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Nonexistence of interior bubbling solutions for slightly supercritical elliptic problems

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Abstract

In this paper, we consider the Neumann elliptic problem $(\mathcal{P}_\varepsilon)$: $-\Delta u + \mu u = u^{((n+2)/(n-2))+\varepsilon}$, $u > 0$ in Ω, $\partial u/\partial \nu = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 4$, ε is a small positive real, and μ is a fixed positive number. We show that, in contrast with the three dimensional case, $(\mathcal{P}_\varepsilon)$ has no solution blowing up at only interior points as ε goes to zero. The proof strategy consists in testing the equation by appropriate vector fields and then using refined asymptotic estimates in the neighborhood of bubbles, we obtain equilibrium conditions satisfied by the concentration parameters. The careful analysis of these balancing conditions allows us to obtain our results.

Mathematics Subject Classification: 35A15; 35J20; 35J25

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1 Introduction and main results

In this paper, we consider the following nonlinear elliptic equation:

$$(\mathcal{P}_{\mu,p}): \quad \begin{cases} -\Delta u + \mu u = u^p, & u > 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < \infty$, Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, and μ is a positive number.

The interest in Problem $(\mathcal{P}_{\mu,p})$ grew up from the fact that it models several phenomena in applied sciences. For example it can be seen as a steady-state problem for parabolic problems in chemotaxis, e.g., Keller–Segel model [13], or for the shadow system of the Gierer–Meinhardt system in biological pattern formation [10, 16].

Many works have been devoted to problem $(\mathcal{P}_{\mu,p})$. It is well known that the situation depends on both the parameter μ and the exponent p. When μ is small and p is subcritical, i.e., $1 < p < (n+2)/(n-2)$, the only solution is the constant one [13]. For large μ and p subcritical, it is known that solutions exist and concentrate at one or several points located in the interior of the domain, on the boundary, or some of them on the boundary and others in the interior (see the review in [17]). In the critical case, i.e., $p = (n+2)/(n-2)$, when μ is small, $n = 3$ and Ω is convex, the only solution is the constant one [31, 32].

However, for $n \in \{4,5,6\}$ and μ small, nonconstant solutions exist (see [4] when Ω is a ball and [25, 30] for general domains). When μ is large, nonconstant solutions also exist (see [1, 26]) and, as in the subcritical case, solutions blow up at one or several boundary points as μ goes to infinity (see [2, 3, 11, 14, 15, 18, 21, 27–29]). The question of the existence of interior blow-up points is still open. However, in contrast with the subcritical case, we know that at least one point must lie on the boundary [22]. In the supercritical case, very little is known. When Ω is a ball, the uniqueness of the radial solution is proved for small μ [12]. For a general smooth bounded domain and a slightly supercritical p, i.e., $p = ((n+2)/(n-2)) + \varepsilon$, where $\varepsilon > 0$ $\varepsilon \to 0$, a single boundary bubble solution exists for fixed $\mu > 0$ and $n \geq 4$ [9, 24]. Furthermore, a single interior bubble solution has been constructed in [23] for $n = 3$. Notice that the slightly supercritical pure Neumann problem, that is, $\mu = 0$ and ε is a small positive real, has been studied recently in [19], and the authors proved the existence and multiplicity of bubbling solutions in a ball. In this paper, we focus on a new phenomenon, which is the nonexistence of interior bubbling solutions for slightly supercritical case when $n \geq 4$. Thus, in what remains of this paper, we consider the slightly supercritical problem

$$(\mathcal{P}_\varepsilon): \quad \begin{cases} -\Delta u + \mu u = u^{p+\varepsilon}, & u > 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where ε is a small positive real, Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 4$, μ is a positive fixed number, and $p + 1 = 2n/(n-2)$ is the critical Sobolev exponent for the embedding $H^1(\Omega) \to L^q(\Omega)$.

Before we state our main result, we need to introduce some notation. Let us define the following family of functions called *bubbles*:

$$\delta_{a,\lambda}(x) = c_0 \frac{\lambda^{(n-2)/2}}{(1+\lambda^2|x-a|^2)^{(n-2)/2}}, \quad \lambda > 0, a, x \in \mathbb{R}^n, c_0 = \big(n(n-2)\big)^{(n-2)/4}, \tag{1}$$

which are the only solutions to the problem [7]

$$-\Delta u = u^{(n+2)/(n-2)}, \quad u > 0 \text{ in } \mathbb{R}^n.$$

We first exclude, in contrast with the three dimensional case [23], the existence of solutions which blow up at a single point lying in the interior of the domain as ε goes to 0. Notice that if (u_ε) is a sequence of nonconstant solutions to $(\mathcal{P}_\varepsilon)$, then there are several and equivalent ways to define blow-up points of (u_ε). For example, $a \in \bar{\Omega}$ will be said to be a blow-up point of (u_ε) if

$$\liminf_{r\to 0} \limsup_{\varepsilon \to 0} \int_{B(a,r)\cap\Omega} |\nabla u_\varepsilon|^2 \left(\text{or} \int_{B(a,r)\cap\Omega} |u_\varepsilon^{\frac{2n}{n-2}} \right) > 0.$$

Our first result is the following.

Theorem 1.1 *Let $n \geq 4$ and let μ be a fixed positive number. Then $(\mathcal{P}_\varepsilon)$ has no solution u_ε that blows up, as $\varepsilon \to 0$, at a single interior point in the sense that*

$$\|u_\varepsilon - \delta_{a_\varepsilon,\lambda_\varepsilon}\|_{H^1(\Omega)} \to 0,$$

with $a_\varepsilon \to \overline{a} \in \Omega$ and $\lambda_\varepsilon \to \infty$ as $\varepsilon \to 0$.

To study the question of interior blow-up points without any assumption on the number of these points, we need to get some information about such possible solutions. This is the goal of the following result.

Theorem 1.2 *Let $n \geq 4$ and μ be a fixed positive number. Let (u_ε) be a sequence of solutions of $(\mathcal{P}_\varepsilon)$ such that*

$$u_\varepsilon = \sum_{i=1}^{N} \delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}} + v_\varepsilon$$

with $N \geq 2$, $\|v_\varepsilon\|_{H^1(\Omega)} \to 0$, $\lambda_{i,\varepsilon} \to \infty$ and $a_{i,\varepsilon} \to \overline{a}_i \in \Omega$ as $\varepsilon \to 0$ for all $i \in \{1,\dots,N\}$.

Then the following facts hold:

(i) *For each $j \in \{1,\dots,N\}$ satisfying $\frac{\lambda_{j,\varepsilon}}{\lambda_{\min,\varepsilon}} \not\to \infty$ as $\varepsilon \to 0$, there exists $k \neq j$ such that $|a_{j,\varepsilon} - a_{k,\varepsilon}| \to 0$ as $\varepsilon \to 0$.*

(ii) *In addition, if $n \geq 5$, then $\frac{\lambda_{\max,\varepsilon}}{\lambda_{\min,\varepsilon}} \to \infty$ as $\varepsilon \to 0$.*

Theorem 1.2 allows us to generalize Theorem 1.1. More precisely, our next result shows the nonexistence of solutions with two or three interior blow-up points.

Theorem 1.3 *Let $n \geq 5$ and $N = 2$ or $n \geq 6$ and $N = 3$. Let μ be a fixed positive number. Then $(\mathcal{P}_\varepsilon)$ has no solution u_ε that blows up, as $\varepsilon \to 0$, at N interior points $a_{1,\varepsilon},\dots, a_{N,\varepsilon}$ in the sense that*

$$\left\| u_\varepsilon - \sum_{i=1}^{N} \delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}} \right\|_{H^1(\Omega)} \to 0$$

with, for each $i \in \{1,\dots,N\}$, $a_{i,\varepsilon} \to \overline{a}_i \in \Omega$ and $\lambda_{i,\varepsilon} \to \infty$ as $\varepsilon \to 0$.

In the case of N interior blow-up points, with $N \geq 4$, the situation becomes more delicate. However, we note that Theorem 1.2 easily gives the following partial nonexistence result.

Corollary 1.4 *Let $n \geq 4$, $N \geq 4$, and μ be a fixed positive number. Then $(\mathcal{P}_\varepsilon)$ has no solution u_ε such that*

$$u_\varepsilon = \sum_{i=1}^{N} \delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}} + v_\varepsilon$$

with $\|v_\varepsilon\|_{H^1(\Omega)} \to 0$, $\lambda_{i,\varepsilon} \to \infty$, $a_{i,\varepsilon} \to \overline{a}_i \in \Omega$ as $\varepsilon \to 0$ for all $i \in \{1,\dots,N\}$, and one of the following two conditions holds:

(i) *$n \geq 5$ and $\frac{\lambda_{\max,\varepsilon}}{\lambda_{\min,\varepsilon}} \not\to \infty$ as $\varepsilon \to 0$,*

(ii) *$n \geq 4$ and there exists $j \in \{1,\dots,N\}$ satisfying $\frac{\lambda_{j,\varepsilon}}{\lambda_{\min,\varepsilon}} \not\to \infty$ as $\varepsilon \to 0$ and $|a_{j,\varepsilon} - a_{k,\varepsilon}| \geq C > 0$ as $\varepsilon \to 0$ for all $k \in \{1,\dots,N\}$ with $k \neq j$.*

To prove our results, we test the equation by appropriate vector fields and then, using refined asymptotic estimates in the neighborhood of bubbles, we obtain equilibrium conditions satisfied by the concentration parameters. The careful analysis of these balancing conditions allows us to obtain our results.

The remainder of the paper is organized as follows: in Sect. 2 we give some basic tools that we use in our proofs. In Sect. 3 we provide an accurate estimate of the gradient terms in the neighborhood of bubbles. Section 4 is devoted to the proof of our results. In Sect. 5, we discuss some future perspectives. Finally, we collect in Sect. 5 some useful estimates needed in this paper

2 Some basic tools

For $v, w \in H^1(\Omega)$, we set

$$\langle v, w\rangle = \int_\Omega \nabla v \cdot \nabla w + \mu \int_\Omega vw, \qquad \|v\|^2 = \langle v, v\rangle. \tag{2}$$

Throughout the sequel we assume that $n \geq 3$, u_ε is a sequence of solutions of $(\mathcal{P}_\varepsilon)$ written in the form

$$u_\varepsilon = \sum_{i=1}^N \delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}} + v_\varepsilon$$

with $N \geq 1$, $\|v_\varepsilon\|_{H^1(\Omega)} \to 0$, $\lambda_{i,\varepsilon} \to \infty$ and $a_{i,\varepsilon} \to \overline{a}_i \in \Omega$ as $\varepsilon \to 0$ for all i.

To simplify the notation, throughout the sequel we set $\delta_i = \delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}}$, $a_i = a_{i,\varepsilon}$, and $\lambda_i = \lambda_{i,\varepsilon}$.

We know that there is a unique way to choose $a_{i,\varepsilon}$, $\lambda_{i,\varepsilon}$, and v_ε such that

$$u_\varepsilon = \sum_{i=1}^N \alpha_{i,\varepsilon}\delta_{a_{i,\varepsilon},\lambda_{i,\varepsilon}} + v_\varepsilon \tag{3}$$

with

$$\begin{cases} \alpha_{i,\varepsilon} \to 1, \qquad a_{i,\varepsilon} \in \Omega, \qquad a_{i,\varepsilon} \to \overline{a}_i \in \Omega, \qquad \lambda_{i,\varepsilon} \to \infty, \\ \varepsilon_{ij} := (\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2)^{(2-n)/2} \to 0, \\ v_\varepsilon \to 0 \quad \text{in } H^1(\Omega), v_\varepsilon \in E_{a,\lambda}, \end{cases} \tag{4}$$

where, for any $(a, \lambda) \in \Omega^N \times (0,\infty)^N$, $E_{a,\lambda}$ denotes

$$E_{a,\lambda} = \biggl\{ v \in H^1(\Omega) : \int_\Omega \nabla v \cdot \nabla \delta_i = \int_\Omega \nabla v \cdot \nabla \frac{\partial \delta_i}{\partial \lambda_i} = \int_\Omega \nabla v \cdot \nabla \frac{\partial \delta_i}{\partial a_i^j} = 0, \\ \forall 1 \leq i \leq N, \forall 1 \leq j \leq n \biggr\}.$$

For the proof of this fact, see [20]. In what follows, we always assume that u_ε is written as in (3) and (4). We start by proving the following crucial lemma.

Lemma 2.1 *Let $n \geq 3$. For all $j \in \{1,\dots,N\}$, it holds*

$$\varepsilon \ln \lambda_j \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof Multiplying $(\mathcal{P}_\varepsilon)$ by δ_i and integrating on Ω, we obtain

$$-\sum_{j=1}^N \alpha_j \int_\Omega \Delta \delta_j \delta_i - \int_\Omega \Delta v_\varepsilon \delta_i + \mu \sum_{j=1}^N \alpha_j \int_\Omega \delta_j \delta_i + \mu \int_\Omega v_\varepsilon \delta_i$$

$$= \int_\Omega \left(\sum_{j=1}^N \alpha_j \delta_j + v_\varepsilon \right)^{p+\varepsilon} \delta_i. \tag{5}$$

Using Lemma 6.6, we have

$$-\int_\Omega \Delta\delta_j \delta_i = \int_\Omega \delta_j^p \delta_i = O(\varepsilon_{ij}) = o(1) \quad \forall j \neq i,$$

$$-\int_\Omega \Delta\delta_i \delta_i = \int_\Omega \delta_i^{p+1} = \int_{\mathbb{R}^n} \delta_i^{p+1} - \int_{\mathbb{R}^n \setminus \Omega} \delta_i^{p+1} = S_n + O\left(\frac{1}{\lambda_i^n}\right),$$

where

$$S_n = c_0^{p+1} \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^n}. \tag{6}$$

Now, since $\partial u_\varepsilon / \partial \nu = 0$, we observe that

$$-\int_\Omega \Delta v_\varepsilon \delta_i = \int_\Omega \nabla v_\varepsilon \nabla \delta_i - \int_{\partial\Omega} \frac{\partial v_\varepsilon}{\partial \nu} \delta_i = \sum_{j=1}^N \alpha_j \int_{\partial\Omega} \frac{\partial \delta_j}{\partial \nu} \delta_i = O\left(\sum_{k=1}^N \frac{1}{\lambda_k^{n-2}} \right) = o(1),$$

and using Lemma 6.6, we get

$$\int_\Omega \delta_j \delta_i + \int_\Omega \delta_i^2 + \int_\Omega |v_\varepsilon| \delta_i = O\left(\varepsilon_{ij} + \frac{\ln^{\sigma_n} \lambda_i}{\lambda_i^{\min(n-2,2)}} + \|v_\varepsilon\| \left(\int_\Omega \delta_i^{2n/(n+2)} \right)^{(n+2)/(2n)} \right) = o(1),$$

where $\sigma_4 = 1$ and $\sigma_n = 0$ if $n \neq 4$.

It remains to estimate the right-hand side of (5). Using Lemma 6.1, we get

$$\int_\Omega \left(\sum_{j=1}^N \alpha_j \delta_j + v_\varepsilon \right)^{p+\varepsilon} \delta_i$$

$$= \int_\Omega (\alpha_i \delta_i)^{p+\varepsilon} \delta_i + O\left(\sum_{j \neq i} \int \left(\delta_j^{p+\varepsilon} \delta_i + \delta_i^{p+\varepsilon} \delta_j \right) + \int \left(|v_\varepsilon|^{p+\varepsilon} \delta_i + \delta_i^{p+\varepsilon} |v_\varepsilon| \right) \right). \tag{7}$$

Concerning the first integral on the right-hand side of (7), it holds

$$\int_\Omega \delta_i^{p+\varepsilon} \delta_i = c_0^{p+1+\varepsilon} \int_\Omega \frac{\lambda_i^{n+\varepsilon\frac{n-2}{2}} \, dx}{(1+\lambda_i^2|x-a_i|^2)^{n+\varepsilon\frac{n-2}{2}}}$$

$$= c_0^{p+1+\varepsilon} \int_{\mathbb{R}^n} \frac{\lambda_i^{n+\varepsilon\frac{n-2}{2}} \, dx}{(1+\lambda_i^2|x-a_i|^2)^{n+\varepsilon\frac{n-2}{2}}} - c_0^{p+1+\varepsilon} \int_{\mathbb{R}^n \setminus \Omega} \frac{\lambda_i^{n+\varepsilon\frac{n-2}{2}} \, dx}{(1+\lambda_i^2|x-a_i|^2)^{n+\varepsilon\frac{n-2}{2}}}.$$

But we have

$$\int_{\mathbb{R}^n} \frac{c_0^{p+1+\varepsilon} \lambda_i^{n+\varepsilon\frac{n-2}{2}}}{(1+\lambda_i^2|x-a_i|^2)^{n+\varepsilon\frac{n-2}{2}}} \, dx = \lambda_i^{\varepsilon\frac{n-2}{2}} c_0^\varepsilon \int_{\mathbb{R}^n} \frac{c_0^{p+1} \, dx}{(1+|x|^2)^{n+\varepsilon\frac{n-2}{2}}} = \lambda_i^{\varepsilon\frac{n-2}{2}} \left(S_n + O(\varepsilon) \right),$$

$$\int_{\mathbb{R}^n \setminus \Omega} \frac{\lambda_i^{n+\varepsilon\frac{n-2}{2}} \, dx}{(1+\lambda_i^2|x-a_i|^2)^{n+\varepsilon\frac{n-2}{2}}} \le \lambda_i^{\varepsilon\frac{n-2}{2}} c \int_{\mathbb{R}^n \setminus \Omega} \frac{\lambda_i^n \, dx}{(1+\lambda_i^2|x-a_i|^2)^n} \le c \frac{\lambda_i^{\varepsilon\frac{n-2}{2}}}{\lambda_i^n}.$$

For the second integral on the right-hand side of (7), using the fact that $\delta_i^\varepsilon \le c\lambda_i^{\varepsilon(n-2)/2}$, we get for $j \neq i$

$$\int_\Omega \bigl(\delta_j^{p+\varepsilon}\delta_i + \delta_j\delta_i^{p+\varepsilon}\bigr) \le c\lambda_i^{\varepsilon\frac{n-2}{2}} \int_\Omega \bigl(\delta_j^{p+\varepsilon}\delta_i^{1-\varepsilon} + \delta_j\delta_i^{p}\bigr)\,dx \le c\lambda_i^{\varepsilon\frac{n-2}{2}}\varepsilon_{ij}^{1-\varepsilon} = o\bigl(\lambda_i^{\varepsilon\frac{n-2}{2}}\bigr),$$
$$\int_\Omega \bigl(|v_\varepsilon|^{p+\varepsilon}\delta_i + |v_\varepsilon|\delta_i^{p+\varepsilon}\bigr) \le c\lambda_i^{\varepsilon\frac{n-2}{2}} \int_\Omega \bigl(|v_\varepsilon|^{p+\varepsilon}\delta_i^{1-\varepsilon} + |v_\varepsilon|\delta_i^{p}\bigr) \le c\|v_\varepsilon\|\lambda_i^{\varepsilon\frac{n-2}{2}} = o\bigl(\lambda_i^{\varepsilon\frac{n-2}{2}}\bigr).$$

Combining the above estimates, we obtain

$$S_n + o(1) = S_n\lambda_i^{\varepsilon\frac{n-2}{2}} + o\bigl(\lambda_i^{\varepsilon\frac{n-2}{2}}\bigr) = \bigl(S_n + o(1)\bigr)\lambda_i^{\varepsilon\frac{n-2}{2}}.$$

Hence $\lambda_i^{\varepsilon(n-2)/2} = 1 + o(1)$, which completes the proof of the lemma. □

Notice that since $|u_\varepsilon|_\infty$ is of the same order as $\lambda_{\max}^{(n-2)/2}$, Lemma 2.1 implies the following important remark.

Remark 2.2 There is $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, we have

$$|u_\varepsilon|_\infty^\varepsilon \le C \quad \text{and} \quad |v_\varepsilon|_\infty^\varepsilon \le C,$$

where C is a positive constant independent of ε.

Now, we are going to estimate the v_ε-part in (3). To this aim, we need to prove the coercivity of the following quadratic form:

$$Q(v) = \int_\Omega |\nabla v|^2 + \mu\int_\Omega v^2 - p\sum_{i=1}^N \int_\Omega \delta_{a_i,\lambda_i}^{p-1} v^2, \quad v \in E_{a,\lambda}. \tag{8}$$

In the case of Dirichlet boundary conditions, this kind of coercivity is proved by Bahri [5]. Such a result was adapted in [20] to our case when the concentration points do not approach each other. What we need here is a result which holds even if the points are close to each other. More precisely, we will give some general formulae for future use. To this aim, let

$$\begin{aligned}\mathcal{V}(N,\varepsilon,\eta) := \bigl\{(\alpha,a,\lambda) \in (0,\infty)^N \times \Omega^N \times (0,\infty)^N : |\alpha_i - 1| < \eta; \varepsilon\ln\lambda_i < \eta;\\ \lambda_i d(a_i,\partial\Omega) > \eta^{-1}\forall i; \varepsilon_{ij} < \eta\ \forall i \neq j\bigr\},\end{aligned} \tag{9}$$

where $N \in \mathbb{N}$ and $\eta > 0$ is a small parameter.

Proposition 2.3 *Let $n \ge 3$ and $(\alpha,a,\lambda) \in \mathcal{V}(N,\varepsilon,\eta)$. Then there exists $\rho_0 > 0$ such that*

$$Q(v) \ge \rho_0\|v\|^2 \quad \forall v \in E_{a,\lambda}.$$

Proof Let $v \in E_{a,\lambda}$ and v_1 be its projection onto $H_0^1(\Omega)$ defined by

$$\Delta v_1 = \Delta v \quad \text{in } \Omega, \qquad v_1 = 0 \quad \text{on } \partial\Omega$$

and define $v_2 = v - v_1$. It is easy to obtain

$$\int_\Omega \nabla v_1 \nabla v_2 = -\int_\Omega v_1 \Delta v_2 + \int_{\partial\Omega} v_1 \frac{\partial v_2}{\partial \nu} = 0, \tag{10}$$

$$\int_\Omega |\nabla v|^2 = \int_\Omega |\nabla v_1|^2 + \int_\Omega |\nabla v_2|^2. \tag{11}$$

For $y \in \Omega$, we denote by $d_y := d(y, \partial\Omega)$. Since v_2 is a harmonic function in Ω, we see that

$$\left|v_2(y)\right| = \left|\int_{\partial\Omega} \frac{\partial G_0}{\partial \nu}(x,y) v_2(x)\, dx\right| \le c \int_{\partial\Omega} \frac{|v_2(x)|}{|x-y|^{n-1}}\, dx \le c \|v_2\|_{H^1} / d_y^{(n-2)/2}, \tag{12}$$

where G_0 is the Green's function of the Laplace operator with Dirichlet boundary conditions and where we have used, in the last inequality, Holder's inequality and Lemma 6.7. In the same way, we have

$$\left|\nabla v_2(y)\right| \le c \|v_2\|_{H^1} / d_y^{n/2} \quad \forall y \in \Omega. \tag{13}$$

Next, for $1 \le i \le N$, taking

$$\psi_i \in \left\{\delta_i, \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}, \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j}, 1 \le j \le n\right\} \tag{14}$$

and $B_i := B(a_i, d_i/2)$, we observe that, for each $y \in B_i$, it holds that $d_i/2 \le d_y \le 2d_i$. Therefore, we get

$$\begin{aligned}\int_\Omega \nabla v_1 \nabla \psi_i &= \int_\Omega \nabla v \nabla \psi_i - \int_\Omega \nabla v_2 \nabla \psi_i \\ &= -\int_{B_i} \nabla v_2 \nabla \psi_i + O\left(\frac{1}{\lambda_i^{(n-2)/2}} \int_{\Omega \setminus B_i} |\nabla v_2| \frac{1}{|y-a_i|^{n-1}}\, dy\right) \\ &= O\left(\frac{\|v_2\|_{H^1}}{d_i^{n/2}} \int_{B_i} |\nabla \psi_i| + \frac{\|v_2\|_{H^1}}{(\lambda_i d_i)^{(n-2)/2}}\right) = O\left(\frac{\|v_2\|_{H^1}}{(\lambda_i d_i)^{(n-2)/2}}\right),\end{aligned} \tag{15}$$

where we have used (13), the fact that $\int_{B_i} |\nabla \psi_i| = O(d_i/\lambda_i^{(n-2)/2})$ (for the first integral), and Holder's inequality (for the second integral). Now, we write

$$Q(v) = Q_0(v_1) + \int_\Omega |\nabla v_2|^2 + \mu \int_\Omega v^2 - p \sum_{i=1}^N \int_\Omega \delta_i^{p-1} (2v_1 + v_2) v_2, \tag{16}$$

where

$$Q_0(v_1) = \int_\Omega |\nabla v_1|^2 - p \sum_{i=1}^N \int_\Omega \delta_i^{p-1} v_1^2.$$

However, using (12), we obtain

$$\int_\Omega \delta_i^{p-1} \big(2|v_1| + |v_2|\big)|v_2|$$

$$
\begin{aligned}
&= \int_{B_i} \delta_i^{p-1}\big(2|v_1| + |v_2|\big)|v_2| + \int_{\Omega\setminus B_i} \delta_i^{p-1}\big(2|v_1| + |v_2|\big)|v_2| \\
&\leq c\|v_2\|\big(\|v_1\|_{H^1} + \|v_2\|_{H^1}\big)\left[\frac{1}{d_i^{(n-2)/2}}\left(\int_{B_i} \delta_i^{\frac{8n}{n^2-4}}\right)^{\frac{n+2}{2n}} + \left(\int_{\Omega\setminus B_i} \delta_i^{p+1}\right)^{\frac{2}{n}}\right] \\
&= o\big(\|v_1\|_{H^1}^2 + \|v_2\|_{H^1}^2\big). \qquad (17)
\end{aligned}
$$

To proceed further, let

$$
\tilde{v}_1 = v_1 \quad \text{in } \Omega, \qquad \tilde{v}_1 = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega.
$$

Clearly, $\tilde{v}_1 \in \mathcal{D}^{1,2}(\mathbb{R}^n)$. Now, we write

$$
\begin{aligned}
&\tilde{v}_1 = \sum_{i=1}^{N(n+2)} \gamma_i \psi_i + \overline{v}_1 \quad \text{where} \\
&\psi_i \in \left\{\delta_k, \lambda_k \frac{\partial \delta_k}{\partial \lambda_k}, \frac{1}{\lambda_k}\frac{\partial \delta_k}{\partial (a_k)_j}, 1 \leq k \leq N; 1 \leq j \leq n\right\} \quad \text{and} \quad \int_{\mathbb{R}^n} \nabla \overline{v}_1 \nabla \psi_i = 0 \quad \forall i.
\end{aligned}
$$

Using (15), we obtain

$$
\int_{\mathbb{R}^n} \nabla \tilde{v}_1 \nabla \psi_i = c\gamma_i + o\Big(\sum \gamma_j\Big) = \int_\Omega \nabla v_1 \nabla \psi_i = o\big(\|v_2\|_{H^1}\big) \quad \forall i. \qquad (18)
$$

This implies that

$$
\int_{\mathbb{R}^n} |\nabla \overline{v}_1|^2 = \int_{\mathbb{R}^n} |\nabla \tilde{v}_1|^2 + o\big(\|v_2\|_{H^1}^2\big). \qquad (19)
$$

Using (19), we get

$$
\begin{aligned}
Q_0(v_1) &= \int_{\mathbb{R}^n} |\nabla \tilde{v}_1|^2 - p\sum \int_{\mathbb{R}^n} \delta_i^{p-1}\tilde{v}_1^2 \\
&= \int_{\mathbb{R}^n} |\nabla \overline{v}_1|^2 - p\sum \int_{\mathbb{R}^n} \delta_i^{p-1}\overline{v}_1^2 + o\big(\|\nabla \overline{v}_1\|_{L^2}^2 + \|v_2\|_{H^1}^2\big). \qquad (20)
\end{aligned}
$$

Combining (18) and Proposition 3.1 of [5], we obtain

$$
\begin{aligned}
Q_0(v_1) &\geq \rho \int_{\mathbb{R}^n} |\nabla \overline{v}_1|^2 + o\big(\|\nabla \overline{v}_1\|_{L^2}^2 + \|v_2\|_{H^1}^2\big) \\
&\geq \frac{\rho}{2}\int_{\mathbb{R}^n} |\nabla \tilde{v}_1|^2 + o\big(\|v_2\|_{H^1}^2\big) = \frac{\rho}{2}\int_\Omega |\nabla v_1|^2 + o\big(\|v_2\|_{H^1}^2\big). \qquad (21)
\end{aligned}
$$

Therefore (21) and (16) imply that

$$
Q(v) \geq \frac{\rho}{2}\int_\Omega |\nabla v_1|^2 + \int_\Omega |\nabla v_2|^2 + \mu \int_\Omega v^2 + o\big(\|v_1\|^2 + \|v_2\|_{H^1}^2\big) \geq C\|v\|^2,
$$

which completes the proof of Proposition 2.3. □

Next, we are going to estimate the norm of v_ε-part of u_ε when ε goes to zero.

Proposition 2.4 *Let $n \geq 3$ and let v_ε be the remainder term defined in* (3). *Then there is $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, the following fact holds*:

$$\begin{aligned}
&\|v_\varepsilon\| \leq CR(\varepsilon, a, \lambda) \quad \textit{with} \\
&R(\varepsilon, a, \lambda) := \varepsilon + \sum_{i=1}^{N} \frac{1}{(\lambda_i d_i)^{\frac{n-2}{2}}} + \sum_{1\leq i,j\leq N, j\neq i} \left(\varepsilon_{ij}^{\frac{n+2}{2(n-2)}} \left(\ln \varepsilon_{ij}^{-1}\right)^{\frac{n+2}{2n}} + \varepsilon_{ij}\right) \\
&\qquad + \sum_{i=1}^{N} \begin{cases} \lambda_i^{-2} \ln^{2/3} \lambda_i & \textit{if } n = 6, \\ \lambda_i^{-\min(2, \frac{n-2}{2})} & \textit{if } n \neq 6. \end{cases}
\end{aligned}$$

Proof Taking $U = \sum_{i=1}^{N} \alpha_i \delta_i$, multiplying $(\mathcal{P}_\varepsilon)$ by v_ε, and integrating on Ω, we obtain

$$\int_\Omega |\nabla v_\varepsilon|^2 + \mu \int_\Omega v_\varepsilon^2 + \mu \sum \alpha_j \int_\Omega \delta_j v_\varepsilon = \int_\Omega (U + v_\varepsilon)^{p+\varepsilon} v_\varepsilon. \tag{22}$$

For the right-hand side in (22), we write

$$\begin{aligned}
\int_\Omega (U + v_\varepsilon)^{p+\varepsilon} v_\varepsilon &= \int_\Omega U^{p+\varepsilon} v_\varepsilon + (p+\varepsilon) \int_\Omega U^{p+\varepsilon-1} v_\varepsilon^2 \\
&\quad + O\left(\int_\Omega |v_\varepsilon|^{p+1}\right) + O_{(n\leq 5)}\left(\int_\Omega U^{p-2} |v_\varepsilon|^3\right) \\
&= \int_\Omega U^{p+\varepsilon} v_\varepsilon + (p+\varepsilon) \int_\Omega U^{p+\varepsilon-1} v_\varepsilon^2 + o\left(\|v_\varepsilon\|^2\right),
\end{aligned} \tag{23}$$

where we have used Remark 2.2 and where the notation $O_{(n\leq 5)}$ means that the term appears only if $n \leq 5$.

However, using Lemma 2.1 and (4), we have

$$(p+\varepsilon) \int_\Omega U^{p+\varepsilon-1} v_\varepsilon^2 = p \sum_{j=1}^{N} \int_\Omega \delta_j^{\frac{4}{n-2}} v_\varepsilon^2 + o\left(\|v_\varepsilon\|^2\right). \tag{24}$$

For the other integral in (23), using Lemma 6.1, we see that, for $n \geq 6$, we have

$$\int_\Omega U^{p+\varepsilon} v_\varepsilon = \sum_{i=1}^{N} \alpha_i^{p+\varepsilon} \int_\Omega \delta_i^{p+\varepsilon} v_\varepsilon + \sum_{i\neq j} O\left(\int_\Omega (\delta_i \delta_j)^{\frac{p+\varepsilon}{2}} |v_\varepsilon|\right). \tag{25}$$

Using Lemmas 2.1, 6.2, and 6.4, we obtain

$$\begin{aligned}
\int_\Omega (\delta_i \delta_j)^{\frac{p+\varepsilon}{2}} |v_\varepsilon| &\leq c \int_\Omega (\delta_i \delta_j)^{\frac{p}{2}} |v_\varepsilon| \\
&\leq c \|v_\varepsilon\| \left(\int_\Omega (\delta_i \delta_j)^{\frac{n}{n-2}}\right)^{\frac{n+2}{2n}} \leq c \|v_\varepsilon\| \left[\varepsilon_{ij}^{\frac{n}{n-2}} \ln \varepsilon_{ij}^{-1}\right]^{\frac{n+2}{2n}}.
\end{aligned} \tag{26}$$

For the first term on the right-hand side of (25), using again Lemmas 6.2 and 6.4, we obtain

$$\int_\Omega \delta_i^{p+\varepsilon} v_\varepsilon = c_0^\varepsilon \lambda_i^{\varepsilon \frac{n-2}{2}} \int_\Omega \delta_i^p v_\varepsilon + O\left(\varepsilon \int_\Omega \delta_i^p |v_\varepsilon| \ln\left(1 + \lambda_i^2 |x - a_i|^2\right)\right)$$

$$= c_0^\varepsilon \lambda_i^{\varepsilon\frac{n-2}{2}} \int_\Omega \delta_i^p v_\varepsilon + O\biggl(\varepsilon \|v_\varepsilon\| \biggl(\int_{\mathbb{R}^n} \delta_i^{p+1} \ln^{\frac{2n}{n+2}}\bigl(1+\lambda_i^2|x-a_i|^2\bigr)\biggr)^{\frac{n+2}{2n}}\biggr)$$
$$= c_0^\varepsilon \lambda_i^{\varepsilon\frac{(n-2)}{2}} \int_\Omega \delta_i^p v_\varepsilon + O\bigl(\varepsilon\|v_\varepsilon\|\bigr). \tag{27}$$

However, since $v_\varepsilon \in E_{a,\lambda}$, using Holder's inequality and Lemma 6.7, it holds

$$\biggl|\int_\Omega \delta_i^p v_\varepsilon\biggr| = \biggl|\int_\Omega -\Delta\delta_i v_\varepsilon\biggr|$$
$$= \biggl|\int_{\partial\Omega} \frac{\partial\delta_i}{\partial\nu} v_\varepsilon\biggr| \le \frac{c}{\lambda_i^{(n-2)/2}} \int_{\partial\Omega} \frac{|v_\varepsilon|}{|x-a_i|^{n-1}} \le \frac{c}{(\lambda_i d_i)^{(n-2)/2}} \|v_\varepsilon\|. \tag{28}$$

For $n \le 5$, using Lemma 6.1, we write

$$\int_\Omega U^{p+\varepsilon} v_\varepsilon = \sum_{i=1}^N \alpha_i^{p+\varepsilon} \int_\Omega \delta_i^{p+\varepsilon} v_\varepsilon + \sum_{i\neq j} O\biggl(\int_\Omega \delta_i^{p-1+\varepsilon}\delta_j|v_\varepsilon| + \int_\Omega \delta_i\delta_j^{p-1+\varepsilon}|v_\varepsilon|\biggr). \tag{29}$$

Using Lemmas 6.2 and 6.6, we get (since $1 \le 2n/(n+2) < 8n/(n^2-4)$ for $n \le 5$)

$$\int_\Omega \delta_i^{p-1+\varepsilon}\delta_j|v_\varepsilon| \le c\int_\Omega \delta_i^{p-1}\delta_j|v_\varepsilon| \le c\|v_\varepsilon\|\biggl(\int \delta_i^{8n/(n^2-4)}\delta_j^{2n/(n+2)}\biggr)^{(n+2)/(2n)} \le c\|v_\varepsilon\|\varepsilon_{ij}. \tag{30}$$

Lastly, by easy computations, we have

$$\int_\Omega \delta_j|v_\varepsilon| \le C\|v_\varepsilon\|\biggl(\int_\Omega \delta_j^{2n/(n+2)}\biggr)^{(n+2)/(2n)} \le C\|v_\varepsilon\|T(\lambda_j) \tag{31}$$

with

$$T(\lambda) := \biggl(\frac{1}{\lambda^{(n-2)/2}} \text{ if } n \le 5; \frac{\ln^{2/3}(\lambda)}{\lambda^2} \text{ if } n = 6; \frac{1}{\lambda^2} \text{ if } n \ge 7\biggr). \tag{32}$$

Combining Proposition 2.3 and the above estimates, we easily obtain Proposition 2.4. □

3 Estimate of the gradient in the neighborhood of bubbles

As our proof is based on an argument by contradiction, we will assume that problem $(\mathcal{P}_\varepsilon)$ has a solution $u := u_\varepsilon$ in the form (3) and satisfying (4), and we will need to give careful estimates of some integrals involved in our proof. In fact, for future use, we are going to give some crucial estimates in a more general situation than ours. To this aim, for $N \in \mathbb{N}$, ε and η small positive reals, taking $(\alpha, a, \lambda) \in \mathcal{V}(N, \varepsilon, \eta)$, where $\mathcal{V}(N, \varepsilon, \eta)$ is defined by (9), and $u = \sum_{i=1}^N \alpha_i \delta_{a_i,\lambda_i} + v$ with $v \in E_{a,\lambda}$, we need to evaluate the following expressions:

$$\int_\Omega \nabla u \nabla\psi_i + \mu\int_\Omega u\psi_i - \int_\Omega |u|^{p-1+\varepsilon}u\psi_i \quad \psi_i \in \biggl\{\delta_i, \lambda_i\frac{\partial\delta_i}{\partial\lambda_i}, \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i}\biggr\}, \quad 1 \le i \le N. \tag{33}$$

We start by dealing with the nonlinear integrals in (33).

Proposition 3.1 *Let $n \geq 3$ and $u := \sum_{i=1}^{N} \alpha_i \delta_{a_i,\lambda_i} + v$ be such that $(\alpha, a, \lambda) \in \mathcal{V}(N, \varepsilon, \eta)$ and $v \in E_{a,\lambda}$. Then, for $1 \leq i \leq N$, we have*

$$\int_\Omega |u|^{p-1+\varepsilon} u \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i}$$
$$= \overline{c}_2 \sum_{j\neq i} \alpha_j \frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} \Bigg[\sum_{k=i,j} \alpha_k^{p-1+\varepsilon} \lambda_k^{\varepsilon \frac{n-2}{2}} \Bigg] + O\bigg(R_{3,i} + \sum_{k\neq r} \varepsilon_{kr}^{\frac{n}{n-2}} \ln \varepsilon_{kr}^{-1} + \varepsilon_{kr}^2 \bigg),$$

where

$$R_{3,i} := \sum_{j\neq i} \big[\lambda_j |a_i - a_j| \varepsilon_{ij}^{\frac{n+1}{n-2}} + \varepsilon \varepsilon_{ij} \big] + \frac{1}{(\lambda_i d_i)^{n/2}} \sum_{k=1}^{N} \frac{1}{(\lambda_k d_k)^{(n+2)/2}}$$
$$+ \|v\| \bigg(\|v\| + \varepsilon + \frac{1}{(\lambda_i d_i)^{n/2}} + \sum_{j\neq i} \big[\varepsilon_{ij} + \varepsilon_{ij}^{(n+2)/(2(n-2))} \ln \big(\varepsilon_{ij}^{-1} \big)^{(n+2)/(2n)} \big] \bigg).$$

Proof To simplify notation, we write $U = \sum_{j=1}^{N} \alpha_j \delta_j$ and $\psi_i = \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i}$. We set

$$\Omega_v = \big\{ x \in \Omega : \big| v(x) \big| \leq U(x) \big\} \quad \text{and} \quad \Omega_v^c = \Omega \setminus \Omega_v.$$

Using Lemma 6.1, we get

$$\int_\Omega |u|^{p-1+\varepsilon} u \psi_i$$
$$= \int_\Omega U^{p+\varepsilon} \psi_i + (p+\varepsilon) \int_\Omega U^{p-1+\varepsilon} v \psi_i + O\bigg(\int_{\Omega_v} U^{p-2+\varepsilon} v^2 |\psi_i| + \int_{\Omega_v^c} |v|^{p+\varepsilon} |\psi_i| \bigg). \tag{34}$$

First, from Lemmas 6.3 and 6.2, we obtain

$$\int_{\Omega_v} U^{p-2+\varepsilon} v^2 |\psi_i| + \int_{\Omega_v^c} |v|^{p+\varepsilon} |\psi_i| \leq c \int_\Omega U^{p-1} v^2 + c \int_\Omega |v|^{p+1} \leq c \|v\|^2. \tag{35}$$

Second, Lemmas 6.1 and 6.2 imply that

$$\int_\Omega U^{p-1+\varepsilon} v \psi_i = \int_\Omega (\alpha_i \delta_i)^{p-1+\varepsilon} v \psi_i + O\bigg(\int_{\Omega_i} (\alpha_i \delta_i)^{p-2} \bigg(\sum_{j\neq i} \alpha_j \delta_j \bigg) |v| |\psi_i| \bigg)$$
$$+ O\bigg(\int_{\Omega_i^c} \bigg(\sum_{j\neq i} \alpha_j \delta_j \bigg)^{p-1} |v| |\psi_i| \bigg), \tag{36}$$

where $\Omega_i = \{ x \in \Omega : \sum_{j\neq i} \alpha_j \delta_j(x) \leq \alpha_i \delta_i(x) \}$ and $\Omega_i^c = \Omega \setminus \Omega_i$.

To deal with the remaining term in (36), we distinguish two cases. For $n \geq 6$, observe that $p - 1 := 4/(n-2) \leq 1$. Thus, using Lemmas 6.3 and 6.4, it follows that

$$\int_{\Omega_i} (\alpha_i \delta_i)^{p-2} \bigg(\sum_{j\neq i} \alpha_j \delta_j \bigg) |v| |\psi_i| + \int_{\Omega_i^c} \bigg(\sum_{j\neq i} \alpha_j \delta_j \bigg)^{p-1} |v| |\psi_i|$$
$$\leq C \int_{\Omega_i} (\alpha_i \delta_i)^{p-1} \bigg(\sum_{j\neq i} \alpha_j \delta_j \bigg) |v| + C \int_{\Omega_i^c} \bigg(\sum_{j\neq i} \alpha_j \delta_j \bigg)^{p-1} |v| \delta_i$$

$$\leq C\sum_{j\neq i}\int_\Omega(\delta_i\delta_j)^{p/2}|v|\leq c\|v\|\left(\int_\Omega(\delta_i\delta_j)^{n/(n-2)}\right)^{(n+2)/(2n)}$$
$$\leq C\|v\|\sum\varepsilon_{ij}^{(n+2)/(2(n-2))}\ln\left(\varepsilon_{ij}^{-1}\right)^{(n+2)/(2n)}. \tag{37}$$

For $n\leq 5$, we have $p-1>1$, and therefore, using (30), it holds

$$\int_{\Omega_i}(\alpha_i\delta_i)^{p-2}\left(\sum_{j\neq i}\alpha_j\delta_j\right)|v||\psi_i|+\int_{\Omega_i^c}\left(\sum_{j\neq i}\alpha_j\delta_j\right)^{p-1}|v||\psi_i|$$
$$\leq c\sum_{j\neq i}\left(\int_\Omega\delta_i^{p-1}\delta_j|v|+\int_\Omega\delta_j^{p-1}|v|\delta_i\right)$$
$$\leq c\|v\|\sum\varepsilon_{ij}. \tag{38}$$

To complete estimate (36), using Lemmas 6.2. 6.3, 6.4, 6.7 and $v\in E_{a,\lambda}$, we obtain

$$\int_\Omega\delta_i^{p-1+\varepsilon}v\psi_i=c_0^\varepsilon\lambda_i^{\varepsilon(n-2)/2}\int_\Omega\delta_i^{p-1}v\psi_i+O\left(\varepsilon\int_\Omega\delta_i^{p-1}|v||\psi_i|\ln\left(1+\lambda_i^2|x-a_i|^2\right)\right)$$
$$=c_0^\varepsilon\lambda_i^{\varepsilon(n-2)/2}\left[\frac{1}{p}\langle\nabla\psi_i,\nabla v\rangle_{L^2}-\int_{\partial\Omega}\frac{\partial\psi_i}{\partial\nu}v\right]+O\left(\varepsilon\|v\|\right) \tag{39}$$
$$=O\left(\frac{1}{\lambda_i^{n/2}}\int_{\partial\Omega}\frac{|v|}{|x-a_i|^n}+\varepsilon\|v\|\right)$$
$$=O\left(\|v\|\left[\varepsilon+\frac{1}{(\lambda_id_i)^{n/2}}\right]\right). \tag{40}$$

It remains to estimate the first integral on the right-hand side of (34). To this aim, by Lemma 6.1, we write

$$\int_\Omega U^{p+\varepsilon}\psi_i=\int_\Omega(\alpha_i\delta_i)^{p+\varepsilon}\psi_i+(p+\varepsilon)\int_\Omega(\alpha_i\delta_i)^{p+\varepsilon-1}\left(\sum_{j\neq i}\alpha_j\delta_j\right)\psi_i+\int_\Omega\left(\sum_{j\neq i}\alpha_j\delta_j\right)^{p+\varepsilon}\psi_i$$
$$+O\left(\int_{\Omega_i}\left[(\alpha_i\delta_i)^{p-2+\varepsilon}\left(\sum_{j\neq i}\alpha_j\delta_j\right)^2+\left(\sum_{j\neq i}\alpha_j\delta_j\right)^{p+\varepsilon}\right]|\psi_i|\right)$$
$$+O\left(\int_{\Omega_i^c}\left[\alpha_i\delta_i\left(\sum_{j\neq i}\alpha_j\delta_j\right)^{p+\varepsilon-1}+(\alpha_i\delta_i)^{p-1+\varepsilon}\left(\sum_{j\neq i}\alpha_j\delta_j\right)\right]|\psi_i|\right). \tag{41}$$

We are going to estimate each term of (41). First, for $n\geq 4$, it follows that $p-1\leq 2$. Using Lemmas 6.2, 6.3, and 6.4, we obtain

$$\int_{\Omega_i}[\cdots]|\psi_i|+\int_{\Omega_i^c}[\cdots]|\psi_i|\leq c\sum_{j\neq i}\int_\Omega(\delta_i\delta_j)^{\frac{n}{n-2}}\leq c\sum_{j\neq i}\varepsilon_{ij}^{\frac{n}{n-2}}\ln\varepsilon_{ij}^{-1}, \tag{42}$$

and for $n=3$, using Lemmas 6.2, 6.3, and 6.6, it follows that

$$\int_{\Omega_i}[\cdots]|\psi_i|+\int_{\Omega_i^c}[\cdots]|\psi_i|\leq c\sum_{j\neq i}\int_\Omega\delta_i^4\delta_j^2+\delta_i^2\delta_j^4\leq c\sum_{j\neq i}\varepsilon_{ij}^2.$$

Second, since ψ_i is odd and δ_i is even with respect to $x - a_i$, using Lemmas 6.2, 6.3, and 6.4, we get

$$\begin{aligned}\int_\Omega (\alpha_i\delta_i)^{p+\varepsilon}\psi_i &= O\left(\int_{\Omega\setminus B(a_i,d_i)} \delta_i^p|\psi_i|\right)\\ &= O\left(\int_{\Omega\setminus B(a_i,d_i)} \frac{\delta_i^{p+1}}{\lambda_i|x-a_i|}\right) = O\left(\frac{1}{(\lambda_i d_i)^{n+1}}\right),\end{aligned} \tag{43}$$

where $d_i := d(a_i, \partial\Omega)$.

Third, using Lemmas 6.2, 6.3, 6.4, and 6.6, we obtain

$$\begin{aligned}&\int_\Omega \delta_i^{p+\varepsilon-1}\psi_i\delta_j\\ &= c_0^\varepsilon\lambda_i^{\frac{\varepsilon(n-2)}{2}} \int_\Omega \delta_i^{p-1}\psi_i\delta_j + O\left(\varepsilon\int_\Omega \delta_i^p\delta_j \ln\left(1+\lambda_i^2|x-a_i|^2\right)\right)\\ &= c_0^\varepsilon\lambda_i^{\frac{\varepsilon(n-2)}{2}} \int_{\mathbb{R}^n} \delta_i^{p-1}\psi_i\delta_j\\ &\quad + O\left(\int_{\mathbb{R}^n\setminus\Omega} \frac{\delta_i^p\delta_j}{\lambda_i|x-a_i|} + \varepsilon\int_\Omega \left(\delta_i^{\frac{n}{n-2}}\delta_j\right)\delta_i^{\frac{2}{n-2}} \ln\left(1+\lambda_i^2|x-a_i|^2\right)\right)\\ &= c_0^\varepsilon\lambda_i^{\frac{\varepsilon(n-2)}{2}} \frac{\overline{c}_2}{p\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i} + O\left(\lambda_j|a_i-a_j|\varepsilon_{ij}^{\frac{n+1}{n-2}} + \frac{1}{(\lambda_i d_i)^{(n+4)/2}}\frac{1}{(\lambda_j d_j)^{(n-2)/2}} + \varepsilon\varepsilon_{ij}\right).\end{aligned} \tag{44}$$

Finally, using Lemmas 6.1, 6.2, and 6.3, we get

$$\begin{aligned}&\int_\Omega \left(\sum_{j\neq i}\alpha_j\delta_j\right)^{p+\varepsilon}\psi_i\\ &= \sum_{j\neq i}\alpha_j^{p+\varepsilon}\int_\Omega \delta_j^{p+\varepsilon}\psi_i + O\left(\sum_{k\notin\{i,j\}}\int_\Omega \delta_k\delta_j^{p-1}|\psi_i|\right)\\ &= \sum_{j\neq i}\alpha_j^{p+\varepsilon}c_0^\varepsilon\lambda_j^{\frac{\varepsilon(n-2)}{2}}\int_\Omega \delta_j^p\psi_i\\ &\quad + O\left(\varepsilon\int_\Omega \delta_j^p\delta_i\ln\left(1+\lambda_j^2|x-a_j|^2\right) + \sum_{k\notin\{i,j\}}\int_\Omega \delta_k\delta_j^{p-1}|\psi_i|\right).\end{aligned} \tag{45}$$

On the other hand, using Lemmas 6.3, 6.4, and 6.6 we see that

$$\sum_{k\notin\{i,j\}}\int_\Omega \delta_k\delta_j^{p-1}|\psi_i| \le \sum_{l\neq r}\int_\Omega (\delta_l\delta_r)^{n/(n-2)} \le c\sum_{l\neq r}\varepsilon_{lr}^{n/(n-2)}\ln\varepsilon_{lr}^{-1}, \quad \text{if } n\ge 4, \tag{46}$$

$$\sum_{k\notin\{i,j\}}\int_\Omega \delta_k\delta_j^{p-1}|\psi_i| \le \sum_{r\neq j}\int_\Omega \delta_j^4\delta_r^2 \le c\sum_{l\neq r}\varepsilon_{lr}^2, \quad \text{if } n=3,$$

$$\int_\Omega \delta_j^p|\delta_i|\ln\left(1+\lambda_j^2|x-a_j|^2\right) \le \int_\Omega \left(\delta_i\delta_j^{n/(n-2)}\right)\delta_j^{2/(n-2)}\ln\left(1+\lambda_j^2|x-a_j|^2\right) \le c\varepsilon_{ij}, \tag{47}$$

$$\int_\Omega \delta_j^p\psi_i = \int_{\mathbb{R}^n}\delta_j^p\psi_i + O\left(\int_{\mathbb{R}^n\setminus\Omega}\frac{\delta_j^p\delta_i}{\lambda_i|x-a_i|}\right)$$

$$= \frac{\overline{c}_2}{\lambda_i}\frac{\partial \varepsilon_{ij}}{\partial a_i} + O\left(\lambda_j|a_i - a_j|\varepsilon_{ij}^{\frac{n+1}{n-2}} + \frac{1}{(\lambda_j d_j)^{\frac{(n+2)}{2}}}\frac{1}{(\lambda_i d_i)^{\frac{n}{2}}}\right). \tag{48}$$

Combining the above estimates, the proof of Proposition 3.1 follows. □

Next, we are going to improve Proposition 3.1 in some particular cases. More precisely, we need to improve the term $\sum \varepsilon_{kr}^{n/(n-2)} \ln \varepsilon_{kr}^{-1}$ in these cases.

Proposition 3.2 *Let $n \geq 5$, and for $1 \leq i \leq N$, let $N_i := \{j : 1 \leq j \leq N, |a_i - a_j| \to 0\}$, $\gamma := \min\{|a_j - a_k|, j \neq k, j,k \in N_i\}$, and $\sigma := \max\{|a_j - a_k|, j \neq k, j,k \in N_i\}$. Assume that $\gamma/\sigma \geq c > 0$ and $\gamma \min\{\lambda_k : k \in N_i\} \geq c > 0$. Then, for $1 \leq i \leq N$, we have*

$$\int_\Omega |u|^{p-1+\varepsilon} u \frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial a_i} = \overline{c}_2 \sum_{j \neq i} \alpha_j \frac{1}{\lambda_i}\frac{\partial \varepsilon_{ij}}{\partial a_i}\left[\alpha_j^{p-1+\varepsilon}\lambda_j^{\frac{\varepsilon(n-2)}{2}} + \alpha_i^{p-1+\varepsilon}\lambda_i^{\frac{\varepsilon(n-2)}{2}}\right] + O(R_{3,i} + R_{4,i}),$$

where $R_{3,i}$ is defined in Proposition 3.1 and

$$R_{4,i} = \frac{1}{\gamma\lambda_i}\sum_{k \neq r} \varepsilon_{kr}^{\frac{n}{n-2}} + \sum_{j \neq i} \varepsilon_{ij}^{\frac{n+1/2}{n-2}} + \frac{1}{\gamma\lambda_i}\sum_{j \neq i} \varepsilon_{ij}^{\frac{n-1/2}{n-2}}.$$

Proof The proof follows the proof of Proposition 3.1, but we need to improve estimates (42) and (46). We remark that, since the distances $|a_k - a_j|$s are of the same order (that is $\gamma/\sigma \geq c > 0$) and $\gamma \min\{\lambda_k : k \in N_i\} \geq c$ (by the assumption of the proposition), it follows from Assertion (4) of Lemma 6.5 that ε_{kj} and $(\lambda_k\lambda_j\gamma^2)^{(2-n)/2}$ are of the same order.

We start by improving (46). Let $B_\ell = B(a_\ell, \gamma/4)$, we write for $k \notin \{i,j\}$ and $i \neq j$

$$\int_\Omega \delta_k \delta_j^{p-1}|\psi_i| = \int_{B_i} \cdots + \int_{B_j} \cdots + \int_{B_k} \cdots + \int_{\Omega \setminus (B_i \cup B_i \cup B_k)} \cdots := (I) + (II) + (III) + (IV).$$

For the last one, using Lemma 6.3, it holds

$$\begin{aligned}(IV) &\leq \frac{C}{\lambda_k^{(n-2)/2}\lambda_i^{n/2}\lambda_j^2}\int_{\Omega \setminus (B_i \cup B_i \cup B_k)} \frac{dx}{|x - a_k|^{n-2}|x - a_i|^{n-1}|x - a_j|^4} \\ &\leq \frac{C}{\lambda_k^{(n-2)/2}\lambda_i^{n/2}\lambda_j^2\gamma^{n+1}} \leq \frac{C}{\lambda_i\gamma}\sum_{l \neq r} \varepsilon_{lr}^{n/(n-2)}.\end{aligned}$$

Concerning the first one, it holds

$$(I) \leq \frac{C}{(\lambda_k\gamma^2)^{\frac{n-2}{2}}(\lambda_j\gamma^2)^2}\int_{B_i} \frac{dx}{\lambda_i^{\frac{n}{2}}|x - a_i|^{n-1}} \leq \frac{C}{\lambda_k^{\frac{n-2}{2}}\lambda_i^{\frac{n}{2}}\lambda_j^2\gamma^{n+1}} \leq \frac{C}{\lambda_i\gamma}\sum_{l \neq r} \varepsilon_{lr}^{\frac{n}{n-2}},$$

and in the same way we obtain

$$(II) + (III) \leq \frac{C}{\lambda_i\gamma}\sum_{l \neq r} \varepsilon_{lr}^{\frac{n}{n-2}}.$$

This completes the desired improvement for (46).

Now, we will focus on the improvement for (42). Using Lemmas 6.2 and 6.3, we get

$$\begin{aligned}
&\int_{\Omega_i}[\cdots]|\psi_i| + \int_{\Omega_i^c}[\cdots]|\psi_i| \\
&\quad\le c\sum_{j\neq i}\int_\Omega \delta_j^{\frac{n+1/2}{n-2}}\delta_i^{\frac{n-1/2}{n-2}}\frac{1}{\lambda_i|x-a_i|} \\
&\quad\le \sum_{j\neq i}\frac{c}{\gamma\lambda_i}\int_{\Omega\setminus B_i}\delta_j^{\frac{n+1/2}{n-2}}\delta_i^{\frac{n-1/2}{n-2}} + c\sum_{j\neq i}\frac{c}{(\gamma^2\lambda_j)^{\frac{n+1/2}{2}}}\int_{B_i}\frac{1}{\lambda_i|x-a_i|}\frac{\lambda_i^{\frac{n-1/2}{n-2}}}{(1+\lambda_i^2|x-a_i|^2)^{\frac{n-1/2}{n-2}}} \\
&\quad\le \sum_{j\neq i}\frac{c}{\gamma\lambda_i}\varepsilon_{ij}^{\frac{n-1/2}{n-2}} + c\sum_{j\neq i}\frac{c}{(\gamma^2\lambda_j)^{\frac{n+1/2}{2}}}\frac{1}{\lambda_i^{\frac{n+1/2}{n-2}}} \\
&\quad\le \sum_{j\neq i}\frac{c}{\gamma\lambda_i}\varepsilon_{ij}^{\frac{n-1/2}{n-2}} + c\varepsilon_{ij}^{\frac{n+1/2}{n-2}},
\end{aligned}$$

where we have used Lemma 6.6. This completes the improvement of (42). Hence the proof of Proposition 3.2 follows. □

Next, we are going to deal with the linear terms in (33). We start by the second one, namely, we prove the following.

Proposition 3.3 *Let $n\ge 3$ and $u := \sum_{i=1}^N \alpha_i\delta_{a_i,\lambda_i} + v$ be such that $(\alpha,a,\lambda)\in\mathcal{V}(N,\varepsilon,\eta)$. Then, for $1\le i\le N$, the following fact holds*:

$$\left|\int_\Omega u\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i}\right| \le cR_{5,i}\quad \textit{with } R_{5,i} := \frac{1}{\lambda_i}\sum_{j\neq i}\varepsilon_{ij} + \begin{cases} \frac{|\ln d_i|}{\lambda_i^2} + \|v\|\frac{1}{\lambda_i^{3/2}} & \textit{if } n=3,\\ \frac{1}{\lambda_i^3 d_i} + \|v\|\frac{(\ln\lambda_i)^{3/4}}{\lambda_i^2} & \textit{if } n=4,\\ \frac{1}{\lambda_i^2(\lambda_i d_i)^{n-3}} + \|v\|\frac{1}{\lambda_i^2} & \textit{if } n\ge 5.\end{cases}$$

Proof Let $\psi_i := \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i}$. Observe that, for $j\neq i$, using Lemmas 6.3 and 6.6, it holds

$$\int_\Omega \delta_j|\psi_i| \le c\int_\Omega\frac{1}{\lambda_i|x-a_i|}\delta_j\delta_i \le \frac{c}{\lambda_i}\left(\int_\Omega(\delta_j\delta_i)^{\frac{n-1}{n-2}}\right)^{\frac{n-2}{n-1}}\left(\int_\Omega\frac{1}{|x-a_i|^{n-1}}\right)^{\frac{1}{n-1}} \le \frac{c}{\lambda_i}\varepsilon_{ij}.$$

For $j=i$, let $R>0$ be such that $\Omega\subset B(a_i,R)$, it holds that

$$\begin{aligned}
\left|\int_\Omega\delta_i\psi_i\right| &= \left|\int_{B(a_i,d_i)}\delta_i\psi_i + \int_{\Omega\setminus B(a_i,d_i)}\delta_i\psi_i\right| \le c\int_{\Omega\setminus B(a_i,d_i)}\frac{1}{\lambda_i|x-a_i|}\delta_i^2 \\
&\le \frac{c}{\lambda_i^2}\int_{\lambda_i d_i}^{\lambda_i R}\frac{t^{n-2}}{(1+t^2)^{n-2}} \le c\left(\frac{|\ln d_i|}{\lambda_i^2}\text{ if } n=3;\ \frac{1}{\lambda_i^2(\lambda_i d_i)^{n-3}}\text{ if } n\ge 4\right).
\end{aligned}$$

Finally, using again Lemma 6.3, it holds

$$\int_\Omega|v||\psi_i| \le c\|v\|\left(\int_\Omega\left(\frac{1}{\lambda_i|x-a_i|}\delta_i\right)^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}}$$

$$\le c\|v\|\frac{1}{\lambda_i^2}\left(\int_0^{\lambda_i R}\frac{t^{n-1-\frac{2n}{n+2}}}{(1+t^2)^{n(n-2)/(n+2)}}\right)^{\frac{n+2}{2n}} \le c\|v\| \times \begin{cases} 1/\lambda_i^{3/2} & \text{if } n=3, \\ (\ln\lambda_i)^{3/4}/\lambda_i^2 & \text{if } n=4, \\ 1/\lambda_i^2 & \text{if } n\ge 5. \end{cases}$$

Thus the proof follows. □

Next, we deal with the first linear term in (33).

Proposition 3.4 *Let $n \ge 3$ and $u := \sum_{i=1}^N \alpha_i\delta_{a_i,\lambda_i} + v$ be such that $(\alpha, a, \lambda) \in \mathcal{V}(N, \varepsilon, \eta)$ and $v \in E_{a,\lambda}$. Then, for $1 \le i \le N$, the following fact holds*:

$$\int_\Omega \nabla u \cdot \nabla \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i} = \overline{c}_2\frac{1}{\lambda_i}\sum_{j\neq i}\alpha_j\frac{\partial\varepsilon_{ij}}{\partial a_i} + O(R_{6,i}) \quad \textit{with}$$

$$R_{6,i} := \frac{1}{(\lambda_i d_i)^{\frac{n}{2}}}\sum_k \frac{1}{(\lambda_k d_k)^{\frac{n-2}{2}}} + \sum_{j\neq i}\lambda_j|a_i - a_j|\varepsilon_{ij}^{\frac{n+1}{n-2}}.$$

Proof Since $v \in E_{a,\lambda}$, it follows that

$$\begin{aligned} &\int_\Omega \nabla u \cdot \nabla \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i} \\ &\quad = \sum_{1\le j\le N}\alpha_j\int_\Omega \nabla\delta_j \cdot \nabla\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i} \\ &\quad = \sum_{1\le j\le N}\alpha_j\int_{\mathbb{R}^n}\nabla\delta_j\nabla\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i} + O\left[\left(\int_{\mathbb{R}^n\setminus\Omega}|\nabla\delta_j|^2\right)^{1/2}\left(\int_{\mathbb{R}^n\setminus\Omega}\left|\nabla\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i}\right|^2\right)^{1/2}\right] \\ &\quad = \sum_{j\neq i}\alpha_j\left(\overline{c}_2\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i} + O\left(\lambda_j|a_i-a_j|\varepsilon_{ij}^{\frac{n+1}{n-2}}\right)\right) + O\left(\frac{1}{(\lambda_i d_i)^{\frac{n}{2}}}\sum_j\frac{1}{(\lambda_j d_j)^{\frac{n-2}{2}}}\right), \end{aligned}$$

where we have used Lemma 6.4. This completes the proof. □

Combining Propositions 3.1, 3.3, and 3.4, we obtain the following balancing expression involving the point of concentration a_i.

Proposition 3.5 *Let $n \ge 3$ and $u = \sum_{j\le N}\alpha_j\delta_{a_j,\lambda_j} + v$ be such that $(\alpha, a, \lambda) \in \mathcal{V}(N, \varepsilon, \eta)$ and $v \in E_{a,\lambda}$. Then, for $1 \le i \le N$, the following fact holds*:

$$\begin{aligned} &\int_\Omega \nabla u \cdot \nabla\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i} + \mu\int_\Omega u\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i} - \int_\Omega |u|^{p-1+\varepsilon}u\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial a_i} \\ &\quad = \overline{c}_2\sum_{j\neq i}\alpha_j\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}\left[1 - \alpha_j^{p-1+\varepsilon}\lambda_j^{\frac{\varepsilon(n-2)}{2}} - \alpha_i^{p-1+\varepsilon}\lambda_i^{\frac{\varepsilon(n-2)}{2}}\right] + O(R_{35,i}) \\ &\quad \textit{where } R_{35,i} := R_{3,i} + R_{5,i} + R_{6,i} + \sum_{k\neq r}\left(\varepsilon_{kr}^{\frac{n}{n-2}}\ln\left(\varepsilon_{kr}^{-1}\right) + \varepsilon_{kr}^2\right), \end{aligned}$$

and where $\overline{c}_2$ is defined in Lemma 6.4 and $R_{3,i}$, $R_{5,i}$, $R_{6,i}$ are defined in Propositions 3.1, 3.3, 3.4, respectively.

When the concentration points satisfy some properties, we can improve the previous proposition. More precisely, combining Propositions 3.2, 3.3, and 3.4, we obtain the following.

Proposition 3.6 *Let $n \geq 5$ and $u = \sum_{j\leq N} \alpha_j \delta_{a_j,\lambda_j} + v$ be such that $(\alpha, a, \lambda) \in \mathcal{V}(N, \varepsilon, \eta)$ and $v \in E_{a,\lambda}$. For $i \leq N$, let $N_i := \{j : |a_i - a_j| \to 0\}$, $\gamma := \min\{|a_j - a_k|, j \neq k, j, k \in N_i\}$ and $\sigma := \max\{|a_j - a_k|, j \neq k, j, k \in N_i\}$. Assume that $\gamma/\sigma \geq c > 0$ and $\gamma \min\{\lambda_k : k \in N_i\} \geq c > 0$. Then, for $1 \leq i \leq N$, the following fact holds*:

$$\begin{aligned}
&\int_\Omega \nabla u \cdot \nabla \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i} + \mu \int_\Omega u \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i} - \int_\Omega |u|^{p-1+\varepsilon} u \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i} \\
&\quad = \overline{c}_2 \sum_{j\neq i} \alpha_j \frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} \big[1 - \alpha_j^{p-1+\varepsilon} \lambda_j^{\frac{\varepsilon(n-2)}{2}} - \alpha_i^{p-1+\varepsilon} \lambda_i^{\frac{\varepsilon(n-2)}{2}}\big] + O(R_{3,i} + R_{4,i} + R_{5,i} + R_{6,i}),
\end{aligned}$$

where $\overline{c}_2$ is defined in Lemma 6.4 and $R_{3,i}$, $R_{4,i}$, $R_{5,i}$, $R_{6,i}$ are defined in Propositions 3.1, 3.2, 3.3, 3.4, respectively.

In the same way, we prove the following balancing expression involving the rate of the concentration and the mutual interaction of bubbles ε_{ij}.

Proposition 3.7 *Let $n \geq 4$ and $u = \sum_{j\leq N} \alpha_j \delta_{a_j,\lambda_j} + v$ be such that $(\alpha, a, \lambda) \in \mathcal{V}(N, \varepsilon, \eta)$ (with $d_i := d(a_i, \partial\Omega) \geq c$ if $n = 4$) and $v \in E_{a,\lambda}$. Then, for $1 \leq i \leq N$, the following fact holds*:

$$\begin{aligned}
&\int_\Omega \nabla u \cdot \nabla \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} + \mu \int_\Omega u \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} - \int_\Omega |u|^{p-1+\varepsilon} u \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \\
&\quad = -c_1 \varepsilon - c(n) \mu \frac{\ln^{\sigma_n}(\lambda_i)}{\lambda_i^2} - \overline{c}_2 \sum_{j\neq i} \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \\
&\qquad + o\bigg(\varepsilon + \sum_{k\leq N} \frac{\ln^{\sigma_n}(\lambda_k)}{\lambda_k^2} + \sum_{k\neq r} \varepsilon_{kr}\bigg) + O\bigg(\frac{1}{(\lambda_i d_i)^{(n-2)/2}} \sum \frac{1}{(\lambda_k d_k)^{(n-2)/2}} + \|v\|^2\bigg),
\end{aligned}$$

where c_1 is defined in (49), $c(n)$ is defined in (52) if $n \geq 5$ and in (53) if $n = 4$ and $\sigma_4 = 1$ and $\sigma_n = 0$ for $n \geq 5$, and $\overline{c}_2$ is defined in Lemma 6.4.

Proof We will follow the proof of Propositions 3.1, 3.3, and 3.4, and we will precise the estimate of some integrals. In fact, in this case, we will use the function $\psi_i := \lambda_i \partial\delta_i/\partial\lambda_i$. Note that (34), (35), (36), (37), (38), and (39) hold in this case since they are based on the fact that $|\psi_i| \leq c\delta_i$, which is also true with the new ψ_i. Some changes are needed for (40). In fact, using Lemma 6.7, we have

$$\frac{1}{p} \int_\Omega \nabla \psi_i \nabla v - \int_{\partial\Omega} \frac{\partial \psi_i}{\partial \nu} v = 0 + O\bigg(\frac{1}{\lambda_i^{(n-2)/2}} \int_{\partial\Omega} \frac{|v|}{|x - a_i|^{n-1}}\bigg) = O\bigg(\frac{1}{(\lambda_i d_i)^{(n-2)/2}} \|v\|\bigg).$$

Furthermore, (41) and (42) also hold true, but (43) becomes as follows (see (3.4) of [6]):

$$\int_\Omega \delta_i^{p+\varepsilon} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = \lambda_i^{\varepsilon(n-2)/2} c_1 \varepsilon + O\bigg(\varepsilon^2 + \frac{1}{(\lambda_i d_i)^n}\bigg),$$

where

$$c_1 := \frac{(n-2)^2}{4} c_0^{2n/(n-2)} \int_{\mathbb{R}^n} \frac{|x|^2-1}{(1+|x|^2)^{n+1}} \ln\bigl(1+|x|^2\bigr)\,dx > 0. \tag{49}$$

Concerning (44), using Lemmas 6.2 and 6.4, it holds

$$\begin{aligned}\int_\Omega \delta_i^{p+\varepsilon-1}\psi_i\delta_j &= c_0^\varepsilon \lambda_i^{\frac{\varepsilon(n-2)}{2}} \int_{\mathbb{R}^n} \delta_i^{p-1}\psi_i\delta_j + O\biggl(\int_{\mathbb{R}^n\setminus\Omega} \delta_i^p\delta_j + \varepsilon\int_\Omega \delta_j\delta_i^{\frac{n+2}{n-2}} \ln\bigl(1+\lambda_i^2|x-a_i|^2\bigr)\biggr)\\ &= c_0^\varepsilon \lambda_i^{\frac{\varepsilon(n-2)}{2}} \frac{\overline{c}_2}{p}\lambda_i \frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + O\biggl(\varepsilon_{ij}^{\frac{n}{n-2}} \ln\varepsilon_{ij}^{-1} + \frac{1}{(\lambda_i d_i)^{(n+2)/2}}\frac{1}{(\lambda_j d_j)^{(n-2)/2}} + \varepsilon\varepsilon_{ij}\biggr).\end{aligned}$$

In addition, (45), (46), and (47) hold. However, (48) becomes

$$\int_\Omega \delta_j^p\psi_i = \int_{\mathbb{R}^n} \delta_j^p\psi_i + O\biggl(\int_{\mathbb{R}^n\setminus\Omega} \delta_j^p\delta_i\biggr) = \overline{c}_2\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + O\biggl(\varepsilon_{ij}^{\frac{n}{n-2}}\ln\varepsilon_{ij}^{-1} + \frac{1}{(\lambda_j d_j)^{\frac{n+2}{2}}(\lambda_i d_i)^{\frac{n-2}{2}}}\biggr).$$

Hence the analogue of Proposition 3.1 becomes

$$\begin{aligned}\int_\Omega |u|^{p-1+\varepsilon}u\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} &= \overline{c}_2\sum_{j\neq i}\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\bigl(\alpha_i^{p-1+\varepsilon}\lambda_i^{\varepsilon\frac{n-2}{2}} + \alpha_j^{p-1+\varepsilon}\lambda_j^{\varepsilon\frac{n-2}{2}}\bigr)\\ &\quad + O\biggl(\sum_{k\neq r}\varepsilon_{kr}^{\frac{n}{n-2}}\ln\varepsilon_{kr}^{-1} + \|v\|^2 + \varepsilon^2 + \frac{1}{(\lambda_i d_i)^{n-2}} + \sum\frac{1}{(\lambda_k d_k)^n}\biggr).\end{aligned} \tag{50}$$

For the analogue of Proposition 3.4, it holds (since $v \in E_{a,\lambda}$)

$$\begin{aligned}\int_\Omega \nabla u\nabla\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} &= \sum_{j=1}^N \alpha_j\biggl(\int_\Omega \delta_j^p\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} + \int_{\partial\Omega}\frac{\partial\delta_j}{\partial\nu}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\biggr)\\ &= \alpha_i\int_{\mathbb{R}^n\setminus\Omega}\delta_i^p\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} + \sum_{j\neq i}\alpha_j\int_\Omega\delta_j^p\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} + \sum_{j=1}^N\alpha_j\int_{\partial\Omega}\frac{\partial\delta_j}{\partial\nu}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}.\end{aligned}$$

Notice that, using Lemma 6.7, we have

$$\begin{aligned}\int_{\partial\Omega}\biggl|\frac{\partial\delta_j}{\partial\nu}\biggr|\lambda_i\biggl|\frac{\partial\delta_i}{\partial\lambda_i}\biggr| &\leq \frac{c}{(\lambda_i\lambda_j)^{(n-2)/2}}\int_{\partial\Omega}\frac{1}{|x-a_j|^{n-1}}\frac{1}{|x-a_i|^{n-2}}\\ &\leq \frac{c}{(\lambda_i\lambda_j)^{(n-2)/2}}\biggl(\int_{\partial\Omega}\frac{1}{|x-a_j|^{2(n-1)^2/n}}\biggr)^{\frac{n}{2n-2}}\biggl(\int_{\partial\Omega}\frac{1}{|x-a_i|^{2n-2}}\biggr)^{\frac{n-2}{2n-2}}\\ &\leq \frac{c}{(\lambda_j d_j)^{(n-2)/2}}\frac{c}{(\lambda_i d_i)^{(n-2)/2}}.\end{aligned}$$

Thus, using Lemma 6.4, we obtain

$$\begin{aligned}\int_\Omega \nabla u\nabla\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} &= \sum_{j\neq}\alpha_j\overline{c}_2\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\\ &\quad + O\biggl(\sum_{j\neq i}\varepsilon_{ij}^{\frac{n}{n-2}}\ln\varepsilon_{ij}^{-1} + \frac{1}{(\lambda_i d_i)^{(n-2)/2}}\sum\frac{1}{(\lambda_k d_k)^{(n-2)/2}}\biggr).\end{aligned} \tag{51}$$

It remains the analogue of Proposition 3.3. Using Lemma 6.2, we have

$$\int_\Omega u\lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = \alpha_i \int_\Omega \delta_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} + O\biggl(\int_\Omega |v|\delta_i + \sum_{j\neq i}\int_\Omega \delta_j\delta_i\biggr).$$

The last integral is computed in Lemma 6.6, and the second one is computed in (31). Concerning the first one, it depends on the dimension n. If $n \geq 5$, then it holds

$$\begin{aligned}\int_\Omega \delta_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} &= \frac{n-2}{2}c_0^2 \int_{\mathbb{R}^n} \lambda_i^{n-2} \frac{1-\lambda_i^2|x-a_i|^2}{(1+\lambda_i^2|x-a_i|^2)^{n-1}} + O\biggl(\int_{\mathbb{R}^n\setminus\Omega} \frac{1}{\lambda_i^{n-2}}\frac{1}{|x-a_i|^{2n-4}}\biggr)\\ &= -\frac{c(n)}{\lambda_i^2} + O\biggl(\frac{1}{\lambda_i^2(\lambda_i d_i)^{n-4}}\biggr)\\ &\quad \text{with } c(n) := \frac{n-2}{2}c_0^2\int_{\mathbb{R}^n}\frac{|x|^2-1}{(1+|x|^2)^{n-1}} > 0. \end{aligned} \tag{52}$$

However, for $n = 4$, let $r, R > 0$ be such that $B(a_i, r) \subset \Omega \subset B(a_i, R)$. (Note that, if a_i is in a compact set of Ω, then r will be independent of a_i). It holds

$$\begin{aligned}\int_\Omega \delta_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} &= \frac{n-2}{2}c_0^2 \int_{B(a_i,r)} \lambda_i^2 \frac{1-\lambda_i^2|x-a_i|^2}{(1+\lambda_i^2|x-a_i|^2)^3} + O\biggl(\int_{B(a_i,R)\setminus B(a_i,r)} \frac{1}{\lambda_i^2}\frac{1}{|x-a_i|^4}\biggr)\\ &= -c(n)\frac{\ln\lambda_i}{\lambda_i^2} + O\biggl(\frac{1}{\lambda_i^2}\biggr) \quad \text{with } c(n) := \frac{n-2}{2}\operatorname{meas}\bigl(S^3\bigr)c_0^2. \end{aligned} \tag{53}$$

Finally, combining (50), (51), (52), and (53), the proof of Proposition 3.7 follows. □

Lastly, we give the following expression involving the gluing parameters $\alpha_i's$. Namely, we have

Proposition 3.8 *Let $n \geq 4$ and $u = \sum_{j\leq N}\alpha_j\delta_{a_j,\lambda_j} + v$ be such that $(\alpha, a, \lambda) \in \mathcal{V}(N, \varepsilon, \eta)$ and $v \in E_{a,\lambda}$. Then, for $1 \leq i \leq N$, the following fact holds*:

$$\begin{aligned}&\int_\Omega \nabla u \cdot \nabla \delta_i + \mu\int_\Omega u\delta_i - \int_\Omega |u|^{p-1+\varepsilon}u\delta_i\\ &\quad = \alpha_i S_n\bigl(1-\alpha_i^{p-1+\varepsilon}\lambda_i^{\varepsilon\frac{n-2}{2}}\bigr)\\ &\qquad + O\biggl(\varepsilon + \|v\|^2 + \sum \varepsilon_{kr} + \frac{\ln^{\sigma_n}\lambda_i}{\lambda_i^2} + \sum\frac{1}{(\lambda_k d_k)^{n-2}}\biggr).\end{aligned}$$

Proof The proof can be done as the previous ones, and it is more easy. Hence we omit it. □

4 Proof of the main results

This section is devoted to the proof of the main results of the paper. Their proof is basically based on the precise estimates made in Sect. 3. We start by excluding the existence of solutions that concentrate at a single interior point.

4.1 Proof of Theorem 1.1

We argue by contradiction, assume that such a sequence of solutions (u_ε) exists. Thus the solution u_ε will have the form (3) that is $u_\varepsilon = \alpha_\varepsilon \delta_{a_\varepsilon,\lambda_\varepsilon} + v_\varepsilon$ and properties (4) are satisfied. Furthermore, Lemma 2.1 and Proposition 2.4 hold true with $N = 1$. Hence, $(\alpha_\varepsilon, a_\varepsilon, \lambda_\varepsilon) \in \mathcal{V}(1, \varepsilon, \eta)$ for small $\eta > 0$ and $v_\varepsilon \in E_{a_\varepsilon,\lambda_\varepsilon}$. Thus, using Propositions 3.7 and 2.4, we obtain the following:

$$0 = -c_1\varepsilon - c(n)\mu \frac{\ln^{\sigma_n}(\lambda_\varepsilon)}{\lambda_\varepsilon^2} + o\left(\varepsilon + \frac{\ln^{\sigma_n}(\lambda_\varepsilon)}{\lambda_\varepsilon^2}\right),$$

which gives a contradiction. Hence the proof is completed.

In the next subsection, we give a partial characterization of the solutions that concentrate at interior points if there exist.

4.2 Proof of Theorem 1.2

Let (u_ε) be a sequence of solutions of $(\mathcal{P}_\varepsilon)$ satisfying the assumptions of the theorem. Thus the solution u_ε will have the form (3), that is, $u_\varepsilon = \sum_{k=1}^N \alpha_{k,\varepsilon}\delta_{a_{k,\varepsilon},\lambda_{k,\varepsilon}} + v_\varepsilon$ and properties (4) are satisfied. Furthermore, Lemma 2.1 and Proposition 2.4 hold true. In addition, Propositions 3.5–3.7 hold and the left-hand side in each proposition is equal to 0. For the sake of simplicity, we will omit the index ε of the variables. Furthermore, without loss of the generality, we can order the λ_is as follows:

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N.$$

First, multiplying Proposition 3.7 with 2^i and summing over $i = 1, \dots, N$, it holds (by using Proposition 2.4 and Assertion (3) of Lemma 6.5)

$$\sum_{k\neq r} \varepsilon_{kr} \le c\left(\varepsilon + \frac{\ln^{\sigma_n}(\lambda_1)}{\lambda_1^2}\right), \quad \text{where } \sigma_n = \begin{cases} 1 & \text{if } n = 4, \\ 0 & \text{if } n \ge 5. \end{cases} \tag{54}$$

First, we prove Assertion (i) in the theorem, arguing by contradiction. Let j be such that λ_j/λ_1 is bounded and assume that $|a_j - a_k| \ge c > 0$ for each $k \neq j$. Thus, we derive that $\varepsilon_{kj} \le c/(\lambda_j\lambda_k)^{(n-2)/2} \le c/\lambda_1^{n-2}$ for each $k \neq j$. Now, writing Proposition 3.7 with $i = j$ and recalling that the left-hand side is 0, we obtain

$$-c_1\varepsilon - c(n)\mu \frac{\ln^{\sigma_n}(\lambda_j)}{\lambda_j^2} = o\left(\varepsilon + \frac{\ln^{\sigma_n}(\lambda_1)}{\lambda_1^2}\right),$$

which gives a contradiction since λ_j and λ_1 are of the same order. Thus Assertion (i) follows.

Second, we focus on the proof of Assertion (ii). In the sequel, we therefore assume that $n \ge 5$. Now, using Proposition 2.4, we get

$$\begin{aligned} \|v_\varepsilon\|^2 \le{}& C\left(\varepsilon^2 + \sum_{i=1}^N \frac{1}{\lambda_i^{n-2}} + \sum_{1\le i,j\le N, j\neq i} \left(\varepsilon_{ij}^{\frac{n+2}{n-2}} \left(\ln \varepsilon_{ij}^{-1}\right)^{\frac{n+2}{n}} + \varepsilon_{ij}^2\right)\right) \\ &+ C\sum_{i=1}^N \begin{cases} \lambda_i^{-4}\ln^{4/3}\lambda_i & \text{if } n = 6, \\ \lambda_i^{-\min(4,n-2)} & \text{if } n \neq 6. \end{cases} \end{aligned} \tag{55}$$

Note that for $n \geq 6$ we have $2n/(n-2) < n-2$ and for $n \geq 5$ we have $2 > n/(n-2)$. This implies that

$$\varepsilon^2 = o\big(\varepsilon^{n/(n-2)}\big) \quad \forall n \geq 5 \quad \text{and} \quad \lambda_i^{2-n} = o\big(\lambda_1^{-2n/(n-2)}\big) \quad \forall n \geq 6. \tag{56}$$

Now, using (54), (55), and (56), the estimate of $\|v_\varepsilon\|^2$ can be written as

$$\|v_\varepsilon\|^2 = O(R_v) \quad \text{where } R_v := o\big(\varepsilon^{n/(n-2)}\big) + \begin{cases} O(1/\lambda_1^3) & \text{if } n = 5, \\ o(1/\lambda_1^{2n/(n-2)}) & \text{if } n \geq 6. \end{cases}$$

Furthermore, the remaining term in Proposition 3.5 can be written as

$$R_{35,i} = o\bigg(\varepsilon^{(n-1)/(n-2)} + \frac{1}{\lambda_1^{2(n-1)/(n-2)}} + \sum_{j\neq i} \frac{1}{\lambda_i}\bigg|\frac{\partial \varepsilon_{ij}}{\partial a_i}\bigg|\bigg) + O\bigg(\frac{1}{\lambda_i}\sum_{j\neq i}\varepsilon_{ij}\bigg). \tag{57}$$

Now, we will focus on Assertion (ii). Arguing by contradiction, assume that λ_N/λ_1 is bounded. From the smallness of the $\varepsilon'_{ij}s$, we deduce that $\lambda_i|a_i - a_j| \to \infty$ for each $j \neq i$. Let $\gamma := \min\{|a_j - a_1| : j \neq i\} > 0$. Since the $\lambda'_k s$ are of the same order, without loss of generality, we can assume that $\gamma = |a_1 - a_{i_0}|$ for some $i_0 \neq 1$. Let $N_1 := \{j : |a_j - a_1| \to 0\}$. Note that $1 \in N_1$ and Assertion (i) implies that N_1 contains at least another index. Since we have assumed that λ_N/λ_1 is bounded, it follows that

$$\varepsilon_{1j} \leq \frac{1}{(\lambda_1\lambda_j)^{(n-2)/2}|a_1 - a_j|^{n-2}} \leq \frac{c}{(\lambda_1\lambda_{i_0})^{(n-2)/2}|a_1 - a_{i_0}|^{n-2}} \leq c\varepsilon_{1i_0}, \quad \forall 2 \leq j \leq N. \tag{58}$$

Now, we need to introduce the points that are very close to a_1. Let us define $N'_2 := \{j \in N_1 : |a_j - a_1|/\gamma \to \infty\}$ and $N_2 := N_1 \setminus N'_2$. We remark that the ε_{ij}s, for $i, j \in N_2$ with $i \neq j$, are of the same order (in the sense that $c \leq \varepsilon_{ij}/\varepsilon_{kr} \leq c'$ for each $i, j, k, r \in N_2$).

Let $\overline{a}$ be such that $\sum_{j\in N_2}(a_j - \overline{a}) = 0$. It is easy to see that $|a_j - \overline{a}| \leq c\gamma$ for each $j \in N_2$. Hence it follows that (by using the fact that λ_N/λ_1 is bounded and Assertion (8) of Lemma 6.5)

$$\lambda_i|a_i - \overline{a}| \leq c\sqrt{\lambda_1\lambda_{i_0}}|a_{i_0} - a_1| \leq c\varepsilon_{1i_0}^{-1/(n-2)} \quad \forall i \in N_2. \tag{59}$$

Combining Propositions 3.8, 2.4, and 3.5 and using (57), we derive that, for $n \geq 5$ and for each $i \in N_2$,

$$\begin{aligned}&\frac{1}{\lambda_i}\sum_{j\neq i}\frac{\partial \varepsilon_{ij}}{\partial a_i} \\ &\quad = o\bigg(\varepsilon^{(n-1)/(n-2)} + \frac{1}{\lambda_1^{2(n-1)/(n-2)}} + \frac{1}{\lambda_i}\sum_{j\neq i}\bigg|\frac{\partial \varepsilon_{ij}}{\partial a_i}\bigg|\bigg) + O\bigg(\frac{1}{\lambda_i}\sum_{j\neq i}\varepsilon_{ij}\bigg) \quad \forall i \leq N.\end{aligned} \tag{60}$$

Since the λ_js are of the same order, using (54) and Assertions (1) and (8) of Lemma 6.5, we get

$$\frac{1}{\lambda_i}\sum_{j\neq i}\bigg|\frac{\partial \varepsilon_{ij}}{\partial a_i}\bigg| \leq \frac{c}{\lambda_i|a_i - a_j|}\varepsilon_{ij} \leq c\varepsilon_{ij}^{(n-1)/(n-2)} \leq c\varepsilon^{(n-1)/(n-2)} + \frac{c}{\lambda_1^{2(n-1)/(n-2)}},$$

$$\frac{1}{\lambda_i}\varepsilon_{ij} \leq \frac{\gamma\varepsilon_{ij}}{\sqrt{\lambda_1\lambda_{i_0}}|a_1 - a_{i_0}|} \leq c\gamma\left(\varepsilon_{1i_o}^{1/(n-2)}\varepsilon_{ij}\right) = o\left(\varepsilon_{1i_o}^{(n-1)/(n-2)} + \varepsilon_{ij}^{(n-1)/(n-2)}\right).$$

Multiplying (60) by $\lambda_i(\overline{a} - a_i)$ and summing over $i \in N_2$, we obtain

$$\sum_{i\in N_2}\sum_{j\neq i}\frac{\partial\varepsilon_{ij}}{\partial a_i}(\overline{a} - a_i) = o\left(\sum_{i\in N_2}\lambda_i|a_i - \overline{a}|\left(\varepsilon^{(n-1)/(n-2)} + \frac{1}{\lambda_1^{2(n-1)/(n-2)}}\right)\right). \tag{61}$$

To proceed further, we split the above sum on j into three blocks.

Block 1*: $i, j \in N_2$ with $j \neq i$.* In this group, using Assertions (1) and (8) of Lemma 6.5, we observe that

$$\begin{aligned}\frac{\partial\varepsilon_{ij}}{\partial a_i}(\overline{a} - a_i) + \frac{\partial\varepsilon_{ij}}{\partial a_j}(\overline{a} - a_j) &= \frac{\partial\varepsilon_{ij}}{\partial a_i}(a_j - a_i)\\ &= (n-2)\lambda_i\lambda_j|a_i - a_j|^2\varepsilon_{ij}^{\frac{n}{n-2}} = (n-2)\varepsilon_{ij}\big(1 + o(1)\big).\end{aligned} \tag{62}$$

Block 2*: $i \in N_2$ and $j \notin N_1$, that is, $|a_i - a_j| \geq c > 0$.* In this case, using Assertion (1) of Lemma 6.5, we obtain

$$\left|\frac{\partial\varepsilon_{ij}}{\partial a_i}\right||\overline{a} - a_i| = (n-2)\lambda_i\lambda_j|a_i - a_j||\overline{a} - a_i|\varepsilon_{ij}^{n/(n-2)} \leq \frac{|\overline{a} - a_i|}{(\lambda_i\lambda_j)^{(n-2)/2}} = o(\varepsilon_{1i_0}). \tag{63}$$

Block 3*: $i \in N_2$ and $j \in N_1 \setminus N_2$.* In this group, using Assertion (8) of Lemma 6.5, the fact that $|a_i - \overline{a}| \leq C\gamma$ for each $i \in N_2$ and $|a_i - a_j| \gg |a_1 - a_{i_0}|$, we get

$$\left|\frac{\partial\varepsilon_{ij}}{\partial a_i}\right||\overline{a} - a_i| \leq \frac{c|\overline{a} - a_i|}{(\lambda_i\lambda_j)^{(n-2)/2}|a_i - a_j|^{n-1}} = o\left(\frac{|\overline{a} - a_i|}{(\lambda_1\lambda_{i_0})^{(n-2)/2}|a_1 - a_{i_0}|^{n-1}}\right) = o(\varepsilon_{1i_0}). \tag{64}$$

Combining estimates (62), (63), (64), and (61), we deduce that

$$\sum_{k\neq j, k,j\in N_2}\varepsilon_{kj} \leq o\left(\varepsilon_{1i_0}^{-1/(n-2)}\left(\varepsilon^{(n-1)/(n-2)} + \frac{1}{\lambda_1^{2(n-1)/(n-2)}}\right)\right),$$

which implies

$$\varepsilon_{1i_0} = o\left(\varepsilon + \frac{1}{\lambda_1^2}\right). \tag{65}$$

Putting (58) and (65) in Proposition 3.7 with $i = 1$ and using the fact that u_ε is a solution of $(\mathcal{P}_\varepsilon)$ (which implies that the left-hand side of the proposition is 0), we obtain

$$-c_1\varepsilon - c(n)\frac{\mu}{\lambda_1^2} = o\left(\varepsilon + \frac{1}{\lambda_1^2}\right), \tag{66}$$

which presents a contradiction.

Hence the proof of the theorem is completed.

In the next subsection, we use Theorem 1.2 and the precise estimates of Sect. 3 to exclude the case of the existence of solutions with two or three interior blow-up points.

4.3 Proof of Theorem 1.3

We argue by contradiction. Assume that such a sequence of solutions (u_ε) exists. Thus the solution u_ε will have the form (3) that is $u_\varepsilon = \sum_{k=1}^N \alpha_{k,\varepsilon}\delta_{a_{k,\varepsilon},\lambda_{k,\varepsilon}} + v_\varepsilon$ with $N \in \{2,3\}$ and properties (4) are satisfied. Furthermore, Lemma 2.1 and Proposition 2.4 hold true. As in the previous proof, without loss of the generality, we can assume that $\lambda_1 \le \cdots \le \lambda_N$.

We first prove the theorem in the case of two interior blow-up points.

Proof of Theorem 1.3 in the case of $N = 2$ and $n \ge 5$ The proof will be decomposed into three steps. The first one is a direct consequence of Theorem 1.2.

Step 1. $\lambda_2/\lambda_1 \to \infty$ and $|a_1 - a_2| \to 0$.

The second one is as follows.

Step 2. There exists a positive constant $\eta_1 > 0$ such that $\lambda_1|a_1 - a_2| \ge \eta_1$.

To prove Step 2, arguing by contradiction, we assume that $\lambda_1|a_1 - a_2| \to 0$ as $\varepsilon \to 0$. Thus, using Assertion (7) of Lemma 6.5, Proposition 3.7 with $i = 1$ implies

$$-c_1\varepsilon - c(n)\frac{\mu}{\lambda_1^2} - \overline{c}_2\frac{n-2}{2}\varepsilon_{12} = o\left(\varepsilon + \frac{1}{\lambda_1^2} + \varepsilon_{12}\right),$$

which cannot occur (since $\varepsilon > 0$ and $\mu > 0$). Hence Step 2 follows.

Step 3. Proof of the theorem in the case mentioned above: on the one hand, using Assertions (1) and (4) of Lemma 6.5, we get

$$\frac{1}{\lambda_1}\left|\frac{\partial\varepsilon_{12}}{\partial a_1}\right| \ge \frac{c}{\lambda_1|a_1 - a_2|}\varepsilon_{12} \ge c\sqrt{\frac{\lambda_2}{\lambda_1}}\varepsilon_{12}^{(n-1)/(n-2)}. \tag{67}$$

On the other hand, applying Proposition 3.5 and using (57), we obtain

$$\frac{1}{\lambda_1}\left|\frac{\partial\varepsilon_{12}}{\partial a_1}\right| \le cR_{35,1} = o\left(\varepsilon^{(n-1)/(n-2)} + \frac{1}{\lambda_1^{2(n-1)/(n-2)}} + \frac{1}{\lambda_1}\left|\frac{\partial\varepsilon_{12}}{\partial a_1}\right|\right) + O\left(\frac{1}{\lambda_1}\varepsilon_{12}\right).$$

However, using (67), we have

$$\frac{1}{\lambda_1}\varepsilon_{12} = |a_1 - a_2|\frac{1}{\lambda_1|a_1 - a_2|}\varepsilon_{12} = o\left(\frac{1}{\lambda_1}\left|\frac{\partial\varepsilon_{12}}{\partial a_1}\right|\right).$$

We derive that

$$\varepsilon_{12} = o\left(\varepsilon + \frac{1}{\lambda_1^2}\right).$$

Putting this information in Proposition 3.7 with $i = 1$, we derive (66), which gives a contradiction. Hence the proof of the theorem is complete in the case of $N = 2$ and $n \ge 5$. □

Proof of Theorem 1.3 in the case of $N = 3$ and $n \ge 6$ To make the proof clearer, we will split it into several claims. The first one is a direct consequence of Theorem 1.2.

Claim 1. $\lambda_3/\lambda_1 \to \infty$, there exists $k \in \{2,3\}$ such that $|a_1 - a_k| \to 0$ and (54) holds true.

Before stating the second claim, we notice that, since u_ε is a solution of $(\mathcal{P}_\varepsilon)$, the left-hand side of Propositions 3.5, 3.7, and 3.8 becomes 0. Thus, using (54), Propositions 2.4, 3.5, and 3.7, we obtain

$$(E_i) \quad -c_1\varepsilon - c(n)\frac{\mu}{\lambda_i^2} - \overline{c}_2 \sum_{j\neq i} \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = o\left(\varepsilon + \frac{1}{\lambda_1^2}\right) \quad \forall 1 \le i \le N,$$

$$(F_i) \quad -\overline{c}_2 \sum_{j\neq i} \frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i}$$

$$= O\left(\varepsilon^2 + \frac{1}{\lambda_1^4}\right) + o\left(\sum_{j\neq i} \frac{1}{\lambda_i}\left|\frac{\partial \varepsilon_{ij}}{\partial a_i}\right|\right) + O\left(\frac{1}{\lambda_i}\sum_{j\neq i}\varepsilon_{ij}\right) \quad \forall 1 \le i \le N.$$

The second claim is as follows.

Claim 2. There exists a positive constant η_1 such that $\lambda_1|a_1 - a_2| \ge \eta_1$.

To prove Claim 2, arguing by contradiction, we assume that $\lambda_1|a_1 - a_2| \to 0$. The smallness of ε_{12} implies that $\lambda_1/\lambda_2 \to 0$. Computing $(E_3) - (E_1)$ and using Lemma 6.5, it holds

$$c(n)\frac{\mu}{\lambda_1^2} + \frac{n-2}{2}\overline{c}_2\varepsilon_{12} + \frac{n-2}{2}\overline{c}_2\varepsilon_{23} + \overline{c}_2\left(-\lambda_3\frac{\partial\varepsilon_{13}}{\partial\lambda_3} + \lambda_1\frac{\partial\varepsilon_{13}}{\partial\lambda_1}\right) = o\left(\varepsilon + \frac{1}{\lambda_1^2}\right).$$

Now, using Assertion (2) of Lemma 6.5, we derive that

$$\frac{1}{\lambda_1^2}; \varepsilon_{12}; \varepsilon_{23} = o(\varepsilon).$$

Putting this information in (E_2), we obtain a contradiction. Hence the proof of this claim is complete.

Next, we prove the following claim.

Claim 3. There exists a positive constant η_2 such that $\lambda_1|a_1 - a_3| \ge \eta_2$.

Arguing by contradiction, assume that $\lambda_1|a_1 - a_3| \to 0$. Using Claim 2, we see that $|a_1 - a_3|/|a_1 - a_2| \to 0$. We distinguish two cases.

First case: $\lambda_2/\lambda_1 \to \infty$. Observe that, in this case, Claim 2 implies that $\lambda_2|a_1 - a_2| \to \infty$. Therefore $\lambda_2|a_2 - a_3| \to \infty$ (since $|a_1 - a_3| = o(|a_1 - a_2|)$). Using Lemma 6.5, $(E_2) - (E_1)$ implies

$$c(n)\frac{\mu}{\lambda_1^2} + \frac{n-2}{2}\overline{c}_2\varepsilon_{13} + \frac{n-2}{2}\overline{c}_2\varepsilon_{23} + \overline{c}_2\left(-\lambda_2\frac{\partial\varepsilon_{12}}{\partial\lambda_2} + \lambda_1\frac{\partial\varepsilon_{12}}{\partial\lambda_1}\right) = o\left(\varepsilon + \frac{1}{\lambda_1^2}\right).$$

Using Assertion (2) of Lemma 6.5, we derive that

$$\frac{1}{\lambda_1^2}; \varepsilon_{13}; \varepsilon_{23} = o(\varepsilon).$$

Putting this information in (E_3), we obtain a contradiction. Hence the proof of Claim 3 follows in this case.

Second case: λ_2/λ_1 is bounded. In this case, the smallness of ε_{12} implies that $\lambda_j|a_1 - a_2| \to \infty$ for $j = 1, 2$. Therefore $\lambda_2|a_2 - a_3| \to \infty$ (since $|a_1 - a_3| = o(|a_1 - a_2|)$). Note that, using Lemma 6.5, we obtain

$$\varepsilon_{23} \le \frac{1}{(\lambda_2\lambda_3|a_2 - a_3|^2)^{(n-2)/2}} \le c\left(\frac{\lambda_1}{\lambda_3}\right)^{(n-2)/2} \frac{1}{(\lambda_2\lambda_1|a_2 - a_1|^2)^{(n-2)/2}} \le c\varepsilon_{13}\varepsilon_{12},$$

$$\frac{1}{\lambda_2}\left|\frac{\partial \varepsilon_{23}}{\partial a_2}\right| = (n-2)\lambda_3|a_2-a_3|\varepsilon_{23}^{n/(n-2)} \leq \frac{c\varepsilon_{23}}{\lambda_2|a_2-a_3|} \leq \frac{c\varepsilon_{23}}{\lambda_2|a_2-a_1|} \leq C\varepsilon_{12}^{1/(n-2)}\varepsilon_{23}.$$

Putting this information in (F_2) and using (54), we obtain

$$\frac{1}{\lambda_2}\left|\frac{\partial \varepsilon_{12}}{\partial a_2}\right| \leq \frac{c}{\lambda_2}\left|\frac{\partial \varepsilon_{23}}{\partial a_2}\right| + c\left(\varepsilon^2 + \frac{1}{\lambda_1^4}\right) \leq c\left(\varepsilon^2 + \frac{1}{\lambda_1^4}\right).$$

On the other hand, using Lemma 6.5 and the fact that λ_2/λ_1 is bounded, we get

$$\frac{1}{\lambda_2}\left|\frac{\partial \varepsilon_{12}}{\partial a_2}\right| = (n-2)\lambda_1|a_2-a_1|\varepsilon_{12}^{n/(n-2)} \geq c\varepsilon_{12}^{(n-1)/(n-2)}.$$

Thus

$$\varepsilon_{12} = o\left(\varepsilon + \frac{1}{\lambda_1^2}\right). \tag{68}$$

Putting this information in (E_1), using the fact that $\lambda_1|a_1-a_3| \to 0$ and Lemma 6.5, we obtain

$$-c_1\varepsilon - c(n)\frac{\mu}{\lambda_1^2} - \frac{n-2}{2}\bar{c}_2\varepsilon_{13} = o\left(\varepsilon + \frac{1}{\lambda_1^2}\right),$$

which gives a contradiction. Hence the proof of Claim 3 also follows in this case. This completes the proof of Claim 3.

Now, we state and prove Claim 4.

Claim 4. There exists a positive constant η_3 such that $|a_1-a_2|/|a_1-a_3| \geq \eta_3$.

Arguing by contradiction, assume that $|a_1-a_2| = o(|a_1-a_3|)$. Using Lemma 6.5, we observe that

$$\begin{aligned}
&\frac{1}{\lambda_1}\left|\frac{\partial \varepsilon_{12}}{\partial a_1}\right| \geq \frac{c\varepsilon_{12}}{\lambda_1|a_1-a_2]} \geq c\varepsilon_{12}^{(n-1)/(n-2)},\\
&\varepsilon_{13} \leq \frac{1}{(\lambda_1\lambda_3|a_1-a_3|^2)^{(n-2)/2}} = o\left(\frac{1}{(\lambda_2\lambda_1|a_2-a_1|^2)^{(n-2)/2}}\right) = o(\varepsilon_{12}),\\
&\frac{1}{\lambda_1}\left|\frac{\partial \varepsilon_{13}}{\partial a_1}\right| \leq \frac{c\varepsilon_{13}}{\lambda_1|a_1-a_3]} = o\left(\frac{c\varepsilon_{12}}{\lambda_1|a_1-a_2]}\right).
\end{aligned} \tag{69}$$

Thus, (F_1) implies that

$$\frac{1}{\lambda_1}\left|\frac{\partial \varepsilon_{12}}{\partial a_1}\right|\big(1+o(1)\big) = O\left(\varepsilon^2 + \frac{1}{\lambda_1^4}\right),$$

which shows that (68) holds. Putting (69) and (68) in (E_1), we obtain (66), which gives a contradiction. Hence the proof of Claim 4 follows.

Next, we prove the following.

Claim 5. There exists a positive constant $\eta_4 > 0$ such that $|a_1-a_3|/|a_1-a_2| \geq \eta_4$.

Arguing by contradiction, assume that $|a_1-a_3| = o(|a_1-a_2|)$. First, observe that

$$\varepsilon_{23} \leq \frac{1}{(\lambda_2\lambda_3|a_2-a_3|^2)^{(n-2)/2}} = o\left(\frac{1}{(\lambda_2\lambda_1|a_2-a_1|^2)^{(n-2)/2}}\right) = o(\varepsilon_{12}), \tag{70}$$

where we have used in the last inequality Claim ! and Assertion (8) of Lemma 6.5.

Second, we distinguish three cases and we will prove that all these cases cannot occur.

First case: λ_2/λ_1 remains bounded. Using Lemma 6.5 and (70), it holds

$$\frac{1}{\lambda_2}\left|\frac{\partial\varepsilon_{12}}{\partial a_2}\right| \geq \frac{c\varepsilon_{12}}{\lambda_2|a_1-a_2]} \geq c\varepsilon_{12}^{(n-1)/(n-2)},$$
$$\frac{1}{\lambda_2}\left|\frac{\partial\varepsilon_{23}}{\partial a_2}\right| \leq \frac{c\varepsilon_{23}}{\lambda_2|a_2-a_3|} = o\left(\frac{\varepsilon_{12}}{\lambda_2|a_1-a_2|}\right).$$

Therefore, using (F_2), we get

$$\frac{1}{\lambda_2}\left|\frac{\partial\varepsilon_{12}}{\partial a_2}\right|\big(1+o(1)\big) = O\left(\varepsilon^2+\frac{1}{\lambda_1^4}\right),$$

which implies that (68) holds in this case. Now, taking $(E_3-E_2-E_1)$ and using (70), (68), we obtain

$$c_1\varepsilon + c(n)\frac{\mu}{\lambda_1^2} + c(n)\frac{\mu}{\lambda_2^2} + \overline{c}_2\left(-\lambda_3\frac{\partial\varepsilon_{13}}{\partial\lambda_3}+\lambda_1\frac{\partial\varepsilon_{13}}{\partial\lambda_1}\right) = o\left(\varepsilon+\frac{1}{\lambda_1^2}\right). \tag{71}$$

Using Assertion (2) of Lemma 6.5, we obtain a contradiction. Hence this case cannot occur.

Second case: $\lambda_2/\lambda_1 \to \infty$ and $\varepsilon_{13} = o(\varepsilon_{12})$. In this case, taking $(E_2-E_1-E_3)$ and using (70), we get

$$c_1\varepsilon + c(n)\frac{\mu}{\lambda_1^2} + \overline{c}_2\left(-\lambda_2\frac{\partial\varepsilon_{12}}{\partial\lambda_2}+\lambda_1\frac{\partial\varepsilon_{12}}{\partial\lambda_1}\right) = o\left(\varepsilon+\frac{1}{\lambda_1^2}\right).$$

Again, using Assertion (2) of Lemma 6.5, we obtain a contradiction. Hence this case cannot also occur.

Third case: $\lambda_2/\lambda_1 \to \infty$ and $\varepsilon_{13} \geq c\varepsilon_{12}$ for some positive constant c.

Using assertions (1) and (4) of Lemma 6.5, it follows that

$$\begin{aligned}\frac{1}{\lambda_1}\left|\frac{\partial\varepsilon_{13}}{\partial a_1}\right| &\geq c\lambda_3|a_1-a_3|\varepsilon_{13}^{n/(n-2)} \geq \frac{c}{\lambda_1|a_1-a_3|}\varepsilon_{13}\\ &\geq c\sqrt{\frac{\lambda_3}{\lambda_1}}\frac{1}{\sqrt{\lambda_1\lambda_3}|a_1-a_3|}\varepsilon_{13} \gg \varepsilon_{13}^{\frac{n-1}{n-2}}\end{aligned} \tag{72}$$

and

$$\begin{aligned}\frac{1}{\lambda_1}\left|\frac{\partial\varepsilon_{12}}{\partial a_1}\right| &\leq c\lambda_2|a_1-a_2|\varepsilon_{12}^{n/(n-2)} \leq \frac{c}{\lambda_1|a_1-a_2|}\varepsilon_{12} \leq \frac{c}{\lambda_1|a_1-a_2|}\varepsilon_{12}\\ &\leq \frac{c}{\lambda_1|a_1-a_2|}\varepsilon_{13} \ll \frac{c}{\lambda_1|a_1-a_3|}\varepsilon_{13} \leq \frac{c}{\lambda_1}\left|\frac{\partial\varepsilon_{13}}{\partial a_1}\right|,\end{aligned}$$

where we have used in the last line the fact that $|a_1-a_3| = o(|a_1-a_2|)$ and (72).

Therefore, using (F_1), we get

$$\frac{1}{\lambda_1}\left|\frac{\partial\varepsilon_{13}}{\partial a_1}\right|\big(1+o(1)\big) = O\left(\varepsilon^2+\frac{1}{\lambda_1^4}\right),$$

which implies that

$$\varepsilon_{12}; \varepsilon_{13} = o\left(\varepsilon + \frac{1}{\lambda_1^2}\right). \tag{73}$$

Thus, putting this information in (E_1), we easily obtain a contradiction. Thus this case cannot also occur. Therefore, the proof of Claim 5 follows.

Now, we state and prove the following.

Claim 6. There exists a positive constant η_5 such that $|a_2 - a_3|/|a_2 - a_1| \geq \eta_5$.

Arguing by contradiction, assume that $|a_2 - a_3| = o(|a_1 - a_2|)$. Multiplying (F_1) by $\frac{a_1-a_2}{|a_1-a_2|}$, we obtain

$$\begin{aligned}
&\left(-\overline{c}_2 \frac{1}{\lambda_1}\frac{\partial \varepsilon_{12}}{\partial a_1} - \overline{c}_2 \frac{1}{\lambda_1}\frac{\partial \varepsilon_{13}}{\partial a_1}\right)\frac{a_1 - a_2}{|a_1 - a_2|} \\
&\quad = O\left(\varepsilon^2 + \frac{1}{\lambda_1^3}\right) + o\left(\sum_{j\neq 1}\left|\frac{1}{\lambda_1}\frac{\partial \varepsilon_{1j}}{\partial a_1}\right|\right) + O\left(\frac{1}{\lambda_1}\sum_{j\neq 1}\varepsilon_{1j}\right).
\end{aligned} \tag{74}$$

However, by Assertion (1) of Lemma 6.5, we have

$$\begin{aligned}
&\left(-\frac{\overline{c}_2}{\lambda_1}\frac{\partial \varepsilon_{12}}{\partial a_1} - \frac{\overline{c}_2}{\lambda_1}\frac{\partial \varepsilon_{13}}{\partial a_1}\right)\frac{a_1 - a_2}{|a_1 - a_2|} \\
&\quad = (n-2)\left(\lambda_2|a_1 - a_2|\varepsilon_{12}^{\frac{n}{n-2}} + \lambda_3|a_1 - a_3|\left(1 + o(1)\right)\varepsilon_{13}^{\frac{n}{n-2}}\right) \\
&\quad \geq c\varepsilon_{12}^{(n-1)/(n-2)} + c\varepsilon_{13}^{(n-1)/(n-2)}.
\end{aligned} \tag{75}$$

Combining (74) and (75), we derive that

$$\varepsilon_{12}; \varepsilon_{13} = o\left(\varepsilon + \frac{1}{\lambda_1^2}\right).$$

Putting this information in (E_1), we obtain (66), which gives a contradiction. Hence the proof of Claim 6 follows.

Now, we deal with the following claim.

Claim 7. There exists a positive constant η_6 such that $\lambda_2/\lambda_3 \geq \eta_6$.

Arguing by contradiction, assume that $\lambda_2/\lambda_3 \to 0$. Thus it follows that

$$\begin{aligned}
&\varepsilon_{13} \leq \frac{1}{(\lambda_1\lambda_3|a_1 - a_3|^2)^{(n-2)/2}} = o\left(\frac{1}{(\lambda_1\lambda_2|a_2 - a_1|^2)^{(n-2)/2}}\right) = o(\varepsilon_{12}), \\
&\frac{1}{\lambda_1}\left|\frac{\partial \varepsilon_{13}}{\partial a_1}\right| \leq \frac{c\varepsilon_{13}}{\lambda_1|a_1 - a_3|} << \frac{\varepsilon_{12}}{\lambda_1|a_1 - a_2|} \leq \frac{c}{\lambda_1}\left|\frac{\partial \varepsilon_{12}}{\partial a_1}\right|.
\end{aligned} \tag{76}$$

Thus (F_1) implies that

$$\varepsilon_{12}^{(n-1)/(n-2)} \leq \frac{c\varepsilon_{12}}{\sqrt{\lambda_1\lambda_2}|a_2 - a_1|} \leq \frac{c\varepsilon_{12}}{\lambda_1|a_2 - a_1|} \leq \frac{1}{\lambda_1}\left|\frac{\partial \varepsilon_{12}}{\partial a_1}\right| \leq c\left(\varepsilon^2 + \frac{1}{\lambda_1^4}\right),$$

which implies (68). Putting (68) and (76) in (E_1), we get (66), which gives a contradiction. Thus the proof of Claim 7 follows.

Lastly, we are going to prove the theorem in the cases mentioned above. Combining the previous claims, we get

$$\varepsilon_{23} \leq \frac{1}{(\lambda_2\lambda_3|a_2-a_3|^2)^{(n-2)/2}} = o\left(\frac{1}{(\lambda_1\lambda_3|a_1-a_3|^2)^{(n-2)/2}}\right) = o(\varepsilon_{13}) \tag{77}$$

$$\frac{1}{\lambda_3}\left|\frac{\partial\varepsilon_{23}}{\partial a_3}\right| \leq \frac{c\varepsilon_{23}}{\lambda_3|a_2-a_3|} << \frac{c\varepsilon_{13}}{\lambda_3|a_1-a_3|} \leq \frac{c}{\lambda_3}\left|\frac{\partial\varepsilon_{13}}{\partial a_3}\right|. \tag{78}$$

Thus, using Proposition 3.6 for $i = 3$, we derive that

$$\frac{1}{\lambda_3}\left|\frac{\partial\varepsilon_{13}}{\partial a_3}\right|\big(1+o(1)\big) = O(R_{3,3}+R_{4,3}+R_{5,3}+R_{6,3}).$$

Observe that, since $n \geq 6$, easy computations show that

$$R_{3,3}+R_{4,3}+R_{5,3}+R_{6,3} = o\left(\varepsilon^{\frac{n}{n-2}} + \lambda_1^{-\frac{2n}{n-2}} + \frac{1}{\lambda_3}\left|\frac{\partial\varepsilon_{13}}{\partial a_3}\right|\right) \quad \text{and} \quad \frac{1}{\lambda_3}\left|\frac{\partial\varepsilon_{13}}{\partial a_3}\right| \geq c\varepsilon_{13}^{n/(n-2)}.$$

We derive that

$$\varepsilon_{13} = o\left(\varepsilon + \frac{1}{\lambda_1^2}\right).$$

Putting this information and (77) in $(E_1 + E_3 - E_2)$, we obtain (66), which gives a contradiction. Hence the proof of the theorem is complete. □

5 Conclusion

By using delicate estimates near the "standard" bubbles, we have provided some necessary conditions to be satisfied by the concentration parameters. The careful analysis of these balancing conditions allows us to observe a new phenomenon in the higher dimensional case: the nonexistence of solutions of $(\mathcal{P}_\varepsilon)$ that blow up at one or two or three interior points. This stands in strong contrast to the fact that if $n = 3$, then solutions to $(\mathcal{P}_\varepsilon)$ exist with interior blow-up points [23]. However, some questions remain open:

(i) Do the results of Theorem 1.3 remain true for all $n \geq 4$?

(ii) Are there any solutions of $(\mathcal{P}_\varepsilon)$ that blow up, as ε goes to zero, at N interior points with $N \geq 4$ and for all dimension $n \geq 4$?

(iii) What happens if we put in front of the nonlinear term of $(\mathcal{P}_\varepsilon)$ a nonconstant function K?

Appendix

In this appendix we collect several estimates needed throughout the paper. We start with the following auxiliary analysis formulae. Their proofs follow from a Taylor expansion with Lagrange remainder.

Lemma 6.1 *For* $1 < \alpha < 3$ *and* $\beta > 0$, *we have*

(1) $(a+b)^\alpha = a^\alpha + \alpha a^{\alpha-1}b + O(a^{\alpha-2}b^2\chi_{b\leq a} + b^\alpha\chi_{a\leq b})$,

(2) $(a+b)^\alpha = a^a + \alpha a^{\alpha-1}b + b^\alpha + O([a^{\alpha-2}b^2 + b^\alpha]\chi_{b\leq a} + [b^{\alpha-1}a + a^{\alpha-1}b]\chi_{a\leq b})$,

(3) $(a+b)^\alpha = a^\alpha + b^\alpha + O(a^{\alpha-1}b\chi_{b\leq a} + ab^{\alpha-1}\chi_{a\leq b})$,

(4) $(a+b)^\beta = a^\beta + O(a^{\beta-1} b \chi_{b\le a} + b^\beta \chi_{a\le b.})$.

Next, we give the following estimate.

Lemma 6.2 *For $\varepsilon \ln\lambda$ small and $a\in\Omega$, for each $x\in\Omega$, it holds*

$$\delta^{\varepsilon}_{a,\lambda}(x) = c_0^{\varepsilon}\lambda^{\varepsilon\frac{n-2}{2}}\left(1-\varepsilon\frac{n-2}{2}\ln\bigl(1+\lambda^2|x-a|^2\bigr)\right) + O\bigl(\varepsilon^2\ln^2\bigl(1+\lambda^2|x-a|^2\bigr)\bigr) = 1+o(1).$$

Proof By the definition of $\delta_{a,\lambda}$, we have

$$\delta^{\varepsilon}_{a,\lambda}(x) = c_0^{\varepsilon}\lambda^{\varepsilon\frac{n-2}{2}}\exp\left(-\varepsilon\frac{n-2}{2}\ln\bigl(1+\lambda^2|x-a|^2\bigr)\right).$$

Using the fact that Ω is bounded and Taylor's expansion, we easily derive the desired result. □

By easy computations, we easily obtain the following result.

Lemma 6.3 *For all $x\in\Omega$, it holds*

(i) $$\frac{1}{\lambda}\frac{\partial\delta_{a,\lambda}}{\partial a}(x) = (n-2)\frac{\lambda(x-a)}{1+\lambda^2|x-a|^2}\delta_{a,\lambda}(x) = \begin{cases} O(\delta_{a,\lambda}(x)), \\ O(\frac{\delta_{a,\lambda}(x)}{\lambda|x-a|}), \end{cases}$$

(ii) $$\lambda\frac{\partial\delta_{a,\lambda}}{\partial\lambda}(x) = \frac{n-2}{2}\delta_{a,\lambda}(x)\frac{1-\lambda^2|x-a|^2}{1+\lambda^2|x-a|^2} = O\bigl(\delta_{a,\lambda}(x)\bigr).$$

Next, we give the following estimates.

Lemma 6.4 *We have*

(1) $$\int_\Omega (\delta_i\delta_j)^{n/(n-2)} \le C\varepsilon_{ij}^{n/(n-2)}\ln\varepsilon_{ij}^{-1},$$

(2) $$\int_\Omega \delta_i^{\frac{2n}{n-2}-\beta}\ln^\gamma\bigl(1+\lambda^2|x-a|^2\bigr) \le \frac{C}{\lambda_i^{\beta\frac{n-2}{2}}} \quad \forall\beta\in[0,\frac{n}{n-2}) \quad \forall\gamma\ge 0,$$

(3) $$\int_{\mathbb{R}^n\setminus B(a,r)} \delta_{a,\lambda}^{(2n)/(n-2)} \le \frac{C}{(\lambda r)^n},$$

(4) $$\int_{\mathbb{R}^n}\delta_i^{\frac{n+2}{n-2}}\frac{1}{\lambda_j}\frac{\partial\delta_j}{\partial a_j} = \frac{n+2}{n-2}\int_{\mathbb{R}^n}\delta_j^{\frac{4}{n-2}}\frac{1}{\lambda_j}\frac{\partial\delta_j}{\partial a_j}\delta_i = \overline{c}_2\frac{1}{\lambda_j}\frac{\partial\varepsilon_{ij}}{\partial a_j} + O\bigl(\lambda_i|a_i-a_j|\varepsilon_{ij}^{\frac{n+1}{n-2}}\bigr),$$

(5) $$\int_{\mathbb{R}^n}\delta_j^{\frac{n+2}{n-2}}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} = \frac{n+2}{n-2}\int_{\mathbb{R}^n}\delta_j\delta_i^{\frac{4}{n-2}}\lambda_i\frac{\partial\delta_i}{\partial\lambda_i} = \overline{c}_2\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + O\bigl(\varepsilon_{ij}^{\frac{n}{n-2}}\ln\varepsilon_{ij}^{-1}\bigr),$$

where $\overline{c}_2 := \int_{\mathbb{R}^n}\frac{c_0^{2n/(n-2)}}{(1+|x|^2)^{(n+2)/2}}$.

Proof (1), (4), and (5) are extracted from estimates $E2$ (page 4), $F11$ (page 22), and $F16$ (page 23) of [5] respectively. However, (2) and (3) follow by using standard computations. □

Now, we state the following properties.

Lemma 6.5 *We have*

(1) $\frac{1}{\lambda_j}\frac{\partial\varepsilon_{ij}}{\partial a_j} = (n-2)\lambda_i(a_i - a_j)\varepsilon_{ij}^{n/(n-2)}$ *and* $\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} = -\frac{n-2}{2}\varepsilon_{ij}\left(1 - 2\frac{\lambda_i}{\lambda_j}\varepsilon_{ij}^{2/(n-2)}\right)$,

(2) $-\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} + \lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} \geq 0$ *if* $\lambda_i \leq \lambda_j$,

(3) $-2\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} - \lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} \geq c\varepsilon_{ij}$ *if* $\lambda_i \leq \lambda_j$ *and* $-\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} - \lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} \geq 0$,

(4) $\frac{c}{(\lambda_i\lambda_j|a_i - a_j|^2)^{\frac{n-2}{2}}} \leq \varepsilon_{ij} \leq \frac{1}{(\lambda_i\lambda_j|a_i - a_j|^2)^{\frac{n-2}{2}}}$ *if* $\lambda_i \leq \lambda_j$ *and* $\lambda_i|a_i - a_j| \geq C$,

(5) $c\left(\frac{\lambda_i}{\lambda_j}\right)^{\frac{n-2}{2}} \leq \varepsilon_{ij} \leq \left(\frac{\lambda_i}{\lambda_j}\right)^{\frac{n-2}{2}}$ *if* $\lambda_i \leq \lambda_j$ *and* $\lambda_i|a_i - a_j| \leq C$,

(6) $-\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} = \frac{n-2}{2}\varepsilon_{ij} + O\left(\varepsilon_{ij}^{n/(n-2)}\right)$ *if* $\lambda_i/\lambda_j \not\to \infty$,

(7) $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j}\right)^{\frac{n-2}{2}}\left(1 + o(1)\right)$ *and* $\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} = \frac{n-2}{2}\varepsilon_{ij}\left(1 + o(1)\right)$ *if* $\lambda_i \leq \lambda_j$ *and* $\lambda_i|a_i - a_j| \to 0$,

(8) $\varepsilon_{ij} = \frac{1 + o(1)}{(\lambda_i\lambda_j|a_i - a_j|^2)^{(n-2)/2}}$ *if* λ_i *and* λ_j *are of the same order*,

(9) $-\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} = \frac{n-2}{2}\varepsilon_{ij}\left(1 + o(1)\right)$ *if* $\lambda_j|a_i - a_j| \to \infty$.

Proof Claim (1) follows immediately from the definition of ε_{ij} (see (4)). Concerning Claim (2), using the second assertion of Claim (1), we get, for $\lambda_i \leq \lambda_j$,

$$-\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} + \lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} = (n-2)\varepsilon_{ij}^{n/(n-2)}\left(\frac{\lambda_j}{\lambda_i} - \frac{\lambda_i}{\lambda_g}\right) \geq 0,$$

which completes the proof of Claim (2). In the same way, we have

$$-\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} - \lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} = (n-2)\varepsilon_{ij}^{n/(n-2)}\left(\lambda_i\lambda_j|a_i - a_j|^2\right),$$

which implies the second assertion of Claim (3). Furthermore, for $\lambda_i \leq \lambda_j$, we have

$$-\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} = \frac{n-2}{2}\varepsilon_{ij}\left(1 - \frac{\lambda_i}{\lambda_j}\varepsilon_{ij}^{2/(n-2)}\right) = \frac{n-2}{2}\varepsilon_{ij}\left(1 + o(1)\right),$$

which completes the proof of Claim (3).

Now, assuming that $\lambda_i \leq \lambda_j$ and $\lambda_i|a_i - a_j| \geq C$, we see that

$$1 \leq \frac{\varepsilon_{ij}^{-2/(n-2)}}{\lambda_i\lambda_j|a_i - a_j|^2} = \frac{1}{\lambda_j^2|a_i - a_j|^2} + \frac{1}{\lambda_i^2|a_i - a_j|^2} + 1 \leq c.$$

Hence the proof of Claim (4) is completed.

Concerning Claim (5), we observe that if $\lambda_i \le \lambda_j$ and $\lambda_i|a_i - a_j| \le C$, we have

$$1 \le \frac{\lambda_i}{\lambda_j}\varepsilon_{ij}^{-2/(n-2)} = 1 + \frac{\lambda_i^2}{\lambda_j^2} + \lambda_i^2|a_i - a_j|^2 \le c.$$

Thus Claim (5) follows.

Notice that Claim (6) follows immediately from the second assertion of Claim (1).

Concerning Claim (7), observe that if $\lambda_i \le \lambda_j$ and $\lambda_i|a_i - a_j| \to 0$, then from the smallness of ε_{ij} it follows that λ_i/λ_j is very small. Thus

$$\lambda_i\lambda_j|a_i - a_j|^2 = \left(\lambda_i^2|a_i - a_j|^2\right)\frac{\lambda_j}{\lambda_i} = o\left(\frac{\lambda_j}{\lambda_i}\right).$$

Hence Claim (7) follows.

Now, note that if λ_i and λ_j are of the same order, then the smallness of ε_{ij} implies that $\lambda_i\lambda_j|a_i - a_j|^2$ is very large. Hence Claim (8) follows.

Finally, if $\lambda_i|a_i - a_j| \to \infty$, then we have

$$\frac{\lambda_i}{\lambda_j}\varepsilon_{ij}^{2/(n-2)} = \frac{1}{1 + \frac{\lambda_j^2}{\lambda_i^2} + \lambda_j^2|a_i - a_j|^2} \to 0,$$

which implies Claim (9), and therefore the proof of the lemma is completed. □

We end this appendix by proving the following two useful results.

Lemma 6.6 *Let* $n \ge 3$.

(1) *For* $1 \le \beta < n/(n-2)$ *and* $\lambda_i \le \lambda_j$, *it holds*

$$\int_\Omega (\delta_i\delta_j)^\beta \le C\varepsilon_{ij}^\beta \begin{cases} \frac{1}{\lambda_i^{\beta(n-2)}} + |a_i - a_j|^{\beta(n-2)} & \text{if } 2\beta(n-2) < n, \\ |a_i - a_j|^{\beta(n-2)}|\ln|a_i - a_j|| + \frac{1}{\lambda_i^{\beta(n-2)}}\ln\lambda_i & \text{if } 2\beta(n-2) = n, \\ |a_i - a_j|^{n-\beta(n-2)} + \frac{1}{\lambda_i^{n-\beta(n-2)}} & \text{if } 2\beta(n-2) > n. \end{cases}$$

(2) *Let* $1 \le \alpha$ *and* $1 \le \beta$ *be such that* $\alpha \ne \beta$ *and* (*i*): $\alpha + \beta = 2n/(n-2)$ *or* (*ii*): $\alpha + \beta < 2n/(n-2)$. *Then it holds*

$$\int_\Omega \delta_i^\alpha\delta_j^\beta \le C\varepsilon_{ij}^{\min(\alpha,\beta)}; \qquad \int_\Omega \delta_i^\alpha\delta_j^\beta = o\left(\varepsilon_{ij}^{\min(\alpha,\beta)}\right) \quad \textit{in case (ii)}.$$

Proof We remark that, for $\beta = n/(n-2)$, the estimate of Claim (1) is already given in Lemma 6.4. Here, we need to improve this estimate when $\beta < n/(n-2)$. Furthermore, the case $2\beta(n-2) < n$ occurs only when $n = 3$ and $\beta < 3/2$, and the case $2\beta(n-2) = n$ occurs only if $n = 4$ and $\beta = 1$ or $n = 3$ and $\beta = 3/2$.

First, we focus on proving Claim (1).

Note that, if $2\beta(n-2) < n$ (this case can occur only if $n = 3$ and $\beta < 3/2$), in this case, it holds

$$\int_\Omega (\delta_i\delta_j)^\beta \le \frac{c}{(\lambda_i\lambda_j)^{\beta(n-2)/2}}\left(\sum_{k=i,j}\int_\Omega \frac{dx}{|x-a_k|^{2\beta(n-2)}}\right) \le c\varepsilon_{ij}^\beta\left(\frac{1}{\lambda_i^{\beta(n-2)}} + |a_i - a_j|^{\beta(n-2)}\right).$$

Thus, in the sequel of the proof, we consider $2\beta(n-2) \geq n$. We distinguish two cases.

Case 1. $\lambda_i|a_i - a_j| \leq M$, where M is a large constant. In this case, by Lemma 6.5, we know that ε_{ij} and $(\lambda_i/\lambda_j)^{(n-2)/2}$ have the same order. Let $B_j := B(a_j, 4M/\lambda_i)$. Observe that, for $x \in \Omega \setminus B_j$, we have $|x - a_i| \geq c|x - a_j|$. Thus

$$\int_\Omega (\delta_i\delta_j)^\beta \leq C\left(\frac{\lambda_i}{\lambda_j}\right)^{\beta\frac{n-2}{2}} \int_{B_j} \frac{dx}{|x-a_j|^{\beta(n-2)}} + \frac{c}{(\lambda_i\lambda_j)^{\beta\frac{n-2}{2}}} \int_{\Omega\setminus B_j} \frac{dx}{|x-a_j|^{2\beta(n-2)}}$$

$$\leq C\varepsilon_{ij}^\beta \frac{1}{\lambda_i^{n-\beta(n-2)}} + C\varepsilon_{ij}^\beta \frac{1}{\lambda_i^{\beta(n-2)}} \begin{cases} \ln\lambda_i & \text{if } 2\beta(n-2) = n, \\ \lambda_i^{2\beta(n-2)-n} & \text{if } 2\beta(n-2) > n. \end{cases}$$

Hence the result in this case.

Case 2. $\lambda_i|a_i - a_j| \geq M$. In this case, Lemma 6.5 implies that $(\lambda_i\lambda_j|a_i - a_j|^2)^{(2-n)/2}$ and ε_{ij} have the same order. For $k = i, j$, let $B_k = B(a_k, |a_i - a_j|/4)$. Observe that

$$\int_\Omega (\delta_i\delta_j)^\beta \leq \frac{1}{(\lambda_i\lambda_j|a_i-a_j|^2)^{\beta(n-2)/2}} \sum_{k\in\{i,j\}} \int_{B_k} \frac{dx}{|x-a_k|^{\beta(n-2)}} + \left(\int_{\Omega\setminus B_i} \delta_i^{2\beta}\right)^{\frac{1}{2}} \left(\int_{\Omega\setminus B_j} \delta_j^{2\beta}\right)^{\frac{1}{2}}$$

$$\leq C\varepsilon_{ij}^\beta |a_i - a_j|^{n-\beta(n-2)} + \prod_{k=i,j}\left(\frac{1}{\lambda_k^{\beta(n-2)}} \int_{\Omega\setminus B_k} \frac{1}{|x-a_k|^{2\beta(n-2)}}\right)^{1/2}$$

$$\leq C\varepsilon_{ij}^\beta |a_i - a_j|^{n-\beta(n-2)}$$

$$+ C\varepsilon_{ij}^\beta |a_i - a_j|^{\beta(n-2)} \begin{cases} |\ln|a_i - a_j|| & \text{if } 2\beta(n-2) = n, \\ |a_i - a_j|^{n-2\beta(n-2)} & \text{if } 2\beta(n-2) > n. \end{cases}$$

Hence the result follows in this case also.

Hence the proof of Claim (1) is complete. Concerning Claim (2), it follows from Claim (d) of Lemma 2.2 of [8] when the assumption (i) is satisfied. However, when (ii) is satisfied (assume that $\alpha < \beta$), let $\gamma := (\alpha + \beta)/2$, it follows that $\gamma < n/(n-2)$ and $\gamma - \alpha = (\beta - \alpha)/2$. Thus, using Holder's inequality and Claim (1), it holds

$$\int_\Omega \delta_i^\alpha \delta_j^\beta = \int_\Omega (\delta_i\delta_j)^\alpha \delta_j^{\beta-\alpha} \leq \left(\int_\Omega (\delta_i\delta_j)^\gamma\right)^{\alpha/\gamma} \left(\int_\Omega \delta_j^{\gamma/2}\right)^{(\gamma-\alpha)/\gamma} = o\left(\varepsilon_{ij}^\alpha\right).$$

Hence the proof is complete. □

Lemma 6.7 *For $a \in \Omega$ and $\beta > 0$, it holds*

$$\int_{\partial\Omega} \frac{1}{|a-y|^{n-1+\beta}}\, dy \leq \frac{c}{d(a,\partial\Omega)^\beta}.$$

Proof We remark that if a is far away from the boundary, then the result is immediate. Hence, we focus on the case where $d_a := d(a, \partial\Omega)$ is small. Let $\overline{a}$ be the projection of a at the boundary. Thus, we have

$$\int_{\partial\Omega} \frac{1}{|a-y|^{n-1+\beta}}\, dy \leq c\int_{\partial\Omega} \frac{1}{(d_a^2 + |\overline{a} - y|^2)^{(n-1+\beta)/2}}\, dy$$

$$
\begin{aligned}
&\leq \frac{c}{d_a^{n-1+\beta}} \int_{\partial\Omega\cap B(\overline{a},1)} \frac{1}{(1+d_a^{-2}|\overline{a}-y|^2)^{(n+1)/2}}\,dy + c \\
&\leq \frac{c}{d_a^{\beta}} \int_{\mathbb{R}^{n-1}} \frac{1}{(1+|y|^2)^{(n-1+\beta)/2}}\,dy + c \leq \frac{c}{d_a^{\beta}}. \qquad \square
\end{aligned}
$$

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