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# The fundamental solution and blow-up problem of an anisotropic parabolic equation 

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#### Abstract

This paper is devoted to the study of anisotropic parabolic equation related to the $p_{i}$-Laplacian with a source term $f(u)$. If $f(u)=0$, then the fundamental solution of the equation is constructed. If there are some restrictions on the growth order of $u$ in the source term, the initial energy $E(0)$ is positive and has a super boundedness, which depends on the Sobolev imbedding index, then the local solution may blow up in finite time.

Mathematics Subject Classification: 35K15; 35B35; 35K55; 35G31 Keywords: Fundamental solution; Anisotropic parabolic equation; Initial energy; The growth order; Blow-up


## 1 Introduction

An anisotropic parabolic equation

$$
\begin{equation*}
u_{t}-\sum_{i=1}^{N} \frac{\partial}{\partial x_{i}}\left(\left|u_{x_{i}}\right|^{p_{i}-2} u_{x_{i}}\right)=f(u), \quad(x, t) \in Q_{T}=\Omega \times(0, T) \tag{1}
\end{equation*}
$$

with the initial value condition

$$
\begin{equation*}
u(x, 0)=u_{0}(x), \quad x \in \Omega \tag{2}
\end{equation*}
$$

and the Dirichlet boundary value condition

$$
\begin{equation*}
u(x, t)=0, \quad x \in \partial \Omega \times(0, T) \tag{3}
\end{equation*}
$$

is considered in this paper, where $\Omega \subset \mathbb{R}^{N}$ is a bounded domain with smooth boundary $\partial \Omega, T$ is any given positive constant, $p_{i}$ is a positive constant, $i=1, \ldots, N$, and the function $f(s)$ is continuous. As usual, we call problem (1)-(2)-(3) the (first) initial boundary value problem. We denote

$$
p^{+}=\max \left\{p_{1}, p_{2}, \ldots, p_{N}\right\}, \quad p^{-}=\min \left\{p_{1}, p_{2}, \ldots, p_{N}\right\},
$$

and assume that $p^{-}>1$.
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Equation (1) appears in several places in the literature. For instance, in biology, it acts as a model describing the spread of an epidemic disease in heterogeneous environments, and in fluid mechanics, it emerges as a mathematical description of the dynamics of fluids with different conductivities in different direction; one can refer to $[1,4,5]$ for details.

People are more familiar with the evolution $p$-Laplacian equation with a source term

$$
\begin{equation*}
u_{t}=\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)+f(u) . \tag{4}
\end{equation*}
$$

As is well known, equation (4) arises from non-Newtonian fluids as a fast diffusion equation if $p<2$, and a slow diffusion equation if $p>2$. When $p=2$, equation (4) becomes the classical heat conduction equation. When $f(u)<0$, it is called an absorption term, equation (4) may have a global solution. While $f(u) \geq 0$, it is called a source term, and equation (4) generally only has a local weak solution. This kind of equation has important physical significance and has received much attention for a long time. Sometimes, the divergence $\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$ is replaced by $\sum_{i=1}^{N} \frac{\partial}{\partial x_{i}}\left(\left|u_{x_{i}}\right|^{p-2} u_{x_{i}}\right)$. Some details are given in what follows.

Consider the slow diffusion equation

$$
\begin{equation*}
u_{t}=\sum_{i=1}^{N} \frac{\partial}{\partial x_{i}}\left(\left|u_{x_{i}}\right|^{p-2} u_{x_{i}}\right)+f(u, x, t) . \tag{5}
\end{equation*}
$$

When $f(u, x, t)=f(x, t)$, Nakao studied the existence of periodic solutions in [16]. If $f(u, x, t)=|u|^{\alpha} u$, then the energy functional is set as

$$
J(u)=\frac{1}{p}\|\nabla u\|_{p}^{p}-\frac{1}{2+\alpha}\|u\|_{2+\alpha}^{2+\alpha},
$$

the potential well is defined as

$$
W=\left\{u \in W_{0}^{1, p}(\Omega) \mid I(u)>0, J(u)<d\right\} \cup\{0\},
$$

where

$$
I(u)=\|\nabla u\|_{p}^{p}-\|u\|_{2+\alpha}^{2+\alpha}
$$

and

$$
d=\inf _{u \in W_{0}^{1, p}(\Omega), u \neq 0} \sup _{\lambda \geq 0} J(\lambda u),
$$

which is called the depth of potential well. By the potential well theory it means that we can study the existence of the global solution and the blow-up of local solutions to equation (5) by means of analyzing the relationship between $I(u)$ and $J(u)$.

Under the assumption

$$
\begin{equation*}
2 \leq p, \quad p<2+\alpha<\infty \quad \text { if } N \leq p \quad \text { and } \quad p<2+\alpha \leq \frac{N p}{N-p} \quad \text { if } N>p \tag{6}
\end{equation*}
$$

and $u_{0}(x) \in W$, that is, $J\left(u_{0}\right)<d$ and $I\left(u_{0}\right)>0$, Tsutsumi [23] showed that equation (5) admits a global solution. While, for the case of negative initial energy $J\left(u_{0}\right)<0$, the solutions of equation (5) can blow up in finite time [23]. If $I\left(u_{0}\right)>0$ or $I\left(u_{0}\right)=0,0<J\left(u_{0}\right) \leq d$, Liu and Zhao [13] showed that problem (5)-(2)-(3) admits a global solution. If $J\left(u_{0}\right)>d$, then using the comparison principle and by the variational method, the global existence and finite time blow-up of solutions were approached by Xu in [25]. Along this way, i.e., by the potential well theory, $\mathrm{Liu}, \mathrm{Yu}$, and Li made a new progress in the study of the finite time blow-up phenomena to equation (5) with the subcritical initial energy and the critical initial energy in [12] recently. The excellent insights into the theory of blow-up of solutions to parabolic equations with constant nonlinearity can be found in the monographs [ $9,14,19]$. In these monographs, instead of the energy function, the proofs of the main results are based on the reduction of the problem to the study of a nonlinear ordinary differential inequality for a suitably chosen function associated with the solution. It happens so that every function satisfying such an inequality becomes unbounded in a finite time, which yields the finite time blow-up of the corresponding weak solution.

Let us come back to the anisotropic parabolic equation (1). For its stationary case, i.e., if $u(x, t)=u(x)$ satisfies the following degenerate elliptic equation:

$$
\begin{equation*}
\sum_{i=1}^{N} \frac{\partial}{\partial x_{i}}\left(\left|u_{x_{i}}\right|^{p_{i}-2} u_{x_{i}}\right)=\lambda f(u, x, t), \quad p^{-} \geq 2 \tag{7}
\end{equation*}
$$

then there are many papers devoted to its multiple solutions under different conditions of $f(u, x, t)$, one can refer to $[7,8,10,17,18]$ for details. Here, $\lambda>0$ is a constant. Naturally, there have been some important results of the anisotropic parabolic equation (1) itself in recent years. For example, if $\Omega \subset\left\{x \in \mathbb{R}^{N} \mid-l \leq x_{i} \leq l\right\}$, by denoting $m=\left\|u_{0}\right\|_{L^{\infty}(\Omega)}$,

$$
\mathbf{m}=\left\{M \in(0, \infty) \left\lvert\, f(M+m)<\left(p^{+}-1\right)\left(\frac{2 M}{3 l^{2}+2 l}\right)^{p^{+}-1}\right.\right\}
$$

and

$$
M_{*}=\inf \mathbf{m},
$$

the solvability of the initial boundary value problem (1)-(3) was approached in [20] by Starovoitov and Tersenov, provided that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $|f(\xi)| \leq$ $f(\eta)$ for all $\xi$ and $\eta$ such that $|\xi| \leq \eta$, and $\mathbf{m} \neq \emptyset$. Moreover, if $f$ is a Lipschitz continuous function, then the solution $u \in L^{\infty}\left(Q_{T}\right) \cap V\left(Q_{T}\right) \cap C\left([0, T] ; L^{s}(\Omega)\right)$ is unique, where $s \in$ $[1, \infty)$ and $V\left(Q_{T}\right)$ is a Banach space; one can refer to [20] for details. At the same time, we have noted with pleasure that the well-posedness of the initial boundary value problem of an anisotropic parabolic equation with the exponent variables was studied by Antontsev and Shmarev in [2, 3] and by Alkis S. Tersenov and Aris S. Tersenov in [21].
However, different from the evolution $p$-Laplacian equation, the fundamental solution of the anisotropic parabolic equation

$$
\begin{equation*}
u_{t}-\sum_{i=1}^{N} \frac{\partial}{\partial x_{i}}\left(\left|u_{x_{i}}\right|^{p_{i}-2} u_{x_{i}}\right)=0, \quad(x, t) \in Q_{T} \tag{8}
\end{equation*}
$$

cannot be found in the references at our hand. So in this paper, we first discuss the local solutions of initial boundary value problem (1)-(3) and give a fundamental solution of equation (8), and then we pay our main attention to the blow-up phenomena of weak solutions to equation (1).
To describe the methods used in this paper, we prefer to give more information about the existence and nonexistence of global weak solutions to equation (4). If

$$
\begin{equation*}
E(0):=\frac{1}{p} \int_{\Omega}\left|\nabla u_{0}\right|^{p} d x-\int_{\Omega} F\left(u_{0}\right) d x \leq-\frac{4(p-1)}{p T(p-2)^{2}} \int_{\Omega} u_{0}^{2}(x) d x, \tag{9}
\end{equation*}
$$

then the weak solution may blow up in finite time, this is an innovative work by Zhao [28] in 1993. Here and in what follows, $F(u)=\int_{0}^{u} f(s) d x$. After 2000, by setting the energy function

$$
E(t)=\frac{1}{p} \int_{\Omega}|\nabla u|^{p} d x-\int_{\Omega} F(u) d x,
$$

imposing the conditions

$$
\begin{equation*}
s f(s) \geq r F(s) \geq|s|^{r}, \quad r>p>2 \tag{10}
\end{equation*}
$$

and

$$
\begin{equation*}
E(0)<E_{1}, \quad\left\|\nabla u_{0}\right\|_{p}>\alpha_{1}, \tag{11}
\end{equation*}
$$

Liu and Wang [11] showed that the weak solution of problem (4)-(2)-(3) blows up in finite time. Here, $\alpha_{1}, E_{1}$ are two constants defined as

$$
\begin{equation*}
\alpha_{1}=B_{1}^{-\frac{r}{r-p}}, \tag{12}
\end{equation*}
$$

$B_{1}$ is the usual Sobolev embedding constant, and

$$
\begin{equation*}
E_{1}=\left(\frac{1}{p}-\frac{1}{r}\right) B_{1}^{-\frac{r p}{r-p}}=B_{1}^{r}\left(\frac{1}{p}-\frac{1}{r}\right) \alpha_{1}^{r} . \tag{13}
\end{equation*}
$$

The latest progress can be found in [6] by Chung and Choi, in which $f(u)$ satisfies the condition

$$
\left(C_{p}\right): \alpha \int_{0}^{u} f(s) d s \leq u f(u)+\beta u^{p}+\gamma, \quad u>0
$$

for some $\alpha, \beta, \gamma>0$, where $0<\beta \leq \frac{(\alpha-p) \lambda_{1, p}}{p}$ and $\lambda_{1, p}$ is the first eigenvalue of $p$-Laplacian.
In this paper, we give a new energy functional matching up with an anisotropic equation, employ some different techniques to study the blow-up phenomena of weak solution to equation (1). Since $p_{i} \neq p_{j}$ when $i \neq j$, although we are using some ideas of $[3,11]$, how to constructed a suitable functional becomes more difficult, how to obtain the corresponding inequalities to obtain the maximum value becomes a very complicated analysis.

The paper is arranged as follows. In Sect. 2, the definition of weak solution and the main results are given. In Sect. 3, the existence of the local solution to equation (1) is described
and the fundamental solution to equation (8) is given. In Sect. 4, some lemmas related to the initial energy are provided. In Sects. 4 and 5, the theorems on blow-up phenomena are proved.

## 2 The main results

For simplicity, sometimes we denote that $\left\|u_{x_{i}}\right\|_{L^{p_{i}}(\Omega)}=\left\|u_{x_{i}}\right\|_{p_{i}}, \ldots$, and so on.

Lemma 1 If $p_{i}$ is a constant, $p_{i}>1$, let $1 \leq r<\frac{N \bar{q}}{N-\bar{q}}$ and $\frac{1}{\bar{q}}=\frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_{i}}$ when $N-\bar{q}>0$, and it is true for all $r \geq 1$ when $\bar{q} \geq N$. Then $W_{0}^{1, \vec{p}}(\Omega) \hookrightarrow L^{r}(\Omega)$ and $\|u\|_{r} \leq M\left(\prod_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}}\right)^{\frac{1}{N}}$ for all $u \in W_{0}^{1, \vec{p}}(\Omega)$, where $M$ is a constant independent of $u, \vec{p}=\left\{p_{i}\right\}$.

This lemma can be found in $[15,22,27]$, and $W_{0}^{1, \vec{p}}(\Omega)=\left\{u:\left\|u_{x_{i}}\right\|_{p_{i}} \leq c, i=1,2, \ldots, N\right.$, $\left.\left.u\right|_{\partial \Omega}=0\right\}$.

As $[3,11]$, let $F(u)=\int_{0}^{u} f(s) d s$, and

$$
\begin{equation*}
\inf \left\{\int_{\Omega} F(u) d x:\|u\|_{r}=1\right\}>0 \tag{14}
\end{equation*}
$$

Then, combining Lemma 1 with (14), similar to the Sobolev embedding inequality, we can show that

$$
\begin{equation*}
\left(\int_{\Omega} r F(u) d x\right)^{\frac{1}{r}} \leq B\left(\prod_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}}\right)^{\frac{1}{N}}, \quad u \in W_{0}^{1, \vec{p}}(\Omega) \tag{15}
\end{equation*}
$$

where $r \in\left(1, \frac{N \tilde{p}}{N-\tilde{p}}\right)$ and $\frac{1}{\tilde{p}}=\frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_{i}}$, and $B$ is the optimal constant satisfying

$$
B^{-1}=\inf _{u \in W_{0}^{1, \vec{p}}(\Omega)} \frac{\left(\prod_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}}\right)^{1 / N}}{\left(\int_{\Omega} r F(u) d x\right)^{1 / r}} .
$$

The weak solution is defined as follows.

Definition 2 Function $u(x, t)$ is said to be a weak solution of problem (1)-(3) if

$$
\begin{aligned}
& u \in C\left([0, T), L^{2}(\Omega)\right) \cap L^{\infty}\left(Q_{T}\right), \quad u_{t} \in L^{2}\left(Q_{T}\right), \\
& u_{x_{i}} \in L^{p_{i}}\left(Q_{T}\right) \cap W^{1,2}\left([0, T) ; L^{2}(\Omega)\right), \\
& \iint_{Q_{T}} u_{t} \varphi d x d t+\iint_{Q_{T}} \sum_{i=1}^{N}\left(\left|u_{x_{i}}\right|^{p_{i}-2} u_{x_{i}} \varphi_{x_{i}}+f(u) \varphi_{x_{i}}\right) d x d t=0, \quad \forall \varphi \in C_{0}^{1}\left(Q_{T}\right),
\end{aligned}
$$

the initial value condition (2) is true in the sense

$$
\lim _{t \rightarrow 0} \int_{\Omega}\left|u(x, t)-u_{0}(x)\right| d x=0
$$

and the boundary value condition (3) is true in the sense of trace.

From this definition, let

$$
\begin{equation*}
E(t)=\sum_{i=1}^{N} \int_{\Omega} \frac{\left|u_{x_{i}}\right|^{p_{i}}}{p_{i}} d x-\int_{\Omega} F(u(x, t)) d x . \tag{16}
\end{equation*}
$$

Then, obviously, we have

$$
\begin{equation*}
E^{\prime}(t)=-\left\|u_{t}\right\|_{2}^{2} \leq 0, \quad t \geq 0 \tag{17}
\end{equation*}
$$

Now, for a given $u$ being a solution of the initial-boundary problem (1)-(3), we define

$$
A_{i}=A_{i}(t)=\left\|u_{x_{i}}\right\|_{L^{p_{i}}(\Omega)}, \quad A=A(t)=\sum_{i=1}^{N} A_{i}(t)
$$

We set the constant $\alpha_{1}$ by

$$
\begin{equation*}
\alpha_{1}^{r-2 p^{+}+\frac{\left(p^{+}\right)^{2}}{p^{-}}}=\frac{2^{\left(1-p^{+}\right)(N-1)}}{p^{+}}\left(2 p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}}\right) \frac{N^{r}}{B^{r}} \tag{18}
\end{equation*}
$$

and define here and in what follows

$$
\begin{equation*}
E_{1}=\frac{B^{r}}{N^{r}}\left(\left(2 p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}}\right)^{-1}-\frac{1}{r}\right) \alpha_{1}^{r} . \tag{19}
\end{equation*}
$$

By these definitions, we have the following.
Theorem 3 Let u be a solution of initial boundary value problem (1)-(3). Suppose that $f(s)$ in $C(\mathbb{R})$ satisfies (14)-(15),

$$
\begin{align*}
& \frac{2^{\left(1-p^{+}\right)(N-1)}}{p^{+}}\left(2 p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}}\right) \frac{N^{r}}{B^{r}} \geq 1  \tag{20}\\
& 2 p^{-}-p^{+}>0, \quad r>2 p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}} \tag{21}
\end{align*}
$$

and there exist positive constants $\sigma, r$, and $\alpha_{2}$ such that $\alpha_{2}>\alpha_{1}$, and

$$
\begin{align*}
& |s|^{r} \leq \sigma r F(s) \leq s f(s), \quad 1<p^{-} \leq p^{+}<r  \tag{22}\\
& \sigma r-p^{+}-r\left(\frac{p^{+} p^{-}}{2 p^{+} p^{-}-\left(p^{+}\right)^{2}}-\frac{p^{+}}{r}\right) \frac{\alpha_{1}^{r}}{\alpha_{2}^{r}}>0 \tag{23}
\end{align*}
$$

If the initial energy satisfies

$$
E(0)<E_{1}
$$

as well as

$$
\sum_{i=1}^{N}\left\|u_{0 x_{i}}\right\|_{p_{i}}>\alpha_{1}
$$

then the solution $u(x, t)$ of problem (1)-(3) blows up in finite time.

Here, condition (20) assures that $\alpha_{1} \geq 1$.
If $\alpha_{1} \leq 1$, then we define

$$
\begin{equation*}
E_{2}=\frac{B^{r}}{N^{r}}\left(\frac{1}{p^{+}}-\frac{1}{r}\right) \tag{24}
\end{equation*}
$$

and let $\alpha_{10}$ be a constant satisfying

$$
\begin{equation*}
\alpha_{10}^{r-p^{+}}=2^{\left(1-p^{+}\right)(N-1)} \frac{N^{r}}{B^{r}} \tag{25}
\end{equation*}
$$

Theorem 4 Let $u$ be a solution of initial boundary value problem (1)-(3). Suppose that $f(s)$ in $C(\mathbb{R})$ satisfies (14)-(15), and there exist positive constants $\sigma, r$, and $\alpha_{20}$ such that $\alpha_{20}>\alpha_{10}$, inequality (22) is true. If

$$
\begin{align*}
& \sigma r-p^{+}-p^{+} r\left(\frac{1}{2 p^{+}-p_{+}}-\frac{1}{r}\right) \frac{\alpha_{10}^{r}}{\alpha_{20}^{r}}>0,  \tag{26}\\
& r \geq \max \left\{p^{+}\left(2-\frac{p^{+}}{p^{1}}\right), p^{+}\right\},  \tag{27}\\
& \frac{B^{r}}{N^{r}}\left(\frac{1}{p^{+}}-\frac{1}{r}\right) \leq 1, \tag{28}
\end{align*}
$$

and the initial energy satisfies

$$
\begin{equation*}
\frac{1}{p^{+}} 2^{\left(1-p^{+}\right)(N-1)}-\frac{1}{N^{r} r} B^{r}<E(0)<E_{2}, \tag{29}
\end{equation*}
$$

as well as

$$
\sum_{i=1}^{N}\left\|u_{0 x_{i}}\right\|_{p_{i}}>\alpha_{10}
$$

then the solution $u(x, t)$ of problem (1)-(3) blows up in finite time.
Here, condition (28) assures that $\alpha_{1} \leq 1$. Also, one can see that condition (21) is in opposition to condition (27).

In fact, besides conditions (10) and (11), by setting (12)(13), Liu and Wang showed that the solution $u(x, t)$ of equation (4) blows up in finite time [11].

One can see that if $p^{+}=p^{-}$and $\sigma=1$, then the growth order condition (22) is just the same as (13). However, in our paper, $\sigma$ can be less than 1 . So our paper improves the results of [11]. Actually, the blow-up phenomena shown in [13, 16, 19, 24, 25, 28] also arise in the anisotropic equation (1). If the readers are interested in these problems, to generalize the results in $[13,16,19,24,25,28]$, they will find that the complexity mainly reflects in the index relationships just like our theorems obtained in this paper.

## 3 The fundamental solution and the existence of weak solution

About two years ago, my graduate student Miss Yang first constructed the fundamental solution of equation (8) in her master thesis [26]. Since it has not been published in any magazine, we extract a part of [26] here.

Proposition 5 There is a fundamental solution of equation (8).

Proof Let us choose

$$
\Phi_{\tau, \alpha}\left(x, t, x_{0}, t_{0}\right)=\sum_{i=1}^{N} K_{i} S^{-\frac{1}{\lambda_{i}}}(t)\left\{\alpha-\left[\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right]^{\frac{p_{i}}{p_{i}-1}}\right\}_{+}^{\frac{p_{i}-1}{p_{i}-2}} .
$$

Here, $\tau$ and $\alpha$ are any given nonnegative constants, $x=\left(x_{1}, x_{2}, \ldots, x_{N}\right) \in \Omega, x^{0}=\left(x_{1}^{0}, x_{2}^{0}, \ldots\right.$, $\left.x_{N}^{0}\right) \in \Omega, t_{0} \geq 0$, and

$$
\begin{aligned}
& S(t)=\tau+\left(t-t_{0}\right), \\
& K_{i}=\left(\frac{p_{i}-2}{p_{i}}\right)^{\frac{p_{i}-1}{p_{i}-2}}\left(\frac{1}{\lambda_{i}}\right)^{\frac{1}{p_{i}-2}}, \\
& \lambda_{i}=2\left(p_{i}-1\right) .
\end{aligned}
$$

We have

$$
\begin{align*}
\frac{\partial \Phi_{\tau, \alpha}}{\partial t}= & -\sum_{i=1}^{N} \frac{1}{\lambda_{i}} K_{i} S^{-\frac{1}{\lambda_{i}}-1}(t)\left\{\alpha-\left[\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right]^{\frac{p_{i}}{p_{i}-1}}\right\}_{+}^{\frac{p_{i}-1}{p_{i}-2}} \\
& +\sum_{i=1}^{N} \frac{1}{\lambda_{i}} \frac{p_{i}}{p_{i}-2} K_{i} S^{-\frac{1}{\lambda_{i}}-1}(t)\left\{\alpha-\left[\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right]^{\frac{p_{i}}{p_{i}-1}}\right\}_{+}^{\frac{p_{i}-1}{p_{i}-2}-1}\left(\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right)^{\frac{p_{i}}{p_{i}-1}}  \tag{30}\\
= & -\sum_{i=1}^{N} H_{i} S^{-\frac{1}{\lambda_{i}}-1}(t)\left\{\alpha-\left[\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right]^{\frac{p_{i}}{p_{i}-1}}\right\}_{+}^{\frac{p_{i}-1}{p_{i}-2}} \\
& +\sum_{i=1}^{N} G_{i} S^{-\frac{1}{\lambda_{i}}-1}(t)\left\{\alpha-\left[\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right]^{\frac{p_{i}}{p_{i}-1}}\right\}_{+}^{\frac{1}{p_{i}-2}}\left(\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right)^{\frac{p_{i}}{p_{i}-1}}
\end{align*}
$$

where

$$
\begin{aligned}
& H_{i}=\left(\frac{p_{i}-2}{p_{i}}\right)^{\frac{p_{i}-1}{p_{i}-2}}\left(\frac{1}{\lambda_{i}}\right)^{\frac{p_{i}-1}{p_{i}-2}}, \\
& G_{i}=\left(\frac{p_{i}-2}{p_{i}}\right)^{\frac{1}{p_{i}-2}}\left(\frac{1}{\lambda_{i}}\right)^{\frac{p_{i}-1}{p_{i}-2}} .
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\frac{\partial \Phi_{\tau, \alpha}}{\partial x_{i}}= & -H_{i} \lambda_{i} \frac{p_{i}}{p_{i}-2} S^{-\frac{1}{\lambda_{i}}}(t)\left\{\alpha-\left[\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right]^{\frac{p_{i}}{p_{i}-1}}\right\}_{+}^{\frac{p_{i}-1}{p_{i}-2}-1} \\
& \times\left(\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right)^{\frac{p_{i}}{p_{i}-1}-1} \frac{\left(x_{i}-x_{i}^{0}\right)}{\left|x_{i}-x_{i}^{0}\right| S^{\frac{1}{\lambda_{i}}}(t)} \\
= & -H_{i} \lambda_{i} \frac{p_{i}}{p_{i}-2} S^{-\frac{2}{\lambda_{i}}}(t)\left\{\alpha-\left[\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right]^{\frac{p_{i}}{p_{i}-1}}\right\}_{+}^{\frac{1}{p_{i}-2}}\left(\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right)^{\frac{1}{p_{i}-1}} \frac{\left(x_{i}-x_{i}^{0}\right)}{\left|x_{i}-x_{i}^{0}\right|} .
\end{aligned}
$$

Then

$$
\begin{align*}
\sum_{i=1}^{N} & \frac{\partial}{\partial x_{i}}\left(\left|\frac{\partial \Phi_{\tau, \alpha}}{\partial x_{i}}\right|^{p_{i}-2} \frac{\partial \Phi_{\tau, \alpha}}{\partial x_{i}}\right) \\
= & -\sum_{i=1}^{N} \frac{\partial}{\partial x_{i}}\left\{H_{i} S^{-\frac{1}{\lambda_{i}}-1}(t)\left\{\alpha-\left[\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{-\frac{1}{\lambda_{i}}}(t)}\right]^{\frac{p_{i}}{p_{i}-1}}\right\}_{+}^{\frac{p_{i}-1}{p_{i}-2}}\left(x_{i}-x_{i}^{0}\right)\right\} \\
= & -\sum_{i=1}^{N} H_{i} S^{-\frac{1}{\lambda_{i}}-1}(t)\left\{\alpha-\left[\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right]^{\frac{p_{i}}{p_{i}-1}}\right\}_{+}^{\frac{p_{i}-1}{p_{i}-2}} \\
& +\sum_{i=1}^{N} H_{i} \frac{p_{i}}{p_{i}-2} S^{-\frac{1}{\lambda_{i}}-1}(t)\left\{\alpha-\left[\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right]^{\frac{p_{i}}{p_{i}-1}}\right\}_{+}^{\frac{p_{i}-1}{p_{i}-2}-1}  \tag{31}\\
& \times\left(\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right)^{\frac{p_{i}}{p_{i}-1}-1} \frac{\left(x_{i}-x_{i}^{0}\right)^{2}}{\left|x_{i}-x_{i}^{0}\right| S^{\frac{1}{\lambda_{i}}}(t)} \\
= & -\sum_{i=1}^{N} H_{i} S^{-\frac{1}{\lambda_{i}}-1}(t)\left\{\alpha-\left[\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right]^{\frac{p_{i}}{p_{i}-1}}\right\}_{+}^{\frac{p_{i}-1}{p_{i}-2}} \\
& +\sum_{i=1}^{N} G_{i} S^{-\frac{1}{\lambda_{i}}-1}(t)\left\{\alpha-\left[\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right]^{\frac{p_{i}}{p_{i}-1}}\right\}_{+}^{\frac{1}{p_{i}-2}}\left(\frac{\left|x_{i}-x_{i}^{0}\right|}{S^{\frac{1}{\lambda_{i}}}(t)}\right)^{\frac{p_{i}}{p_{i}-1}} .
\end{align*}
$$

At last, by combining (30) with (31), we have

$$
\begin{equation*}
\frac{\partial \Phi_{\tau, \alpha}}{\partial t}=\sum_{i=1}^{N} \frac{\partial}{\partial x_{i}}\left(\left|\frac{\partial \Phi_{\tau, \alpha}}{\partial x_{i}}\right|^{p_{i}-2} \frac{\partial \Phi_{\tau, \alpha}}{\partial x_{i}}\right) . \tag{32}
\end{equation*}
$$

Remark 6 If $\left(x_{0}, t_{0}\right)=(0,0)$, then $\Phi_{\tau, \alpha}(x, t, 0,0)$ is the solution of equation (32) with the initial value condition

$$
\Phi_{\tau, \alpha}(x, 0,0,0)=M \delta(x)
$$

where $\delta(x)$ is the Dirac function and $M=\left\|\Phi_{\tau, \alpha}(\cdot, t, 0,0)\right\|_{1, \mathbb{R}^{N}}$. Moreover, by choosing two suitable constants $\tau_{1}, \alpha_{1}$ such that $M=1$, the corresponding $\Phi_{\tau_{1}, \alpha_{1}}(x, t, 0,0)$ is the fundamental solution of equation (32).

Now, consider the regularization equation

$$
\begin{equation*}
u_{t}=\sum_{i=1}^{N} \frac{\partial}{\partial x_{i}}\left(\left(\left|u_{x_{i}}\right|^{2}+\frac{1}{n}\right)^{\frac{p_{i}-2}{2}} u_{x_{i}}\right)+f_{n}(u), \quad(x, t) \in Q_{T}, \tag{33}
\end{equation*}
$$

with the initial boundary value conditions

$$
\begin{align*}
& u(x, t)=0, \quad x \in \partial \Omega \times(0, T),  \tag{34}\\
& u(x, 0)=u_{0 n}(x), \quad x \in \Omega, \tag{35}
\end{align*}
$$

where $f_{n} \in C^{1}(\mathbb{R}),\left|f_{n}\right| \leq f, f_{n} \rightarrow f$ uniformly on bounded subsets of $[a, b]$, and $|f(u)| \leq g(u)$; $u_{0 n} \in C_{0}^{\infty}(\Omega)$ and $\left\|u_{0 n}\right\|_{L^{\infty}(\Omega)} \leq\left\|u_{0}\right\|_{L^{\infty}(\Omega)},\left\|u_{0 n x_{i}}\right\|_{L^{p_{i}}(\Omega)} \leq\left\|u_{0 x_{i}}\right\|_{L^{p_{i}}(\Omega)}$, and

$$
u_{0 n}(x) \rightarrow u_{0}(x), \quad \text { in } W_{0}^{1, p^{+}}(\Omega)
$$

By a similar method as that in [28] and using the classical weakly convergent method, we can prove the following existence theorem.

Theorem $7 \operatorname{Let} f(s) \in C(\mathbb{R})$ and there exist a function $g(s) \in C^{1}(\mathbb{R})$ such that $|f(s)| \leq g(s)$, $p^{-} \geq 2$, and $u_{0}(x) \in W_{0}^{1, \vec{p}}(\Omega) \cap L^{\infty}(\Omega)$. Then the initial boundary value problem (1)-(3) has a solution $u \in L^{\infty}\left(Q_{T_{1}}\right)$, where $T_{1}$ is a small positive constant. Iff $(s)$ is a Lipschitz function, then the local solution is unique.

For simplicity of the paper, we omit the details of the proof here.

## 4 Proof of Theorem 3

By the mathematical induction, we easily show the following inequality:

$$
\begin{equation*}
\left(\beta_{1}+\beta_{2}+\cdots+\beta_{N}\right)^{p} \leq 2^{(p-1)(N-1)}\left(\beta_{1}^{p}+\beta_{2}^{p}+\cdots+\beta_{N}^{p}\right) \tag{36}
\end{equation*}
$$

where $p \geq 1, \beta_{i}>0, i=1,2, \ldots, N, N \geq 2$.
Now, we define two functions $g_{i}:[0, \infty) \rightarrow \mathbb{R}, i=1,2$, by

$$
g_{1}(\alpha)=\frac{1}{p^{+}} 2^{\left(1-p^{+}\right)(N-1)} \alpha^{2 p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}}}-\frac{1}{N^{r} r} B^{r} \alpha^{r}
$$

and

$$
g_{2}(\alpha)=\frac{1}{p^{+}} 2^{\left(1-p^{+}\right)(N-1)} \alpha^{p^{+}}-\frac{1}{N^{r} r} B^{r} \alpha^{r} .
$$

Let $\alpha_{1}$ be defined by (18). One can see that, when $\alpha<\alpha_{1}$, then $g_{1}^{\prime}(\alpha)>0$. While $\alpha>\alpha_{1}$, $g_{1}^{\prime}(\alpha)<0$. Accordingly, we have

$$
\begin{equation*}
E_{1}=g_{1}\left(\alpha_{1}\right)=\max g_{1}(\alpha) \tag{37}
\end{equation*}
$$

At the same time, let $\alpha_{10}$ be defined as (25), i.e.,

$$
\alpha_{10}^{r-p^{+}}=2^{\left(1-p^{+}\right)(N-1)} \frac{N^{r}}{B^{r}}
$$

One can see that when $\alpha<\alpha_{10}, g_{2}^{\prime}(\alpha)>0$; while $\alpha>\alpha_{1}, g_{2}^{\prime}(\alpha)<0$. According to (26), we have

$$
\begin{equation*}
E_{2}=g_{2}\left(\alpha_{10}\right)=\max g_{2}(\alpha)=\frac{B^{r}}{N^{r}}\left(\frac{1}{p^{+}}-\frac{1}{r}\right) \tag{38}
\end{equation*}
$$

Since $\frac{1}{p^{+}}\left(2 p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}}\right) \leq 1$, we know that the inequality

$$
\alpha_{1}^{r-2 p^{+}+\frac{\left(p^{+}\right)^{2}}{p^{-}}} \leq \alpha_{10}^{r-p^{+}}
$$

is always true. By (20)

$$
\frac{2^{\left(1-p^{+}\right)(N-1)}}{p^{+}}\left(2 p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}}\right) \frac{N^{r}}{B^{r}} \geq 1
$$

we have $\alpha_{1} \geq 1$. Now, we define

$$
g(\alpha)= \begin{cases}g_{2}(\alpha), & \alpha<1 \\ g_{1}(\alpha), & \alpha \geq 1\end{cases}
$$

Then $g(\alpha)$ is a continuous function on $[0, \infty)$ and has the following properties:
(1) It is a differential function except at the point $\alpha=1$;
(2) It is increasing for $0<\alpha<\alpha_{1}$; decreasing for $\alpha>\alpha_{1}$;
(3) $g(\alpha) \rightarrow-\infty$ as $\alpha \rightarrow+\infty$;
(4) $g\left(\alpha_{1}\right)=E_{1}=\max g(\alpha)$.

Lemma 8 If $u(x, t)$ is a solution of problem (1)-(3), regarding $g(A)$ as a composite function of variable $t$, we have

$$
\begin{equation*}
E(t) \geq g(A) \tag{39}
\end{equation*}
$$

Proof By (14), (15), and (36), when $A \geq 1$, we have

$$
\begin{align*}
E(t) & \geq \sum_{i=1}^{N} \int_{\Omega} \frac{\left|u_{x_{i}}\right|^{p_{i}}}{p_{i}} d x-\frac{1}{r} B^{r}\left(\prod_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}}\right)^{\frac{r}{N}} \\
& \geq \frac{1}{p^{+}} A^{p^{+}} \sum_{i=1}^{N}\left(\frac{A_{i}}{A^{\frac{p^{+}}{p_{i}}}}\right)^{p_{i}}-\frac{1}{r} B^{r}\left(\prod_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}}\right)^{\frac{r}{N}} \\
& \geq \frac{1}{p^{+}} A^{p^{+}} \sum_{i=1}^{N}\left(\frac{A_{i}}{A^{p^{+}}}\right)^{p^{+}}-\frac{1}{r} B^{r}\left(\prod_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}}\right)^{\frac{r}{N}} \\
& \geq \frac{1}{p^{+}} 2^{\left(1-p^{+}\right)(N-1)} A^{p^{+}}\left(\sum_{i=1}^{N} \frac{A_{i}}{A^{p^{+}}} A^{p^{+}}-\frac{1}{N^{r} r} B^{r}\left(\sum_{i=1}^{N} A_{i}\right)^{r}\right.  \tag{40}\\
& =\frac{1}{p^{+}} 2^{\left(1-p^{+}\right)(N-1)} A^{p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}}}\left(\sum_{i=1}^{N} A_{i}\right)^{p^{+}}-\frac{1}{N^{r} r} B^{r}\left(\sum_{i=1}^{N} A_{i}\right)^{r} \\
& =\frac{1}{p^{+}} 2^{\left(1-p^{+}\right)(N-1)} A^{2 p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}}}-\frac{1}{N^{r} r} B^{r} A^{r} \\
& =g_{1}(A) \\
& =g(\alpha) .
\end{align*}
$$

When

$$
\alpha=\sum_{i=1}^{N} A_{i}<1,
$$

$$
A_{i}<1, \quad i=1,2, \ldots, N
$$

By (14), (15), and (36), we have

$$
\begin{align*}
E(t) & \geq \sum_{i=1}^{N} \int_{\Omega} \frac{\left|u_{x_{i}}\right|^{p_{i}}}{p_{i}} d x-\frac{1}{r} B^{r}\left(\prod_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}}\right)^{\frac{r}{N}} \\
& \geq \frac{1}{p^{+}} \sum_{i=1}^{N} \int_{\Omega} \left\lvert\, u_{x_{i}} p^{p_{i}} d x-\frac{1}{r} B^{r}\left(\prod_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}}\right)^{\frac{r}{N}}\right. \\
& \geq \frac{1}{p^{+}} \sum_{i=1}^{N} \int_{\Omega}\left|\frac{u_{x_{i}}}{A_{i}}\right|^{p_{i}} A_{i}^{p_{i}} d x-\frac{1}{r} B^{r}\left(\prod_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}}\right)^{\frac{r}{N}} \\
& \geq \frac{1}{p^{+}} \sum_{i=1}^{N} A_{i}^{p^{+}}\left\|\frac{u_{x_{i}}}{A_{i}}\right\|_{p_{i}}^{p_{i}}-\frac{1}{N^{r} r} B^{r}\left(\sum_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}}\right)^{r} \\
& \geq \frac{1}{p^{+}} \sum_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}}^{p_{i}}-\frac{1}{N^{r} r} B^{r}\left(\sum_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}}\right)^{r}  \tag{41}\\
& \geq \frac{1}{p^{+}} \sum_{i=1}^{N} A_{i}^{p^{+}}-\frac{1}{N^{r} r} B_{1}^{r}\left(\sum_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}}\right)^{r} \\
& \geq \frac{1}{p^{+}} 2^{\left(1-p^{+}\right)(N-1)}\left(\sum_{i=1}^{N} A_{i}\right)^{p^{+}}-\frac{1}{N^{r} r} B^{r}\left(\sum_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}(x)}\right)^{r} \\
& =\frac{1}{p^{+}} 2^{\left(1-p^{+}\right)(N-1)} A^{p^{+}}-\frac{1}{N^{r} r} B^{r} A^{r} \\
& =g_{2}(A) \\
& =g(\alpha) .
\end{align*}
$$

Combining (40) with (41), we obtain inequality (39). Here and in what follows, we use the denotation $g(\alpha)$ to replace $g(A)$ if we regard $g(\alpha)$ is a function defined on $\alpha \in[0,+\infty)$.

Lemma 9 Suppose that $u(x, t)$ is a solution of problem (1)-(3), the initial value satisfying $E(0)<E_{1}$, as well as $\sum_{i=1}^{N}\left\|u_{0_{x_{i}}}\right\|_{p_{i}(x)}>\alpha_{1}$. Then, for any $t \geq 0$, there exists a positive constant $\alpha_{2}, \alpha_{2}>\alpha_{1}$ such that

$$
\begin{equation*}
\sum_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}} \geq \alpha_{2}, \quad \forall t \geq 0 \tag{42}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(N^{r} \int_{\Omega} r F(u) d x\right)^{1 / r} \geq B \alpha_{2} \tag{43}
\end{equation*}
$$

Proof As we have said before, the function $g(\alpha)$ gets the maximum value at $\alpha_{1}$, moreover, $g\left(\alpha_{1}\right)=E_{1}$. Since $E(0)<E_{1}$, then there is $\alpha_{2}>\alpha_{1}$ such that $g\left(\alpha_{2}\right)=E(0)$.

Let $\alpha_{0}=\sum_{i=1}^{N}\left\|u_{0 x_{i}}\right\|_{p_{i}}$. Then, by (39), we possess $g\left(\alpha_{0}\right) \leq E(0)=g\left(\alpha_{2}\right)$, and so $\alpha_{0} \geq \alpha_{2}$.

Next, inequality (42) can be proved by contradiction. If there is $t_{0}>0$ such that $\sum_{i=1}^{N}\left\|u_{x_{i}}\left(\cdot, t_{0}\right)\right\|_{p_{i}}<\alpha_{2}$, then according to the continuity of $\sum_{i=1}^{N}\left\|u_{x_{i}}(\cdot, t)\right\|_{p_{i}}$, we can choose appropriate $t_{0}$ such that $\sum_{i=1}^{N}\left\|u_{x_{i}}\left(\cdot, t_{0}\right)\right\|_{p_{i}}>\alpha_{1}$. On the premise of (39), we have

$$
E\left(t_{0}\right) \geq g\left(\sum_{i=1}^{N}\left\|u_{x_{i}}\left(\cdot, t_{0}\right)\right\|_{p_{i}}\right)>g\left(\alpha_{2}\right)=E(0)
$$

This contradicts the decreasing function $E(t)$. So we have (42).
Again, as (39) shows, when $A \geq 1$, we have

$$
\sum_{i=1}^{N} \int_{\Omega} \frac{1}{p_{i}}\left|u_{x_{i}}\right|^{p_{i}} d x \geq \frac{1}{p^{+}} 2^{\left(1-p^{+}\right)(N-1)} A^{2 p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}}}
$$

Now, since (42), we know $A \geq \alpha_{2}>\alpha_{1} \geq 1$. Then the definition of $E(t)$ (16) yields

$$
\begin{align*}
\int_{\Omega} F(u) d x & =\sum_{i=1}^{N} \int_{\Omega} \frac{1}{p_{i}}\left|u_{x_{i}}\right|^{p_{i}} d x-E(t) \\
& \geq \frac{1}{p^{+}} 2^{\left(1-p^{+}\right)(N-1)} A^{2 p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}}}-E(0)  \tag{44}\\
& \geq \frac{1}{p^{+}} 2^{\left(1-p^{+}\right)(N-1)} \alpha_{2}^{2 p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}}}-g\left(\alpha_{2}\right) \\
& =\frac{1}{N^{r} r} B^{r} \alpha_{2}^{r} .
\end{align*}
$$

So (43) is true.

Lemma 10 Let

$$
\begin{equation*}
H_{1}(t)=E_{1}-E(t), \quad \forall t \geq 0 . \tag{45}
\end{equation*}
$$

Then there holds

$$
\begin{equation*}
0<H_{1}(0) \leq H_{1}(t) \leq \int_{\Omega} F(u) d x, \quad \forall t \geq 0 . \tag{46}
\end{equation*}
$$

Proof Since $E^{\prime}(t) \leq 0, t \geq 0$, we have $H_{1}^{\prime}(t)=-E^{\prime}(t) \geq 0$, and

$$
H_{1}(t) \geq H_{1}(0)=E_{1}-E(0)>0, \quad \forall t \geq 0 .
$$

At the same time, by (16), we know

$$
H_{1}(t)=E_{1}-\sum_{i=1}^{N} \int_{\Omega} \frac{1}{p_{i}}\left|u_{x_{i}}\right|^{p_{i}} d x+\int_{\Omega} F(u) d x,
$$

and from Lemma 9, there holds

$$
\begin{aligned}
E_{1} & -\sum_{i=1}^{N} \int_{\Omega} \frac{1}{p_{i}}\left|u_{x_{i}}\right|^{p_{i}} d x \\
& \leq E_{1}-g\left(\alpha_{2}\right) \\
& \leq E_{1}-g\left(\alpha_{1}\right)=-\frac{1}{N^{r} r} C_{1}^{r} \alpha_{1}^{r} \\
& \leq 0 .
\end{aligned}
$$

Thus,

$$
\begin{equation*}
H_{1}(t) \leq \int_{\Omega} F(u) d x, \quad t \geq 0 \tag{47}
\end{equation*}
$$

By these lemmas, we are able to prove Theorem 3.

Proof of Theorem 3 Let

$$
\begin{equation*}
G(t)=\frac{1}{2} \int_{\Omega} u^{2}(x, t) d x \tag{48}
\end{equation*}
$$

By (16), (45), and (46), equation (1) yields

$$
\begin{align*}
G^{\prime}(t) & =\int_{\Omega} u u_{t} d x \\
& =\int_{\Omega} u\left[\sum_{i=1}^{N} \frac{\partial}{\partial x_{i}}\left(\left|u_{x_{i}}\right|^{p_{i}-2} u_{x_{i}}\right)+f(u)\right] d x \\
& =\int_{\Omega} u f(u) d x-\int_{\Omega} \sum_{i=1}^{N}\left|u_{x_{i}}\right|^{p_{i}} d x  \tag{49}\\
& =\int_{\Omega} u f(u) d x-\sum_{i=1}^{N} \int_{\Omega} p_{i} \frac{\left|u_{x_{i}}\right|^{p_{i}}}{p_{i}} d x \\
& \geq \int_{\Omega} u f(u) d x-p^{+} \int_{\Omega} F(u) d x+p^{+} H_{1}(t)-p^{+} E_{1}
\end{align*}
$$

Then, since (18)

$$
\alpha_{1}^{r-2 p^{+}+\frac{\left(p^{+}\right)^{2}}{p^{-}}}=\frac{2^{\left(1-p^{+}\right)(N-1)}}{p^{+}}\left(2 p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}}\right) \frac{N^{r}}{B^{r}},
$$

by the definition of $E_{1}(19), E_{1}=g\left(\alpha_{1}\right)$, and Lemma 9, we have

$$
\begin{aligned}
p^{+} E_{1} & =p^{+} \frac{B^{r}}{N^{r}}\left(\left(2 p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}}\right)^{-1}-\frac{1}{r}\right) \alpha_{1}^{r} \\
& =p^{+}\left[\frac{1}{p^{+}} 2^{\left(1-p^{+}\right)(N-1)} \alpha_{1}^{2 p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}}}-\frac{1}{N^{r} r} B^{r} \alpha_{1}^{r}\right]
\end{aligned}
$$

$$
\begin{align*}
& =p^{+}\left(\frac{p^{-}}{2 p^{+} p^{-}-\left(p^{+}\right)^{2}}-\frac{1}{r}\right) \frac{B^{r}}{N^{r}} \alpha_{1}^{r}  \tag{50}\\
& =p^{+}\left(\frac{p^{-}}{2 p^{+} p^{-}-\left(p^{+}\right)^{2}}-\frac{1}{r}\right) \frac{B^{r}}{N^{r}} \alpha_{2}^{r} \frac{\alpha_{1}^{r}}{\alpha_{2}^{r}} \\
& \leq r\left(\frac{p^{+} p^{-}}{2 p^{+} p^{-}-\left(p^{+}\right)^{2}}-\frac{p^{+}}{r}\right) \frac{\alpha_{1}^{r}}{\alpha_{2}^{r}} \int_{\Omega} F(u) d x .
\end{align*}
$$

Then, by combining (23)(22), (49) with (50), we have

$$
\begin{align*}
G^{\prime}(t) & \geq\left(\delta r-p^{+}-r\left(\frac{p^{+} p^{-}}{2 p^{+} p^{-}-\left(p^{+}\right)^{2}}-\frac{p^{+}}{r}\right) \frac{\alpha_{1}^{r}}{\alpha_{2}^{r}}\right) \int_{\Omega} F(u) d x+p^{+} H_{1}(t)  \tag{51}\\
& \geq C^{*} \int_{\Omega} F(u) d x
\end{align*}
$$

where $C^{*}=\delta r-p^{+}-r\left(\frac{p^{+} p^{-}}{2 p^{+} p^{-}-\left(p^{+}\right)^{2}}-\frac{p^{+}}{r}\right) \frac{\alpha_{1}^{r}}{\alpha_{2}^{r}}>0$.
Moreover, we need to make an estimate of $G^{r / 2}(t)$. By inequality (22) and the Hölder inequality, we can obtain

$$
\begin{equation*}
G^{r / 2}(t)=\left(\frac{1}{2} \int_{\Omega} u^{2}(x, t) d x\right)^{r / 2} \leq 2^{-\frac{r}{2}}|\Omega|^{\frac{r-2}{2}}\|u\|_{r}^{r} \leq r 2^{-\frac{r}{2}}|\Omega|^{\frac{r-2}{2}} \int_{\Omega} F(u) d x . \tag{52}
\end{equation*}
$$

Combining formulas (51) and (52), we have

$$
\begin{equation*}
G^{\prime}(t) \geq \eta G^{r / 2}(t) \tag{53}
\end{equation*}
$$

where $\eta=C^{*} /\left(r 2^{-\frac{r}{2}}|\Omega|^{\frac{r-2}{2}}\right)$.
By integrating (53) over [ $0, t$ ], we have

$$
\begin{equation*}
G^{r / 2-1}(t) \geq \frac{1}{G^{1-r / 2}(0)-(r / 2-1) \eta t} \tag{54}
\end{equation*}
$$

Accordingly, $G$ blows up at the time of $t^{*} \leq G^{r / 2-1}(0) /[(r / 2-1) \eta]$.

## 5 Proof of Theorem 4

In this section, we use a similar method to prove Theorem 4.
Since $\frac{1}{p^{+}}\left(2 p^{+}-\frac{\left(p^{+}\right)^{2}}{p^{-}}\right) \leq 1$, by assumptions (27) and (28), we know

$$
\begin{equation*}
\alpha_{1}^{r-2 p^{+}+\frac{\left(p^{+}\right)^{2}}{p^{-}}} \leq \alpha_{10}^{r-p^{+}}=\frac{B^{r}}{N^{r}}\left(\frac{1}{p^{+}}-\frac{1}{r}\right) \leq 1, \tag{55}
\end{equation*}
$$

and so

$$
\alpha_{1}>1, \quad \alpha_{10}<1
$$

Lemma 8 yields

$$
\begin{equation*}
E(t) \geq g(A), \quad t \in[0, T) \tag{56}
\end{equation*}
$$

is still true. Now, the function $g(\alpha)$ has the following properties.
(1) When $\alpha \in\left[0, \alpha_{10}\right], g(\alpha)=g_{2}(\alpha)$ is increasing; when $\alpha \in\left[\alpha_{10}, 1\right], g(\alpha)=g_{2}(\alpha)$ is decreasing.
(2) When $\alpha \in\left[1, \alpha_{1}\right], g(\alpha)=g_{1}(\alpha)$ is increasing; when $\alpha \in\left[\alpha_{1},+\infty\right), g(\alpha)=g_{2}(\alpha)$ is decreasing.
(3) $g(\alpha) \rightarrow-\infty$ as $\alpha \rightarrow+\infty$;
(4) $g_{2}\left(\alpha_{10}\right)=g\left(\alpha_{10}\right)=E_{2}=\max g(\alpha) \geq g\left(\alpha_{1}\right)=g_{1}\left(\alpha_{1}\right)=E_{1}$.

Since (29),

$$
g(1)=\frac{1}{p^{+}} 2^{\left(1-p^{+}\right)(N-1)}-\frac{1}{N^{r} r} B^{r}<E(0)<E_{2},
$$

by the above properties of $g(\alpha)$, there are two points $\beta_{i}<1, i=1,2$, such that $g\left(\beta_{i}\right)=E(0)$. We now choose the larger one and denote it as $\alpha_{20}$, i.e., then there is $1>\alpha_{20}>\alpha_{10}$ such that $g\left(\alpha_{20}\right)=E(0)$.
Let $\alpha_{0}=\sum_{i=1}^{N}\left\|u_{0 x_{i}}\right\|_{p_{i}}$. Then, by Lemma 8, we possess $g\left(\alpha_{0}\right)=g(A(0)) \leq E(0)=g\left(\alpha_{20}\right)$.
By these facts, similar as Lemma 9, we can show the following lemma, and we omit the details.

Lemma 11 Suppose that $u(x, t)$ is a solution of problem (1)-(3), and the initial value satisfies (29) as well as $\sum_{i=1}^{N}\left\|u_{0 x_{i}}\right\|_{p_{i}(x)}>\alpha_{10}$. Then, for any $t \geq 0$, we have

$$
\begin{equation*}
\sum_{i=1}^{N}\left\|u_{x_{i}}\right\|_{p_{i}} \geq \alpha_{20} \tag{57}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(N^{r} \int_{\Omega} r F(u) d x\right)^{1 / r} \geq B \alpha_{20} \tag{58}
\end{equation*}
$$

Now, let us introduce the following function:

$$
\begin{equation*}
H_{2}(t)=E_{2}-E(t), \quad \forall t \geq 0 . \tag{59}
\end{equation*}
$$

Similar as Lemma 10, we have the following.

Lemma $12 H(t)$ satisfies

$$
\begin{equation*}
0<H_{2}(0) \leq H_{2}(t) \leq \int_{\Omega} F(u) d x, \quad \forall t \geq 0 . \tag{60}
\end{equation*}
$$

After these preparations, we can give the proof of Theorem 4.

Proof of Theorem 4 Let

$$
\begin{equation*}
G(t)=\frac{1}{2} \int_{\Omega} u^{2}(x, t) d x \tag{61}
\end{equation*}
$$

Then, from (60) and (61), we have

$$
\begin{align*}
G^{\prime}(t) & =\int_{\Omega} u u_{t} d x \\
& =\int_{\Omega} u\left[\sum_{i=1}^{N} \frac{\partial}{\partial x_{i}}\left(\left|u_{x_{i}}\right|^{p_{i}-2} u_{x_{i}}\right)+f(u)\right] d x \\
& =\int_{\Omega} u f(u) d x-\int_{\Omega} \sum_{i=1}^{N}\left|u_{x_{i}}\right|^{p_{i}} d x  \tag{62}\\
& =\int_{\Omega} u f(u) d x-\sum_{i=1}^{N} \int_{\Omega} p_{i} \frac{\left|u_{x_{i}}\right|^{p_{i}}}{p_{i}} d x \\
& \geq \int_{\Omega} u f(u) d x-p^{+} \int_{\Omega} F(u) d x+p^{+} H_{2}(t)-p^{+} E_{2} .
\end{align*}
$$

Moreover, since (26)

$$
\alpha_{10}^{r-p^{+}}=2^{\left(1-p^{+}\right)(N-1)} \frac{N^{r}}{B^{r}}
$$

by (22) and Lemma 11, we have

$$
\begin{align*}
p^{+} E_{2} & =p^{+}\left[\frac{1}{p^{+}} 2^{\left(1-p^{+}\right)(N-1)} A^{p^{+}}-\frac{1}{N^{r} r} B^{r} A^{r}\right] \\
& =p^{+}\left(\frac{1}{p^{+}}-\frac{1}{r}\right) \frac{B^{r}}{N^{r}} \alpha_{10}^{r}  \tag{63}\\
& \leq p^{+} r\left(\frac{1}{p^{+}}-\frac{1}{r}\right) \frac{\alpha_{10}^{r}}{\alpha_{20}^{r}} \int_{\Omega} F(u) d x .
\end{align*}
$$

Thus, by conditions (22),(26) and using (62) and (63), we can obtain

$$
\begin{align*}
G^{\prime}(t) & \geq p^{+} r\left(\frac{1}{p^{+}}-\frac{1}{r}\right) \frac{\alpha_{10}^{r}}{\alpha_{20}^{r}} \int_{\Omega} F(u) d x+p^{+} H_{2}(t)  \tag{64}\\
& \geq C^{*} \int_{\Omega} F(u) d x
\end{align*}
$$

where $C^{*}=p^{+} r\left(\frac{1}{p^{+}}-\frac{1}{r}\right) \frac{\alpha_{10}^{r}}{\alpha_{20}^{r}}>0$.
Once more, we want to give an estimate of $G^{r / 2}(t)$. By inequality (22) and the Hölder inequality, we can extrapolate that

$$
\begin{equation*}
G^{r / 2}(t)=\left(\frac{1}{2} \int_{\Omega} u^{2}(x, t) d x\right)^{r / 2} \leq 2^{-\frac{r}{2}}|\Omega|^{\frac{r-2}{2}}\|u\|_{r}^{r} \leq r 2^{-\frac{r}{2}}|\Omega|^{\frac{r-2}{2}} \int_{\Omega} F(u) d x \tag{65}
\end{equation*}
$$

By (64) and (65), we have

$$
\begin{equation*}
G^{\prime}(t) \geq \eta G^{r / 2}(t) \tag{66}
\end{equation*}
$$

where $\eta=C^{*} /\left(r 2^{-\frac{r}{2}}|\Omega|^{\frac{r-2}{2}}\right)$.

By integrating (66) over $[0, t]$, we have

$$
G^{r / 2-1}(t) \geq \frac{1}{G^{1-r / 2}(0)-(r / 2-1) \eta t}
$$

which implies that $G$ blows up at the time of $t^{*} \leq G^{r / 2-1}(0) /[(r / 2-1) \eta]$.

## Funding

The paper is partially supported by NSF of Fujian Province (No. 2022J011242), China.

## Availability of data and materials

There is not any data in this paper

## Declarations

## Ethics approval and consent to participate

No applicable.

## Competing interests

The authors declare no competing interests.

## Author contributions

I am the unique author of this paper.
Received: 30 March 2023 Accepted: 7 September 2023 Published online: 15 September 2023

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