^2\Psi_*(x_0,r,l_{x_0,r}). \tag{4.6}
$$

Proof We re-scale the map

$$
\tilde{w} = \frac{u - l_{x_0,\theta r}}{C_2\gamma(1 + |Xl_{x_0,r}|)}, \quad \text{where } \gamma = \left(\frac{\delta}{2}\right)^{-1}\sqrt{\Psi_*(x_0,2r,l)},
$$

and

$$
X\tilde{w} = \frac{Xu - Xl_{x_0,\theta r}}{C_2\gamma(1 + |Xl_{x_0,r}|)}, \quad l_{x_0,r} = u_{x_0,r} + Xl_{x_0,r}(\bar{x} - \bar{x}_0), C_2 = \max\{C_1, \sqrt{C_c}, 1\}.
$$

We claim that $\tilde{w}$ satisfies the $\mathcal{A}$-harmonic approximation Lemma 5. First denote

$$
\mathcal{A} := \frac{D_P A_i^\alpha(x, l(x_0), Xl)}{(1 + |Xl|)^{p-1}},
$$

for our choice of the bilinear form, conditions (2.6) are clearly valid. Next, we verify that $\tilde{w}$ satisfies (2.7) in Lemma 5,

$$\begin{aligned}
&\left|\fint_{B_r(x_0)} \mathcal{A}(X\tilde{w}, X\varphi)\,dx\right| \\
&\quad= \left|\fint_{B_r(x_0)} \frac{D_p A_i^\alpha(x, l(x_0), Xl)}{(1+|Xl|)^{p-1}} \cdot \frac{Xu - Xl_{x_0,\theta r}}{C_2\gamma(1+|Xl|)} \cdot X\varphi\,dx\right| \\
&\quad\le \frac{1}{C_2\gamma}\left|\fint_{B_r(x_0)} \frac{D_p A_i^\alpha(x, l(x_0), Xl)}{(1+|Xl|)^{p}} \cdot (Xu - Xl) \cdot X\varphi\,dx\right| \\
&\quad\le \frac{1}{C_2\gamma}\fint_{B_r(x_0)} \mathcal{A}(Xw, X\varphi)\,dx \\
&\quad\le \frac{C_1}{C_2}\frac{\mu^{\frac{1}{p}}(\Psi_*^{\frac{1}{p}}(x_0,2r,l))\Psi_*^{\frac{1}{2}}(x_0,2r,l) + \Psi_*(x_0,2r,l)}{(\frac{\delta}{2})^{-1}\Psi_*^{\frac{1}{2}}(x_0,2r,l)} \sup_{B_r(x_0)} |X\varphi| \\
&\quad= \frac{C_1}{C_2}\frac{\delta}{2}\left[\mu^{\frac{1}{p}}\left(\Psi_*^{\frac{1}{p}}(x_0,2r,l)\right) + \Psi_*^{\frac{1}{2}}(x_0,2r,l)\right] \sup_{B_r(x_0)} |X\varphi| \\
&\quad\le \delta \sup_{B_r(x_0)} |X\varphi|.
\end{aligned}$$

Moreover, $\tilde{w}$ satisfies the following energy bound:

$$\begin{aligned}
&\fint_{B_r(x_0)} |X\tilde{w}|^2\,dx \\
&\quad= \fint_{B_r(x_0)} \frac{|Xu - Xl_{x_0,\theta r}|^2}{C_2^2\gamma^2(1+|Xl|)^2}\,dx \\
&\quad\le \frac{1}{C_2^2\gamma^2}\fint_{B_r(x_0)} \frac{|Xu - Xl|^2}{(1+|Xl|)^2}\,dx \\
&\quad\le \frac{1}{C_2^2\gamma^2}\Phi(x_0,r,l) \\
&\quad= \left(\frac{\delta}{2}\right)^2 \frac{\Phi(x_0,r,l)}{C_2^2\Psi_*(x_0,2r,l)} \\
&\quad\le \left(\frac{\delta}{2}\right)^2 \frac{C_c\Psi_*(x_0,2r,l)}{C_2^2\Psi_*(x_0,2r,l)} \le \frac{C_c}{C_2^2} \le 1.
\end{aligned}$$

According to Lemma 6, it yields that $\fint_{B_r(x_0)} |X\tilde{w}|^2\,dx \le 1$ meets (2.8) in Lemma 5. So, there is an $\mathcal{A}$-harmonic function $h \in C^\infty(B_r(x_0), \mathbb{R}^N)$ to satisfy (2.4), and due to $\theta \in (0, \frac{1}{8}]$, it infers

$$\fint_{B_{2\theta r}(x_0)} \left|X^2 h(x)\right|^2 dx \le \sup_{B_{\frac{r}{4}}(x_0)} \left|X^2 h\right|^2.$$

By Proposition 3.1 in [21] with $k = 2$, the following inequality

$$\int_{B_{\frac{r}{8}}(x_0)} \left|X^2 h(x)\right|^2 dx \le Cr^{-4}\int_{B_r(x_0)} |h|^2\,dx$$

is true. Replace h with $h - h_{x_0,r}$ and combine with Poincaré's inequality, then one has

$$\int_{B_{\frac{r}{8}}(x_0)} \left|X^2 h\right|^2 dx \le Cr^{-4} \int_{B_r(x_0)} |h - h_{x_0,r}|^2 dx \le CC_p r^{-2} \int_{B_r(x_0)} |Xh|^2 dx.$$

According to (2.9),

$$\sup_{B_{\frac{r}{8}}(x_0)} \left|X^2 h\right|^2 \le 2^{3Q} CC_p r^{-2} \fint_{B_r(x_0)} |Xh|^2 dx \le 2^{3Q} CC_p r^{-2} \cdot 2^{Q+1} = 2^{4Q+1} CC_p r^{-2},$$

i.e.,

$$r^2 \sup_{B_{\frac{r}{8}}(x_0)} \left|X^2 h\right|^2 \le 2^{4Q+1} C_3,$$

where denote $C_3 = CC_p$. Moreover, it can be deduced

$$\fint_{B_{2\theta r}(x_0)} \left|X^2 h(x)\right|^2 dx \le \sup_{B_{\frac{r}{4}}(x_0)} \left|X^2 h\right|^2 \le 2^{4Q+1} C_3 r^{-2}. \tag{4.7}$$

When $p > 2$, we can get

$$\sup_{B_{\frac{r}{8}}(x_0)} \left|X^2 h\right|^p \le \left(\sup_{B_{\frac{r}{8}}(x_0)} \left|X^2 h\right|^2\right)^{\frac{p}{2}} \le \left(2^{4Q+1} C_3 r^{-2}\right)^{\frac{p}{2}} = 2^{(4Q+1)\frac{p}{2}} C_3^{\frac{p}{2}} r^{-p}. \tag{4.8}$$

So, take

$$l^h = h_{x_0,\theta r} + (Xh)_{x_0,\theta r}(x - x_0),$$

then let us estimate $\fint_{B_{\theta r}(x_0)} |\frac{\tilde{w} - l^h(x)}{\theta r}|^s dx$ when $s = 2, p$, respectively. For $s = 2$, it shows

$$\begin{aligned} &\fint_{B_{\theta r}(x_0)} \left|\frac{\tilde{w} - l^h(x)}{\theta r}\right|^2 dx \\ &\quad = \fint_{B_{\theta r}(x_0)} \left|\frac{\tilde{w} - h + h - l^h(x)}{\theta r}\right|^2 dx \\ &\quad \le 2 \fint_{B_{\theta r}(x_0)} \left|\frac{\tilde{w} - h}{\theta r}\right|^2 dx + 2 \fint_{B_{\theta r}(x_0)} \left|\frac{h - l^h(x)}{\theta r}\right|^2 dx, \end{aligned} \tag{4.9}$$

then

$$\begin{aligned} &2 \fint_{B_{\theta r}(x_0)} \left|\frac{\tilde{w} - h}{\theta r}\right|^2 dx \\ &\quad = \frac{2}{\theta^2} \fint_{B_{\theta r}(x_0)} \left|\frac{\tilde{w} - h(x)}{r}\right|^2 dx \\ &\quad \le \frac{2}{\theta^2} \fint_{B_r(x_0)} \left|\frac{\tilde{w} - h(x)}{r}\right|^2 dx \\ &\quad \le 2C\theta^{-Q-2}\varepsilon, \end{aligned} \tag{4.10}$$

and by Poincaré's inequality, we can obtain

$$\begin{aligned}
&2\int_{B_{\theta r}(x_0)}\left|\frac{h-l^h(x)}{\theta r}\right|^2 dx \\
&\quad = 2\int_{B_{\theta r}(x_0)}\left|\frac{h-h_{x_0,\theta r}-(Xh)_{x_0,\theta r}(x-x_0)}{\theta r}\right|^2 dx \\
&\quad \le \frac{2}{(\theta r)^2}C_p(\theta r)^2 \fint_{B_{\theta r}(x_0)}\left|Xh-(Xh)_{x_0,\theta r}\right|^2 dx \\
&\quad \le 2C_p^2(\theta r)^2 \sup_{B_{\frac{r}{8}}(x_0)}\left|X^2h\right|^2 \\
&\quad \le 2^{4Q+2}C_p^2C_3^{\frac{p}{2}}\theta^2.
\end{aligned} \tag{4.11}$$

So, take (4.10) and (4.11) into (4.9), it derives

$$\begin{aligned}
&\fint_{B_{\theta r}(x_0)}\left|\frac{\tilde{w}-l^h(x)}{\theta r}\right|^2 dx \\
&\quad \le 2\fint_{B_{\theta r}(x_0)}\left|\frac{\tilde{w}-h}{\theta r}\right|^2 dx + \frac{2}{|B_{\theta r}(x_0)|}\int_{B_{\theta r}(x_0)}\left|\frac{h-l^h(x)}{\theta r}\right|^2 dx \\
&\quad \le 2C\theta^{-Q-2}\varepsilon + 2^{4Q+2}C_p^2C_3^{\frac{p}{2}}\theta^2 \\
&\quad \le C_4\bigl(\theta^{-Q-2}\varepsilon+\theta^2\bigr),
\end{aligned} \tag{4.12}$$

where $C_4 = 2C\cdot 2^{4Q+2}C_p^2C_3^{\frac{p}{2}} = 2^{4Q+3}C_p^2C_3^{\frac{p}{2}}$. Using the fact of this inequality, we have

$$\begin{aligned}
&\fint_{B_{\theta r}(x_0)}\left|\frac{\tilde{w}-l^h(x)}{\theta r}\right|^2 dx \\
&\quad = \frac{1}{C_2^2\gamma^2(1+|Xl_{x_0,r}|)^2}\fint_{B_{\theta r}(x_0)}\left|\frac{u-l_{x_0,\theta r}-C_2\gamma(1+|Xl_{x_0,r}|)l^h(x)}{\theta r}\right|^2 dx \\
&\quad \le C_4\bigl(\theta^{-Q-2}\varepsilon+\theta^2\bigr).
\end{aligned}$$

After the simplification, we thus conclude

$$\fint_{B_{\theta r}(x_0)}\left|\frac{u-l_{x_0,\theta r}-C_2\gamma(1+|Xl_{x_0,r}|)l^h(x)}{(\theta r)(1+|Xl_{x_0,r}|)}\right|^2 dx \le C_4C_2^2\gamma^2\bigl(\theta^{-Q-2}\varepsilon+\theta^2\bigr). \tag{4.13}$$

Similar to $s=2$, the result for condition of $s=p$ is as follows:

$$\begin{aligned}
&\fint_{B_{\theta r}(x_0)}\left|\frac{\tilde{w}-l^h(x)}{\theta r}\right|^p dx \\
&\quad \le 2^{p-1}\fint_{B_{\theta r}(x_0)}\left|\frac{\tilde{w}-h}{\theta r}\right|^p dx + 2^{p-1}\fint_{B_{\theta r}(x_0)}\left|\frac{h-l^h(x)}{\theta r}\right|^p dx \\
&\quad \le \frac{2^{p-1}}{\theta^p}\theta^{-Q}\varepsilon + 2^{p-1}C_p^p(\theta r)^p\fint_{B_{\theta r}(x_0)}\left|X^2h\right|^p dx \\
&\quad \le 2^{p-1}\theta^{-Q-p}\varepsilon + 2^{p-1}C_p^p(\theta r)^p\sup_{B_{\frac{r}{8}}(x_0)}\left|X^2h\right|^p dx
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&\le 2^{p-1}\theta^{-Q-p}\varepsilon + 2^{(4Q+1)\frac{p}{2}+p-1}C_p^p C_3^{\frac{p}{2}}\theta^p \\
&\le C_5\big(\theta^{-Q-p}\varepsilon + \theta^p\big),
\end{aligned}$$

where $C_5 = 2^{p-1}C \cdot 2^{(4Q+2)\frac{p}{2}+p-1}C_p^p C_3^{\frac{p}{2}} = 2^{(4Q+1)\frac{p}{2}+2(p-1)}C_p^p C_3^{\frac{p}{2}}$. And it infers that

$$\fint_{B_{\theta r}(x_0)}\left|\frac{u - l_{x_0,\theta r} - C_2\gamma(1+|Xl_{x_0,r}|)l^h(x)}{(\theta r)(1+|Xl_{x_0,r}|)}\right|^p dx \le C_5 C_2^p \gamma^p\big(\theta^{-Q-p}\varepsilon + \theta^p\big).$$

It follows by $0 < \gamma = (\frac{\delta}{2})^{-1}\sqrt{\Psi_*(x_0, 2r, l)} \le 1$, $\theta \in (0, \frac{1}{8}]$, $p \ge 2$, and take $\varepsilon = \theta^{Q+2p}$, then

$$\begin{aligned}
&\Psi(x_0, \theta r, l_{x_0,\theta r}) \\
&\quad = \fint_{B_{\theta r}(x_0)}\left|\frac{u - l_{x_0,\theta r}}{(\theta r)(1+|Xl_{x_0,\theta r}|)}\right|^2 dx + \fint_{B_{\theta r}(x_0)}\left|\frac{u - l_{x_0,\theta r}}{(\theta r)(1+|Xl_{x_0,\theta r}|)}\right|^p dx \\
&\quad \le \fint_{B_{\theta r}(x_0)}\left|\frac{u - l_{x_0,\theta r} - C_2\gamma(1+|Xl_{x_0,r}|)l^h(x)}{(\theta r)(1+|Xl_{x_0,r}|)}\right|^2 dx \\
&\qquad + \fint_{B_{\theta r}(x_0)}\left|\frac{u - l_{x_0,\theta r} - C_2\gamma(1+|Xl_{x_0,r}|)l^h(x)}{(\theta r)(1+|Xl_{x_0,r}|)}\right|^p dx \\
&\quad \le C_4 C_2^2\gamma^2\big(\theta^{-Q-2}\varepsilon + \theta^2\big) + C_5 C_2^p\gamma^p\big(\theta^{-Q-p}\varepsilon + \theta^p\big) \\
&\quad \le 2(C_4 + C_5)C_2^p\theta^2\gamma^2 \\
&\quad = 4(C_4 + C_5)C_2^p\theta^2\left(\frac{\delta}{2}\right)^{-2}\Psi_*(x_0, 2r, l) \\
&\quad := C_6\theta^2\Psi_*(x_0, 2r, l),
\end{aligned} \tag{4.15}$$

again for a constant $C_6 = 4(C_4 + C_5)C_2^p(\frac{\delta}{2})^{-2}$. □

In the following lemma, we iterate the excess improvement estimate from Lemma 8.

Lemma 9 *Suppose that the assumptions of Theorem 1 are satisfied, for $\mu = Q + 2\gamma > 2$, there exist $\varepsilon_*, k_*, \rho_* > 0$, $\theta \in (0, \frac{1}{8}]$ such that*

$$\Psi(x_0, r, l_{x_0,r}) < \varepsilon_*, \qquad \Upsilon_\mu(x_0, r) < \varepsilon_* \tag{A_0}$$

for $r \in (0, \rho_*)$ with $B_r(x_0) \subset\subset \Omega$ imply

$$\Psi\big(x_0, \theta^k r, l_{x_0,\theta^k r}\big) < \varepsilon_*, \qquad \Upsilon_\mu\big(x_0, \theta^k r\big) < \varepsilon_*, \tag{A_k}$$

respectively, for every $k \in \mathbb{N}$, where

$$\Upsilon_\mu(x_0, r) = r^{-\mu}\fint_{B_r(x_0)}|u - u_{x_0,r}|^2\,dx.$$

Proof Let

$$\theta < \frac{1}{2\sqrt{C_6}} \le \frac{1}{8},$$
$$\omega(\varepsilon_*) < \varepsilon_* < 1,$$
$$f\big(x_0, \theta^k r\big) < \varepsilon_*,$$

fix $\rho_* \in (0,1)$ small enough to guarantee

$$\rho_* \le \big\{\rho_0, \varepsilon_*^{\frac{1}{2-\mu}}, 1\big\}, \qquad V(\rho_*) < \varepsilon_*.$$

Next we will prove assertion (A_k) by induction. We assume that we have already established (A_k) up to some $k \in \mathbb{N}_0$. We begin with proving the first part of assertion (A_{k+1}), that is, the one concerning $\Psi(x_0, \theta^k r, l_{x_0,\theta^{k+1}r})$. For this we want to show that the assumptions for the excess improvement in Lemma 8 are satisfied. First, we note that

$$\fint_{B_{\theta^k r}(x_0)} \big|u - l(x_0)\big|^2\,dx = \big(\theta^k r\big)^\mu \Upsilon_\mu\big(x_0, \theta^k r\big) \le \Upsilon_\mu\big(x_0, \theta^k r\big).$$

So, it infers with $\Upsilon_\mu(x_0, \theta^k r) < \varepsilon_*$ that

$$\begin{aligned}
&\Psi_*\big(x_0, \theta^k r, l_{x_0,\theta^k r}\big) \\
&\quad = \Psi\big(x_0, \theta^k r, l_{x_0,\theta^k r}\big) + \omega\bigg(\fint_{B_{\theta^k r}(x_0)} \big|u - l(x_0)\big|^2\,dx\bigg) \\
&\qquad + V\big(\theta^k r\big) + \big(\theta^k r\big)^2 f\big(x_0, \theta^k r\big) \\
&\quad \le \Psi\big(x_0, \theta^k r, l_{x_0,\theta^k r}\big) + \omega\big[\Upsilon_\mu(x_0, r)\big] + V\big(\theta^k r\big) + \big(\theta^k r\big)^2 f\big(x_0, \theta^k r\big) \\
&\quad \le 2\varepsilon_* + \omega(\varepsilon_*) + V(\rho_*) \le 4\varepsilon_*.
\end{aligned} \tag{4.16}$$

We can thus apply Lemma 8 with the radius $\theta^k r$ instead of r, which yields

$$\Psi\big(x_0, \theta^{k+1} r, l_{x_0,\theta^{k+1}r}\big) \le C_6\theta^2 \Psi_*\big(x_0, \theta^{k+1} r, l_{x_0,\theta^{k+1}r}\big) \le 4C_6\theta^2\varepsilon_* < \varepsilon_*. \tag{4.17}$$

We have thus established the first part of assertion (A_{k+1}), and it remains to prove the second one, that is, the one concerning $\Upsilon_\mu(x_0, \theta^{k+1} r)$. For this aim, we first compute by the definition of $\Psi(x_0, \theta^k r, l_{x_0,\theta^k r})$ and (4.17),

$$\fint_{B_{\theta^k r}(x_0)} |u - l_{x_0,\theta^k r}|^2\,dx \le 2\big(\theta^k r\big)^2 \varepsilon_* \big(1 + |X l_{x_0,\theta^k r}|^2\big). \tag{4.18}$$

Abbreviating $l_{x_0,\theta^k r} = u_{x_0,\theta^k r} + X l_{x_0,\theta^k r}(x - x_0)$, and then, the latter part of (A_{k+1}) is estimated:

$$\begin{aligned}
\Upsilon_\mu\big(x_0, \theta^{k+1} r\big) &= \big(\theta^{k+1} r\big)^{-\mu} \fint_{B_{\theta^{k+1} r}(x_0)} |u - u_{x_0,\theta^{k+1} r}|^2\,dx \\
&\le \big(\theta^{k+1} r\big)^{-\mu} \fint_{B_{\theta^{k+1} r}(x_0)} |u - u_{x_0,\theta^k r}|^2\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le 2\big(\theta^{k+1}r\big)^{-\mu} \fint_{B_{\theta^{k+1}r}(x_0)} \big[|u - l_{x_0,\theta^k r}|^2\,dx + \big(\theta^{k+1}r\big)^2 |Xl_{x_0,\theta^k r}|^2\big] \\
&\le 2\big(\theta^{k+1}r\big)^{-\mu} [\theta^{-Q} \fint_{B_{\theta^k r}(x_0)} \big[|u - l_{x_0,\theta^k r}|^2\,dx + \big(\theta^{k+1}r\big)^2 |Xl_{x_0,\theta^k r}|^2\big] \\
&\le 2\big(\theta^{k+1}r\big)^{-\mu} \big[2\theta^{-Q}\big(\theta^k r\big)^2 \varepsilon_* \big(1 + |Xl_{x_0,\theta^k r}|^2\big) + \big(\theta^{k+1}r\big)^2 |Xl_{x_0,\theta^k r}|^2\big] \\
&\le 4\big(\theta^{k+1}r\big)^{-\mu} \big(\theta^k r\big)^2 \big[\big(\theta^{-Q}\varepsilon_* + \theta^2\big)|Xl_{x_0,\theta^k r}|^2 + \theta^{-Q}\varepsilon_*\big] \\
&\le 4\big(\theta^{k+1}r\big)^{2-\mu} \big[\big(\theta^{-Q}\varepsilon_* + (\theta r)^2\big)|Xl_{x_0,\theta^k r}|^2 + \theta^{-Q}\varepsilon_*\big],
\end{aligned}
\tag{4.19}
$$

where we use $(\theta^k r)^{-\mu} \le (\theta^{k+1}r)^{-\mu}$ in the last inequality.

Now, we need to estimate $|Xl_{x_0,\theta^k r}|$. Owing to (2.3) and $(\theta^k r)^\mu < 1$, we have

$$
|Xl_{x_0,\theta^k r}|^2 \le C \fint_{B_{\theta^k r}(x_0)} |u - u_{x_0,\theta^k r}|^2\,dx \le C\big(\theta^k r\big)^\mu \Upsilon_\mu(x_0, r) < C\varepsilon_*. \tag{4.20}
$$

Combined with the above estimations, take ε_* sufficiently small so that $4\theta^{-Q}\varepsilon_*^2 + 4C\theta^{-Q}\varepsilon_*^3 + 4C(\theta r)^2\varepsilon_*^2 < 1$ is true, then the following can be obtained:

$$
\begin{aligned}
\Upsilon_\mu\big(x_0, \theta^{k+1}r\big) &\le 4\big(\theta^k r\big)^{2-\mu} \big[\theta^{-Q}\varepsilon_* + \big(\theta^{-Q}\varepsilon_* + (\theta r)^2\big)|Xl_{x_0,\theta^k r}|^2\big] \\
&\le 4\rho_*^{2-\mu}\theta^{-Q}\varepsilon_* + 4C\rho_*^{2-\mu}\big(\theta^{-Q}\varepsilon_* + (\theta r)^2\big)\varepsilon_* \\
&\le \big(4\theta^{-Q}\varepsilon_*^2 + 4C\theta^{-Q}\varepsilon_*^3 + 4C(\theta r)^2\varepsilon_*^2\big)\varepsilon_* \\
&< \varepsilon_*.
\end{aligned}
\tag{4.21}
$$

This proves the second part of assertion (A_{k+1}) and finally concludes the proof of the lemma. □

Proof of Theorem 1 In fact, by Lebesgue's differentiation theorem, one gets $|\sum_1 \cup \sum_2| = 0$. Consequently, it suffices to show that every $x_0 \in \Omega \setminus (\sum_1 \cup \sum_2)$ is a regular point. For this, we note first that from Poincaré's inequality it implies

$$
\begin{aligned}
\Psi(x_0, r_0, l_{x_0,r}) &\le r^{-2} \fint_{B_{r_0}(x_0)} |u - l_{x_0,r_0}|^2\,dx + r^{-p} \fint_{B_{r_0}(x_0)} |u - l_{x_0,r_0}|^p\,dx \\
&\le C_p^2 \fint_{B_{r_0}(x_0)} |Xu - Xl_{x_0,r_0}|^2\,dx + C_p^p \fint_{B_{r_0}(x_0)} |Xu - Xl_{x_0,r_0}|^p\,dx.
\end{aligned}
$$

Because of $r_0 \le 1$, using

$$
l_{x_0,r_0} = u_{x_0,r_0} + Xl_{x_0,r_0}(\bar{x} - \bar{x}_0),
$$

namely, $u_{x_0,r_0} = l_{x_0,r_0} - Xl_{x_0,r_0}(\bar{x} - \bar{x}_0)$, $|l_{x_0,r_0}| + |Xl_{x_0,r_0}| \le M_0$, there holds

$$
\begin{aligned}
\Upsilon_\mu(x_0, r) &= r^{-\mu} \fint_{B_r(x_0)} |u - u_{x_0,r}|^2\,dx \\
&= r^{2-\mu} \fint_{B_r(x_0)} \frac{|u - u_{x_0,r}|^2}{r^2}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\le r^{2-\mu} \fint_{B_r(x_0)} \left(\frac{|u - u_{x_0,r}|}{r} + |Xl_{x_0,r_0}| \right)^2 dx \\
&\le 2r^{2-\mu} \left[\fint_{B_r(x_0)} \frac{|u - u_{x_0,r}|^2}{r^2} dx + |Xl_{x_0,r_0}|^2 \right] \\
&\le C(M_0) r^{2-\mu} \left[\fint_{B_r(x_0)} \frac{|u - l_{x_0,r}|^2}{r^2} dx + 1 \right] \\
&\le C(M_0) r^{2-\mu} \left[\Psi(x_0, r, l_{x_0,r}) + 1 \right],
\end{aligned}
$$

and by the definition of $\sum_1$ and $\sum_2$, there exists $0 < r < \min\{\rho_*, \operatorname{dist}(x_0, \partial\Omega)\}$ satisfying

$$
\Psi(x_0, r, l_{x_0,r}) < \varepsilon_*, \qquad \Upsilon_\mu(x_0, r) < \varepsilon_*.
$$

The continuity of integrals implies that there exists a neighborhood $U \subseteq \Omega$ of x_0 so that for any $x \in U$,

$$
\Psi(x, r, l_{x_0,r}) < \varepsilon_*, \qquad \Upsilon_\mu(x, r) < \varepsilon_*,
$$

there for all $x \in U$ and $k \in \mathbb{N}$ that

$$
\Psi\left(x, \theta^k r, l_{x_0,\theta^k r}\right) < \varepsilon_*, \qquad \Upsilon_\mu\left(x, \theta^k r\right) < \varepsilon_*,
$$

and it follows that

$$
\sup_{x \in U, \sigma \in (0,r)} \sigma^{-\mu} \fint_{B_\sigma(x)} |u - u_{x,\sigma}|^2 dx = \sup_{x \in U, \sigma \in (0,r)} \Upsilon_\mu(x, \sigma) < \varepsilon_* < \infty,
$$

i.e., $u \in \mathcal{L}^{p,\mu}(B_\sigma(x))$. Therefore, by Hölder's properties of Campanato type for a continuous function, we have

$$
u \in \Gamma^{0,\gamma}_{\mathrm{loc}}\left(U, \mathbb{R}^N\right). \qquad \square
$$

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