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New generalized Halanay inequalities and relative applications to neural networks with variable delays

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Abstract

The asymptotic behavior of solutions for a new class of generalized Halanay inequalities is studied via the fixed point method. This research provides a new approach to the study of the stability of Halanay inequality. To make the application of fixed point method in stability research more flexible and feasible, we introduce corresponding functions to construct an operator according to different characteristics of coefficients. The results obtained in this paper are applied to the stability study of a neural network system, which has high value in application. Moreover, three examples and simulations are given to illustrate the results. The conclusions in this paper greatly improve and generalize the relative results in the current literature.

Keywords: Generalized Halanay inequalities; Asymptotic stability; Fixed point method; Neural network

1 Introduction

The delay dynamical systems have been applied in a lot of fields such as medicine biology, neural networks, physics, electrical engineering, and other fields of engineering and science. Stability has always been the most widely studied in the theory of dynamical systems. Therefore, research on the stability of delay dynamical systems has been very fruitful, see for instance [1–22]. Recently, as a generalization of dynamical systems, many authors have studied the stability of Halanay inequality systems.

To discuss the asymptotic stability of the following dynamical systems with delay τ:

$$y'(t) = -ay(t) + by(t-\tau), \quad t \geq t_0,$$

Halanay ([4] and [5]) proved the so-called Halanay inequalities

$$y'(t) \leq -ay(t) + b \sup_{t-\tau \leq s \leq t} y(s), \quad (t \geq t_0), \qquad y(t) = \psi(t), \quad (t \leq t_0) \tag{1.1}$$

and the following lemma.

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Lemma 1.1 (*Halanay* 1975 [5]) *Let* $a > b > 0$ *and* $\psi(t) \geq 0$ *be continuous and bounded. If* $y(t)$ *satisfies* (1.1), *then there are* $\varrho, c > 0$ *such that* $y(t) \leq ce^{-\varrho(t-t_0)}$. *Hence, when* $t \to +\infty$, $y(t) \to 0$.

In view of the above properties of the Halanay inequality, many authors have studied the stability of various generalized types of delay dynamical systems. Let $R = (-\infty, +\infty)$, $R^+ = (0, +\infty)$, $C(A, \Omega)$ be a continuous function from A to Ω, $BC(A, \Omega)$ be a bounded continuous function from A to Ω, $I_m = 1, 2, \dots, m$. Baker and Tang [6] gave one generalization of (1.1) and obtained Lemma 1.2.

Lemma 1.2 (*Baker and Tang* 1996 [6]) *Let* $y(t) > 0$ *satisfy*

$$y'(t) \leq -a(t)y(t) + b(t) \sup_{t-\tau(t)\leq s\leq t} y(s) \quad \text{for } t \geq t_0 \quad \text{and} \quad y(t) = \psi(t) \quad \text{for } t \leq t_0, \tag{1.2}$$

where $\psi(t) \in BC((-\infty, t_0], R^+)$. *When* $t \geq t_0$, $a(t), b(t), \tau(t) \geq 0$. *And* $\tau(t)$ *satisfies* $t - \tau(t) \to +\infty$ *as* $t \to +\infty$. *There is* $\theta > 0$ *such that* $b(t) - a(t) \leq -\theta < 0$ *for* $t \geq t_0$, *then* $y(t) \to 0$ *as* $t \to +\infty$.

Some authors further presented the generalized Halanay inequality

$$\begin{aligned} &D^+y(t) \leq -a(t)y(t) + b(t) \sup_{t-\tau(t)\leq s\leq t} y(s) \quad \text{for } t \geq t_0 \quad \text{and} \\ &y(t) = \psi(t) \quad \text{for } t \leq t_0, \end{aligned} \tag{1.3}$$

where the upper-right Dini derivative $D^+y(t)$ is defined as

$$D^+y(t) = \limsup_{\sigma\to 0^+} \frac{y(t+\sigma) - y(t)}{\sigma}, \tag{1.4}$$

$a(t)$, $b(t)$ and $\tau(t)$ are defined by Lemma 1.2.

As applications of generalized Halanay inequality (1.3), Tian [7] researched the stability and boundedness of inequality (1.3) with constant delays. Wen [8] obtained the dissipativity results of Volterra functional differential equations. Based on Wen [8], Liu ([9] and [10]) considered boundedness, asymptotic stability, and exponential stability of inequality (1.3) and obtained the following lemma.

Lemma 1.3 (*Liu* 2012 [10]) *If* $y(t)$ *satisfies* (1.3), $b(t) \geq 0$, *then there are* $\lambda_a > 0$, $\lambda_b > 0$, $\tilde{\tau} > 0$ *such that* $|a(t)| \leq \lambda_a$, $b(t) \leq \lambda_b$, $\tau(t) \leq \tilde{\tau}$, *then* $y(t) \to 0$ *as* $t \to +\infty$ *if* $\lim_{t\to+\infty} \int_o^t [a(s) - b(s)e^{M_a\tilde{\tau}}]\,ds = +\infty$.

Then, Ruan [11] studied the stability and boundedness of inequality (1.3) by integral inequalities and obtained the following lemma.

Lemma 1.4 (*Ruan* [11]) *If* $y(t)$ *satisfies* (1.3), $b(t) \geq 0$, *and*

$$\lim_{t\to+\infty} \int_o^t \left[-a(s) + b(s)e^{\sup_{t-\tau(t)\leq s\leq t} \int_s^t a(v)\,dv}\right] ds = -\infty,$$

then $y(t) \to 0$ *as* $t \to +\infty$.

Many authors used the stability results of Halanay inequality to study the synchronization and stability of neural networks. When studying the stability of dynamical systems, most of the authors (such as [12, 13]) used Lyapunov's direct method. Yet, there are many problems which make this method invalid. For solving the problems encountered in the study of Lyapunov's direct method, Burton and other authors [14–18] investigated the stability of stochastic dynamical systems driven by Brownian motion using fixed point theory. Later, Shahram Rezapour and his collaborators [19–21] used the fixed point method to study the properties associated with the solution of stochastic fractional differential system. The results showed that the fixed point method can overcome many problems in the study of the stability of dynamical systems.

However, when using Halanay inequalities to discuss the stability of dynamical systems, the fixed point method is seldom used. In this paper, we study the asymptotic stability of dynamical system with variable delays via generalized Halanay inequalities by the fixed point method. In particular, the obtained conclusions improve and promote the results of some existing papers. See the examples in Sect. 4.

The remaining parts of the paper are designed as follows. The main theoretical conclusions are firstly proposed and then proved in Sect. 2. The conclusions in Sect. 2 are applied to study the global stability of neural networks in Sect. 3. Examples with numerical simulations are illustrated in Sect. 4. The conclusions are given in Sect. 5.

2 Main results

Consider the following generalized Halanay's inequality with multiple delays:

$$\begin{cases} D^+y(t) \le -\sum_{i=1}^m a_i(t)y(t) + \sum_{i=1}^m b_i(t)y(g_i(t)) \\ \qquad\qquad + \sum_{k=1}^m c_k(t)\sup_{r_k(t)\le s\le t} y(s), \quad t \ge 0, \\ y(t) = |\psi(t)| \in C([\psi(0),0],R^+), \quad t \le 0. \end{cases} \tag{2.1}$$

Here,

$$\psi(0) = \max_{1\le i,k\le m}\bigl\{\inf\bigl(g_i(s), s\ge 0\bigr), \inf\bigl(r_k(s), s \ge 0\bigr)\bigr\},$$

$D^+y(t)$ is defined by (1.4), $a_i(t), b_i(t), c_k(t), g_i(t), r_k(t) \in C(R^+,R)$ satisfy $g_i(t) \to \infty$, $r_k(t) \to \infty$ as $t \to \infty$. Besides, let $g_i(t) \le t$ be differentiable and $r_k(t) \le t$ $(1 \le i, k \le m < \infty)$.

Theorem 2.1 *Assume that there are some functions $f_i(t) \in C(R^+,R^+)$, $(i \in I_m)$ and a positive constant $\alpha < 1$ such that, for $t \ge 0$,*

(i)

$$f(t) = \sum_{i=1}^m f_i(t) \quad \textit{and} \quad \liminf_{t\to\infty} \int_0^t f(s)\,ds > -\infty,$$

(ii)

$$\sup_{t\ge t_0}\Biggl\{\sum_{i=1}^m \int_{g_i(t)}^t \bigl|f_i(s) - a_i(s)\bigr|\,ds$$

$$
\begin{aligned}
&+\int_0^t e^{-\int_s^t f(v)\,dv}\sum_{i=1}^m \bigl|b_i(s)+\bigl[f_i\bigl(g_i(s)\bigr)-a_i\bigl(g_i(s)\bigr)\bigr]g_i'(s)\bigr|\,ds\\
&+\int_0^t f(s)e^{-\int_s^t f(v)\,dv}\left(\sum_{i=1}^m\int_{g_i(s)}^s \bigl|f_i(v)-a_i(v)\bigr|\,dv\right)ds\\
&+\int_0^t e^{-\int_s^t f(v)\,dv}\sum_{k=1}^m \bigl|c_k(s)\bigr|\,ds\Biggr\}\\
\le{}& \alpha<1.
\end{aligned}
$$

Then $y(t)\to 0$ as $t\to+\infty$ if and only if

(iii)

$$\int_0^t f(s)\,ds\to\infty \quad as\ t\to\infty.$$

Proof Define the following delay dynamic system:

$$
\begin{cases}
dy(t)=[\sum_{i=1}^m(-a_i(t))y(t)+\sum_{i=1}^m b_i(t)y(g_i(t))\\
\qquad +\sum_{k=1}^m c_k(t)\sup_{r_k(t)\le s\le t}y(s)]\,dt, \quad t\ge 0,\\
y(t)=|\psi(t)|\in C([\psi(0),0],R^+), \quad t\le 0.
\end{cases}
\tag{2.2}
$$

From (2.2), we obtain

$$
\begin{aligned}
y(t)={}&\phi(0)e^{-\int_0^t f(v)\,dv}+\int_0^t\left[f(s)-\sum_{i=1}^m a_i(s)\right]e^{-\int_s^t f(v)\,dv}y(s)\,ds\\
&+\int_0^t e^{-\int_s^t f(v)\,dv}\sum_{i=1}^m b_i(s)y\bigl(g_i(s)\bigr)\,ds\\
&+\int_0^t e^{-\int_s^t f(v)\,dv}\sum_{k=1}^m c_k(s)\sup_{r_k(s)\le u\le s}y(u)\,ds.
\end{aligned}
\tag{2.3}
$$

Among

$$
\begin{aligned}
&\int_0^t\left[f(s)-\sum_{i=1}^m a_i(s)\right]e^{-\int_s^t f(v)\,dv}y(s)\,ds\\
&=\sum_{i=1}^m\int_0^t e^{-\int_s^t f(v)\,dv}\,d\int_{g_i(s)}^s\bigl[f_i(u)-a_i(u)\bigr]y(u)\,du\\
&\quad+\sum_{i=1}^m\int_0^t e^{-\int_s^t f(v)\,dv}\bigl[f_i\bigl(g_i(s)\bigr)-a_i\bigl(g_i(s)\bigr)\bigr]g_i'(s)y\bigl(g_i(s)\bigr)\,ds\\
&=\sum_{i=1}^m\int_{g_i(t)}^t\bigl[f_i(s)-a_i(s)\bigr]y(s)\,ds-e^{-\int_0^t f(v)\,dv}\sum_{i=1}^m\int_{g_i(0)}^0\bigl[f_i(s)-a_i(s)\bigr]\psi(s)\,ds\\
&\quad+\sum_{i=1}^m\int_0^t e^{-\int_s^t f(v)\,dv}\bigl[f_i\bigl(g_i(s)\bigr)-a_i\bigl(g_i(s)\bigr)\bigr]g_i'(s)y\bigl(g_i(s)\bigr)\,ds
\end{aligned}
\tag{2.4}
$$

$$-\int_0^t f(s)e^{-\int_s^t f(v)\,dv}\left(\sum_{i=1}^m \int_{g_i(s)}^s \bigl[f_i(u)-a_i(u)\bigr]y(u)\,du\right)ds.$$

So, combining (2.3) and (2.4), we know

$$\begin{aligned} y(t) &= \left(\psi(0)-\sum_{i=1}^m \int_{g_i(0)}^0 \bigl[f_i(s)-a_i(s)\bigr]\psi(s)\,ds\right)e^{-\int_0^t f(v)\,dv} + \sum_{i=1}^m \int_{g_i(t)}^t \bigl[f_i(s)-a_i(s)\bigr]y(s)\,ds \\ &\quad + \int_0^t e^{-\int_s^t f(v)\,dv}\sum_{i=1}^m \bigl(\bigl[f_i\bigl(g_i(s)\bigr)-a_i\bigl(g_i(s)\bigr)\bigr]g_i'(s)+b_i(s)\bigr)f\bigl(g_i(s)\bigr)\,ds \\ &\quad - \int_0^t f(s)e^{-\int_s^t f(v)\,dv}\left(\sum_{i=1}^m \int_{g_i(s)}^s \bigl[f_i(u)-a_i(u)\bigr]y(u)\,du\right)ds \\ &\quad + \int_0^t e^{-\int_s^t f(v)\,dv}\sum_{k=1}^m c_k(s)\sup_{r_k(s)\le u\le s} y(u)\,ds. \end{aligned} \tag{2.5}$$

Define the operator $\Psi : S \to S$ as follows:

$$\begin{aligned} (\Psi y)(t) &= \left(\psi(0)-\sum_{i=1}^m \int_{g_i(0)}^0 \bigl[f_i(s)-a_i(s)\bigr]\psi(s)\,ds\right)e^{-\int_0^t f(v)\,dv} \\ &\quad + \sum_{i=1}^m \int_{g_i(t)}^t \bigl[f_i(s)-a_i(s)\bigr]y(s)\,ds \\ &\quad + \int_0^t e^{-\int_s^t f(v)\,dv}\sum_{i=1}^m \bigl(\bigl[f_i\bigl(g_i(s)\bigr)-a_i\bigl(g_i(s)\bigr)\bigr]g_i'(s)+b_i(s)\bigr)y\bigl(g_i(s)\bigr)\,ds \\ &\quad - \int_0^t f(s)e^{-\int_s^t f(v)\,dv}\left(\sum_{i=1}^m \int_{g_i(s)}^s \bigl[f_i(u)-a_i(u)\bigr]y(u)\,du\right)ds \\ &\quad + \int_0^t e^{-\int_s^t f(v)\,dv}\sum_{k=1}^m c_k(s)\sup_{r_k(s)\le u\le s} y(u)\,ds := \sum_{j=1}^5 I_j(t),\ t\ge 0. \end{aligned} \tag{2.6}$$

The initial value is $(\Psi y)t) = \psi(t)$ for $t \in [\psi(0),0]$. Denote by $\mathcal{S}$ the Banach space of all functions $\varphi \in BC(R,R)$. Then $\mathcal{S}$ is a complete metric space with metric $\rho(x,y) = \sup_{t\ge 0}|x(t)-y(t)|$. Moreover, $\varphi(s) = |\psi(s)|$ for $s \in (-\infty,0]$, and when $t \ge 0$, we have $|\varphi(t)| \to 0$ as $t \to +\infty$. Then it is obvious that Ψ is continuous on $[0,\infty)$.

Next, we show that $\Psi(S) \in S$. From condition (iii), we know, when $t \to \infty$,

$$\bigl|I_1(t)\bigr| = \left|\phi(0)-\sum_{i=1}^m \int_{g_i(0)}^0 \bigl[f_i(s)-a_i(s)\bigr]\psi(s)\,ds\right|e^{-\int_0^t f(v)\,dv} \to 0. \tag{2.7}$$

As $t \to \infty$, $g_i(t) \to \infty$, and $|y(t)| \to 0$, then for any $\epsilon > 0$, there exists $T_1 > 0$ such that $t \ge T_1$ implies $|y(t)| < \epsilon$ and $|y(g_i(t))| < \epsilon$, $i \in I_m$. Hence, when $t \ge T_1$, from condition (ii), we have

$$\bigl|I_2(t)\bigr| = \left|\sum_{i=1}^m \int_{g_i(t)}^t \bigl[f_i(s)-a_i(s)\bigr]y(s)\,ds\right| \le \epsilon\left(\sum_{i=1}^m \int_{g_i(t)}^t \bigl|f_i(s)-a_i(s)\bigr|\,ds\right) < \epsilon. \tag{2.8}$$

Then $|I_2(t)| \to 0$ as $t \to \infty$. Meanwhile, from condition (ii), we have

$$\begin{aligned}
|I_3(t)| &= \left| \int_0^t e^{-\int_s^t f(v)\,dv} \sum_{i=1}^m \bigl[\bigl(f_i\bigl(g_i(s)\bigr) - a_i\bigl(g_i(s)\bigr)\bigr)g_i'(s) + b_i(s)\bigr] y\bigl(g_i(s)\bigr)\,ds \right| \\
&= \left| \int_0^{T_1} e^{-\int_s^t f(v)\,dv} \sum_{i=1}^m \bigl[\bigl(f_i\bigl(g_i(s)\bigr) - a_i\bigl(g_i(s)\bigr)\bigr)g_i'(s) + b_i(s)\bigr] y\bigl(g_i(s)\bigr)\,ds \right. \\
&\quad \left. + \int_{T_1}^t e^{-\int_s^t f(v)\,dv} \sum_{i=1}^m \bigl[\bigl(f_i\bigl(g_i(s)\bigr) - a_i\bigl(g_i(s)\bigr)\bigr)g_i'(s) + b_i(s)\bigr] y\bigl(g_i(s)\bigr)\,ds \right| \\
&\le \left| \int_0^{T_1} e^{-\int_s^t f(v)\,dv} \sum_{i=1}^m \bigl[\bigl(f_i\bigl(g_i(s)\bigr) - a_i\bigl(g_i(s)\bigr)\bigr)g_i'(s) + b_i(s)\bigr] y\bigl(g_i(s)\bigr)\,ds \right| \\
&\quad + \left| \int_{T_1}^t e^{-\int_s^t f(v)\,dv} \sum_{i=1}^m \bigl[\bigl(f_i\bigl(g_i(s)\bigr) - a_i\bigl(g_i(s)\bigr)\bigr)g_i'(s) + b_i(s)\bigr] y\bigl(g_i(s)\bigr)\,ds \right| \\
&\le \sup_{t\in[0,T_1]} \bigl|y(t)\bigr| \left(\int_0^{T_1} e^{-\int_s^t f(v)\,dv} \sum_{i=1}^m \bigl[\bigl(f_i\bigl(g_i(s)\bigr) - a_i\bigl(g_i(s)\bigr)\bigr)g_i'(s) + b_i(s)\bigr]\,ds \right) \\
&\quad + \epsilon \left(\int_{T_1}^t e^{-\int_s^t f(v)\,dv} \sum_{i=1}^m \bigl[\bigl(f_i\bigl(g_i(s)\bigr) - a_i\bigl(g_i(s)\bigr)\bigr)g_i'(s) + b_i(s)\bigr]\,ds \right).
\end{aligned} \tag{2.9}$$

By condition (iii), there exists $T_2 \ge T_1$ when $t \ge T_2$ such that

$$\sup_{t\in[0,T_1]} \bigl|y(t)\bigr| \left(\int_0^{T_1} e^{-\int_s^t f(v)\,dv} \sum_{i=1}^m \bigl[\bigl(f_i\bigl(g_i(s)\bigr) - a_i\bigl(g_i(s)\bigr)\bigr)g_i'(s) + b_i(s)\bigr]\,ds \right) < \epsilon. \tag{2.10}$$

We easily know that $|I_3(t)| < 2\epsilon$ by condition (ii). Therefore, $|I_3(t)| \to 0$, as $t \to \infty$. Similarly, we can get $|I_4(t)| \to 0$ as $t \to \infty$.

As $t \to \infty$, $r_k(t) \to \infty$ and $|y(t)| \to 0$. Then there is $T_3 > 0$ such that $r_k(t) \ge T_3$, $k \in I_m$, implies $|y(t)| < \epsilon$ for any $\epsilon > 0$. Hence, when $r_k(t) \ge T_3$, we have

$$\begin{aligned}
\bigl|I_5(t)\bigr| &\le \left| \int_0^{T_3} e^{-\int_s^t f(v)\,dv} \sum_{k=1}^m c_k(s) \sup_{r_k(s)\le u\le s} y(u)\,ds \right| \\
&\quad + \left| \int_{T_3}^t e^{-\int_s^t f(v)\,dv} \sum_{k=1}^m c_k(s) \sup_{r_k(s)\le u\le s} y(u)\,ds \right| \\
&\le \sup_{r_k(0)\le t\le T_3} \bigl|y(t)\bigr| \left| \int_0^{T_3} e^{-\int_s^t f(v)\,dv} \sum_{k=1}^m c_k(s)\,ds \right| \\
&\quad + \epsilon \left| \int_{T_3}^t e^{-\int_s^t f(v)\,dv} \sum_{k=1}^m c_k(s)\,ds \right|.
\end{aligned} \tag{2.11}$$

As can be seen from the above proof, $|I_5(t)| \to 0$ as $t \to \infty$. Then we have $|(\Psi y)(t)| \to 0$ as $t \to \infty$. Therefore, we get the conclusion of $\Psi(S) \subset S$.

For $\xi \in S$ and $\varphi \in S$, we have

$$\begin{aligned}
&\sup_{s\in[0,t]} \left|(\Psi\xi)(s) - (\Psi\varphi)(s)\right| \\
&\quad \le \sup_{s\in[0,t]} \left|\xi(s) - \varphi(s)\right| \sup_{s\in[0,t]} \left\{ \sum_{i=1}^{m} \int_{g_i(s)}^{s} \left|f_i(v) - a_i(v)\right| dv \right. \\
&\qquad + \int_0^s e^{-\int_v^s f(v)\,dv} \sum_{i=1}^{m} \left|b_i(v) + \left(f_i\left(g_i(s)\right) - a_i\left(g_i(s)\right)\right) g_i'(v)\right| dv \\
&\qquad + \int_0^s f(v) e^{-\int_v^s f(v)\,dv} \left(\sum_{i=1}^{m} \int_{g_i(v)}^{v} \left|f_i(u) - a_i(u)\right| du \right) dv \\
&\qquad \left. + \int_0^s e^{-\int_v^s f(v)\,dv} \sum_{k=1}^{m} \left|c_k(v)\right| dv \right\} \le \alpha \sup_{s\in[0,t]} \left|\xi(s) - \varphi(s)\right|.
\end{aligned} \tag{2.12}$$

Therefore, we obtain that Ψ is a contraction mapping according to the contraction mapping principle. Ψ has a unique fixed point $y(t)$ in $\mathcal{S}$ by the contraction mapping principle. The fixed point is a solution of (2.2) with $y(s) = |\psi(s)|$ on $[\psi(0), 0)$ and $|y(t)| \to 0$ as $t \to \infty$.

Then, we need to prove that the zero solution of (2.2) is stable. Suppose that $\sigma > 0$ is given and choose a positive constant $\sigma(\theta < \sigma)$ satisfying

$$\theta \left(1 + \sum_{i=1}^{m} \int_{g_i(0)}^{0} \left|f_i(s) - a_i(s)\right| ds \right) e^{-\int_0^t f(v)\,dv} + \alpha\sigma < \sigma.$$

If $y(t) = y(t, 0, |\psi|)$ is a solution of (2.2) with $|\psi| < \theta$, then $y(t) = (\Psi y)(t)$ is defined in (2.6). We have $|y(t)| < \sigma$ for all $t \ge 0$. Notice that $|y(t)| < \sigma$ on $[\psi(0), 0)$. Suppose that there is $t^* > 0$ such that $|y(t^*)| = \sigma$ and $|y(s)| < \sigma$ for $\psi(0) \le s < t^*$. From (2.6), we obtain

$$\begin{aligned}
\left|y(t^*)\right| &\le |\psi| \left(1 + \sum_{i=1}^{m} \int_{g_i(0)}^{0} \left|f_i(s) - a_i(s)\right| ds \right) e^{-\int_0^{t^*} f(v)\,dv} \\
&\quad + \sigma \left\{ \sum_{i=1}^{m} \int_{g_i(s)}^{s} \left|f_i(v) - a_i(v)\right| dv \right. \\
&\quad + \int_0^s e^{-\int_v^s f(v)\,dv} \sum_{i=1}^{m} \left|b_i(v) + \left(f_i\left(g_i(s)\right) - a_i\left(g_i(s)\right)\right) g_i'(v)\right| dv \\
&\quad + \int_0^s f(v) e^{-\int_v^s f(v)\,dv} \left(\sum_{i=1}^{m} \int_{g_i(v)}^{v} \left|f_i(u) - a_i(u)\right| du \right) dv \\
&\quad \left. + \int_0^s e^{-\int_v^s f(v)\,dv} \sum_{k=1}^{m} \left|c_k(v)\right| dv \right\} \\
&\le |\psi| \left(1 + \sum_{i=1}^{m} \int_{g_i(0)}^{0} \left|f_i(s) - a_i(s)\right| ds \right) e^{-\int_0^{t^*} f(v)\,dv} + \alpha\sigma < \sigma.
\end{aligned} \tag{2.13}$$

This is contradictory to the definition of t^*. Thus the zero solution of (2.2) is asymptotically stable if condition (iii) is established.

On the contrary, assume that condition (iii) is not met, then there is a sequence t_l, $t_l \to \infty$ as $l \to \infty$ such that $\lim_{l\to\infty} \int_0^{t_l} f(s)\,ds = p$ for some $p \in R$ by condition (i). We can select a constant $Q > 0$ satisfying $0 < \int_0^{t_l} f(s)\,ds \le Q$ for all $l \ge 1$. We define $A(s)$ as follows for simplification:

$$\begin{aligned} A(s) &:= \sum_{i=1}^{m} \bigl| b_i(v) + \bigl[f_i\bigl(g_i(s)\bigr) - a_i\bigl(g_i(s)\bigr)\bigr] g_i'(v) \bigr| \\ &\quad + f(v) \left(\sum_{i=1}^{m} \int_{g_i(v)}^{v} \bigl| f_i(u) - a_i(u) \bigr|\,du \right), \quad s \ge 0. \end{aligned}$$

By condition (ii), we have

$$\int_0^{t_l} e^{-\int_s^{t_l} f(v)\,dv} A(s)\,ds \le \alpha.$$

This yields

$$\int_0^{t_l} e^{\int_0^s f(v)\,dv} A(s)\,ds \le \alpha e^{\int_0^{t_l} f(v)\,dv} \le e^{Q}.$$

From the above, there is a convergent subsequence as $\{\int_0^{t_l} e^{\int_0^s f(v)\,dv} A(s)\,ds\}$ is bounded. For the convenience, we may suppose that there exists some $\gamma \in R^+$ such that

$$\lim_{l\to\infty} \int_0^{t_l} e^{\int_0^s f(v)\,dv} A(s)\,ds = \gamma.$$

Then we can find an integer $\tilde{k} > 0$ large enough such that, for all $l \ge \tilde{k}$,

$$\lim_{l\to\infty} \int_{t_{\tilde{k}}}^{t_l} e^{\int_0^s f(v)\,dv} A(s)\,ds < \frac{\theta}{8\beta},$$

where $\beta = \sup_{t\in[0,+\infty)} e^{-\int_0^t f(v)\,dv}$, $\theta > 0$ satisfies $8\theta\beta e^{Q} + \alpha < 1$.

Next, we will discuss the zero solution $y(t) = y(t, t_{\tilde{k}}, |\psi|)$ of system (2.2) with $|\psi(t_{\tilde{k}})| = \theta$ and $|\psi(s)| \le \theta$ for $s \le t_{\tilde{k}}$. Then $|y(t)| \le 1$ for $t \ge t_{\tilde{k}}$. We may select ψ such that

$$B(t_{\tilde{k}}) := \psi(t_{\tilde{k}}) - \sum_{i=1}^{m} \int_{g_i(t_{\tilde{k}})}^{t_{\tilde{k}}} \bigl[f_i(s) - a_i(s) \bigr] \psi(s)\,ds \ge \frac{1}{2}\theta.$$

From (2.6), we obtain

$$\begin{aligned} &\left| y(t_l) - \sum_{i=1}^{m} \int_{g_i(t_l)}^{t_l} \bigl[f_i(s) - a_i(s) \bigr] y(s)\,ds \right| \\ &\quad \ge B(t_{\tilde{k}}) e^{-\int_{t_{\tilde{k}}}^{t_l} f(v)\,dv} \left\{ B(t_{\tilde{k}}) e^{-\int_{t_{\tilde{k}}}^{t_l} f(v)\,dv} - 2\int_{t_{\tilde{k}}}^{t_l} e^{-\int_s^{t_l} f(v)\,dv} A(s)\,ds \right\} \\ &\quad \ge \frac{1}{2}\theta e^{-\int_{t_{\tilde{k}}}^{t_l} f(v)\,dv} \left\{ \frac{1}{2}\theta e^{-\int_{t_{\tilde{k}}}^{t_l} f(v)\,dv} - 2\int_{t_{\tilde{k}}}^{t_l} e^{-\int_s^{t_l} f(v)\,dv} A(s)\,ds \right\} \end{aligned}$$

$$= \frac{1}{2}\theta e^{-\int_{t_{\tilde{k}}}^{t_l} f(v)\,dv}\left\{\frac{1}{2}\theta e^{-\int_{t_{\tilde{k}}}^{t_l} f(v)\,dv} - 2e^{-\int_0^{t_l} f(v)\,dv}\int_{t_{\tilde{k}}}^{t_l} e^{\int_0^s f(v)\,dv}A(s)\,ds\right\} \tag{2.14}$$

$$\geq \frac{1}{2}\theta e^{-2\int_{t_{\tilde{k}}}^{t_l} f(v)\,dv}\left\{\frac{1}{2}\theta - 2\beta\int_{t_{\tilde{k}}}^{t_l} e^{\int_0^s f(v)\,dv}A(s)\,ds\right\}$$

$$\geq \frac{1}{8}\theta^2 e^{-2\int_{t_{\tilde{k}}}^{t_l} f(v)\,dv}$$

$$\geq \frac{1}{8}\theta^2 e^{-2Q} > 0.$$

However, provided $g_i(t_l) \to \infty$ as $l \to \infty$ holds. From condition (ii), we have $|y(t_l) - \sum_{i=1}^m \int_{g_i(t_l)}^{t_l}[f_i(s) - a_i(s)]y(s)\,ds| \to 0$ as $l \to \infty$ for $|y(t)| = |y(t, t_{\tilde{k}}, |\psi|)| \to 0$, which is contradictory to (2.14). Therefore, for the asymptotic stability of system (2.2), condition (iii) is essential. Thus, system (2.1) is asymptotically stable if and only if condition (iii) holds. The proof is complete. □

Apparently, if we set $m = k = 1$, $a_1(t) = a(t)$, $b_i(t) = 0$, $c_1(t) = b(t)$, $g_1(t) = r_1(t) = t - \tau(t)$ in Theorem 2.1, we have Theorem 2.2.

Theorem 2.2 *Let $\tau(t)$ be differentiable. Assume that $y(t)$ satisfies* (1.3), *there are $f(t) \in C(R^+, R^+)$ and a positive constant $\alpha < 1$ such that, for $t \geq 0$,*

(i)

$$\liminf_{t\to\infty} \int_0^t f(s)\,ds > -\infty,$$

(ii)

$$\begin{aligned}\sup_{t\geq t_0}\bigg\{&\int_{t-\tau(t)}^t \big|f(s) - a(s)\big|\,ds \\ &+ \int_0^t e^{-\int_s^t f(v)\,dv}\big|\big[f\big(s-\tau(s)\big) - a\big(s-\tau(s)\big)\big]\big(1-\tau'(s)\big)\big|\,ds \\ &+ \int_0^t f(s)e^{-\int_s^t f(v)\,dv}\bigg(\int_{s-\tau(s)}^s \big|f(u) - a(u)\big|\,du\bigg)\,ds \\ &+ \int_0^t e^{-\int_s^t f(v)\,dv}\big|b(s)\big|\,ds\bigg\} \leq \alpha < 1.\end{aligned}$$

Then $y(t) \to 0$ as $t \to +\infty$ if and only if

(iii)

$$\int_0^t f(s)\,ds \to \infty \quad \text{as } t \to \infty.$$

Remark 2.1 We do not require bounded delay $\tau(t)$ nor inverse function of delay, which improves the results in a lot of literature works, for example, [9, 10, 15].

Remark 2.2 In Theorem 2.2, we do not require $a(t) > b(t)$. This greatly improves the conclusions of studies [5, 8–11].

3 Applications

Consider the Grossberg–Hopfield neural network with multiple time-varying delays as follows:

$$\begin{cases} dy_i(t) = [\sum_{j=1}^m (-a_{ij}(t))y_i(t) + \sum_{j=1}^m b_{ij}(t)h_j(y_j(t)) \\ \qquad + \sum_{j=1}^m c_{ij}(t)g_j(y_j(k_{ij}(t))) + I_i(t)]\,dt, \quad t \geq 0, \\ y_i(t) = |\psi_i(t)| \in C([\psi(0),0],R), \quad t \leq 0. \end{cases} \tag{3.1}$$

Here, self-inhibition $a_{ij}(t)$, the interconnection weights $b_{ij}(t)$, $c_{ij}(t)$ and $h_j(t)$, $g_j(t)$, $k_{ij}(t)$ are scalar integrable functions for $t \in [0,+\infty)$, inputs $I_i(t) : R^+ \longrightarrow R$ are continuously functions, $i \in I_m$. $\psi(0)$ is defined as above.

Definition 3.1 (Gopalsam [22]) The solution $u(t) = (u_1(t),\dots,u_n(t))$ of (3.1) is globally asymptotically stable if and only if every other solution $v(t) = (v_1(t),\dots,v_n(t))$ of (3.1) with $v_i(0) > 0$ $(i \in I_m)$ is defined for all $t > 0$ and satisfies

$$\lim_{t\to\infty} \bigl|u_i(t) - v_i(t)\bigr| = 0, \quad i = l,2,\dots.n. \tag{3.2}$$

Theorem 3.1 *The functions $h_j(t)$, $g_j(t)$ satisfying the Lipschitz condition with Lipschitz's constant L_j, P_j are differentiable $(j \in I_m)$. Assume that there is a positive constant $\alpha < 1$ and some functions $f_{ij}(t) \in C(R^+,R^+)$ $(i,j \in I_M)$ such that, for $t \geq 0$,*

(i)

$$f_i(t) = \sum_{j=1}^m f_{ij}(t) \quad \text{and} \quad \liminf_{t\to\infty} \int_0^t f_i(t)\,ds > -\infty,$$

(ii)

$$\sup_{t\geq t_0} \left\{ \int_0^t e^{-\int_s^t f_i(v)\,dv} \sum_{j=1}^m \bigl|b_{ij}(s)L_j + f_{ij}(s) - a_{ij}(s)\bigr|\,ds + \int_0^t e^{-\int_s^t f_i(v)\,dv} \sum_{j=1}^m \bigl|c_{ij}(s)P_j\bigr|\,ds \right\}$$
$$\leq \alpha < 1.$$

Then the neural network system (3.1) is globally asymptotically stable if and only if

(iii)

$$\int_0^t f_i(s)\,ds \to \infty \quad \text{as } t \to \infty.$$

Proof For system (3.1), we know

$$\begin{aligned} \frac{d[u_i(t) - v_i(t)]}{dt} &= -\sum_{j=1}^m a_{ij}(t)\bigl[u_i(t) - v_i(t)\bigr] + \sum_{j=1}^m b_{ij}(t)\bigl[h_j\bigl(u_j(t)\bigr) - h_j\bigl(v_j(t)\bigr)\bigr] \\ &\quad + \sum_{j=1}^m c_{ij}(t)\bigl[g_j\bigl(u_j\bigl(k_{ij}(t)\bigr)\bigr) - g_j\bigl(v_j\bigl(k_{ij}(t)\bigr)\bigr)\bigr], \quad t \geq 0, i \in I_m. \end{aligned} \tag{3.3}$$

Define $x_i(t) = u_i(t) - v_i(t)$, $t \geq 0$, $k(t) = \min\{k_{ij}(t)\}, i, j \in I_m$. From Theorem 3.1, we can obtain the following inequalities:

$$\begin{cases} D^+ x_i(t) \leq \sum_{j=1}^m (-a_{ij}(t)) x_i(t) + \sum_{j=1}^m b_{ij}(t) L_j x_j(t) \\ \qquad\qquad + \sum_{j=1}^m c_{ij}(t) P_j \sup_{k_{ij}(t) \leq s \leq t} x_j(s), \quad t \geq 0, \\ x_i(t) = \sup_{k(0) \leq s \leq 0} x_i(s), \quad t \leq 0. \end{cases} \tag{3.4}$$

For $t \geq 0$, define $y(t) := \max\{x_i(t), i \in I_m\}$. For all $t \in [0, +\infty)$, let $i_{\tilde{t}}$ stand for the index such that $y(t) = |x_{i_{\tilde{t}}}(t)|$. So, we have for $t \geq 0$

$$\begin{aligned} D^+ y(t) &\leq \sum_{j=1}^m \bigl(-a_{i_{\tilde{t}}j}(t)\bigr) y(t) + \sum_{j=1}^m b_{i_{\tilde{t}}j}(t) L_j x_j(t) \\ &\quad + \sum_{j=1}^m c_{i_{\tilde{t}}j}(t) P_j \sup_{k_{i_{\tilde{t}}j}(t) \leq s \leq t} x_j(s) \\ &\leq \sum_{j=1}^m \bigl(-a_{i_{\tilde{t}}j}(t)\bigr) y(t) + \sum_{j=1}^m b_{i_{\tilde{t}}j}(t) L_j y(t) \\ &\quad + \sum_{j=1}^m c_{i_{\tilde{t}}j}(t) P_j \sup_{k(t) \leq s \leq t} y(s), \quad i \in I_m. \end{aligned}$$

Let $a_j(t) = a_{i_{\tilde{t}}j}(t)$, $b_j(t) = b_{i_{\tilde{t}}j}(t)$, $c_j(t) = c_{i_{\tilde{t}}j}(t)$, then we get

$$\begin{cases} D^+ y(t) \leq \sum_{j=1}^m (-a_j(t)) y(t) + \sum_{j=1}^m b_j(t) L_j y(t) \\ \qquad\qquad + \sum_{j=1}^m c_j(t) P_j \sup_{k(t) \leq s \leq t} y(s), \quad t \geq 0 \\ y(t) = \sup_{k(0) \leq s \leq 0} x(s), \quad t \leq 0. \end{cases} \tag{3.5}$$

From Theorem 2.1, we can get the conclusion of Theorem 3.1. The proof is complete. □

4 Examples

In this section, we present some examples and numerical simulations to test and verify our main conclusions.

Example 4.1 Consider a delay dynamical system

$$\begin{aligned} dx(t) = &-(3+3t)^{-1} x(t) - (6+6t)^{-1} x(t) + (8+6t)^{-1} x\left(t - \frac{t}{3}\right) \\ &+ (9+6t)^{-1} x\left(t - \frac{2t}{3}\right) \\ &+ (19+18t)^{-1} \sup_{3t/4 \leq s \leq t} x(s) + (10+9t)^{-1} \sup_{4t/5 \leq s \leq t} x(s) \quad \text{for } t \geq 0. \end{aligned} \tag{4.1}$$

The initial value is $x(t) = 10$ for $t \in [-2, 0]$. In Theorem 2.1, let $f_1(t) \equiv a_1(t) = (3+3t)^{-1}$, $f_2(t) \equiv a_2(t) = (6+6t)^{-1}$, $f(t) = f_1(t) + f_2(t) = (2+2t)^{-1}$, $b_1(t) = (8+6t)^{-1}$, $b_2(t) = (9+6t)^{-1}$, $c_1(t) = (19+18t)^{-1}$, $c_2(t) = (10+9t)^{-1}$. Because $\sup_{t \geq 0} \int_0^t e^{-\int_s^t (2+2u)^{-1} \, d\mu} |(9+6s)^{-1} + (8+6s)^{-1}| \, ds < \frac{2}{3}$, $\sup_{t \geq 0} \int_0^t e^{-\int_s^t (2+2u)^{-1} \, d\mu} |(10+9s)^{-1} + (19+18s)^{-1}| \, ds < \frac{1}{3}$. So, we know $x(t) \to 0$

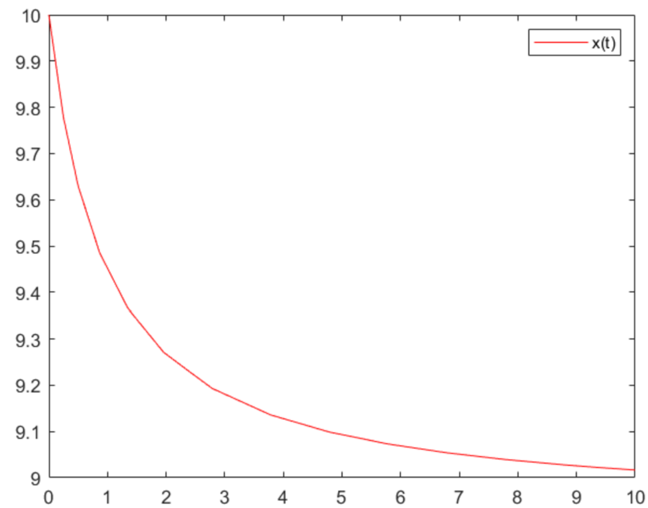

Figure 1 Example 4.1

as $t \to +\infty$ from Theorem 2.1. Simulation result presented in Fig. 1 shows the validity of our theoretical result. Figure 1 is the graph of the system

$$\begin{aligned} dx(t) = & -(3+3t)^{-1}x(t) - (6+6t)^{-1}x(t) \\ & + (8+6t)^{-1}x\left(t - \frac{t}{3}\right) + (9+6t)^{-1}x\left(t - \frac{2t}{3}\right) \\ & + (19+18t)^{-1}x\left(t - \frac{t}{4}\right) + (10+9t)^{-1}x\left(t - \frac{t}{5}\right) \quad \text{for } t \geq 0. \end{aligned}$$

Remark 4.1 Because $\tau_1(t) = \frac{t}{3}$, $\tau_2(t) = \frac{2t}{3}$, $\tau_3(t) = \frac{t}{4}$, $\tau_4(t) = \frac{t}{5}$ are unbounded, [9, 10, 15] are invalid.

Example 4.2 Consider a delay dynamic system

$$\begin{aligned} & dx(t) = \left[-2tx(t) + 6e^{-1.2}tx\left(t - \frac{1}{2+t}\right)\right]dt, \quad t \geq 0, \quad \text{and} \\ & x(t) = 10, \quad t \in [-1, 0]. \end{aligned} \tag{4.2}$$

In Theorem 2.2, let $f(t) = \alpha(t) = 2t$, $\beta(t) = 6e^{-1.2}t$. Because

$$\sup_{t\geq 0} \int_0^t e^{-\int_s^t 2\mu\, d\mu} \left|6e^{-1.2}s\right| ds \leq 3e^{-1.2} < 1,$$

we know $x(t) \to 0$ as $t \to +\infty$ from Theorem 2.2. The system (4.2) is asymptotically stable, as shown in Figs. 2.

Remark 4.2 In [5, 6, 8–10], and [11], the authors required $-\lambda(t) + \delta(t) \leq -\vartheta < 0$ for $t \geq 0$ and positive constant ϑ. Obviously, our Example 4.2 does not require such a restriction.

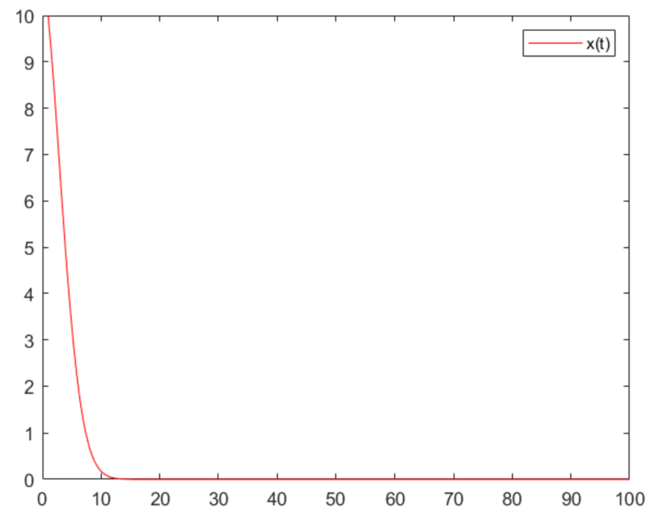

Figure 2 Example 4.2

Example 4.3 Consider a 2-dimensional Grossberg–Hopfield neural network as follows:

$$\begin{cases} dy_i(t) = [-a_i(t)x_i(t) + \sum_{j=1}^m b_{ij}(t)h_j(x_j(t)) \\ \qquad + \sum_{j=1}^m c_{ij}(t)g_j(x_j(k_{ij}(t))) + I_i(t)]\,dt, \quad t \geq 0, \\ x_i(t) = |\varphi_i(t)|, \quad t \leq 0, i = 1, 2. \end{cases} \tag{4.3}$$

We consider the dynamical behavior of two solutions $x^{(1)}(t) = (x_1^{(1)}(t), x_2^{(1)}(t))$, $x^{(2)}(t) = (x_1^{(2)}(t), x_2^{(2)}(t))$ of (4.3) with different initial values $\varphi^{(1)}(t) = (\varphi_1^{(1)}(t), \varphi_2^{(1)}(t))$, $\varphi^{(2)}(t) = (\varphi_1^{(2)}(t), \varphi_2^{(2)}(t))$ for $t \in [-2, 0]$, which have the following definition:

$$\varphi_1^{(1)}(t) = 20, \qquad \varphi_2^{(1)}(t) = 30, \qquad \varphi_1^{(2)}(t) = 40, \quad \text{and} \quad \varphi_2^{(2)}(t) = 50.$$

We further set $a_1(t) = a_2(t) = t$, $b_{11}(t) = b_{21}(t) = 0.2t$, $b_{12}(t) = b_{22}(t) = 0.3t$, $c_{11}(t) = c_{21}(t) = 0.15t$, $c_{12}(t) = c_{22}(t) = 0.25t$, $I_1(t) = \cos t$, $I_2(t) = \sin t$, and $k_{11}(t) = k_{12}(t) = k_{21}(t) = k_{22}(t) = 0.4t$. For each $s \in R$, $h_1(s) = h_2(s) = \arctan(s)$, $g_1(s) = g_2(s) = \sqrt{s+1}$. It is easy to know that $L_1 = L_1 = P_1 = P_2 = 1$. Let $f_1(t) = f_2(t) = a_1(t) = a_2(t) = t$, define $y_1(t) = |x_1^{(1)}(t) - x_1^{(2)}(t))|$, $y_2(t) = |x_2^{(1)}(t) - x_2^{(2)}(t)|$, $y(t) = (y_1(t), y_2(t))^\intercal$. From Theorem 3.1,

$$\sup_{t\geq 0} \int_0^t e^{-\int_s^t f_1(\mu)\,d\mu} \sum_{j=1}^2 \bigl|b_{1j}(s)L_j\bigr|\,ds + \int_0^t e^{-\int_s^t h_1(\mu)\,d\mu} \sum_{j=1}^2 \bigl|c_{1j}(s)P_j\bigr|\,ds = 0.9 < 1,$$

$$\sup_{t\geq 0} \int_0^t e^{-\int_s^t f_2(\mu)\,d\mu} \sum_{j=1}^2 \bigl|b_{2j}(s)L_j\bigr|\,ds + \int_0^t e^{-\int_s^t h_2(\mu)\,d\mu} \sum_{j=1}^2 \bigl|c_{2j}(s)P_j\bigr|\,ds = 0.9 < 1.$$

The neural network system (4.3) is globally asymptotically stable, as shown in Figs. 3–5.

Remark 4.3 Because $a_i(t), b_{ij}(t), c_{ij}(t)$ $(i, j \in I_m)$ are unbounded, Theorem 3 in [9] and Proposition 3 in [10] cannot be applied to system (4.3). Besides, because delays are unbounded, Theorem 3 in [11] will be invalid.

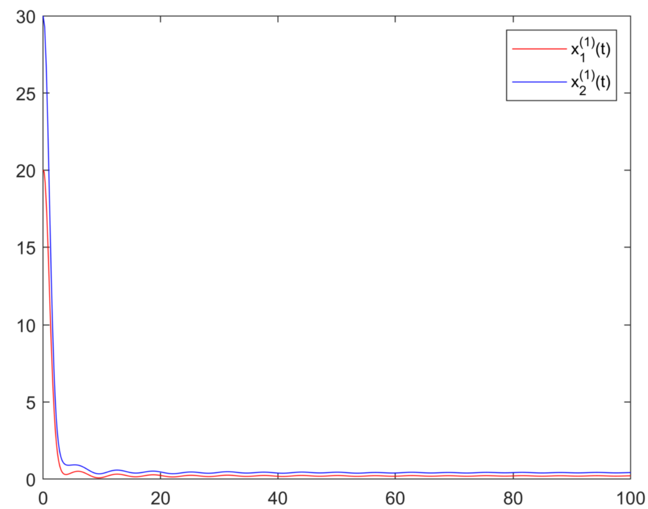

Figure 3 The state response of $x_1^{(1)}(t)$ and $x_2^{(1)}(t)$ in system (4.3)

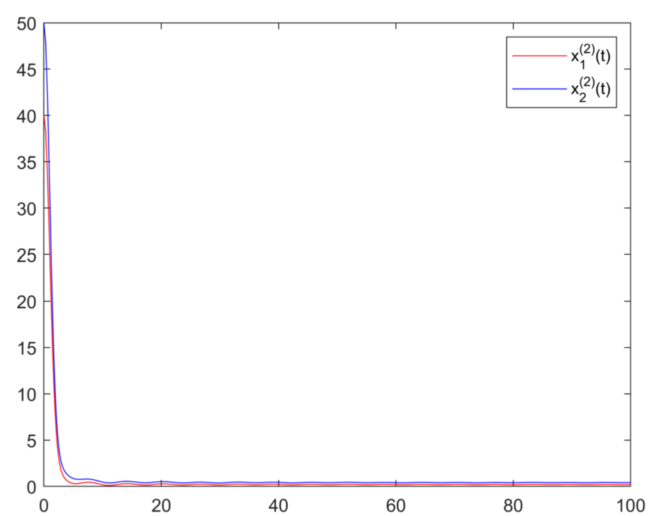

Figure 4 The state response of $x_1^{(2)}(t)$ and $x_2^{(2)}(t)$ in system (4.3)

5 Conclusion

In this note, we first used the fixed point method to study a new kind of generalized Halanay inequalities and obtained some sufficient conditions of asymptotic behavior. Then, we applied our conclusions to the study of the asymptotic synchronization and convergence of neural network systems. Finally, we presented some examples and numerical simulations to test and verify our main conclusions. The conclusions in this note improve and generalize the relative results in [4–11]. Also, to the authors' knowledge, the study of stochastic differential systems with time lag driven jointly by Brownian and fractional Brownian motions is rare, and only the existence of uniqueness and convergence of solutions are studied. In addition, the study of stochastic time-lagged partial differential systems jointly driven by Brownian and fractional Brownian motion is even rarer at present. Therefore, it is our future research goal to study the properties associated with the so-

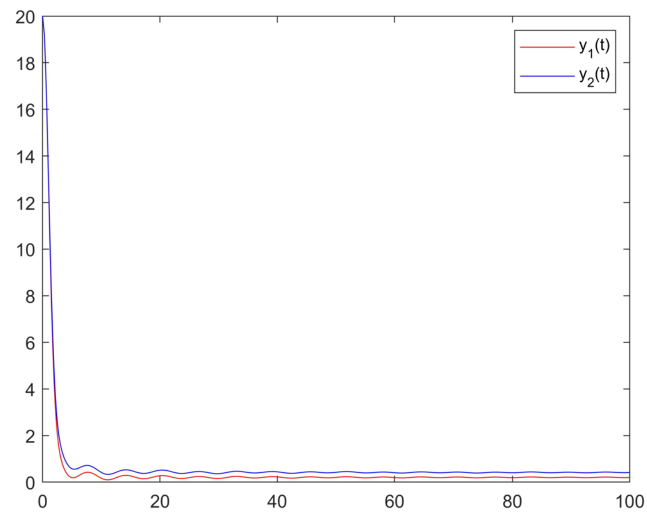

Figure 5 The state response of $y_1(t)$ and $y_2(t)$ in system (4.3)

lutions of stochastic fractional dynamical systems or doubly-driven stochastic dynamical systems by using the immobile point method as well as Halanay inequalities, based on the studies [19–21].

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Availability of data and materials
All data used to support the findings of this study are included within the article.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
C.W. and F. J performed the research. H. C. and R.L. prepared Figs. 1-5. Y.S. and F.J. designed the research and wrote the manuscript. F.J. supervised this study. All authors reviewed the manuscript.

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