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Applying periodic and anti-periodic boundary conditions in existence results of fractional differential equations via nonlinear contractive mappings

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Abstract

We introduce a notion of nonlinear cyclic orbital $(\xi - \mathscr{F})$-contraction and prove related results. With these results, we address the existence and uniqueness results with periodic/anti-periodic boundary conditions for:

1. The nonlinear multi-order fractional differential equation

$$\mathcal{L}(\mathcal{D})\theta(\varsigma) = \sigma(\varsigma, \theta(\varsigma)), \quad \varsigma \in \mathscr{J} = [0, \mathscr{A}], \mathscr{A} > 0,$$

where

$$\begin{aligned} &\mathcal{L}(\mathcal{D}) = \gamma_w \, {}^c\mathcal{D}^{\delta_w} + \gamma_{w-1} \, {}^c\mathcal{D}^{\delta_{w-1}} + \cdots + \gamma_1 \, {}^c\mathcal{D}^{\delta_1} + \gamma_0 \, {}^c\mathcal{D}^{\delta_0}, \\ &\gamma_\flat \in \mathbb{R} \quad (\flat = 0, 1, 2, 3, \ldots, w), \qquad \gamma_w \neq 0, \\ &0 \leq \delta_0 < \delta_1 < \delta_2 < \cdots < \delta_{w-1} < \delta_w < 1; \end{aligned}$$

2. The nonlinear multi-term fractional delay differential equation

$$\begin{aligned} &\mathcal{L}(\mathcal{D})\theta(\varsigma) = \sigma(\varsigma, \theta(\varsigma), \theta(\varsigma - \tau)), \quad \varsigma \in \mathscr{J} = [0, \mathscr{A}], \mathscr{A} > 0; \\ &\theta(\varsigma) = \bar{\sigma}(\varsigma), \quad \varsigma \in [-\tau, 0], \end{aligned}$$

where

$$\begin{aligned} &\mathcal{L}(\mathcal{D}) = \gamma_w \, {}^c\mathcal{D}^{\delta_w} + \gamma_{w-1} \, {}^c\mathcal{D}^{\delta_{w-1}} + \cdots + \gamma_1 \, {}^c\mathcal{D}^{\delta_1} + \gamma_0 \, {}^c\mathcal{D}^{\delta_0}, \\ &\gamma_\flat \in \mathbb{R} \quad (\flat = 0, 1, 2, 3, \ldots, w), \qquad \gamma_w \neq 0, \\ &0 \leq \delta_0 < \delta_1 < \delta_2 < \cdots < \delta_{w-1} < \delta_w < 1; \end{aligned}$$

moreover, here ${}^c\mathcal{D}^{\delta}$ is predominantly called Caputo fractional derivative of order δ.

Keywords: Nonlinear cyclic orbital $(\xi - \mathscr{F})$-contraction; Fractional differential equations; Fractional delay differential equations; Green function; Periodic/anti-periodic boundary conditions; Fixed point

1 Introduction

The field of fractional differential equations (FDEs) has gained recognition and significance in recent years as a result of its practical implications with physics, ecology and banking [1, 2]. It is commonly known that classical calculus can be used to model and analyse important and complicated phenomena across many different scientific fields. However, FDEs can provide a deeper examination of many complex natural systems. Diffusion for picture restoration, the spread of viral diseases and other situations fall under this category. In the overwhelming of the aforementioned situations, such types of described anomalous techniques have macroscopically complicated dynamics, and regular derivative frameworks are inadequate to characterise actual behaviour. Therefore, fractional differential equations are preferred over the use of ordinary differential equations [3]. While many conventional methods do not necessarily require explicit mention of fractional differential equations, some research results for fractional differential equations can be attained in a similar manner. As a result, new methods and scientific discoveries are created specifically for fractional differential equations. In light of this, many academicians concentrate on initial and boundary value issues involving various derivative types, such as Atangana–Baleanu, Caputo–Fabrizio and Caputo. The amount of research conducted on the subject has significantly increased over the last few years, with a range of fascinating and useful results (see [4–16]).

The very first statement to the fixed point theory was in a paper proving the existence result. Subsequently, this technique was enhanced as a successive approximating method, and in the context of complete normed space, it was shown and given as a fixed point theorem. It offers a rough technique for precisely characterising the fixed point. It also ensures that a fixed point will exist and be unique. This approach allows us to guarantee that it provides a solution to the initial problem by specifying the conditions that apply when a fixed point resists a particular function. In many different fields of mathematics, the existence result is analogous to the fixed point existence for an applicable function in a variety of mathematical challenges [17–24]. Fixed point results are some of those mathematical notations that show that at least one point still remains fixed when a set's points are adapted into points of the same set.

Scientific theories of fixed points are very useful in determining whether an equation has a solution. The differential operator, for instance, in differential equations transforms one function into another. It is possible to find the solution of a fractional derivative for a function that has not experienced a substantial progress.

The exponential function e^q is significant to the theory of differential equations with integer orders. Its one-parameter extension, the function that is presently depicted by [25]

$$\mathcal{E}_\delta(q) = \sum_{s=0}^{\infty} \frac{q^s}{\Gamma(\delta s + 1)},$$

was exemplified by Mittag-Leffler in his research articles [26–28].

In particular, Agarwal [29] is credited for the development of the two parameter Mittag-Leffler type function, which is fundamental to the fractional calculus. Humbert and Agarwal [30] explored a variety of correlations for this function by employing the Laplace transform method. The Agarwal function might have been a better name for this function. The two-parameter function is now known as the Mittag-Leffler function, while Humbert and Agarwal cordially left almost the same representation as used for the one-parameter Mittag-Leffler function.

The two-parametric Mittag-Leffler function is defined by (see Podlubny [31])

$$\mathcal{E}_{\delta,\varrho}(a) = \sum_{j=0}^{\infty} \frac{a^j}{\Gamma(j\delta+\varrho)}, \quad \delta > 0, \varrho > 0.$$

The Laplace transform for Mittag-Leffler function in two parameters is presented as $\mathfrak{E}_j(a,l;\delta,\varrho) := a^{j\delta+\varrho-1}\mathcal{E}^{(j)}_{\delta,\varrho}(la^{\delta})$ along with the derivatives given by (refer to Podlubny [31])

$$\mathscr{L}\left\{a^{j\delta+\varrho-1}\mathcal{E}^{(j)}_{\delta,\varrho}\left(\pm l\varsigma^{\delta}\right)\right\} = \frac{j!\gamma^{\delta-\varrho}}{(\gamma^{\delta} \mp l)^{j+1}}, \quad \mathrm{Re}(\gamma) > |l|^{\frac{1}{\delta}},$$

where $\mathcal{E}^{(j)}_{\delta,\varrho}(y) = \frac{d^j}{dy^j}\mathcal{E}_{\delta,\varrho}(y) = \sum_{u=0}^{\infty} \frac{(u+j)!y^u}{u!\Gamma(\delta u+\delta j+\varrho)}$, $j = 0, 1, 2, \ldots$.

A similar conceptual study can be found in [32] and [33].

The novelty of this article is to explore the connections between fractional Green's functions, multi-term fractional order differential equations and metric fixed point theory. We prove the existence of a solution and the uniqueness of nonlinear multi-order fractional differential equations via *nonlinear cyclic orbital* $(\xi - \mathscr{F})$*-contraction*. An interesting feature of our result is that continuity is no longer needed.

2 Nonlinear contractive mappings

Definition 2.1 [34] Let $\mathbb{X}$ be a set that is nonempty. Consider $\zeta : \mathbb{X} \times \mathbb{X} \to [1,+\infty)$. The mapping $\mathbb{r} : \mathbb{X} \times \mathbb{X} \to [0,+\infty)$ is said to be a controlled rectangular metric if the following conditions hold:

- $\mathbb{r}(x,y) = 0 \Leftrightarrow x = y$;
- $\mathbb{r}(x,y) = \mathbb{r}(y,x)$;
- $\mathbb{r}(x,y) \leq \zeta(x,\alpha)\mathbb{r}(x,\alpha) + \zeta(\alpha,\mu)\mathbb{r}(\alpha,\mu) + \zeta(\mu,y)\mathbb{r}(\mu,y)$

for all $x, y \in \mathbb{X}$ and for all distinct points $\alpha, \mu \in \mathbb{X}$, each distinct from x and y respectively. As in [34], $(\mathbb{X},\mathbb{r})$ denotes a controlled rectangular metric space (for our convenience, it is called CRMS). The topological properties such as Cauchy, completeness and convergence of controlled rectangular metric space can be seen in [34].

Definition 2.2 [35] $\mathscr{T} : \mathfrak{B} \cup \mathfrak{C} \to \mathfrak{B} \cup \mathfrak{C}$ is said to be a cyclic map if $\mathscr{T}(\mathfrak{B}) \subseteq \mathfrak{C}$ and $\mathscr{T}(\mathfrak{C}) \subseteq \mathfrak{B}$, where $\mathfrak{B}$ and $\mathfrak{C}$ are nonempty closed subsets of a complete metric space $(\mathbb{X},\mathbb{r})$.

Definition 2.3 [36] With respect to a complete metric space, represented by $\mathcal{X}$, let $\mathcal{A}$ and $\mathcal{B}$ be nonempty closed subsets. Using the distance function d and the cyclic mapping $\mathfrak{H}$, there are some $k_x \in (0,1)$ such that

$$d\left(\mathfrak{H}^{2n}x, \mathfrak{H}y\right) \leq k_x d\left(\mathfrak{H}^{2n-1}x, y\right).$$

Then ς is called a cyclic orbital contraction.

Property-$\mathcal{F}^{\star}$: Let $\mathcal{F}^{\star}$ be the family of all functions $\mathscr{F} : (0,\infty) \to \mathbb{R}$ and $\xi : (0,\infty) \to (0,\infty)$. We say that $\mathcal{F}^{\star}$ satisfies **Property**-$\mathcal{F}^{\star}$ if the following conditions hold:

- $\mathscr{F}$ is strictly increasing;

- For every positive sequence $\{b_w\} \in \mathbb{R}$, we have $\lim_{w\to\infty} b_w = 0$ iff $\lim_{w\to\infty} \mathscr{F}(b_w) = -\infty$;
- There exists $z \in (0,1)$ such that $\lim_{b\to 0^+} b^z \mathscr{F}(b) = 0$;
- $\liminf_{u\to v^+} \xi(u) > 0$ for all $v \geq 0$.

The above-mentioned conditions can be seen in [37] and [38, 39]. In very recent years, Wardowski [37] introduced a novel perspective of contraction and established a fixed point theorem that, in comparison to earlier research findings, generalises the Banach principle of contraction. The conditions mentioned above are a few of the prerequisites that writers must meet in order to formulate statements of specific F-contractions and are also used as key strategies to achieve at fixed point problems (see [40, 41]). The elegance of the $\mathscr{F}$-contraction is noteworthy; all that is needed is a complete metric space and an easily verifiable nonlinear condition. The $\mathscr{F}$-contraction also has an impact on how conventional methods are modelled. The $\mathscr{F}$-contraction continues to be a topic of extra motivation and research due to its broad range of application and usefulness, which constitutes for the emphasis to its generalisations. This illustrates why mathematics has so many implementations in other fields.

We now present our next concept.

Definition 2.4 Suppose that $\mathfrak{B}$ and $\mathfrak{C}$ are two nonempty subsets of the CRM-space $(\mathbb{X}, \mathbb{r})$ and $\mathcal{F}^\star$ is the family of mappings satisfying the *property-$\mathcal{F}^\star$*. Assume that $\mathscr{W}$ is a cyclic mapping from $\mathfrak{B} \cup \mathfrak{C}$ to $\mathfrak{B} \cup \mathfrak{C}$ such that, for some $x \in \mathfrak{B}$,

$$\xi\left(\mathbb{r}\left(\mathscr{W}^{2w-1}x, y\right)\right) + \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^{2w}x, \mathscr{W}y\right)\right) \leq \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^{2w-1}x, y\right)\right) \tag{1}$$

for all $w \in \mathbb{N}$ and $y \in \mathfrak{B}$. Then $\mathscr{W}$ is called a *nonlinear cyclic orbital* $(\xi - \mathscr{F})$*-contraction*.

Theorem 2.1 *Let* $(\mathbb{X}, \mathbb{r})$ *be a controlled rectangular metric space. Suppose that* $\mathfrak{B}$ *and* $\mathfrak{C}$ *are two nonempty closed subsets of* $(\mathbb{X}, \mathbb{r})$. *Let* $\mathscr{W}$ *be a* nonlinear cyclic orbital $(\xi - \mathscr{F})$-contraction. *For* $x_0 \in \mathfrak{B}$, *take* $x_w = \mathscr{W}^w x_0$. *For* $x \in \mathbb{X}$, $\lim_{w\to+\infty} \zeta(x_w, x)$, $\lim_{w\to+\infty} \zeta(x, x_w)$ *and* $\lim_{w\to+\infty} \zeta(x_w, x_m)$ *exist and are finite for all* $w, m \in \mathbb{N}$, $w \neq m$. *Then* $\mathscr{W}$ *has a unique fixed point in* $\mathfrak{B} \cap \mathfrak{C}$.

Proof Let $x = x_0 \in \mathfrak{B}$. Define

$$x_w = \mathscr{W}^w x_0.$$

Since $\{x_0\}$ in $\mathfrak{B}$ and $x_{2w+1} \in \mathfrak{C}$ for $w \geq 0$.

From (1), we have

$$\mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^2 x, \mathscr{W}x\right)\right) \leq \mathscr{F}\left(\mathbb{r}(\mathscr{W}x, x)\right) - \xi\left(\mathbb{r}(\mathscr{W}x, x)\right).$$

This can be written as follows:

$$\mathscr{F}\left(\mathbb{r}(x_2, x_1)\right) \leq \mathscr{F}\left(\mathbb{r}(x_1, x_0)\right) - \xi\left(\mathbb{r}(x_1, x_0)\right).$$

Again,

$$\begin{aligned}
\mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^3x,\mathscr{W}^2x\right)\right) &= \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^2x,\mathscr{W}^3x\right)\right) \\
&= \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^2x,\mathscr{W}\left(\mathscr{W}^2x\right)\right)\right) \\
&\leq \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}x,\mathscr{W}^2x\right)\right) - \xi\left(\mathbb{r}\left(\mathscr{W}x,\mathscr{W}^2x\right)\right) \\
&= \mathscr{F}\left(\mathbb{r}(x_1,x_2)\right) - \xi\left(\mathbb{r}(x_1,x_2)\right) \\
\Rightarrow \quad \mathscr{F}\left(\mathbb{r}(x_3,x_2)\right) &\leq \mathscr{F}\left(\mathbb{r}(x_2,x_1)\right) - \xi\left(\mathbb{r}(x_2,x_1)\right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^4x,\mathscr{W}^3x\right)\right) &= \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^2\left(\mathscr{W}^2x\right),\mathscr{W}\left(\mathscr{W}^2x\right)\right)\right) \\
&\leq \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}\left(\mathscr{W}^2x\right),\mathscr{W}^2x\right)\right) - \xi\left(\mathbb{r}\left(\mathscr{W}\left(\mathscr{W}^2x\right),\mathscr{W}^2x\right)\right) \\
&\leq \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^3x,\mathscr{W}^2x\right)\right) - \xi\left(\mathbb{r}\left(\mathscr{W}^3x,\mathscr{W}^2x\right)\right) \\
&= \mathscr{F}\left(\mathbb{r}(x_3,x_2)\right) - \xi\left(\mathbb{r}(x_3,x_2)\right), \\
\text{which implies} \quad \mathscr{F}\left(\mathbb{r}(x_4,x_3)\right) &\leq \mathscr{F}\left(\mathbb{r}(x_3,x_2)\right) - \xi\left(\mathbb{r}(x_3,x_2)\right).
\end{aligned}$$

By repeating the same process, we get

$$\begin{aligned}
\mathscr{F}\left(\mathbb{r}(x_w,x_{w+1})\right) &\leq \mathscr{F}\left(\mathbb{r}(x_{w-1},x_w)\right) - \xi\left(\mathbb{r}(x_{w-1},x_w)\right) \\
&< \mathscr{F}\left(\mathbb{r}(x_{w-1},x_w)\right).
\end{aligned} \tag{2}$$

Since $\mathscr{F}$ is increasing, then $\mathbb{r}(x_w,x_{w+1}) \leq \mathbb{r}(x_{w-1},x_w)$ for all $w \in \mathbb{N}$, that is, the positive sequence $\{\mathbb{r}(x_w,x_{w+1})\}$ is decreasing. Therefore it converges to a limit $r \geq 0$. Inequality (2) becomes

$$\begin{aligned}
\mathscr{F}\left(\mathbb{r}(x_w,x_{w+1})\right) &\leq \mathscr{F}\left(\mathbb{r}(x_{w-1},x_w)\right) - \xi\left(\mathbb{r}(x_{w-1},x_w)\right) \\
&\leq \mathscr{F}\left(\mathbb{r}(x_{w-2},x_{w-1})\right) - \xi\left(\mathbb{r}(x_{w-2},x_{w-1})\right) - \xi\left(\mathbb{r}(x_{w-1},x_w)\right) \\
&\;\vdots \\
&< \mathscr{F}\left(\mathbb{r}(x_0,x_1)\right) - \sum_{\flat=0}^{w-1} \xi\left(\mathbb{r}(x_\flat,x_{\flat+1})\right).
\end{aligned} \tag{3}$$

Since $\liminf_{\sigma\to\theta^+} \xi(\sigma) > 0$, we have $\liminf_{w\to\infty} \xi(\mathbb{r}(x_w,x_{w+1})) > 0$. Thus there exist $w_0 \in \mathbb{N}$ and $a > 0$ such that, for all $w \geq w_0$, $\xi(\mathbb{r}(x_w,x_{w+1})) > a$. Hence (3) becomes

$$\begin{aligned}
\mathscr{F}\left(\mathbb{r}(x_w,x_{w+1})\right) &\leq \mathscr{F}\left(\mathbb{r}(x_0,x_1)\right) - \sum_{\flat=0}^{w_0-1} \xi\left(\mathbb{r}(x_\flat,x_{\flat+1})\right) - \sum_{\flat=w_0}^{w-1} \xi\left(\mathbb{r}(x_\flat,x_{\flat+1})\right) \\
&\leq \mathscr{F}\left(\mathbb{r}(x_0,x_1)\right) - \sum_{\flat=w_0}^{w-1} a \\
&= \mathscr{F}\left(\mathbb{r}(x_0,x_1)\right) - (w-w_0)a.
\end{aligned} \tag{4}$$

For all $w \geq w_0$, taking limit as $n \to \infty$ in (4), we get

$$\lim_{w\to\infty} \mathscr{F}\big(\mathbb{r}(x_w, x_{w+1})\big) \leq \lim_{w\to\infty}\big[\mathscr{F}\big(\mathbb{r}(x_0, x_1)\big) - (w - w_0)a\big],$$

which gives $\lim_{w\to\infty} \mathscr{F}(\mathbb{r}(x_w, x_{w+1})) = -\infty$, and hence condition (2) of *property-$\mathcal{F}^\star$*,

$$\lim_{w\to\infty} \mathbb{r}(x_w, x_{w+1}) = 0. \tag{5}$$

By using condition (3) of *property-$\mathcal{F}^\star$*, there exists $k \in (0,1)$ such that

$$\lim_{w\to\infty}\big[\mathbb{r}(x_w, x_{w+1})\big]^k \mathscr{F}\big(\mathbb{r}(x_w, x_{w+1})\big) = 0.$$

From (4), we get

$$\big[\mathbb{r}(x_w, x_{w+1})\big]^k \mathscr{F}\big(\mathbb{r}(x_w, x_{w+1})\big) \leq \big[\mathbb{r}(x_w, x_{w+1})\big]^k\big[\mathscr{F}\big(\mathbb{r}(x_0, x_1)\big) - (w - w_0)a\big],$$

and hence,

$$\big[\mathbb{r}(x_w, x_{w+1})\big]^k\big[\mathscr{F}\big(\mathbb{r}(x_w, x_{w+1})\big) - \mathscr{F}\big(\mathbb{r}(x_0, x_1)\big)\big] \leq -\big[\mathbb{r}(x_w, x_{w+1})\big]^k (w - w_0)a \leq 0.$$

Taking limit as $n \to \infty$, we get

$$\lim_{w\to\infty}\big[\mathbb{r}(x_w, x_{w+1})\big]^k (w - w_0)a = 0.$$

Then there exists $w_1 \in \mathbb{N}$ such that, for all $w \geq w_1$,

$$\mathbb{r}(x_w, x_{w+1}) \leq \frac{1}{[(w - w_0)a]^{\frac{1}{k}}}. \tag{6}$$

Now we shall prove that $\lim_{w\to\infty} \mathbb{r}(x_w, x_{w+2}) = 0$.

For all $w, m \in \mathbb{N}$, we assume that $x_w \neq x_m$. Consider the possibility that $x_w = x_m$ in the case of some $w = m + \gamma$, where $\gamma > 0$, thus $\mathscr{W} x_w = \mathscr{W} x_m$.

Consider

$$\begin{aligned}\mathscr{F}\big(\mathbb{r}(x_m, x_{m+1})\big) &= \mathscr{F}\big(\mathbb{r}(x_w, x_{w+1})\big)\\ &\leq \mathscr{F}\big(\mathbb{r}(x_{w-1}, x_w)\big) - \xi\big(\mathbb{r}(x_{w-1}, x_w)\big).\end{aligned} \tag{7}$$

Since $\liminf_{\sigma\to\theta^+} \xi(\sigma) > 0$, we have $\liminf_{w\to\infty} \xi(\mathbb{r}(x_{w-1}, x_w)) > 0$.

Thus (7) will become

$$\begin{aligned}\mathscr{F}\big(\mathbb{r}(x_m, x_{m+1})\big) &\leq \mathscr{F}\big(\mathbb{r}(x_{w-1}, x_w)\big) - \xi\big(\mathbb{r}(x_{w-1}, x_w)\big)\\ &< \mathscr{F}\big(\mathbb{r}(x_{w-1}, x_w)\big).\end{aligned}$$

By continuing this process, we get

$$\mathscr{F}\left(\mathbb{r}(x_m, x_{m+1})\right) < \mathscr{F}\left(\mathbb{r}(x_m, x_{m+1})\right), \quad \text{a contradiction.}$$

Therefore, for all $m, w \in \mathbb{N}$, $\mathbb{r}(x_m, x_w) > 0$.

Now we will prove $\lim_{w\to\infty} \mathbb{r}(x_w, x_{w+2}) = 0$.

In order to this, let us suppose

$$\begin{aligned}
\mathscr{F}\left(\mathbb{r}(x_1, x_3)\right) &= \mathscr{F}\left(\mathbb{r}(x_3, x_1)\right) \\
&= \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^2(\mathscr{W}x), \mathscr{W}x\right)\right) \\
&\leq \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}(\mathscr{W}x), x\right)\right) - \xi\left(\mathbb{r}\left(\mathscr{W}(\mathscr{W}x), x\right)\right) \\
&= \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^2x, x\right)\right) - \xi\left(\mathbb{r}\left(\mathscr{W}^2x, x\right)\right) \\
&= \mathscr{F}\left(\mathbb{r}(x_2, x_0)\right) - \xi\left(\mathbb{r}(x_2, x_0)\right).
\end{aligned}$$

This can be written as

$$\mathscr{F}\left(\mathbb{r}(x_1, x_3)\right) \leq \mathscr{F}\left(\mathbb{r}(x_0, x_2)\right) - \xi\left(\mathbb{r}(x_0, x_2)\right).$$

Again,

$$\begin{aligned}
\mathscr{F}\left(\mathbb{r}(x_2, x_4)\right) &= \mathscr{F}\left(\mathbb{r}(x_4, x_2)\right) \\
&= \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^2\left(\mathscr{W}^2x\right), \mathscr{W}(\mathscr{W}x)\right)\right) \\
&\leq \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}\left(\mathscr{W}^2x\right), \mathscr{W}x\right)\right) - \xi\left(\mathbb{r}\left(\mathscr{W}\left(\mathscr{W}^2x\right), \mathscr{W}x\right)\right) \\
&= \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^3x, \mathscr{W}x\right)\right) - \xi\left(\mathbb{r}\left(\mathscr{W}^3x, \mathscr{W}x\right)\right) \\
&= \mathscr{F}\left(\mathbb{r}(x_3, x_1)\right) - \xi\left(\mathbb{r}(x_3, x_1)\right).
\end{aligned}$$

This can be written as

$$\mathscr{F}\left(\mathbb{r}(x_2, x_4)\right) \leq \mathscr{F}\left(\mathbb{r}(x_1, x_3)\right) - \xi\left(\mathbb{r}(x_1, x_3)\right).$$

By repeating the same process, we get

$$\begin{aligned}
\mathscr{F}\left(\mathbb{r}(x_w, x_{w+2})\right) &\leq \mathscr{F}\left(\mathbb{r}(x_{w-1}, x_{w+1})\right) - \xi\left(\mathbb{r}(x_{w-1}, x_{w+1})\right) \\
&\leq \mathscr{F}\left(\mathbb{r}(x_{w-2}, x_w)\right) - \xi\left(\mathbb{r}(x_{w-2}, x_w)\right) - \xi\left(\mathbb{r}(x_{w-1}, x_{w+1})\right) \\
&\vdots \\
&\leq \mathscr{F}\left(\mathbb{r}(x_0, x_2)\right) - \sum_{\flat=0}^{w-1} \xi\left(\mathbb{r}(x_\flat, x_{\flat+2})\right).
\end{aligned} \tag{8}$$

Since $\liminf_{\sigma\to\theta^+}\xi(\sigma)>0$, we have $\liminf_{w\to\infty}\xi(\mathbb{r}(x_w,x_{w+2}))>0$. Thus there exists $w_0\in\mathbb{N}$ and $b>0$ such that, for all $w\ge w_0$, $\xi(\mathbb{r}(x_w,x_{w+2}))>b$. Hence (8) becomes

$$\begin{aligned}\mathscr{F}\big(\mathbb{r}(x_w,x_{w+2})\big)&\le\mathscr{F}\big(\mathbb{r}(x_0,x_2)\big)-\sum_{\flat=0}^{w_0-1}\xi\big(\mathbb{r}(x_\flat,x_{\flat+2})\big)-\sum_{\flat=w_0}^{w-1}\xi\big(\mathbb{r}(x_\flat,x_{\flat+2})\big)\\&\le\mathscr{F}\big(\mathbb{r}(x_0,x_2)\big)-\sum_{\flat=w_0}^{w-1}b\\&=\mathscr{F}\big(\mathbb{r}(x_0,x_2)\big)-(w-w_0)b.\end{aligned}\tag{9}$$

For all $w\ge w_0$, taking limit as $w\to\infty$ in (9) and by using condition (2) of *property-$\mathcal{F}^\star$*, and doing the same process as we did for (4), we get

$$\lim_{w\to\infty}\mathbb{r}(x_w,x_{w+2})=0.\tag{10}$$

Now our aim is to prove that $\{x_w\}$ is a Cauchy. In other words, it is represented as

$$\lim_{w,m\to\infty}\mathbb{r}(x_w,x_m)=0,\quad\forall w,m\in\mathbb{N}.$$

Denote $\mathbb{r}_p=\mathbb{r}(x_p,x_{p+1})$ for all $p\in\mathbb{N}$. Now split it into two cases.

Case 1: Assume that $m=w+2\lambda+1$ with $\lambda\ge1$. By hypothesis, we have

$$\begin{aligned}\mathbb{r}(x_w,x_m)&=\mathbb{r}(x_w,x_{w+2\lambda+1})\\
&\le\zeta(x_w,x_{w+1})\mathbb{r}(x_w,x_{w+1})+\zeta(x_{w+1},x_{w+2})\mathbb{r}(x_{w+1},x_{w+2})\\
&\quad+\zeta(x_{w+2},x_{w+2\lambda+1})\mathbb{r}(x_{w+2},x_{w+2\lambda+1})\\
&\le\zeta(x_w,x_{w+1})\mathbb{r}(x_w,x_{w+1})+\zeta(x_{w+1},x_{w+2})\mathbb{r}(x_{w+1},x_{w+2})\\
&\quad+\zeta(x_{w+2},x_{w+2\lambda+1})\zeta(x_{w+2},x_{w+3})\mathbb{r}(x_{w+2},x_{w+3})\\
&\quad+\zeta(x_{w+2},x_{w+2\lambda+1})\zeta(x_{w+3},x_{w+4})\mathbb{r}(x_{w+3},x_{w+4})\\
&\quad+\zeta(x_{w+2},x_{w+2\lambda+1})\zeta(x_{w+4},x_{w+2\lambda+1})\mathbb{r}(x_{w+4},x_{w+2\lambda+1})\\
&\le\zeta(x_w,x_{w+1})\mathbb{r}(x_w,x_{w+1})+\zeta(x_{w+1},x_{w+2})\mathbb{r}(x_{w+1},x_{w+2})\\
&\quad+\zeta(x_{w+2},x_{w+2\lambda+1})\zeta(x_{w+2},x_{w+3})\mathbb{r}(x_{w+2},x_{w+3})\\
&\quad+\zeta(x_{w+2},x_{w+2\lambda+1})\zeta(x_{w+3},x_{w+4})\mathbb{r}(x_{w+3},x_{w+4})\\
&\quad+\zeta(x_{w+2},x_{w+2\lambda+1})\zeta(x_{w+4},x_{w+2\lambda+1})\zeta(x_{w+4},x_{w+5})\mathbb{r}(x_{w+4},x_{w+5})\\
&\quad+\zeta(x_{w+2},x_{w+2\lambda+1})\zeta(x_{w+4},x_{w+2\lambda+1})\zeta(x_{w+5},x_{w+6})\mathbb{r}(x_{w+5},x_{w+6})\\
&\quad+\zeta(x_{w+2},x_{w+2\lambda+1})\zeta(x_{w+4},x_{w+2\lambda+1})\zeta(x_{w+6},x_{w+2\lambda+1})\mathbb{r}(x_{w+6},x_{w+2\lambda+1})\\
&\le\zeta(x_w,x_{w+1})\mathbb{r}(x_w,x_{w+1})+\zeta(x_{w+1},x_{w+2})\mathbb{r}(x_{w+1},x_{w+2})\\
&\quad+\zeta(x_{w+2},x_{w+2\lambda+1})\zeta(x_{w+2},x_{w+3})\mathbb{r}(x_{w+2},x_{w+3})\\
&\quad+\zeta(x_{w+2},x_{w+2\lambda+1})\zeta(x_{w+3},x_{w+4})\mathbb{r}(x_{w+3},x_{w+4})\\
&\quad+\zeta(x_{w+2},x_{w+2\lambda+1})\zeta(x_{w+4},x_{w+2\lambda+1})\zeta(x_{w+4},x_{w+5})\mathbb{r}(x_{w+4},x_{w+5})\end{aligned}$$

$$
\begin{aligned}
&+ \zeta(x_{w+2}, x_{w+2\lambda+1})\zeta(x_{w+4}, x_{w+2\lambda+1})\zeta(x_{w+5}, x_{w+6})\mathbb{r}(x_{w+5}, x_{w+6}) \\
&\vdots \\
&+ \zeta(x_{w+2}, x_{w+2\lambda+1})\zeta(x_{w+4}, x_{w+2\lambda+1}) \times \cdots \times \zeta(x_{w+2\lambda-2}, x_{w+2\lambda+1}) \\
&\times \big[\zeta(x_{w+2\lambda-2}, x_{w+2\lambda+1})\mathbb{r}(x_{w+2\lambda-2}, x_{w+2\lambda+1}) \\
&+ \zeta(x_{w+2\lambda-1}, x_{w+2\lambda})\mathbb{r}(x_{w+2\lambda-1}, x_{w+2\lambda})\big] \\
&+ \zeta(x_{w+2}, x_{w+2\lambda+1})\zeta(x_{w+4}, x_{w+2\lambda+1}) \times \cdots \\
&\times \zeta(x_{w+2\lambda-2}, x_{w+2\lambda+1})\zeta(x_{w+2\lambda}, x_{w+2\lambda+1})\mathbb{r}(x_{w+2\lambda}, x_{w+2\lambda+1}) \\
\le{}& \zeta(x_w, x_{w+1})\mathbb{r}_w + \zeta(x_{w+1}, x_{w+2})\mathbb{r}_{w+1} \\
&+ \zeta(x_{w+2}, x_{w+2\lambda+1})\zeta(x_{w+2}, x_{w+3})\mathbb{r}_{w+2} \\
&+ \zeta(x_{w+2}, x_{w+2\lambda+1})\zeta(x_{w+3}, x_{w+4})\mathbb{r}_{w+3} \\
&+ \zeta(x_{w+2}, x_{w+2\lambda+1})\zeta(x_{w+4}, x_{w+2\lambda+1})\zeta(x_{w+4}, x_{w+5})\mathbb{r}_{w+4} \\
&+ \zeta(x_{w+2}, x_{w+2\lambda+1})\zeta(x_{w+4}, x_{w+2\lambda+1})\zeta(x_{w+5}, x_{w+6})\mathbb{r}_{w+5} \\
&\vdots \\
&+ \zeta(x_{w+2}, x_{w+2\lambda+1}) \times \cdots \times \zeta(x_{w+2\lambda-2}, x_{w+2\lambda+1}) \\
&\times \big[\zeta(x_{w+2\lambda-2}, x_{w+2\lambda+1})\mathbb{r}_{w+2\lambda-2} + \zeta(x_{w+2\lambda-1}, x_{w+2\lambda})\mathbb{r}_{w+2\lambda-1}\big] \\
&+ \zeta(x_{w+2}, x_{w+2\lambda+1})\zeta(x_{w+4}, x_{w+2\lambda+1}) \times \cdots \\
&\times \zeta(x_{w+2\lambda-2}, x_{w+2\lambda+1})\zeta(x_{w+2\lambda}, x_{w+2\lambda+1})\mathbb{r}_{w+2\lambda}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\mathbb{r}(x_w, x_m) \\
&\le \zeta(x_w, x_{w+1})\frac{1}{[(w-w_0)a]^{\frac{1}{k}}} + \zeta(x_{w+1}, x_{w+2})\frac{1}{[((w+1)-w_0)a]^{\frac{1}{k}}} \\
&\quad + \zeta(x_{w+2}, x_{w+2\lambda+1})\Bigg[\zeta(x_{w+2}, x_{w+3})\frac{1}{[((w+2)-w_0)a]^{\frac{1}{k}}} \\
&\quad + \zeta(x_{w+3}, x_{w+4})\frac{1}{[((w+3)-w_0)a]^{\frac{1}{k}}}\Bigg] \\
&\quad \vdots \\
&\quad + \zeta(x_{w+2}, x_{w+2\lambda+1})\zeta(x_{w+4}, x_{w+2\lambda+1})\cdots\zeta(x_{w+2\lambda-2}, x_{w+2\lambda+1}) \\
&\quad \times \Bigg[\zeta(x_{w+2\lambda-2}, x_{w+2\lambda-1})\frac{1}{[((w+2\lambda-2)-w_0)a]^{\frac{1}{k}}} \\
&\quad + \zeta(x_{w+2\lambda-1}, x_{w+2\lambda})\frac{1}{[((w+2\lambda-1)-w_0)a]^{\frac{1}{k}}}\Bigg] \\
&\quad + \zeta(x_{w+2}, x_{w+2\lambda+1})\zeta(x_{w+4}, x_{w+2\lambda+1})\cdots\zeta(x_{w+2\lambda-2}, x_{w+2\lambda+1})\zeta(x_{w+2\lambda}, x_{w+2\lambda+1})
\end{aligned}
$$

$$\begin{aligned}
&\times \left[\zeta(x_{w+2\lambda}, x_{w+2\lambda+1}) \frac{1}{[((w+2\lambda)-w_0)a]^{\frac{1}{k}}}\right.\\
&\left.+ \zeta(x_{w+2\lambda+1}, x_{w+2\lambda+2}) \frac{1}{[((w+2\lambda+1)-w_0)a]^{\frac{1}{k}}}\right]\\
&\le \zeta(x_w, x_{w+1}) \frac{1}{[(w-w_0)a]^{\frac{1}{k}}} + \zeta(x_{w+1}, x_{w+2}) \frac{1}{[((w+1)-w_0)a]^{\frac{1}{k}}}\\
&+ \sum_{\flat=w+2}^{w+2\lambda} \prod_{j=w+2}^{\flat} \zeta(x_j, x_{w+2\lambda+1}) \left[\zeta(x_\flat, x_{\flat+1}) \frac{1}{[(\flat-w_0)a]^{\frac{1}{k}}}\right.\\
&\left.+ \zeta(x_{\flat+1}, x_{\flat+2}) \frac{1}{[((\flat+1)-w_0)a]^{\frac{1}{k}}}\right]. \qquad (11)
\end{aligned}$$

We simply utilise that $\zeta(x,y) \ge 1$.

Assume

$$\mathcal{S}_{\mathcal{Z}} = \sum_{\flat=0}^{\mathcal{Z}} \prod_{j=0}^{j=\flat} \zeta(x_j, x_{w+2\lambda+1}) \left[\zeta(x_\flat, x_{\flat+1}) \frac{1}{[(\flat-w_0)a]^{\frac{1}{k}}} + \zeta(x_{\flat+1}, x_{\flat+2}) \frac{1}{[((\flat+1)-w_0)a]^{\frac{1}{k}}}\right].$$

Following that, we are able to express (11) as

$$\mathbb{r}(x_w, x_m) \le \zeta(x_w, x_{w+1}) \frac{1}{[(w-w_0)a]^{\frac{1}{k}}} + \zeta(x_{w+1}, x_{w+2}) \frac{1}{[((w+1)-w_0)a]^{\frac{1}{k}}} + \mathcal{S}_{m-1} - \mathcal{S}_{w+1}.$$

Now, let

$$a_\flat = \prod_{j=0}^{\flat} \zeta(x_j, x_m) \left[\zeta(x_\flat, x_{\flat+1}) \frac{1}{[(\flat-w_0)a]^{\frac{1}{k}}} + \zeta(x_{\flat+1}, x_{\flat+2}) \frac{1}{[((\flat+1)-w_0)a]^{\frac{1}{k}}}\right].$$

Since $0 < k < 1$, $\frac{1}{[(\flat-w_0)a]^{\frac{1}{k}}}$ and $\frac{1}{[((\flat+1)-w_0)a]^{\frac{1}{k}}}$ converge, which yields $a_\flat$ converges.

Thus the series

$$\sum_{\flat=w+2}^{w+2\lambda} \prod_{j=w+2}^{\flat} \zeta(x_j, x_{w+2\lambda+1}) \left[\zeta(x_\flat, x_{\flat+1}) \frac{1}{[(\flat-w_0)a]^{\frac{1}{k}}} + \zeta(x_{\flat+1}, x_{\flat+2}) \frac{1}{[((\flat+1)-w_0)a]^{\frac{1}{k}}}\right]$$

converges.

On the other hand, $\zeta(x_w, x_{w+1}) \frac{1}{[(w-w_0)a]^{\frac{1}{k}}}$, $\zeta(x_{w+1}, x_{w+2}) \frac{1}{[((w+1)-w_0)a]^{\frac{1}{k}}}$ converges as $w \to \infty$. From (11), we conclude that $\lim_{w,m\to\infty} \mathbb{r}(x_w, x_m) = 0$.

Case 2: Take $\rho = 2\lambda$ ($\lambda \ge 1$), thus

$$\begin{aligned}
\mathbb{r}(x_w, x_{w+2\lambda}) &\le \zeta(x_w, x_{w+2}) \mathbb{r}(x_w, x_{w+2}) + \zeta(x_{w+2}, x_{w+3}) \mathbb{r}(x_{w+2}, x_{w+3})\\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda}) \mathbb{r}(x_{w+3}, x_{w+2\lambda})\\
&\le \zeta(x_w, x_{w+2}) \mathbb{r}(x_w, x_{w+2}) + \zeta(x_{w+2}, x_{w+3}) \mathbb{r}(x_{w+2}, x_{w+3})\\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda}) \big[\zeta(x_{w+3}, x_{w+4}) \mathbb{r}(x_{w+3}, x_{w+4})\\
&\quad + \zeta(x_{w+4}, x_{w+5}) \mathbb{r}(x_{w+4}, x_{w+5}) + \zeta(x_{w+5}, x_{w+2\lambda}) \mathbb{r}(x_{w+5}, x_{w+2\lambda})\big]
\end{aligned}$$

$$
\begin{aligned}
&= \zeta(x_w, x_{w+2})\mathbb{r}(x_w, x_{w+2}) + \zeta(x_{w+2}, x_{w+3})\mathbb{r}(x_{w+2}, x_{w+3}) \\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+3}, x_{w+4})\mathbb{r}(x_{w+3}, x_{w+4}) \\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+4}, x_{w+5})\mathbb{r}(x_{w+4}, x_{w+5}) \\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+5}, x_{w+2\lambda})\mathbb{r}(x_{w+5}, x_{w+2\lambda}) \\
&\le \zeta(x_w, x_{w+2})\mathbb{r}(x_w, x_{w+2}) + \zeta(x_{w+2}, x_{w+3})\mathbb{r}(x_{w+2}, x_{w+3}) \\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda})\big[\zeta(x_{w+3}, x_{w+4})\mathbb{r}(x_{w+3}, x_{w+4}) \\
&\quad + \zeta(x_{w+4}, x_{w+5})\mathbb{r}(x_{w+4}, x_{w+5}) + \zeta(x_{w+5}, x_{w+2\lambda})\mathbb{r}(x_{w+5}, x_{w+2\lambda})\big] \\
&\le \zeta(x_w, x_{w+2})\mathbb{r}(x_w, x_{w+2}) + \zeta(x_{w+2}, x_{w+3})\mathbb{r}(x_{w+2}, x_{w+3}) \\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+3}, x_{w+4})\mathbb{r}(x_{w+3}, x_{w+4}) \\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+4}, x_{w+5})\mathbb{r}(x_{w+4}, x_{w+5}) \\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+5}, x_{w+2\lambda})\big[\zeta(x_{w+5}, x_{w+6})\mathbb{r}(x_{w+5}, x_{w+6}) \\
&\quad + \zeta(x_{w+6}, x_{w+7})\mathbb{r}(x_{w+6}, x_{w+7}) + \zeta(x_{w+7}, x_{w+2\lambda})\mathbb{r}(x_{w+7}, x_{w+2\lambda})\big] \\
&\le \zeta(x_w, x_{w+2})\mathbb{r}(x_w, x_{w+2}) + \zeta(x_{w+2}, x_{w+3})\mathbb{r}(x_{w+2}, x_{w+3}) \\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+3}, x_{w+4})\mathbb{r}(x_{w+3}, x_{w+4}) \\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+4}, x_{w+5})\mathbb{r}(x_{w+4}, x_{w+5}) \\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+5}, x_{w+2\lambda})\zeta(x_{w+5}, x_{w+6})\mathbb{r}(x_{w+5}, x_{w+6}) \\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+5}, x_{w+2\lambda})\zeta(x_{w+6}, x_{w+7})\mathbb{r}(x_{w+6}, x_{w+7}) \\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+5}, x_{w+2\lambda})\zeta(x_{w+7}, x_{w+2\lambda})\mathbb{r}(x_{w+7}, x_{w+2\lambda}) \\
&\le \zeta(x_w, x_{w+2})\mathbb{r}(x_w, x_{w+2}) + \zeta(x_{w+2}, x_{w+3})\mathbb{r}(x_{w+2}, x_{w+3}) \\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda})\big[\zeta(x_{w+3}, x_{w+4})\mathbb{r}(x_{w+3}, x_{w+4}) \\
&\quad + \zeta(x_{w+4}, x_{w+5})\mathbb{r}(x_{w+4}, x_{w+5})\big] \\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+5}, x_{w+2\lambda})\big[\zeta(x_{w+5}, x_{w+6})\mathbb{r}(x_{w+5}, x_{w+6}) \\
&\quad + \zeta(x_{w+6}, x_{w+7})\mathbb{r}(x_{w+6}, x_{w+7})\big] \\
&\quad + \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+5}, x_{w+2\lambda})\zeta(x_{w+7}, x_{w+2\lambda})\mathbb{r}(x_{w+7}, x_{w+2\lambda}).
\end{aligned}
$$

Repeating this process and using the hypothesis, we get

$$
\begin{aligned}
&\mathbb{r}(x_w, x_{w+2\lambda}) \\
&\quad \le \zeta(x_w, x_{w+2})\mathbb{r}(x_w, x_{w+2}) + \zeta(x_{w+2}, x_{w+3})\frac{1}{[(w+2-w_0)a]^{\frac{1}{k}}} \\
&\qquad + \zeta(x_{w+3}, x_{w+2\lambda})\left[\zeta(x_{w+3}, x_{w+4})\frac{1}{[(w+3-w_0)a]^{\frac{1}{k}}}\right. \\
&\qquad \left. + \zeta(x_{w+4}, x_{w+5})\frac{1}{[(w+4-w_0)a]^{\frac{1}{k}}}\right] \\
&\qquad + \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+5}, x_{w+2\lambda})\left[\zeta(x_{w+5}, x_{w+6})\frac{1}{[(w+5-w_0)a]^{\frac{1}{k}}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \zeta(x_{w+6}, x_{w+7}) \frac{1}{[(w+6-w_0)a]^{\frac{1}{k}}}\Bigg] + \cdots \\
&+ \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+5}, x_{w+2\lambda})\zeta(x_{w+7}, x_{w+2\lambda})\cdots\zeta(x_{w+2\lambda-3}, x_{w+2\lambda}) \\
&\times \Bigg[\zeta(x_{w+2\lambda-3}, x_{w+2\lambda-2}) \frac{1}{[(w+2\lambda-3-w_0)a]^{\frac{1}{k}}} \\
&+ \zeta(x_{w+2\lambda-2}, x_{w+2\lambda-1}) \frac{1}{[(w+2\lambda-2-w_0)a]^{\frac{1}{k}}}\Bigg] \\
&+ \zeta(x_{w+3}, x_{w+2\lambda})\zeta(x_{w+5}, x_{w+2\lambda})\zeta(x_{w+7}, x_{w+2\lambda})\cdots\zeta(x_{w+2\lambda-3}, x_{w+2\lambda}) \\
&\times \zeta(x_{w+2\lambda-1}, x_{w+2\lambda})\Bigg[\zeta(x_{w+2\lambda-1}, x_{w+2\lambda}) \frac{1}{[(w+2\lambda-1-w_0)a]^{\frac{1}{k}}} \\
&+ \zeta(x_{w+2\lambda}, x_{w+2\lambda+1}) \frac{1}{[(w+2\lambda-w_0)a]^{\frac{1}{k}}}\Bigg].
\end{aligned}
\tag{12}
$$

Thus we conclude

$$
\begin{aligned}
\mathbb{r}(x_w, x_m) \le{}& \zeta(x_w, x_{w+2})\mathbb{r}(x_w, x_{w+2}) + \zeta(x_{w+2}, x_{w+3}) \frac{1}{[(w+2-w_0)a]^{\frac{1}{k}}} \\
&+ \sum_{\flat=w+3}^{\flat=w+2\lambda-1} \prod_{\eta=w+3}^{\eta=\flat} \zeta(x_\eta, x_{w+2\lambda})\Bigg[\zeta(x_\flat, x_{\flat+1}) \frac{1}{[(\flat-w_0)a]^{\frac{1}{k}}} \\
&+ \zeta(x_{\flat+1}, x_{\flat+2}) \frac{1}{[(\flat+1-w_0)a]^{\frac{1}{k}}}\Bigg].
\end{aligned}
\tag{13}
$$

We simply utilise that $\zeta(x, y) \ge 1$.

Assume

$$
\mathcal{S}_{\mathcal{Z}} = \sum_{\flat=0}^{\flat=\mathcal{Z}} \prod_{\eta=0}^{\eta=\flat} \zeta(x_\eta, x_{w+2\lambda})\Bigg[\zeta(x_\flat, x_{\flat+1}) \frac{1}{[(\flat-w_0)a]^{\frac{1}{k}}} + \zeta(x_{\flat+1}, x_{\flat+2}) \frac{1}{[(\flat+1-w_0)a]^{\frac{1}{k}}}\Bigg].
$$

Then we have

$$
\mathbb{r}(x_w, x_m) \le \mathbb{r}(x_w, x_{w+2})\zeta(x_w, x_{w+2}) + \zeta(x_{w+2}, x_{w+3}) \frac{1}{[(w+2-w_0)a]^{\frac{1}{k}}} + \mathcal{S}_{m-1} - \mathcal{S}_{w+2}.
$$

Now let

$$
b_\flat = \prod_{\eta=0}^{\flat} \zeta(x_\eta, x_m)\Bigg[\zeta(x_\flat, x_{\flat+1}) \frac{1}{[(\flat-w_0)a]^{\frac{1}{k}}} + \zeta(x_{\flat+1}, x_{\flat+2}) \frac{1}{[(\flat+1-w_0)a]^{\frac{1}{k}}}\Bigg].
$$

With the help of (5), we can say that $b_\flat$ converges. Thus the series

$$
\sum_{\flat=w+3}^{\flat=w+2\lambda-1} \prod_{\eta=w+3}^{\eta=\flat} \zeta(x_\eta, x_{w+2\lambda})\Bigg[\zeta(x_\flat, x_{\flat+1}) \frac{1}{[(\flat-w_0)a]^{\frac{1}{k}}} + \zeta(x_{\flat+1}, x_{\flat+2}) \frac{1}{[(\flat+1-w_0)a]^{\frac{1}{k}}}\Bigg]
$$

converges.

From (10), $\zeta(x_w, x_{w+2})\mathbb{r}(x_w, x_{w+2})$ converges as $w \to \infty$. Thus, from (13), we can write

$$\lim_{w,m\to\infty} \mathbb{r}(x_w, x_m) = 0.$$

We conclude that the sequence x_w is a Cauchy sequence in the complete controlled rectangular metric space,

which yields $\mathfrak{B} \cap \mathfrak{C} \neq \emptyset$, as $\mathscr{W}^w x \to \sigma$, where $\sigma \in \mathfrak{B} \cap \mathfrak{C}$.

Now we prove $\mathscr{W}\sigma = \sigma$. From *nonlinear cyclic orbital* ($\xi - \mathscr{F}$)*-contraction*, we have

$$\xi\left(\mathbb{r}\left(\mathscr{W}^{2w-1}x, \sigma\right)\right) + \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^{2w}x, \mathscr{W}\sigma\right)\right) \leq \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^{2w-1}x, \sigma\right)\right).$$

Applying $\lim_{w\to\infty}$ to the above inequality, we get

$$\begin{aligned}
&\lim_{w\to\infty} \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^{2w}x, \mathscr{W}\sigma\right)\right) \leq \lim_{w\to\infty}\left[\mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^{2w-1}x, \sigma\right)\right) - \xi\left(\mathbb{r}\left(\mathscr{W}^{2w-1}x, \sigma\right)\right)\right],\\
&\lim_{w\to\infty} \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^{2w}x, \mathscr{W}\sigma\right)\right) = -\infty,
\end{aligned}$$

and hence from the property-$\mathcal{F}^\star$, we have

$$\begin{aligned}
&\lim_{w\to\infty} \mathbb{r}\left(\mathscr{W}^{2w}x\mathscr{W}\sigma\right) = 0\\
&\quad\Rightarrow\quad \mathbb{r}(\sigma, \mathscr{W}\sigma) = 0\\
&\quad\Rightarrow\quad \sigma = \mathscr{W}\sigma.
\end{aligned}$$

Finally, we show that σ is a unique fixed point of $\mathscr{W}$. Suppose on the contrary that there exists a point $\beta \in \mathfrak{B} \cap \mathfrak{C}$ such that $\sigma \neq \beta$ and $\mathscr{W}\beta = \beta$.

$$\begin{aligned}
\mathscr{F}\left(\mathbb{r}(\beta, \sigma)\right) &= \mathscr{F}\left(\mathbb{r}(\mathscr{W}\beta, \sigma)\right)\\
&= \lim_{w\to\infty} \mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^{2w}x, \mathscr{W}\beta\right)\right)\\
&\leq \left[\mathscr{F}\left(\mathbb{r}\left(\mathscr{W}^{2w-1}x, \beta\right)\right) - \xi\left(\mathbb{r}\left(\mathscr{W}^{2w-1}x, \beta\right)\right)\right]\\
&= \mathscr{F}\left(\mathbb{r}(\sigma, \beta)\right) - \xi\left(\mathbb{r}(\sigma, \beta)\right)\\
&< \mathscr{F}\left(\mathbb{r}(\sigma, \beta)\right),
\end{aligned} \tag{14}$$

which is a contradiction. Hence $\sigma = \beta$. This completes the proof. □

Theorem 2.2 *Let $\mathfrak{B}$ and $\mathfrak{C}$ be nonempty subsets of CRMS $(\mathbb{X}, \mathbb{r})$ with property $\mathscr{F}^*$. Suppose that $\mathscr{W}$ is a cyclic mapping from $\mathfrak{B} \cup \mathfrak{C}$ to $\mathfrak{B} \cup \mathfrak{C}$ such that*

$$\text{For every } x \in \mathfrak{B} \text{ and } y \in \mathfrak{C}, \quad \xi\left(\mathbb{r}(x, y)\right) + \mathscr{F}\left(\mathbb{r}(\mathscr{W}x, \mathscr{W}y)\right) \leq \mathscr{F}\left(\mathbb{r}(x, y)\right). \tag{15}$$

Take $x_0 \in \mathfrak{B}$ and $x_w = \mathscr{W}^w x_0$. For $x \in \mathbb{X}$, $\lim_{w\to+\infty} \zeta(x_w, x)$, $\lim_{w\to+\infty} \zeta(x, x_w)$ and $\lim_{w\to+\infty} \zeta(x_w, x_m)$ exist and are finite. Then there will be a fixed point that is unique in $\mathfrak{B} \cap \mathfrak{C}$.

Proof Take $x_0 \in \mathfrak{B} \cup \mathfrak{C}$. Moreover, define $x_w = \mathscr{W}^w x_0$. Thus, for $x_0 \in \mathfrak{B} \cap \mathfrak{C}$, $\xi(\mathbb{r}(x_0, x_1)) + \mathscr{F}(\mathbb{r}(\mathscr{W}x_0, \mathscr{W}x_1)) \leq \mathscr{F}(\mathbb{r}(x_0, x_1))$,

$$\text{i.e.,} \quad \mathscr{F}\big(\mathbb{r}(\mathscr{W}x_1, x_2)\big) \leq \mathscr{F}\big(\mathbb{r}(x_0, x_1)\big) - \xi\big(\mathbb{r}(x_0, x_1)\big).$$

We can determine that, by induction,

$$\mathscr{F}\big(\mathbb{r}(\mathscr{W}x_w, x_{w+1})\big) \leq \mathscr{F}\big(\mathbb{r}(x_0, x_1)\big) - \sum_{\flat=1}^{w} \xi\big(\mathbb{r}(x_{\flat-1}, x_\flat)\big) \quad \text{for all } w \geq 0.$$

By using the same pattern as in the above theorem, one can prove that $\{\mathscr{W}^w x\}$ converges to some point $\vartheta \in \mathfrak{B} \cup \mathfrak{C}$. Note that an infinite number of terms of the sequence $\{\mathscr{W}^w x\}$ lie in $\mathfrak{B}$ and an infinite number of terms lie in $\mathfrak{C}$. Thus $\vartheta \in \mathfrak{B} \cap \mathfrak{C}$. So $\mathfrak{B} \cap \mathfrak{C} \neq \phi$.

Since $\mathscr{W}$ is cyclic, $\mathscr{W}(\mathfrak{B}) \subseteq \mathfrak{C}$ and $\mathscr{W}(\mathfrak{C}) \subseteq \mathfrak{B}$ lead to $\mathscr{W} : \mathfrak{B} \cap \mathfrak{C} \to \mathfrak{B} \cap \mathfrak{C}$. Thus (15) implies that $\mathscr{W}$ restricted to $\mathfrak{B} \cap \mathfrak{C}$ is a cyclic $\mathscr{F}$-contractive mapping. Hence the defined contractive mapping applies to $\mathscr{W}$ on $\mathfrak{B} \cap \mathfrak{C}$. Hence one can easily prove that $\mathscr{W}$ has a unique fixed point in $\mathfrak{B} \cap \mathfrak{C}$. □

3 Connecting fixed point elements to nonlinear multi-order fractional differential equations

We utilise our Theorem 2.2 to investigate the existence and uniqueness of solutions for the nonlinear fractional differential equation of multi-order. An interesting feature about our result is that continuity is no longer needed. The nonlinear multi-order fractional differential equation is studied in the current section:

$$\mathcal{L}(\mathcal{D})\theta(\varsigma) = \sigma\big(\varsigma, \theta(\varsigma)\big), \quad \varsigma \in \mathscr{J} = [0, \mathscr{A}], \mathscr{A} > 0. \tag{16}$$

Here,

$$\mathcal{L}(\mathcal{D}) = \gamma_w \, {}^c\mathcal{D}^{\delta_w} + \gamma_{w-1} \, {}^c\mathcal{D}^{\delta_{w-1}} + \cdots + \gamma_1 \, {}^c\mathcal{D}^{\delta_1} + \gamma_0 \, {}^c\mathcal{D}^{\delta_0},$$
$$\gamma_\flat \in \mathbb{R} \quad (\flat = 0, 1, 2, 3, \dots, w), \qquad \gamma_w \neq 0, \qquad 0 \leq \delta_0 < \delta_1 < \delta_2 < \cdots < \delta_{w-1} < \delta_w < 1,$$

with the periodic boundary condition

$$\theta(0) = \theta(\mathscr{A}) \tag{17}$$

and the anti-periodic boundary condition

$$\theta(0) = -\theta(\mathscr{A}). \tag{18}$$

Theorem 3.1 *Under the following assumptions, boundary value problem* (16)–(17) *and* (16)–(18) *has a unique solution.*

$(\mathcal{A}_1)$. *For all $\varsigma \in [0, \mathscr{A}]$, we have*

$$\psi(\varsigma) \leq \int_0^{\mathscr{A}} \mathscr{G}(\varsigma, \varkappa)\sigma\big(\varkappa, \varphi(\varkappa)\big)\, d\varkappa \tag{19}$$

and

$$\varphi(\varsigma) \geq \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\bigl(\varkappa,\psi(\varkappa)\bigr)\,d\varkappa. \tag{20}$$

($\mathcal{A}_2$). *For all* $\varkappa \in [0,\mathscr{A}]$, σ *is a decreasing function, that is,* $x,y \in \mathbb{R}$, $x \geq y \Rightarrow \sigma(\varkappa,x) \leq \sigma(\varkappa,y)$.

($\mathcal{A}_3$). $\sup_{\varsigma\in[0,\mathscr{A}]} \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\,d\varkappa \leq 1$.

($\mathcal{A}_4$). *For all* $\varkappa \in [0,\mathscr{A}]$, *for all* $x,y \in \mathbb{R}$ *with* ($x \leq \varphi_0$ *and* $y \geq \psi_0$) *or* ($x \geq \psi_0$ *and* $y \leq \varphi_0$) *for* $\psi_0,\varphi_0 \in \mathbb{R}\times\mathbb{R}$.

($\mathcal{A}_5$). $|\sigma(\varsigma,y) - \sigma(\varsigma,x)| \leq \frac{1}{\tilde{\mathscr{G}}}|y-x|e^{-\frac{1}{|y-x|}}$, x, $y \in \mathbb{R}$, *where* $\tilde{\mathscr{G}} = \sup\{\int_0^{\mathscr{A}} |\mathscr{G}(\varsigma,\varkappa)|\,d\varkappa, \varsigma \in [0,\mathscr{A}]\}$; *here the function* $\varsigma \in [0,\mathscr{A}] \mapsto \int_0^{\mathscr{A}} |\mathscr{G}(\varsigma,\varkappa)|\,d\varkappa$ *is continuous on* $[0,\mathscr{A}]$.

($\mathcal{A}_6$). *The mapping* σ *from* $\mathscr{J}\times\mathbb{R}$ *to* $\mathbb{R}$ *is continuous.*

($\mathcal{A}_7$). *Define the set* $\mathcal{C} = \{\theta \in \mathbb{C}([0,\mathscr{A}])/\psi \leq \theta(\varsigma) \leq \varphi$ *for all* $\varsigma \in [0,\mathscr{A}]\}$.

Proof Let $\mathbb{C}(\mathscr{J},\mathbb{R})$ represent the set of all continuous functions from $\mathscr{J} = [0,\mathscr{A}]$ into $\mathbb{R}$ with the norm

$$\|\theta\| = \sup\bigl\{\bigl|\theta(\varsigma)\bigr|^2; \varsigma \in \mathscr{J}\bigr\}, \quad \mathscr{A} > 0.$$

$\mathbb{C}^w(\mathscr{J},\mathbb{R})$ represents the set of all functions characterised on $\mathscr{J} = [0,\mathscr{A}]$, $\mathscr{A} > 0$.

Applying periodic boundary condition:

Case-1: $w = 1$ and $\psi_0 = 0$.

The nonlinear fractional differential equation (16)–(17) reduces to

$$\gamma_1\, {}^c\mathcal{D}^{\delta_1}\theta(\varsigma) + \gamma_0\theta(\varsigma) = \sigma\bigl(\varsigma,\theta(\varsigma)\bigr), \tag{21}$$

$\varsigma \in \mathscr{J} = [0,\mathscr{A}]$, $\mathscr{A} > 0$, $\gamma_0,\gamma_1 \in \mathbb{R}$, $\gamma_1 \neq 0$, the boundary conditions that are periodic $\theta(0) = \theta(\mathscr{A})$ yield the equation for the fractional integral

$$\theta(\varsigma) = \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\bigl(\varkappa,\theta(\varkappa)\bigr)\,d\varkappa,$$

under which $\mathscr{G}(\varsigma,\varkappa)$ is the subsequent Green function

$$\mathscr{G}(\varsigma,\varkappa) = \begin{cases} [\frac{\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\varsigma^{\delta_1})}{1-\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\mathscr{A}^{\delta_1})}]\frac{1}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1-1}\mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1}) & \\ \quad + \frac{1}{\gamma_1}(\varsigma-\varkappa)^{\delta_1-1}\mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\varsigma-\varkappa)^{\delta_1}), & \text{for } 0 \leq \varkappa < \varsigma, \\ [\frac{\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\varsigma^{\delta_1})}{1-\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\mathscr{A}^{\delta_1})}]\frac{1}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1-1}\mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1}), & \text{for } \varsigma \leq \varkappa < \mathscr{A}. \end{cases}$$

Case-2: $w \geq 2$.

The nonlinear fractional differential equation (16)–(17)

$$\mathcal{L}(\mathcal{D})\theta(\varsigma) = \sigma\bigl(\varsigma,\theta(\varsigma)\bigr), \quad \varsigma \in \mathscr{J} = [0,\mathscr{A}], \mathscr{A} > 0,$$

where

$$\mathcal{L}(\mathcal{D}) = \gamma_w\, {}^c\mathcal{D}^{\delta_w} + \gamma_{w-1}\, {}^c\mathcal{D}^{\delta_{w-1}} + \cdots + \gamma_1\, {}^c\mathcal{D}^{\delta_1} + \gamma_0\, {}^c\mathcal{D}^{\delta_0},$$

$$\gamma_\flat \in \mathbb{R} \quad (\flat = 0,1,2,3,\ldots,w), \qquad \gamma_\flat \neq 0, \qquad 0 \leq \delta_0 < \delta_1 < \delta_2 < \cdots < \delta_{w-1} < \delta_w < 1,$$

the boundary conditions that are periodic

$$\theta(0) = \theta(\mathscr{A})$$

yield the equation for the fractional integral

$$\theta(\varsigma) = \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\bigl(\varkappa,\theta(\varkappa)\bigr)\,d\varkappa,$$

under which $\mathscr{G}(\varsigma,\varkappa)$ is the subsequent Green function:

For $0 \leq \varkappa < \varsigma$,

$$\begin{aligned}
\mathscr{G}(\varsigma,\varkappa) = {}& \Biggl[\frac{\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})}{1-\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})} \\
&\times \frac{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\varsigma^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\varsigma^{\delta})}{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\mathscr{A}^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\mathscr{A}^{\delta})} \Biggr] \\
&\times \frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2}) \\
&\times \prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\mathscr{A}-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\mathscr{A}-\varkappa)^{\delta}\right) \\
&+ \frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2}) \\
&\times \prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\varsigma-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\varsigma-\varkappa)^{\delta}\right).
\end{aligned}$$

For $\varsigma \leq \varkappa < \mathscr{A}$,

$$\begin{aligned}
\mathscr{G}(\varsigma,\varkappa) = {}& \Biggl[\frac{\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})}{1-\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})} \\
&\times \frac{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\varsigma^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\varsigma^{\delta})}{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\mathscr{A}^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\mathscr{A}^{\delta})} \Biggr] \\
&\times \frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2}) \\
&\times \prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\mathscr{A}-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\mathscr{A}-\varkappa)^{\delta}\right),
\end{aligned}$$

where the terms $(r; a_0, a_1, a_2, \dots, a_{w-2})$ are the so-called the multinomial coefficients; here, moreover,

$$\delta = \delta_w - \delta_{w-1}, \qquad \varrho = \delta_w + \sum_{j=0}^{w-2} (\delta_{w-1} - \delta_j) a_j - \delta_q + 1,$$

$$\lambda = \delta_w + \sum_{j=0}^{w-2} (\delta_{w-1} - \delta_j) a_j.$$

Applying anti-periodic boundary condition:

Case-1: $w = 1$, $\delta_0 = 0$.

The nonlinear fractional differential equation (16)–(17) reduces to

$$\gamma_1 {}^c\mathcal{D}^{\delta_1} \theta(\varsigma) + \gamma_0 \theta(\varsigma) = \sigma\big(\varsigma, \theta(\varsigma)\big), \tag{22}$$

$\varsigma \in \mathscr{J} = [0, \mathscr{A}]$, $\mathscr{A} > 0, \gamma_0, \gamma_1 \in \mathbb{R}$, $\gamma_1 \neq 0$, the boundary conditions that are anti-periodic $\theta(0) = -\theta(\mathscr{A})$ yield the equation for the fractional integral

$$\theta(\varsigma) = \int_0^{\mathscr{A}} \mathscr{G}(\varsigma, \varkappa) \sigma\big(\varkappa, \theta(\varkappa)\big)\, d\varkappa,$$

under which $\mathscr{G}(\varsigma, \varkappa)$ is the subsequent Green function:

$$\mathscr{G}(\varsigma, \varkappa) = \begin{cases} \big[\frac{(-1)\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\varsigma^{\delta_1})}{1+\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\mathscr{A}^{\delta_1})}\big] \frac{1}{\gamma_1} (\mathscr{A} - \varkappa)^{\delta_1 - 1} \mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\mathscr{A} - \varkappa)^{\delta_1}) & \\ \quad + \frac{1}{\gamma_1}(\varsigma - \varkappa)^{\delta_1 - 1} \mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\varsigma - \varkappa)^{\delta_1}), & \text{for } 0 \le \varkappa < \varsigma, \\ \big[\frac{(-1)\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\varsigma^{\delta_1})}{1+\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\mathscr{A}^{\delta_1})}\big] \frac{1}{\gamma_1} (\mathscr{A} - \varkappa)^{\delta_1 - 1} \mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\mathscr{A} - \varkappa)^{\delta_1}), & \text{for } \varsigma \le \varkappa < \mathscr{A}. \end{cases}$$

Case-2: $w \ge 2$.

The nonlinear fractional differential equation $\mathcal{L}(\mathcal{D})\theta(\varsigma) = \sigma(\varsigma, \theta(\varsigma))$, the boundary conditions that are anti-periodic $\theta(0) = -\theta(\mathscr{A})$ yield the equation for the fractional integral

$$\theta(\varsigma) = \int_0^{\mathscr{A}} \mathscr{G}(\varsigma, \varkappa) \sigma\big(\varkappa, \theta(\varkappa)\big)\, d\varkappa,$$

under which $\mathscr{G}(\varsigma, \varkappa)$ is the subsequent Green function:

For $0 \le \varkappa < \varsigma$,

$$\begin{aligned} \mathscr{G}(\varsigma, \varkappa) = {} & \Bigg[\frac{(-1)\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\ge 0,\dots,a_{w-2}\ge 0}}(r; a_0, a_1, a_2, \dots, a_{w-2})}{1+\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\ge 0,\dots,a_{w-2}\ge 0}}(r; a_0, a_1, a_2, \dots, a_{w-2})} \\ & \times \frac{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\varsigma^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\varsigma^{\delta})}{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\mathscr{A}^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\mathscr{A}^{\delta})}\Bigg] \\ & \times \frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\ge 0,\dots,a_{w-2}\ge 0}}(r; a_0, a_1, a_2, \dots, a_{w-2}) \end{aligned}$$

$$\times \prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\mathscr{A}-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\mathscr{A}-\varkappa)^{\delta}\right)$$
$$+\frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})$$
$$\times \prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\varsigma-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\varsigma-\varkappa)^{\delta}\right).$$

For $\varsigma \leq \varkappa < \mathscr{A}$,

$$\mathscr{G}(\varsigma,\varkappa)=\left[\frac{(-1)\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})}{1+\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})}\right.$$
$$\left.\times\frac{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\varsigma^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\varsigma^{\delta})}{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\mathscr{A}^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\mathscr{A}^{\delta})}\right]$$
$$\times\frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})$$
$$\times \prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\mathscr{A}-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\mathscr{A}-\varkappa)^{\delta}\right)$$
$$+\frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})$$
$$\times \prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\varsigma-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\varsigma-\varkappa)^{\delta}\right),$$

where the terms $(r;a_0,a_1,a_2,\ldots,a_{w-2})$ are the so-called the multinomial coefficients; here, moreover,

$$\delta=\delta_w-\delta_{w-1},\qquad \varrho=\delta_w+\sum_{j=0}^{w-2}(\delta_{w-1}-\delta_j)a_j-\delta_q+1,$$
$$\lambda=\delta_w+\sum_{j=0}^{w-2}(\delta_{w-1}-\delta_j)a_j.$$

Now consider the fractional integral

$$\theta(\varsigma)=\int_0^{\mathscr{A}}\mathscr{G}(\varsigma,\varkappa)\sigma\bigl(\varkappa,\theta(\varkappa)\bigr)\,d\varkappa,$$

where (17) and (18) are the boundary conditions and $\mathscr{G}(\varsigma,\varkappa)$ is the Green function referring to those ailments accordingly, as given above, and the function $\sigma:[0,\mathscr{A}]\times\mathbb{R}\to\mathbb{R}$ is continuous.

Let $\mathbb{X}=\mathbb{C}([0,\mathscr{A}])$ be the set of real-valued continuous functions from $\mathscr{J}=[0,\mathscr{A}]$ into $\mathbb{R}$. We endow $\mathbb{X}$ with $\mathbb{r}(g,h)=\sup_{\varsigma\in\mathscr{J}}|g(\varsigma)-h(\varsigma)|^2=\|g(\varsigma)-h(\varsigma)\|$ for all $g,h\in\mathbb{X}$.

Define $\zeta : \mathbb{X} \times \mathbb{X} \to [1,\infty)$ by

$$\zeta\big(\zeta_1(\varsigma),\zeta_2(\varsigma)\big) = \begin{cases} 3 + \sup_{\varsigma\in\mathscr{J}} |\zeta_1(\varsigma) - \zeta_2(\varsigma)|, & \text{if } \zeta_1(\varsigma) \neq \zeta_2(\varsigma), \\ 3, & \text{if } \zeta_1(\varsigma) = \zeta_2(\varsigma). \end{cases}$$

It is clear that $(\mathbb{X}, \mathbb{r})$ is a CRMS.

Let us define $\mathscr{F} : (0,\infty) \to \mathbb{R}$ and $\xi : (0,\infty) \to (0,\infty)$ by $\mathscr{F}(z) = \log z$, $z > 0$ and $\xi(t) = \frac{1}{t}$, $t \in \mathbb{R}^+$.

Let $(\delta,\varrho) \in \mathbb{X} \times \mathbb{X}$, $(\delta_0,\varrho_0) \in \mathbb{R} \times \mathbb{R}$ such that

$$\delta_0 \le \delta(\varsigma) \le \varrho(\varsigma) \le \varrho_0 \quad \text{for all } \varsigma \in \mathscr{J}. \tag{23}$$

Define the closed subsets of $\mathbb{X}$, $\mathfrak{B}$ and $\mathfrak{C}$ by

$$\mathfrak{B} = \big\{\theta(\varsigma) \in \mathbb{X}/\theta(\varsigma) \le \varrho\big\} \quad \text{and}$$
$$\mathfrak{C} = \big\{\theta(\varsigma) \in \mathbb{X}/\theta(\varsigma) \ge \delta\big\}.$$

Define the mapping $\mathscr{A} : \mathbb{C}(\mathscr{J},\mathbb{R}) \to \mathbb{C}(\mathscr{J},\mathbb{R})$ by

$$\mathscr{A}\theta(\varsigma) = \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\theta(\varkappa)\big)\,d\varkappa \quad \text{for all } \varsigma \in [0,\mathscr{A}].$$

We shall prove that

$$\mathscr{A}(\mathfrak{B}) \subseteq \mathfrak{C} \quad \text{and} \quad \mathscr{A}(\mathfrak{C}) \subseteq \mathfrak{B}. \tag{24}$$

Let $\theta \in \mathfrak{B}$, that is, $\theta(\varkappa) \le \varrho(\varkappa)$ for all $\varkappa \in [0,\mathscr{A}]$. By using $(\mathcal{A}_2)$ of our assumption, we get

$$\mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\theta(\varkappa)\big) \ge \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\varrho(\varkappa)\big) \quad \text{for all } \varsigma,\varkappa \in [0,\mathscr{A}], \tag{25}$$

as $\mathscr{G}(\varsigma,\varkappa) \ge 0$ for all $\varsigma,\varkappa \in [0,\mathscr{A}]$.

The above inequality with hypothesis implies that

$$\int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\theta(\varkappa)\big)\,d\varkappa \ge \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\varrho(\varkappa)\big)\,d\varkappa \ge \delta(\varsigma)$$
$$\text{for all } \varsigma,\varkappa \in [0,\mathscr{A}].$$

Thus $\mathscr{A}\theta \in \mathfrak{C}$.

Similarly, let $\theta \in \mathfrak{C}$, that is, $\theta(\varkappa) \ge \delta(\varkappa)$ for all $\varkappa \in [0,\mathscr{A}]$. By using $(\mathcal{A}_2)$ of our assumption and since $\mathscr{G}(\varsigma,\varkappa) \ge 0$ for all $\varsigma,\varkappa \in [0,\mathscr{A}]$, we get

$$\mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\theta(\varkappa)\big) \le \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\delta(\varkappa)\big) \quad \text{for all } \varsigma,\varkappa \in [0,\mathscr{A}]. \tag{26}$$

The above inequality with hypothesis implies that

$$\int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\theta(\varkappa)\big)\,d\varkappa \le \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\delta(\varkappa)\big)\,d\varkappa \le \delta(\varsigma)$$

for all $\varsigma,\varkappa \in [0,\mathscr{A}]$.

Thus $\mathscr{A}\theta \in \mathfrak{B}$.

Hence $\mathscr{A}(\mathfrak{B}) \subseteq \mathfrak{C}$ and $\mathscr{A}(\mathfrak{C}) \subseteq \mathfrak{B}$.

Now, let $\theta \in \mathfrak{B}$ and $\mu \in \mathfrak{C}$, that is, for all $\varsigma \in [0,\mathscr{A}]$,

$$\theta(\varsigma) \le \varrho(\varsigma), \qquad \mu(\varsigma) \ge \delta(\varsigma).$$

This implies that for all $\varsigma \in [0,\mathscr{A}]$,

$$\theta(\varsigma) \le \varrho_0, \qquad \mu(\varsigma) \ge \delta_0.$$

Now, by using conditions $(\mathcal{A}_3)$ and $(\mathcal{A}_5)$,

$$\begin{aligned}
&\big|(\mathscr{A}\theta)(\varsigma) - (\mathscr{A}\mu)(\varsigma)\big|^2 \\
&\quad = \left|\int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\theta(\varkappa)\big)\,d\varkappa - \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\mu(\varkappa)\big)\,d\varkappa\right|^2 \\
&\quad = \left|\int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\big[\sigma\big(\varkappa,\theta(\varkappa)\big) - \sigma\big(\varkappa,\mu(\varkappa)\big)\big]\,d\varkappa\right|^2 \\
&\quad \le \left|\int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\,d\varkappa\right|^2 \big|\sigma\big(\varkappa,\theta(\varkappa)\big) - \sigma\big(\varkappa,\mu(\varkappa)\big)\big|^2 \\
&\quad \le \left|\int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\,d\varkappa\right|^2 \frac{1}{(\tilde{\mathscr{G}})^2}\big|\theta(\varkappa) - \mu(\varkappa)\big|^2 e^{-\frac{1}{|\theta(\varkappa)-\mu(\varkappa)|^2}} \\
&\quad \le \left|\sup \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\,d\varkappa\right|^2 \frac{1}{(\tilde{\mathscr{G}})^2} \sup_{\varsigma\in\mathscr{J}}\big|\theta(\varsigma) - \mu(\varsigma)\big|^2 e^{-\frac{1}{\sup_{\varsigma\in\mathscr{J}}|\theta(\varsigma)-\mu(\varsigma)|^2}} \\
&\quad \le (\tilde{\mathscr{G}})^2 \frac{1}{(\tilde{\mathscr{G}})^2}\mathbb{r}\big(\theta(\varsigma),\mu(\varsigma)\big)e^{-\frac{1}{\mathbb{r}(\theta(\varsigma),\mu(\varsigma))}} \\
&\quad \le \mathbb{r}\big(\theta(\varsigma),\mu(\varsigma)\big)e^{-\frac{1}{\mathbb{r}(\theta(\varsigma),\mu(\varsigma))}},
\end{aligned}$$

which implies

$$\sup\big|(\mathscr{A}\theta)(\varsigma) - (\mathscr{A}\mu)(\varsigma)\big|^2 \le e^{-\frac{1}{\mathbb{r}(\theta(\varsigma),\mu(\varsigma))}}\,\mathbb{r}\big(\theta(\varsigma),\mu(\varsigma)\big).$$

Thus,

$$\mathbb{r}(\mathscr{A}\theta,\mathscr{A}\mu) \le e^{-\frac{1}{\mathbb{r}(\theta(\varsigma),\mu(\varsigma))}}\,\mathbb{r}\big(\theta(\varsigma),\mu(\varsigma)\big).$$

Applying log on both sides, we get

$$\begin{aligned}
&\log\big[\mathbb{r}(\mathscr{A}\theta,\mathscr{A}\mu)\big] \le \log\big[e^{-\frac{1}{\mathbb{r}(\theta(\varsigma),\mu(\varsigma))}}\,\mathbb{r}\big(\theta(\varsigma),\mu(\varsigma)\big)\big] \\
&\quad \Rightarrow \quad \log\big[\mathbb{r}(\mathscr{A}\theta,\mathscr{A}\mu)\big] \le \log\big[\mathbb{r}\big(\theta(\varsigma),\mu(\varsigma)\big)\big] + \log\big[e^{-\frac{1}{\mathbb{r}(\theta(\varsigma),\mu(\varsigma))}}\big]
\end{aligned}$$

$$\Rightarrow \quad \log\big[\mathbb{r}(\mathscr{A}\theta,\mathscr{A}\mu)\big] \le \log\big[\mathbb{r}\big(\theta(\varsigma),\mu(\varsigma)\big)\big] - \frac{1}{\mathbb{r}(\theta(\varsigma),\mu(\varsigma))}$$

$$\Rightarrow \quad \mathscr{F}\big(\mathbb{r}(\mathscr{A}\theta,\mathscr{A}\mu)\big) \le \mathscr{F}\big(\mathbb{r}(\theta,\mu)\big) - \frac{1}{\mathbb{r}(\theta,\mu)}$$

$$\Rightarrow \quad \frac{1}{\mathbb{r}(\theta,\mu)} + \mathscr{F}\big(\mathbb{r}(\mathscr{A}\theta,\mathscr{A}\mu)\big) \le \mathscr{F}\big(\mathbb{r}(\theta,\mu)\big)$$

$$\Rightarrow \quad \xi\big(\mathbb{r}(\theta,\mu)\big) + \mathscr{F}\big(\mathbb{r}(\mathscr{A}\theta,\mathscr{A}\mu)\big) \le \mathscr{F}\big(\mathbb{r}(\theta,\mu)\big).$$

A similar method can be used to demonstrate that the inequality mentioned above is true if we take $\theta \in \mathfrak{C}$ and $\mu \in \mathfrak{B}$. Thus, by our Theorem 2.2, $\mathscr{A}$ has a unique fixed point, as it satisfied all the conditions of Theorem 2.2. □

We deduce that $\mathscr{A}$ has a unique fixed point $\theta^* \in \mathfrak{B} \cap \mathfrak{C} = \mathcal{C}$, that is, $\theta^* \in \mathcal{C}$ is the unique solution to boundary value problem (16)–(17) and (16)–(18). More importantly, we have utilised our fixed point result to demonstrate the existence of solution to nonlinear multi-order fractional differential equation with boundary conditions that are periodic/anti-periodic in the context of CRMS without considering the property of continuity.

Theorem 3.2 *Under the following assumptions, boundary value problem* (16)–(17) *and* (16)–(18) *has a unique solution.*

($\mathcal{C}_1$). *For all* $\varsigma \in [0,\mathscr{A}]$, *we have*

$$\psi(\varsigma) \le \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\varphi(\varkappa)\big)\,d\varkappa \tag{27}$$

and

$$\varphi(\varsigma) \ge \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\psi(\varkappa)\big)\,d\varkappa. \tag{28}$$

($\mathcal{C}_2$). *For all* $\varkappa \in [0,\mathscr{A}]$, σ *is a decreasing function, that is,* $x,y \in \mathbb{R}$, $x \ge y \Rightarrow \sigma(\varkappa,x) \le \sigma(\varkappa,y)$.

($\mathcal{C}_3$). $\sup_{\varsigma\in[0,\mathscr{A}]} \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\,d\varkappa \le 1$.

($\mathcal{C}_4$). *For all* $\varkappa \in [0,\mathscr{A}]$, *for all* $x,y \in \mathbb{R}$ *with* ($x \le \varphi_0$ *and* $y \ge \psi_0$) *or* ($x \ge \psi_0$ *and* $y \le \varphi_0$) *for* $\psi_0,\varphi_0 \in \mathbb{R}\times\mathbb{R}$.

($\mathcal{C}_5$). *There exists a strictly increasing sequence* $(\ell_w)_{w\in\mathbb{N}\cup\{0\}}$ *satisfying* $w_0 = 0$, $\ell_w \ge 1$, $\ell_w - \ell_{w-1} \le 1$ *for all* $w \in \mathbb{N}$, $\ell_w \to \infty$ *such that, for any* $w \in \mathbb{N}$,

$$\big|\sigma(\varsigma,y) - \sigma(\varsigma,x)\big|^2 \le \frac{1}{(\tilde{\mathscr{G}})^2}\frac{1}{e^{x_w-(x_{w-1})^2}}|y-x|^2$$

for all $x,y \in \mathbb{R}$ *and* $\varsigma \in \mathscr{J}$ *such that* $|y-x| < x_w e^{\mathscr{A}}$ *,where* $\tilde{\mathscr{G}} = \sup\{\int_0^{\mathscr{A}} |\mathscr{G}(\varsigma,\varkappa)|\,d\varkappa$, $\varsigma \in [0,\mathscr{A}]\}$.

($\mathcal{C}_6$). σ *is a mapping from* $\mathscr{J} \times \mathbb{R}$ *to* $\mathbb{R}$ *is continuous.*

($\mathcal{C}_7$). *Define the set* $\mathcal{C} = \{\theta \in \mathbb{C}([0,\mathscr{A}])/\psi \le \theta(\varsigma) \le \varphi$ *for all* $\varsigma \in [0,\mathscr{A}]\}$.

Proof Let $\mathbb{C}(\mathscr{J},\mathbb{R})$ represent the set of all continuous functions from $\mathscr{J}=[0,\mathscr{A}]$ into $\mathbb{R}$ with the norm

$$\|\theta\| = \sup\{|\theta(\varsigma)|^2; \varsigma \in \mathscr{J}\}, \quad \mathscr{A} > 0.$$

$\mathbb{C}^w(\mathscr{J},\mathbb{R})$ represents the set of all real-valued functions characterised on $\mathscr{J}=[0,\mathscr{A}]$, $\mathscr{A}>0$ consisting of continuous derivatives.

Applying periodic boundary condition:

Case-1: $w=1$ and $\psi_0=0$.

The nonlinear fractional differential equation (16)–(17) reduces to

$$\gamma_1\, {}^c\mathcal{D}^{\delta_1}\theta(\varsigma) + \gamma_0\theta(\varsigma) = \sigma\big(\varsigma,\theta(\varsigma)\big), \tag{29}$$

$\varsigma \in \mathscr{J}=[0,\mathscr{A}]$, $\mathscr{A}>0$, $\gamma_0,\gamma_1\in\mathbb{R}$, $\gamma_1\neq 0$, with boundary condition that is periodic $\theta(0)=\theta(\mathscr{A})$ and is equal to the fractional integral

$$\theta(\varsigma) = \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\theta(\varkappa)\big)\,d\varkappa.$$

The following Green function, where $\mathscr{G}(\mathfrak{h},\varkappa)$, is used:

$$\mathscr{G}(\varsigma,\varkappa) = \begin{cases} [\frac{\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\varsigma^{\delta_1})}{1-\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\mathscr{A}^{\delta_1})}]\frac{1}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1-1}\mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1}) & \\ \quad +\frac{1}{\gamma_1}(\varsigma-\varkappa)^{\delta_1-1}\mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\varsigma-\varkappa)^{\delta_1}), & \text{for } 0\le\varkappa<\varsigma, \\ [\frac{\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\varsigma^{\delta_1})}{1-\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\mathscr{A}^{\delta_1})}]\frac{1}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1-1}\mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1}), & \text{for } \varsigma\le\varkappa<\mathscr{A}. \end{cases}$$

Case-2: $w\ge 2$.

The nonlinear fractional differential equation (16)–(17)

$$\mathcal{L}(\mathcal{D})\theta(\varsigma) = \sigma\big(\varsigma,\theta(\varsigma)\big), \quad \varsigma\in\mathscr{J}=[0,\mathscr{A}], \mathscr{A}>0;$$

here,

$$\mathcal{L}(\mathcal{D}) = \gamma_w\,{}^c\mathcal{D}^{\delta_w} + \gamma_{w-1}\,{}^c\mathcal{D}^{\delta_{w-1}} + \cdots + \gamma_1\,{}^c\mathcal{D}^{\delta_1} + \gamma_0\,{}^c\mathcal{D}^{\delta_0},$$

$$\gamma_\flat\in\mathbb{R} \quad (\flat=0,1,2,3,\ldots,w), \qquad \gamma_\flat\neq 0, \qquad 0\le\delta_0<\delta_1<\delta_2<\cdots<\delta_{w-1}<\delta_w<1,$$

with boundary condition that is periodic

$$\theta(0)=\theta(\mathscr{A}),$$

equal to the fractional integral

$$\theta(\varsigma) = \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\theta(\varkappa)\big)\,d\varkappa,$$

where $\mathscr{G}(\varsigma,\varkappa)$ is the following Green function:

For $0 \leq \varkappa < \varsigma$,

$$\begin{aligned}\mathscr{G}(\varsigma,\varkappa) &= \Bigg[\frac{\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\dots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\dots,a_{w-2})}{1-\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\dots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\dots,a_{w-2})}\\
&\quad\times\frac{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\varsigma^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\varsigma^{\delta})}{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\mathscr{A}^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\mathscr{A}^{\delta})}\Bigg]\\
&\quad\times\frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\dots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\dots,a_{w-2})\\
&\quad\times\prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\mathscr{A}-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\mathscr{A}-\varkappa)^{\delta}\right)\\
&\quad+\frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\dots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\dots,a_{w-2})\\
&\quad\times\prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\varsigma-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\varsigma-\varkappa)^{\delta}\right).\end{aligned}$$

For $\varsigma \leq \varkappa < \mathscr{A}$,

$$\begin{aligned}\mathscr{G}(\varsigma,\varkappa) &= \Bigg[\frac{\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\dots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\dots,a_{w-2})}{1-\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\dots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\dots,a_{w-2})}\\
&\quad\times\frac{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\varsigma^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\varsigma^{\delta})}{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\mathscr{A}^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\mathscr{A}^{\delta})}\Bigg]\\
&\quad\times\frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\dots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\dots,a_{w-2})\\
&\quad\times\prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\mathscr{A}-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\mathscr{A}-\varkappa)^{\delta}\right),\end{aligned}$$

where the terms $(r;a_0,a_1,a_2,\dots,a_{w-2})$ are the so-called the multinomial coefficients; here, moreover,

$$\delta=\delta_w-\delta_{w-1},\qquad \varrho=\delta_w+\sum_{j=0}^{w-2}(\delta_{w-1}-\delta_j)a_j-\delta_q+1,$$

$$\lambda=\delta_w+\sum_{j=0}^{w-2}(\delta_{w-1}-\delta_j)a_j.$$

Applying anti-periodic boundary condition:

Case-1: $w=1$, $\delta_0=0$.

The nonlinear fractional differential equation (16)–(17) reduces to

$$\gamma_1\,{}^{c}\mathcal{D}^{\delta_1}\theta(\varsigma)+\gamma_0\theta(\varsigma)=\sigma\big(\varsigma,\theta(\varsigma)\big), \tag{30}$$

$\varsigma \in \mathscr{J} = [0, \mathscr{A}]$, $\mathscr{A} > 0, \gamma_0, \gamma_1 \in \mathbb{R}, \gamma_1 \neq 0$, has a boundary condition that is anti-periodic $\theta(0) = -\theta(\mathscr{A})$ and equivalent to the fractional integral equation

$$\theta(\varsigma) = \int_0^{\mathscr{A}} \mathscr{G}(\varsigma, \varkappa)\sigma\left(\varkappa, \theta(\varkappa)\right) d\varkappa,$$

where $\mathscr{G}(\varsigma, \varkappa)$ is the following Green function:

$$\mathscr{G}(\varsigma, \varkappa) = \begin{cases} [\frac{(-1)\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\varsigma^{\delta_1})}{1+\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\mathscr{A}^{\delta_1})}]\frac{1}{\gamma_1}(\mathscr{A} - \varkappa)^{\delta_1-1}\mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\mathscr{A} - \varkappa)^{\delta_1}) & \\ \quad + \frac{1}{\gamma_1}(\varsigma - \varkappa)^{\delta_1-1}\mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\varsigma - \varkappa)^{\delta_1}), & \text{for } 0 \le \varkappa < \varsigma, \\ [\frac{(-1)\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\varsigma^{\delta_1})}{1+\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\mathscr{A}^{\delta_1})}]\frac{1}{\gamma_1}(\mathscr{A} - \varkappa)^{\delta_1-1}\mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\mathscr{A} - \varkappa)^{\delta_1}), & \text{for } \varsigma \le \varkappa < \mathscr{A}. \end{cases}$$

Case-2: $w \geq 2$.

The nonlinear fractional differential equation $\mathcal{L}(\mathcal{D})\theta(\varsigma) = \sigma(\varsigma, \theta(\varsigma))$ with the boundary value condition that is anti-periodic $\theta(0) = -\theta(\mathscr{A})$ is equal to the fractional integral

$$\theta(\varsigma) = \int_0^{\mathscr{A}} \mathscr{G}(\varsigma, \varkappa)\sigma\left(\varkappa, \theta(\varkappa)\right) d\varkappa,$$

where $\mathscr{G}(\varsigma, \varkappa)$ is the following Green function:

For $0 \le \varkappa < \varsigma$,

$$\begin{aligned} \mathscr{G}(\varsigma,\varkappa) = &\left[\frac{(-1)\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\ge 0,\dots,a_{w-2}\ge 0}}(r;a_0,a_1,a_2,\dots,a_{w-2})}{1+\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\ge 0,\dots,a_{w-2}\ge 0}}(r;a_0,a_1,a_2,\dots,a_{w-2})}\right. \\ &\times \left.\frac{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\varsigma^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\varsigma^{\delta})}{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\mathscr{A}^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\mathscr{A}^{\delta})}\right] \\ &\times \frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\ge 0,\dots,a_{w-2}\ge 0}}(r;a_0,a_1,a_2,\dots,a_{w-2}) \\ &\times \prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\mathscr{A}-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\mathscr{A}-\varkappa)^{\delta}\right) \\ &+ \frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\ge 0,\dots,a_{w-2}\ge 0}}(r;a_0,a_1,a_2,\dots,a_{w-2}) \\ &\times \prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\varsigma-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\varsigma-\varkappa)^{\delta}\right). \end{aligned}$$

For $\varsigma \le \varkappa < \mathscr{A}$,

$$\begin{aligned} \mathscr{G}(\varsigma,\varkappa) = &\left[\frac{(-1)\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\ge 0,\dots,a_{w-2}\ge 0}}(r;a_0,a_1,a_2,\dots,a_{w-2})}{1+\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\ge 0,\dots,a_{w-2}\ge 0}}(r;a_0,a_1,a_2,\dots,a_{w-2})}\right. \\ &\times \left.\frac{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\varsigma^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\varsigma^{\delta})}{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\mathscr{A}^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\mathscr{A}^{\delta})}\right] \end{aligned}$$

$$\times \frac{1}{\gamma_w} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r \\ a_0 \geq 0,\ldots,a_{w-2}\geq 0}} (r; a_0, a_1, a_2, \ldots, a_{w-2})$$

$$\times \prod_{\flat=0}^{w-2} \left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat} (\mathscr{A} - \varkappa)^{\delta r + \lambda - 1} \mathcal{E}_{\delta,\lambda}^{(r)} \left(-\frac{\gamma_{w-1}}{\gamma_w} (\mathscr{A} - \varkappa)^{\delta}\right)$$

$$+ \frac{1}{\gamma_w} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r \\ a_0 \geq 0,\ldots,a_{w-2}\geq 0}} (r; a_0, a_1, a_2, \ldots, a_{w-2})$$

$$\times \prod_{\flat=0}^{w-2} \left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat} (\varsigma - \varkappa)^{\delta r + \lambda - 1} \mathcal{E}_{\delta,\lambda}^{(r)} \left(-\frac{\gamma_{w-1}}{\gamma_w} (\varsigma - \varkappa)^{\delta}\right),$$

where the terms $(r; a_0, a_1, a_2, \ldots, a_{w-2})$ are the so-called the multinomial coefficients; here, moreover,

$$\delta = \delta_w - \delta_{w-1}, \qquad \varrho = \delta_w + \sum_{j=0}^{w-2} (\delta_{w-1} - \delta_j) a_j - \delta_q + 1,$$

$$\lambda = \delta_w + \sum_{j=0}^{w-2} (\delta_{w-1} - \delta_j) a_j.$$

Now assume the fractional integral equation

$$\theta(\varsigma) = \int_0^{\mathscr{A}} \mathscr{G}(\varsigma, \varkappa) \sigma\big(\varkappa, \theta(\varkappa)\big)\, d\varkappa,$$

where $\mathscr{G}(\varsigma, \varkappa)$ is the Green function corresponding to the boundary conditions (17) and (18), respectively, as given above, and the function $\sigma : [0, \mathscr{A}] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Let $\mathbb{X} = \mathbb{C}([0, \mathscr{A}])$ be the set of real-valued continuous functions from $\mathscr{J} = [0, \mathscr{A}]$ into $\mathbb{R}$. We endow $\mathbb{X}$ with the $\mathbb{r}(g, h) = \sup_{\varsigma \in \mathscr{J}} |g(\varsigma) - h(\varsigma)|^2 = \|g(\varsigma) - h(\varsigma)\|$ for all $g, h \in \mathbb{X}$.

Define $\zeta : \mathbb{X} \times \mathbb{X} \to [1, \infty)$ by

$$\zeta\big(\zeta_1(\varsigma), \zeta_2(\varsigma)\big) = \begin{cases} 3 + \sup_{\varsigma \in \mathscr{J}} |\zeta_1(\varsigma) - \zeta_2(\varsigma)|, & \text{if } \zeta_1(\varsigma) \neq \zeta_2(\varsigma), \\ 3, & \text{if } \zeta_1(\varsigma) = \zeta_2(\varsigma). \end{cases}$$

It is clear that $(\mathbb{X}, \mathbb{r})$ is a CRMS.

Let $(\delta, \varrho) \in \mathbb{X} \times \mathbb{X}$, $(\delta_0, \varrho_0) \in \mathbb{R} \times \mathbb{R}$ such that

$$\delta_0 \leq \delta(\varsigma) \leq \varrho(\varsigma) \leq \varrho_0 \quad \text{for all } \varsigma \in \mathscr{J}. \tag{31}$$

Define the closed subsets of $\mathbb{X}$, $\mathfrak{B}$ and $\mathfrak{C}$ by

$$\mathfrak{B} = \big\{\theta(\varsigma) \in \mathbb{X} / \theta(\varsigma) \leq \varrho\big\} \quad \text{and}$$

$$\mathfrak{C} = \big\{\theta(\varsigma) \in \mathbb{X} / \theta(\varsigma) \geq \delta\big\}.$$

Define the mapping $\mathscr{A} : \mathbb{C}(\mathscr{J}, \mathbb{R}) \to \mathbb{C}(\mathscr{J}, \mathbb{R})$ by

$$\mathscr{A}\theta(\varsigma) = \int_0^{\mathscr{A}} \mathscr{G}(\varsigma, \varkappa)\sigma\big(\varkappa, \theta(\varkappa)\big)\, d\varkappa \quad \text{for all } \varsigma \in [0, \mathscr{A}]. \tag{32}$$

We shall prove that

$$\mathscr{A}(\mathfrak{B}) \subseteq \mathfrak{C} \quad \text{and} \quad \mathscr{A}(\mathfrak{C}) \subseteq \mathfrak{B}. \tag{33}$$

Let $\theta \in \mathfrak{B}$, that is, $\theta(\varkappa) \leq \varrho(\varkappa)$ for all $\varkappa \in [0, \mathscr{A}]$. By using $(\mathcal{C}_2)$ of our assumption, we get

$$\mathscr{G}(\varsigma, \varkappa)\sigma\big(\varkappa, \theta(\varkappa)\big) \geq \mathscr{G}(\varsigma, \varkappa)\sigma\big(\varkappa, \varrho(\varkappa)\big) \quad \text{for all } \varsigma, \varkappa \in [0, \mathscr{A}], \tag{34}$$

as $\mathscr{G}(\varsigma, \varkappa) \geq 0$ for all $\varsigma, \varkappa \in [0, \mathscr{A}]$.

The above inequality (34) with hypothesis implies that

$$\int_0^{\mathscr{A}} \mathscr{G}(\varsigma, \varkappa)\sigma\big(\varkappa, \theta(\varkappa)\big)\, d\varkappa \geq \int_0^{\mathscr{A}} \mathscr{G}(\varsigma, \varkappa)\sigma\big(\varkappa, \varrho(\varkappa)\big)\, d\varkappa \geq \delta(\varsigma)$$
$$\text{for all } \varsigma, \varkappa \in [0, \mathscr{A}].$$

Thus $\mathscr{A}\theta \in \mathfrak{C}$.

Similarly, let $\theta \in \mathfrak{C}$, that is, $\theta(\varkappa) \geq \delta(\varkappa)$ for all $\varkappa \in [0, \mathscr{A}]$. By using $(\mathcal{C}_2)$ of our assumption and since $\mathscr{G}(\varsigma, \varkappa) \geq 0$ for all $\varsigma, \varkappa \in [0, \mathscr{A}]$, we get

$$\mathscr{G}(\varsigma, \varkappa)\sigma\big(\varkappa, \theta(\varkappa)\big) \leq \mathscr{G}(\varsigma, \varkappa)\sigma\big(\varkappa, \delta(\varkappa)\big) \quad \text{for all } \varsigma, \varkappa \in [0, \mathscr{A}]. \tag{35}$$

Inequality (35) with hypothesis implies that

$$\int_0^{\mathscr{A}} \mathscr{G}(\varsigma, \varkappa)\sigma\big(\varkappa, \theta(\varkappa)\big)\, d\varkappa \leq \int_0^{\mathscr{A}} \mathscr{G}(\varsigma, \varkappa)\sigma\big(\varkappa, \delta(\varkappa)\big)\, d\varkappa \leq \delta(\varsigma)$$
$$\text{for all } \varsigma, \varkappa \in [0, \mathscr{A}].$$

Thus $\mathscr{A}\theta \in \mathfrak{B}$.

Hence we proved (33), that is, $\mathscr{A}(\mathfrak{B}) \subseteq \mathfrak{C}$ and $\mathscr{A}(\mathfrak{C}) \subseteq \mathfrak{B}$.

Now, let $\theta \in \mathfrak{B}$ and $\mu \in \mathfrak{C}$, that is, for all $\varsigma \in [0, \mathscr{A}]$,

$$\theta(\varsigma) \leq \varrho(\varsigma), \qquad \mu(\varsigma) \geq \delta(\varsigma).$$

This implies from the hypothesis that for all $\varsigma \in [0, \mathscr{A}]$,

$$\theta(\varsigma) \leq \varrho_0, \qquad \mu(\varsigma) \geq \delta_0.$$

A fixed point of the operator $\mathscr{A}$ in (32) will be the solution of (16)–(17) and (16)–(18), *i.e.*, nonlinear multi-order fractional differential equation with periodic/anti-periodic boundary conditions.

Let us define $\mathscr{F} : (0, \infty) \to \mathbb{R}$ by $F(a) = \log a$ for $a > 0$ and $\xi : (0, \infty) \to (0, \infty)$ by

$$\xi(\varsigma) = \begin{cases} -\varsigma + \ell_1, & 0 < \varsigma < \ell_1, \\ -\varsigma + \ell_w, & \ell_{w-1} < \varsigma < \ell_w, w \geq 2. \end{cases}$$

Fix $w \geq 2$.

For any $\theta(\varsigma), \mu(\varsigma) \in \mathbb{C}(\mathscr{J}, \mathbb{R})$ such that $\ell_{w-1} < |\theta(\varsigma) - \mu(\varsigma)| < \ell_w$.

By using $(\mathcal{C}_3)$ and $(\mathcal{C}_5)$,

$$\begin{aligned} \left|\mathscr{A}\theta(\varsigma) - \mathscr{A}\mu(\varsigma)\right|^2 &\leq \left|\int_0^{\mathscr{A}} \mathscr{G}(\varsigma, \varkappa)\, d\varkappa\right|^2 \left|\sigma\left(\varkappa, \theta(\varkappa)\right) - \sigma\left(\varkappa, \mu(\varkappa)\right)\right|^2 \\ &\leq \left|\int_0^{\mathscr{A}} \mathscr{G}(\varsigma, \varkappa)\, d\varkappa\right|^2 \frac{1}{(\tilde{\mathscr{G}})^2} \frac{1}{e^{\ell_w - (\ell_{w-1})^2}} \left|\theta(\varsigma) - \mu(\varsigma)\right|^2 \\ &\leq (\tilde{\mathscr{G}})^2 \frac{1}{(\tilde{\mathscr{G}})^2} \frac{1}{e^{\ell_w - (\ell_{w-1})^2}} \left|\theta(\varsigma) - \mu(\varsigma)\right|^2 \\ &\leq \frac{1}{e^{\ell_w - (\ell_{w-1})^2}} \left|\theta(\varsigma) - \mu(\varsigma)\right|^2 \\ &= \frac{1}{e^{\ell_w - \ell_{w-1}^2}} |\theta - \mu|^2 \\ &< \frac{|\theta - \mu|^2}{e^{\ell_w - |\theta - \mu|^2}}. \end{aligned}$$

Thus,

$$\begin{aligned} &\left|\mathscr{A}\theta(\varsigma) - \mathscr{A}\mu(\varsigma)\right|^2 \leq \frac{|\theta - \mu|^2}{e^{\ell_w - |\theta - \mu|^2}} \\ &\quad \Rightarrow \quad e^{\ell_w - |\theta - \mu|^2} \leq \frac{|\theta - \mu|^2}{|\mathscr{A}\theta(\varsigma) - \mathscr{A}\mu(\varsigma)|^2} \\ &\quad \Rightarrow \quad \ell_w - |\theta - \mu|^2 \leq \log \frac{|\theta - \mu|^2}{|\mathscr{A}\theta(\varsigma) - \mathscr{A}\mu(\varsigma)|^2} \\ &\quad \Rightarrow \quad \ell_w - |\theta - \mu|^2 + \log\left|\mathscr{A}\theta(\varsigma) - \mathscr{A}\mu(\varsigma)\right|^2 \leq \log|\theta - \mu|^2 \\ &\quad \Rightarrow \quad \xi\left(|\theta - \mu|^2\right) + \mathscr{F}\left(|\mathscr{A}\theta - \mathscr{A}\mu|^2\right) \leq \mathscr{F}\left(|\theta - \mu|^2\right), \\ &\xi\left(\mathbb{r}(\theta, \mu)\right) + \mathscr{F}\left(\mathbb{r}(\mathscr{A}\theta, \mathscr{A}\mu)\right) \leq \mathscr{F}\left(\mathbb{r}(\theta, \mu)\right) \end{aligned} \tag{36}$$

for all $\theta, \mu \in \mathcal{C}(\mathscr{J}, \mathbb{R})$ satisfying $\ell_{w-1} \leq |\theta - \mu| < \ell_w$ when $w \geq 2$ for $w = 1$.

One can easily prove as above that (36) is satisfied for all $\theta, \mu \in \mathcal{C}(\mathscr{J}, \mathbb{R})$ such that $0 < |\theta - \mu| < \ell_1$.

Thus all the conditions of Theorem 2.2 are satisfied. Hence $\mathscr{A}$ has a unique solution, *i.e.,* nonlinear multi-order differential equation with periodic/anti-periodic boundary conditions has a unique solution. □

4 Connecting fixed point elements to nonlinear multi-term fractional delay differential equations

In this section our Theorem 2.2 is used to investigate the existence and uniqueness of solutions for the nonlinear multi-term fractional delay differential equations:

$$\mathcal{L}(\mathcal{D})\theta(\varsigma) = \sigma\big(\varsigma,\theta(\varsigma),\theta(\varsigma-\tau)\big), \quad \varsigma \in \mathscr{J} = [0,\mathscr{A}], \mathscr{A} > 0; \tag{37}$$
$$\theta(\varsigma) = \bar{\sigma}(\varsigma), \quad \varsigma \in [-\tau,0].$$

Here,

$$\mathcal{L}(\mathcal{D}) = \gamma_w\, {}^c\mathcal{D}^{\delta_w} + \gamma_{w-1}\, {}^c\mathcal{D}^{\delta_{w-1}} + \cdots + \gamma_1\, {}^c\mathcal{D}^{\delta_1} + \gamma_0\, {}^c\mathcal{D}^{\delta_0},$$
$$\gamma_\flat \in \mathbb{R} \quad (\flat = 0,1,2,3,\dots,w), \qquad \gamma_w \neq 0, \qquad 0 \le \delta_0 < \delta_1 < \delta_2 < \cdots < \delta_{w-1} < \delta_w < 1,$$

and ${}^c\mathcal{D}^\delta$ denotes the Caputo fractional derivative of order δ. Moreover, $\sigma : [0,\mathscr{A}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\bar{\sigma} : [-\tau,0] \to \mathbb{R}$ are continuous with the periodic boundary condition

$$\theta(0) = \theta(\mathscr{A}) \tag{38}$$

and the anti-periodic boundary condition

$$\theta(0) = -\theta(\mathscr{A}). \tag{39}$$

Problem (37)–(38) is equivalent to the integral equations for $w = 1$ and $\delta_0 = 0$ as well as $w \ge 2$.

$$\theta(\varsigma) = \begin{cases} \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma(\varkappa,\theta(\varkappa),\theta(\varkappa-\tau))\,d\varkappa, & \varsigma \in [0,\mathscr{A}], \\ \bar{\sigma}(\varsigma), & \varsigma \in [-\tau,0]. \end{cases}$$

The Green function for problem (37)–(38) when $w = 1$ and $\delta_0 = 0$ is

$$\mathscr{G}(\varsigma,\varkappa) = \begin{cases} [\frac{\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\varsigma^{\delta_1})}{1-\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\mathscr{A}^{\delta_1})}]\frac{1}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1-1}\mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1}) & \\ \quad + \frac{1}{\gamma_1}(\varsigma-\varkappa)^{\delta_1-1}\mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\varsigma-\varkappa)^{\delta_1}), & \text{for } 0 \le \varkappa < \varsigma, \\ [\frac{\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\varsigma^{\delta_1})}{1-\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\mathscr{A}^{\delta_1})}]\frac{1}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1-1}\mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1}), & \text{for } \varsigma \le \varkappa < \mathscr{A}. \end{cases}$$

The Green function for problem (37)–(38) when $w \ge 2$ is, for $0 \le \varkappa < \varsigma$,

$$\begin{aligned} \mathscr{G}(\varsigma,\varkappa) = {} & \Bigg[\frac{\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\ge 0,\dots,a_{w-2}\ge 0}}(r;a_0,a_1,a_2,\dots,a_{w-2})}{1-\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\ge 0,\dots,a_{w-2}\ge 0}}(r;a_0,a_1,a_2,\dots,a_{w-2})} \\ & \times \frac{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\varsigma^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\varsigma^{\delta})}{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\mathscr{A}^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\mathscr{A}^{\delta})}\Bigg] \\ & \times \frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\ge 0,\dots,a_{w-2}\ge 0}}(r;a_0,a_1,a_2,\dots,a_{w-2}) \end{aligned}$$

$$\times \prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\mathscr{A}-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\mathscr{A}-\varkappa)^{\delta}\right)$$

$$+\frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})$$

$$\times \prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\varsigma-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\varsigma-\varkappa)^{\delta}\right).$$

For $\varsigma \leq \varkappa < \mathscr{A}$,

$$\mathscr{G}(\varsigma,\varkappa)=\left[\frac{\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})}{1-\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})}\right.$$

$$\left.\times \frac{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\varsigma^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\varsigma^{\delta})}{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\mathscr{A}^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\mathscr{A}^{\delta})}\right]$$

$$\times\frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})$$

$$\times \prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\mathscr{A}-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\mathscr{A}-\varkappa)^{\delta}\right),$$

where $(r;a_0,a_1,a_2,\ldots,a_{w-2})=\frac{r!}{a_0!,a_1!,a_2!,\ldots,a_{w-2}!}$ are the so-called the multinomial coefficients; here, moreover,

$$\delta=\delta_w-\delta_{w-1},\qquad \varrho=\delta_w+\sum_{j=0}^{w-2}(\delta_{w-1}-\delta_j)a_j-\delta_q+1,$$

$$\lambda=\delta_w+\sum_{j=0}^{w-2}(\delta_{w-1}-\delta_j)a_j.$$

Problem (37)–(39) is equivalent to the integral equation for $w=1$ and $\delta_0=0$ as well as for $w\geq 2$.

$$\theta(\varsigma)=\begin{cases}\int_0^{\mathscr{A}}\mathscr{G}(\varsigma,\varkappa)\sigma(\varkappa,\theta(\varkappa),\theta(\varkappa-\tau))\,d\varkappa, & \varsigma\in[0,\mathscr{A}],\\ \bar{\sigma}(\varsigma), & \varsigma\in[-\tau,0],\end{cases}$$

where

$$\mathscr{G}(\varsigma,\varkappa)=\begin{cases}[\frac{(-1)\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\varsigma^{\delta_1})}{1+\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\mathscr{A}^{\delta_1})}]\frac{1}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1-1}\mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1}) & \\ \quad+\frac{1}{\gamma_1}(\varsigma-\varkappa)^{\delta_1-1}\mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\varsigma-\varkappa)^{\delta_1}), & \text{for } 0\leq\varkappa<\varsigma,\\ [\frac{(-1)\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\varsigma^{\delta_1})}{1+\mathcal{E}_{\delta_1,1}(-\frac{\gamma_0}{\gamma_1}\mathscr{A}^{\delta_1})}]\frac{1}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1-1}\mathcal{E}_{\delta_1,\delta_1}(-\frac{\gamma_0}{\gamma_1}(\mathscr{A}-\varkappa)^{\delta_1}), & \text{for } \varsigma\leq\varkappa<\mathscr{A}.\end{cases}$$

The Green function for problem (37)–(39) when $w \geq 2$ is, for $0 \leq \varkappa < \varsigma$,

$$\begin{aligned}\mathscr{G}(\varsigma,\varkappa) &= \Bigg[\frac{(-1)\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})}{1+\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})} \\ &\quad\times\frac{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\varsigma^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\varsigma^{\delta})}{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\mathscr{A}^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\mathscr{A}^{\delta})}\Bigg] \\ &\quad\times\frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2}) \\ &\quad\times\prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\mathscr{A}-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\mathscr{A}-\varkappa)^{\delta}\right) \\ &\quad+\frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2}) \\ &\quad\times\prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\varsigma-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\varsigma-\varkappa)^{\delta}\right).\end{aligned}$$

For $\varsigma \leq \varkappa < \mathscr{A}$,

$$\begin{aligned}\mathscr{G}(\varsigma,\varkappa) &= \Bigg[\frac{(-1)\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})}{1+\sum_{q=0}^{w}(\frac{\gamma_q}{\gamma_w})\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2})} \\ &\quad\times\frac{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\varsigma^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\varsigma^{\delta})}{\prod_{\flat=0}^{w-2}(\frac{\gamma_\flat}{\gamma_w})^{a_\flat}\mathscr{A}^{\delta r+\varrho-1}\mathcal{E}_{\delta,\varrho}^{(r)}(-\frac{\gamma_{w-1}}{\gamma_w}\mathscr{A}^{\delta})}\Bigg] \\ &\quad\times\frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2}) \\ &\quad\times\prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\mathscr{A}-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\mathscr{A}-\varkappa)^{\delta}\right) \\ &\quad+\frac{1}{\gamma_w}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\sum_{\substack{a_0+a_1+\cdots+a_{w-2}=r\\ a_0\geq 0,\ldots,a_{w-2}\geq 0}}(r;a_0,a_1,a_2,\ldots,a_{w-2}) \\ &\quad\times\prod_{\flat=0}^{w-2}\left(\frac{\gamma_\flat}{\gamma_w}\right)^{a_\flat}(\varsigma-\varkappa)^{\delta r+\lambda-1}\mathcal{E}_{\delta,\lambda}^{(r)}\left(-\frac{\gamma_{w-1}}{\gamma_w}(\varsigma-\varkappa)^{\delta}\right),\end{aligned}$$

where the terms $(r;a_0,a_1,a_2,\ldots,a_{w-2}) = \frac{r!}{a_0!,a_1!,a_2!,\ldots,a_{w-2}!}$ are the so-called multinomial coefficients; here, moreover,

$$\delta = \delta_w - \delta_{w-1}, \qquad \varrho = \delta_w + \sum_{j=0}^{w-2}(\delta_{w-1}-\delta_j)a_j - \delta_q + 1,$$

$$\lambda = \delta_w + \sum_{j=0}^{w-2}(\delta_{w-1} - \delta_j)a_j.$$

Theorem 4.1 *Under the following assumptions, boundary value problem* (37)–(38) *and* (37)–(39) *has a unique solution.*

($\mathcal{D}_1$). *Let* $\tilde{\mathscr{G}} = \sup\{\int_0^{\mathscr{A}} |\mathscr{G}(\varsigma,\varkappa)|\,d\varkappa, \varsigma \in [0,\mathscr{A}]\}$, *where the function* $\varsigma \in [0,\mathscr{A}] \mapsto \int_0^{\mathscr{A}} |\mathscr{G}(\varsigma,\varkappa)|\,d\varkappa$ *is continuous on* $[0,\mathscr{A}]$.

($\mathcal{D}_2$). *Define the set* $\mathcal{C} = \{\theta \in \mathbb{C}([-\tau,\mathscr{A}])/\psi \le \theta(\varsigma) \le \varphi$ *for all* $\varsigma \in [-\tau,\mathscr{A}]\}$.

($\mathcal{D}_3$). *The function* $\sigma : \mathscr{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

($\mathcal{D}_4$). *For all* $\varsigma \in [0,\mathscr{A}]$, *we have*

$$\psi(\varsigma) \le \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\varphi(\varkappa),\varphi(\varkappa-\tau)\big)\,d\varkappa \tag{40}$$

and

$$\varphi(\varsigma) \ge \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\psi(\varkappa),\psi(\varkappa-\tau)\big)\,d\varkappa. \tag{41}$$

($\mathcal{D}_5$). *For all* $\varkappa \in [0,\mathscr{A}]$, σ *is a decreasing function, that is,* $x,y \in \mathbb{R}$, $x \ge y \Rightarrow \sigma(\varkappa,x) \le \sigma(\varkappa,y)$.

($\mathcal{D}_6$). $\sup_{\varsigma\in[0,\mathscr{A}]} \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\,d\varkappa \le 1$.

($\mathcal{D}_7$). *For all* $\varkappa \in [0,\mathscr{A}]$, *for all* $x,y \in \mathbb{R}$ *with* ($x \le \varphi_0$ *and* $y \ge \psi_0$) *or* ($x \ge \psi_0$ *and* $y \le \varphi_0$) *for* $\psi_0,\varphi_0 \in \mathbb{R}\times\mathbb{R}$.

($\mathcal{D}_8$). *Assume that* $|\sigma(\varsigma,x_1,y) - \sigma(\varsigma,x_2,y)| \le |x_1 - x_2| e^{\frac{1}{|x_1-x_2|}} \frac{1}{\tilde{\mathscr{G}}}$.

Proof Let $\mathbb{C}([-\tau,\mathscr{A}],\mathbb{R})$ denote the set of all continuous functions defined on $[-\tau,\mathscr{A}]$ into $\mathbb{R}$ with the norm $\|\theta\|_\infty = \sup\{|\theta(\varsigma)|^2/\varsigma \in [-\tau,\mathscr{A}]\}$, $\mathscr{A} > 0$.

Here, $\mathbb{C}([-\tau,\mathscr{A}],\mathbb{R}) = \{\theta(\varsigma)/\theta : [-\tau,\mathscr{A}] \to \mathbb{R}\}$.

Let $\mathbb{X} = \mathbb{C}([-\tau,\mathscr{A}],\mathbb{R})$, we endow $\mathbb{X}$ with

$$\mathbb{r}(a,b) = \sup_{\varsigma\in[-\tau,\mathscr{A}]} \big|a(\varsigma) - b(\varsigma)\big|^2 = \big\|a(\varsigma) - b(\varsigma)\big\|_\infty \quad \text{for all } a,b \in \mathbb{X}.$$

Define $\zeta : \mathbb{C}([-\tau,\mathscr{A}],\mathbb{R}) \times \mathbb{C}([-\tau,\mathscr{A}],\mathbb{R}) \to [1,\infty)$ by

$$\zeta\big(\zeta_1(\varsigma),\zeta_2(\varsigma)\big) = \begin{cases} 3 + \sup_{\varsigma\in[-\tau,\mathscr{A}]} |\zeta_1(\varsigma),\zeta_2(\varsigma)|, & \text{if } \zeta_1(\varsigma) \ne \zeta_2(\varsigma), \\ 3, & \text{if } \zeta_1(\varsigma) \ne \zeta_2(\varsigma). \end{cases}$$

It is clear that $(\mathbb{X},\mathbb{r})$ is a CRMS. Let us define $\mathscr{F} : (0,\infty) \to \mathbb{R}$ and $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\mathscr{F}(s) = \log s$, $s > 0$ and $\xi(z) = \frac{1}{z}$, $z \in \mathbb{R}^+$.

Let $(\delta,\varrho) \in \mathbb{X}\times\mathbb{X}$, $(\delta_0,\varrho_0) \in \mathbb{R}\times\mathbb{R}$ such that

$$\delta_0 \le \delta(\varsigma) \le \varrho(\varsigma) \le \varrho_0 \quad \text{for all } \varsigma \in \mathscr{J}. \tag{42}$$

Define the closed subsets of $\mathbb{X}$, $\mathfrak{B}$ and $\mathfrak{C}$ by

$$\mathfrak{B} = \big\{\theta(\varsigma) \in \mathbb{X}/\theta(\varsigma) \le \varrho\big\} \quad \text{and}$$

$$\mathfrak{C} = \left\{\theta(\varsigma) \in \mathbb{X}/\theta(\varsigma) \geq \delta\right\}.$$

Define the operator $\mathscr{A} : \mathbb{C}([-\tau,\mathscr{A}],\mathbb{R}) \times \mathbb{C}([-\tau,\mathscr{A}],\mathbb{R})$ as

$$\mathscr{A}\theta(\varsigma) = \begin{cases} \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma(\varkappa,\theta(\varkappa),\theta(\varkappa-\tau))\,d\varkappa, & \text{for all } \varsigma \in [0,\mathscr{A}], \\ \overline{\sigma}(\varsigma), & \varsigma \in [-\tau,0], \end{cases}$$

where $\mathscr{G}(\varsigma,\varkappa)$ is the Green function of the corresponding boundary value problem.

We shall prove that

$$\mathscr{A}(\mathfrak{B}) \subseteq \mathfrak{C} \quad \text{and} \quad \mathscr{A}(\mathfrak{C}) \subseteq \mathfrak{B}. \tag{43}$$

Let $\theta \in \mathfrak{B}$, that is, $\theta(\varkappa) \leq \varrho(\varkappa)$ for all $\varkappa \in [0,\mathscr{A}]$. By using $(\mathcal{D}_2)$ of our assumption, we get

$$\mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\theta(\varkappa),\theta(\varkappa-\tau)\big) \geq \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\varrho(\varkappa),\varrho(\varkappa-\tau)\big) \quad \text{for all } \varsigma,\varkappa \in [0,\mathscr{A}] \tag{44}$$

as $\mathscr{G}(\varsigma,\varkappa) \geq 0$ for all $\varsigma,\varkappa \in [0,\mathscr{A}]$.

The above inequality with hypothesis implies that

$$\int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\theta(\varkappa),\theta(\varkappa-\tau)\big)\,d\varkappa \geq \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\varrho(\varkappa),\varrho(\varkappa-\tau)\big)\,d\varkappa \geq \delta(\varsigma)$$
$$\text{for all } \varsigma,\varkappa \in [0,\mathscr{A}].$$

Thus $\mathscr{A}\theta \in \mathfrak{C}$.

Similarly, let $\theta \in \mathfrak{C}$, that is, $\theta(\varkappa) \geq \delta(\varkappa)$ for all $\varkappa \in [0,\mathscr{A}]$. By using $(\mathcal{D}_2)$ of our assumption and since $\mathscr{G}(\varsigma,\varkappa) \geq 0$ for all $\varsigma,\varkappa \in [0,\mathscr{A}]$, we get

$$\mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\theta(\varkappa),\theta(\varkappa-\tau)\big) \leq \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\delta(\varkappa),\delta(\varkappa-\tau)\big) \quad \text{for all } \varsigma,\varkappa \in [0,\mathscr{A}]. \tag{45}$$

The above inequality with hypothesis implies that

$$\int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\theta(\varkappa),\theta(\varkappa-\tau)\big)\,d\varkappa \leq \int_0^{\mathscr{A}} \mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\delta(\varkappa),\delta(\varkappa-\tau)\big)\,d\varkappa \leq \delta(\varsigma)$$
$$\text{for all } \varsigma,\varkappa \in [0,\mathscr{A}].$$

Thus $\mathscr{A}\theta \in \mathfrak{B}$.

Hence $\mathscr{A}(\mathfrak{B}) \subseteq \mathfrak{C}$ and $\mathscr{A}(\mathfrak{C}) \subseteq \mathfrak{B}$.

Now, let $\theta \in \mathfrak{B}$ and $\mu \in \mathfrak{C}$, that is, for all $\varsigma \in [0,\mathscr{A}]$,

$$\theta(\varsigma) \leq \varrho(\varsigma), \qquad \mu(\varsigma) \geq \delta(\varsigma).$$

This implies that for all $\varsigma \in [0,\mathscr{A}]$,

$$\theta(\varsigma) \leq \varrho_0, \qquad \mu(\varsigma) \geq \delta_0.$$

Let $\theta(\varsigma) \in \mathfrak{B}$ and $\mu(\varsigma) \in \mathfrak{C}$.

By using conditions ($\mathcal{D}_6$) and ($\mathcal{D}_8$),

$$
\begin{aligned}
&\left|\mathscr{A}\theta(\varsigma)-\mathscr{A}\mu(\varsigma)\right|^2\\
&=\left|\int_0^{\mathscr{A}}\mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\theta(\varkappa),\theta(\varkappa-\tau)\big)\,d\varkappa-\int_0^{\mathscr{A}}\mathscr{G}(\varsigma,\varkappa)\sigma\big(\varkappa,\mu(\varkappa),\mu(\varkappa-\tau)\big)\,d\varkappa\right|^2\\
&=\left|\int_0^{\mathscr{A}}\mathscr{G}(\varsigma,\varkappa)\big[\sigma\big(\varkappa,\theta(\varkappa),\theta(\varkappa-\tau)\big)-\sigma\big(\varkappa,\mu(\varkappa),\mu(\varkappa-\tau)\big)\big]\,d\varkappa\right|^2\\
&=\Bigg(\bigg|\int_0^{\mathscr{A}}\mathscr{G}(\varsigma,\varkappa)\big[\sigma\big(\varkappa,\theta(\varkappa),\theta(\varkappa-\tau)\big)-\sigma\big(\varkappa,\mu(\varkappa),\theta(\varkappa-\tau)\big)\\
&\quad+\sigma\big(\varkappa,\mu(\varkappa),\theta(\varkappa-\tau)\big)-\sigma\big(\varkappa,\mu(\varkappa),\mu(\varkappa-\tau)\big)\big]\,d\varkappa\bigg|\Bigg)^2\\
&\le\Bigg(\bigg|\int_0^{\mathscr{A}}\left|\mathscr{G}(\varsigma,\varkappa)\right|\Big[\Big|\sigma\big(\varkappa,\theta(\varkappa),\theta(\varkappa-\tau)\big)-\sigma\big(\varkappa,\mu(\varkappa),\theta(\varkappa-\tau)\big)\Big|\\
&\quad+\Big|\sigma\big(\varkappa,\mu(\varkappa),\theta(\varkappa-\tau)\big)-\sigma\big(\varkappa,\mu(\varkappa),\mu(\varkappa-\tau)\big)\Big|\Big]\bigg|\Bigg)^2\\
&\le\left|\int_0^{\mathscr{A}}|\mathscr{G}(\varsigma,\varkappa)\,d\varkappa\right|^2\left|\theta(\varkappa)-\mu(\varkappa)\right|^2\frac{1}{(\tilde{\mathscr{G}})^2}e^{-\frac{1}{|\theta(\varkappa)-\mu(\varkappa)|^2}}\\
&\le\left|\sup\int_0^{\mathscr{A}}|\mathscr{G}(\varsigma,\varkappa)\,d\varkappa\right|^2\frac{1}{(\tilde{\mathscr{G}})^2}\sup_{\varsigma\in[0,\mathscr{A}]}\left|\theta(\varkappa)-\mu(\varkappa)\right|^2e^{-\frac{1}{\sup_{\varsigma\in[0,\mathscr{A}]}|\theta(\varkappa)-\mu(\varkappa)|^2}}\\
&\le(\tilde{\mathscr{G}})^2\frac{1}{(\tilde{\mathscr{G}})^2}\mathbb{r}\big(\theta(\varkappa),\mu(\varkappa)\big)e^{-\frac{1}{\mathbb{r}(\theta(\varkappa),\mu(\varkappa))}}\\
&=\mathbb{r}\big(\theta(\varkappa),\mu(\varkappa)\big)e^{-\frac{1}{\mathbb{r}(\theta(\varkappa),\mu(\varkappa))}},
\end{aligned}
$$

which implies

$$\sup\left|\mathscr{A}\theta(\varsigma)-\mathscr{A}\mu(\varsigma)\right|^2\le e^{-\frac{1}{\mathbb{r}(\theta(\varkappa),\mu(\varkappa))}}\mathbb{r}\big(\theta(\varkappa),\mu(\varkappa)\big).$$

Applying log on both sides, we get

$$\mathscr{F}\big(\mathbb{r}(\mathscr{A}\theta,\mathscr{A}\mu)\big)+\xi\big(\mathbb{r}(\theta,\mu)\big)\le\mathscr{F}\big(\mathbb{r}(\theta,\mu)\big).$$

Using the same technique, we can show that the above inequality holds also if we take $\theta\in\mathfrak{C}$ and $\mu\in\mathfrak{B}$.

Hence $\mathscr{A}$ has a unique fixed point $\theta^*\in\mathfrak{B}\cap\mathfrak{C}=\mathcal{C}$, *i.e.*, $\theta^*\in\mathcal{C}$ is the unique solution to (37)–(38) and (37)–(39). □

5 Conclusion

In this article, we developed connections between a number of concepts, including Green's functions, multi-term fractional order differential equations and metric fixed point theory. We provided the results of fixed point of nonlinear cyclic orbital ($\xi-\mathscr{F}$)-contraction under controlled rectangular metric space. With the aid of these results, we were able to derive the existence and uniqueness theorems for fractional boundary value problems in terms of Green's function for various multi-order fractional differential equations. We shall attempt to use the techniques described in this article in further work, which may serve as some kind of inspiration for using fixed point theory and fractional calculus in neural network algorithms and machine learning systems.

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
P.S.K., V.V., and K.S.N. were involved with the organizing and execution of the study, as well as in the findings analysis and manuscript writing.

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