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Initial boundary value problem for a viscoelastic wave equation with Balakrishnan–Taylor damping and a delay term: decay estimates and blow-up result

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Abstract

In this paper, we study the initial boundary value problem for the following viscoelastic wave equation with Balakrishnan–Taylor damping and a delay term where the relaxation function satisfies $g'(t) \leq -\xi(t)g^r(t)$, $t \geq 0$, $1 \leq r < \frac{3}{2}$. The main goal of this work is to study the global existence, general decay, and blow-up result. The global existence has been obtained by potential-well theory, the decay of solutions of energy has been established by introducing suitable energy and Lyapunov functionals, and a blow-up result has been obtained with negative initial energy.

Keywords: Wave equation; Balakrishnan–Taylor damping; Delay term; Global existence; Decay estimates; Blow up

1 Introduction

In this paper, we consider the following initial-boundary value problem with a delay term

$$\begin{cases} v_{tt} - (a + b\|\nabla v\|_2^2 + \alpha \int_\Omega \nabla v \nabla v_t \, dx)\Delta v & \\ \quad + \int_0^t g(t-s)\Delta v(s)\, ds + \mu_1 v_t + \mu_2 v_t(t-\tau) = |v|^{p-2}v, & x \in \Omega, t > 0, \\ v(x,t) = 0, & x \in \partial\Omega, t > 0, \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), & x \in \Omega, \\ v_t(x, t-\tau) = f_0(x, t-\tau), & x \in \Omega, t \in [0,\tau), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. $p \geq 4, a, b, \alpha, \mu_1$ are fixed positive constants, μ_2 is a real number, $\tau > 0$ represents the time delay, and g is a positive function.

In the absence of the Balakrishnan–Taylor damping ($\alpha = 0$), Problem (1.1) is reduced to the well-known nonlinear wave equation with $b = g = 0$ and a Kirchhof-type wave equation with $g = 0$, which has been extensively studied, see for instance [5, 8, 13, 24, 30, 31, 35, 38, 41, 42] and the references therein. Balakrishnan–Taylor damping ($\alpha \neq 0$), $g = 0$, and

$\mu_1 = \mu_2 = 0$, was initially proposed by Balakrishnan and Taylor [2], and Bass and Zes [3]. It is related to the panel flutter equation and to the spillover problem. So far, it has been studied by many authors, we refer the interested readers to [12, 15, 32, 39, 43, 44] and the references therein. Zarai and Tatar [44] studied the following problem

$$v_{tt} - \left(a + b\|\nabla v\|_2^2 + \sigma \int_\Omega \nabla v \nabla v_t \, dx\right)\Delta v + \int_0^t h(t-s)\Delta v(s)\, ds = 0. \tag{1.2}$$

They proved the global existence and the polynomial decay of the problem. Exponential decay and blow up of the solution to the problem were established in Tatar and Zarai [39].

It is well known that time-delay effects often appear in many chemical, physical, and economical phenomena because these phenomena depend not only on the present state but also on the past history of the system. Nicaise and Pignotti [33] considered the following wave equation with a delay term

$$v_{tt} - \Delta v + \mu_1 v_t + \mu_2 v_t(t-\tau) = 0. \tag{1.3}$$

They obtained some stability results in the case $0 < \mu_2 < \mu_1$. Then, they extended the result to the time-dependent delay case in the work of Nicaise and Pignotti [34]. Kirane and Said-Houari [23] considered a viscoelastic wave equation with time delay

$$v_{tt} - \Delta v + \int_0^t g(t-s)\Delta v(s)\, ds + \mu_1 v_t + \mu_2 v_t(t-\tau) = 0. \tag{1.4}$$

They proved the global well posedness of solutions and established the decay rate of energy for $0 < \mu_2 < \mu_1$. Kafini et al. [17] investigated the following nonlinear wave equation with delay

$$v_{tt} - \operatorname{div}\left(|\nabla v|^{m-2}\nabla v\right) + \mu_1 v_t + \mu_1 v_t(t-\tau) = b|v|^{p-2}v. \tag{1.5}$$

They proved the blow-up result of solutions with negative initial energy and $p \geq m$, and we refer the interested readers to [9, 10, 18, 27] and the references therein. For the viscoelastic wave equation with Balakrishnan–Taylor damping and time delay, Lee et al. [25] studied the following equation

$$\begin{aligned} v_{tt} &- \left(a + b\|\nabla v\|_2^2 + \sigma \int_\Omega \nabla v \nabla v_t \, dx\right)\Delta v \\ &+ \int_0^t g(t-s)\Delta v(s)\, ds + \mu_0 v_t + \mu_1 v_t(t-\tau) = 0 \end{aligned} \tag{1.6}$$

and established a general energy decay result by suitable Lyapunov functionals. Gheraibia et al. [14] considered the following equation

$$\begin{aligned} v_{tt} &- \left(a + b\|\nabla v\|_2^2 + \alpha \int_\Omega \nabla v \nabla v_t \, dx\right)\Delta v + \sigma(t)\int_0^t g(t-s)\Delta v(s)\, ds + \mu_1 |v_t|^{m-2}v_t \\ &+ \mu_2 \left|v_t(t-\tau)\right|^{m-2} v_t(t-\tau) = 0 \end{aligned} \tag{1.7}$$

and proved the general decay result of the solution in the case $|\mu_2| < \mu_1$. For the related works of PDEs with time delay, see for instance [6, 7, 11, 16, 19–22, 26, 28, 36, 37, 40] and the references therein.

Motivated by the previous work, in this paper, we consider the problem (1.1) and under suitable assumptions on the relaxation functions g, we prove the global existence, general decay and the finite-time blow-up results of the solutions.

The outline of this paper is as follows: In Sect. 2, we give some preliminary results. In Sect. 3, we obtain the global existence of the solution of (1.1). Section 4 and Sect. 5 cover the general decay and blow-up of solutions, respectively.

2 Some preliminaries

In this section, we give some notation for function spaces and preliminary lemmas. Denote by $\|\cdot\|_p$ and $\|\cdot\|_{H^1}$ to the usual $L^p(\Omega)$ norm and $H^1(\Omega)$ norm, respectively.

For the relaxation function g, we assume

(A_1): $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing differentiable function satisfying

$$a - \int_0^\infty g(s)\,ds := l \geq 0. \tag{2.1}$$

(A_2): There exist a nonincreasing differentiable function ξ with $\xi(0) > 0$ satisfying

$$g(t) \geq 0, \qquad g'(t) \leq -\xi(t) g^r(t), \quad t \geq 0, 1 \leq r < \frac{3}{2}. \tag{2.2}$$

(A_3): The constant p satisfies

$$p \geq 4, \quad \text{if } n = 1, 2, \qquad 4 \leq p \leq \frac{2(n-1)}{n-2}, \quad \text{if } n \geq 3. \tag{2.3}$$

(A_4): The constants μ_1 and μ_2 satisfy

$$|\mu_2| < \mu_1.$$

Assume further that g satisfies

$$\int_0^\infty g(s)\,ds < \frac{a(p-2)}{p-2+(1/2\eta)}. \tag{2.4}$$

Lemma 2.1 (*Sobolev–Poincare inequality* [1]). *Let q be a number with $2 \leq q < \infty$ $(n = 1, 2)$ or $2 \leq q < \frac{2n}{n-2}$ $(n \geq 3)$, then, there is a constant $c_* = c_*(\Omega, q)$ such that*

$$\|v\|_q \leq c_* \|\nabla v\|_2 \quad \text{for } v \in H_0^1(\Omega).$$

By using direct calculations, we have

$$\begin{aligned}\int_0^t g(t-s) \int_\Omega v(s)\,ds v_t(t)\,dx = {}& -\frac{1}{2}\frac{d}{dt}\left[(g \circ v)(t) - \|v(t)\|_2^2 \int_0^t g(s)\,ds\right] \\ & -\frac{1}{2} g(t)\|v(t)\|_2^2 + \frac{1}{2}(g' \circ v)(t),\end{aligned} \tag{2.5}$$

where

$$(g \circ v)(t) = \int_0^t g(t-s)\left\| v(t) - v(s) \right\|_2^2 \, ds.$$

To deal with the time-delay term, motivated by Nicaise and Pignotti [33], we introduce a new variable

$$z(x,\rho,t) = v_t(x, t - \tau\rho), \quad x \in \Omega, \rho \in (0,1), t > 0, \tag{2.6}$$

which gives us

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega \times (0,1) \times (0,\infty). \tag{2.7}$$

Then, problem (1.1)is equivalent to

$$\begin{cases} v_{tt} - (a + b\|\nabla v\|_2^2 + \alpha \int_\Omega \nabla v \nabla v_t \, dx)\Delta v & \\ \quad + \int_0^t g(t-s)\Delta v(s) \, ds + \mu_1 v_t + \mu_2 z(1,t) = |v|^{p-2} v, & x \in \Omega, t > 0, \\ \tau z_t(\rho,t) + z_\rho(\rho,t) = 0, & x \in \Omega, \rho \in (0,1), t > 0, \\ z(\rho,0) = f_0(-\tau\rho), & x \in \Omega, \rho \in (0,1), \\ v(x,t) = 0, & x \in \partial\Omega, t > 0, \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), & x \in \Omega. \end{cases} \tag{2.8}$$

Let ζ be a positive constant satisfying

$$\tau|\mu_2| \le \zeta \le \tau\bigl(2\mu_1 - |\mu_2|\bigr). \tag{2.9}$$

We first state a local existence theorem that can be established.

Theorem 2.2 *Let (A_1)–(A_4) hold. Then, for every $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $f_0 \in L^2((\Omega) \times (0,1))$, there exists a unique local solution of the problem (1.1) in the class*

$$v \in C\bigl([0,T]; H_0^1(\Omega)\bigr) \cap C^1\bigl([0,T]; L^2(\Omega)\bigr), \qquad v_t \in C\bigl([0,T]; H_0^1(\Omega)\bigr) \cap L^2\bigl([0,T] \times (\Omega)\bigr).$$

Now, we define the energy associated with problem (2.8) by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}\left(a - \int_0^t g(s)\, ds\right)\|\nabla v\|_2^2 + \frac{b}{4}\|\nabla v\|_2^4 + \frac{1}{2}(g \circ \nabla v)(t) \\ & + \frac{\zeta}{2}\int_0^1 \left\| z(\rho,t) \right\|_2^2 d\rho - \frac{1}{p}\|v\|_p^p. \end{aligned} \tag{2.10}$$

Lemma 2.3 *Let (v,z) be a solution of problem (2.8). Then,*

$$E'(t) \le \frac{1}{2}\bigl(g' \circ \nabla v\bigr)(t) - c_0\bigl(\|v_t\|_2^2 + \left\| z(1,t) \right\|_2^2\bigr). \tag{2.11}$$

Proof Multiplying the first equation in (2.8) by v_t, integrating over Ω, and using (2.5), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|v_t\|_2^2+\frac{1}{2}\left(a-\int_0^t g(s)\,ds\right)\|\nabla v\|_2^2+\frac{b}{4}\|\nabla v\|_2^4+\frac{1}{2}(g\circ\nabla v)(t)-\frac{1}{p}\|v\|_p^p\right]\\
&\quad=-\alpha\left(\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2\right)^2-\frac{1}{2}g(t)\|\nabla v\|_2^2-\frac{1}{2}\left(g'\circ\nabla v\right)(t)\\
&\qquad-\mu_1\|v_t\|_2^2-\mu_2\int_\Omega z(1,t)v_t\,dx. \qquad (2.12)
\end{aligned}$$

Multiplying the second equation in (2.8) by ζz and integrating over $\Omega\times(0,1)$, we obtain

$$\begin{aligned}
\frac{\zeta}{2}\frac{d}{dt}\int_\Omega\int_0^1\left|z(\rho,t)\right|^2 d\rho\,dx&=-\frac{\zeta}{2\tau}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}\left|z(\rho,t)\right|^2 d\rho\,dx\\
&=\frac{\zeta}{2\tau}\left(\|v_t\|_2^2-\left\|z(1,t)\right\|_2^2\right). \qquad (2.13)
\end{aligned}$$

Using Young's inequality, we have

$$-\mu_2\int_\Omega z(1,t)v_t\,dx\le\frac{|\mu_2|}{2}\left\|z(1,t)\right\|_2^2+\frac{|\mu_2|}{2}\|v_t\|_2^2. \qquad (2.14)$$

Combining (2.12), (2.13), and (2.14), we obtain

$$\begin{aligned}
E'(t)&\le-\alpha\left(\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2\right)^2+\frac{1}{2}\left(g'\circ\nabla v\right)(t)-\frac{1}{2}g(t)\|\nabla v\|_2^2\\
&\quad-c_0\left(\|v_t\|_2^2+\left\|z(1,t)\right\|_2^2\right), \qquad (2.15)
\end{aligned}$$

where $c_0=\min\{\mu_1-\frac{\zeta}{2\tau}-\frac{|\mu_2|}{2},\frac{\zeta}{2\tau}-\frac{|\mu_2|}{2}\}$, which is positive by (2.9). The proof is complete. □

Next, we define the functionals

$$\begin{aligned}
I(t)&=\left(a-\int_0^t g(s)\,ds\right)\|\nabla v\|_2^2+\frac{b}{2}\|\nabla v\|_2^4+(g\circ\nabla v)(t)\\
&\quad+\zeta\int_0^1\left\|z(\rho,t)\right\|_2^2 d\rho-\|v\|_p^p \qquad (2.16)
\end{aligned}$$

and

$$\begin{aligned}
J(t)&=\frac{1}{2}\left(a-\int_0^t g(s)\,ds\right)\|\nabla v\|_2^2+\frac{b}{4}\|\nabla v\|_2^4+\frac{1}{2}(g\circ\nabla v)(t)\\
&\quad+\frac{\zeta}{2}\int_0^1\left\|z(\rho,t)\right\|_2^2 d\rho-\frac{1}{p}\|v\|_p^p. \qquad (2.17)
\end{aligned}$$

Then, it is obvious that

$$E(t)=\frac{1}{2}\|v_t\|_2^2+J(t). \qquad (2.18)$$

3 Global existence

In this section, we will prove that the global existence of the solution to (1.1) is in time.

Lemma 3.1 *Assume that* (A_1), (A_3)–(A_4) *hold, and for any* $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *such that*

$$I(0) > 0 \quad and \quad \beta = \frac{c_*^p}{l}\left[\frac{2p}{l(p-2)}E(0)\right]^{\frac{p-2}{2}} < 1, \tag{3.1}$$

then,

$$I(t) > 0, \quad \forall t > 0. \tag{3.2}$$

Proof Since $I(0) > 0$, then by the continuity of v, there exists a time $T_m > 0$ such that

$$I(t) \geq 0, \quad \forall t \in [0, T_m]. \tag{3.3}$$

From (2.16) and (2.17), we have

$$\begin{aligned} J(t) &= \frac{p-2}{2p}\left[\left(a - \int_0^t g(s)\,ds\right)\|\nabla v\|_2^2 + \frac{b}{2}\|\nabla v\|_2^4 + (g \circ \nabla v)(t) + \zeta \int_0^1 \|z(\rho,t)\|_2^2\,d\rho\right] \\ &\quad + \frac{1}{p}I(t) \\ &\geq \frac{p-2}{2p}\left[\left(a - \int_0^t g(s)\,ds\right)\|\nabla v\|_2^2 + \frac{b}{2}\|\nabla v\|_2^4 + (g \circ \nabla v)(t) + \zeta \int_0^1 \|z(\rho,t)\|_2^2\,d\rho\right] \\ &\geq \frac{p-2}{2p}\left[\left(a - \int_0^t g(s)\,ds\right)\|\nabla v\|_2^2\right]. \end{aligned} \tag{3.4}$$

Thus, from (A_1), (2.11), (2.18), and (3.4), we obtain

$$\begin{aligned} l\|\nabla v\|_2^2 &\leq \left(a - \int_0^t g(s)\,ds\right)\|\nabla v\|_2^2 \\ &\leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E(0), \quad \forall t \in [0, T_m]. \end{aligned} \tag{3.5}$$

Exploiting Lemma 2.1, (3.1), and (3.5), we obtain

$$\begin{aligned} \|v\|_p^p &\leq c_*^p\|\nabla v\|_2^p \leq \frac{c_*^p}{l}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{p-2}{2}} l\|\nabla v\|_2^2 \\ &= \beta l\|\nabla v\|_2^2 < \left(a - \int_0^t g(s)\,ds\right)\|\nabla v\|_2^2. \end{aligned} \tag{3.6}$$

Hence, we can obtain

$$I(t) > 0, \quad \forall t \in [0, T_m].$$

By repeating the procedure, T_m is extended to T. The proof is complete. □

Theorem 3.2 *Assume that the conditions of Lemma 3.1 hold, then the solution (1.1) is global and bounded.*

Proof It suffices to show that $\|v_t\|_2^2 + \|\nabla v\|_2^2$ is bounded independently of t. By using (2.11), (2.18), and (3.5), we obtain

$$E(0) \geq E(t) = J(t) + \frac{1}{2}\|v_t\|_2^2 \geq \frac{p-2}{2p}\left(l\|\nabla v\|_2^2\right) + \frac{1}{2}\|v_t\|_2^2. \tag{3.7}$$

Therefore, we have

$$\|v_t\|_2^2 + \|\nabla v\|_2^2 \leq K_1 E(0), \tag{3.8}$$

where K_1 is a positive constant. □

4 General decay

In this section, we prove the general decay result by constructing a suitable Lyapunov functional.

Theorem 4.1 *Let $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Assume that (A_1)–(A_4) hold. Then, there exist two positive constants K and k such that the solution of problem (1.1) satisfies, for all $\forall t \geq t_0$,*

$$E(t) \leq K e^{-k\int_{t_0}^t \xi(s)\,ds}, \quad r = 1, \tag{4.1}$$

$$E(t) \leq K\left[\frac{1}{\int_{t_0}^t \xi^{2r-1}(s)\,ds + 1}\right]^{1/(2r-2)}, \quad r > 1. \tag{4.2}$$

Moreover, if

$$\int_0^{+\infty}\left[\frac{1}{t\xi^{2r-1}(t) + 1}\right]^{1/(2r-2)} dt < +\infty, \quad 1 < r < \frac{3}{2}, \tag{4.3}$$

then

$$E(t) \leq K\left[\frac{1}{\int_{t_0}^t \xi^{r}(s)\,ds + 1}\right]^{1/r-1}, \quad r > 1. \tag{4.4}$$

For this goal, we set

$$F(t) := E(t) + \varepsilon\Psi(t), \tag{4.5}$$

where ε is a positive constant to be specified later and

$$\Psi(t) = \int_\Omega v v_t\,dx + \frac{\alpha}{4}\|\nabla v\|_2^4. \tag{4.6}$$

In order to show our stability result, we need the following lemmas:

Lemma 4.2 *Let (v,z) be a solution of problem (2.8). Then, there exist two positive constants α_1 and α_2 such that*

$$\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t), \tag{4.7}$$

for $\varepsilon > 0$ small enough.

Lemma 4.3 *Assume that g satisfies (A_1) and (A_2), then*

$$\int_0^\infty \xi(t) g^{1-\theta}(t)\,dt \leq +\infty, \quad \forall \theta < 2 - r.$$

Corollary 4.4 ([4]) *Assume that g satisfies (A_1) and (A_2), and v is the solution of (1.1), then*

$$\xi(t)(g \circ \nabla u)(t) \leq \left[-E'(t)\right]^{\frac{1}{2r-1}}.$$

Lemma 4.5 *Let (v,z) be a solution of problem (2.8). Then, the functional $F(t)$ satisfies*

$$F'(t) \leq -k_1 E(t) + k_2 (g \circ \nabla v)(t), \quad \forall t \geq t_0, \tag{4.8}$$

where k_1 and k_2 are some positive constants.

Proof Taking a derivation of (4.5), using (2.8), and Lemma 2.3, we obtain

$$\begin{aligned} F'(t) &= E'(t) + \varepsilon \int_\Omega v_t^2\,dx + \varepsilon \int_\Omega v v_{tt}\,dx + \varepsilon\alpha \|\nabla v\|_2^2 \int_\Omega \nabla v \nabla v_t\,dx \\ &\leq -(c_0 - \varepsilon)\|v_t\|_2^2 - c_0 \left\| z(1,t)\right\|_2^2 - \varepsilon a \|\nabla v\|_2^2 - \epsilon b \|\nabla v\|_2^4 + \varepsilon \|v\|_p^p \\ &\quad + \varepsilon \int_\Omega \nabla v \int_0^t g(t-s)\nabla v(s)\,ds\,dx - \varepsilon\mu_1 \int_\Omega v v_t\,dx - \varepsilon\mu_2 \int_\Omega z(1,t) v\,dx. \end{aligned} \tag{4.9}$$

By using Hölder's, Young's, Sobolev–Poincare inequalities, and (A_1), we obtain

$$\int_\Omega \nabla v \int_0^t g(t-s)\nabla v(s)\,ds\,dx \leq \left(\eta + (a-l)\right)\|\nabla v\|_2^2 + \frac{(a-l)}{4\eta}(g \circ \nabla v)(t) \tag{4.10}$$

and

$$\mu_1 \int_\Omega v v_t\,dx \leq \eta \mu_1^2 c_*^2 \|\nabla v\|_2^2 + \frac{1}{4\eta}\|v_t\|_2^2 \tag{4.11}$$

and

$$\mu_2 \int_\Omega z(1,t) v\,dx \leq \eta \mu_2^2 c_*^2 \|\nabla v\|_2^2 + \frac{1}{4\eta}\left\|z(1,t)\right\|_2^2. \tag{4.12}$$

Combining (4.10)–(4.12) and (4.9), we obtain

$$\begin{aligned} F'(t) &\leq -\left\{c_0 - \varepsilon\left(1 + \frac{1}{4\eta}\right)\right\}\|v_t\|_2^2 - \left\{c_0 - \frac{\varepsilon}{4\eta}\right\}\left\|z(1,t)\right\|_2^2 - \varepsilon b \|\nabla v\|_2^4 \\ &\quad - \varepsilon\left\{l - \eta\left(1 + \mu_1^2 c_*^2 \mu_2^2 c_*^2\right)\right\}\|\nabla v\|_2^2 + \frac{(a-l)}{4\eta}(g \circ \nabla v)(t) + \varepsilon \|v\|_p^p. \end{aligned} \tag{4.13}$$

At this point, we choose η and ε so small that (4.7) remains valid and

$$l-\eta\left(1+\mu_1^2c_*^2\mu_2^2c_*^2\right)>0,\qquad c_0-\varepsilon\left(1+\frac{1}{4\eta}\right)>0,\qquad c_0-\frac{\varepsilon}{4\eta}>0.$$

Consequently, inequality (4.13) becomes

$$F'(t)\le -k_1E(t)+k_2(g\circ\nabla v)(t),\quad \forall t\ge t_0, \tag{4.14}$$

where k_i, $i=1,2.$ are some positive constants. □

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Multiplying (4.14) by $\xi(t)$, we obtain

$$\xi(t)F'(t)\le -k_1\xi(t)E(t)+k_2\xi(t)(g\circ\nabla u)(t),\quad \forall t\ge t_0. \tag{4.15}$$

4.1 Case: $r=1$

Using (A_2) and (2.11), then inequality (4.14) becomes

$$\begin{aligned}\xi(t)F'(t)&\le -k_1\xi(t)E(t)+k_2\xi(t)(g\circ\nabla v)(t)\\&\le -k_1\xi(t)E(t)-k_2\left(g'\circ\nabla v\right)(t)\\&\le -k_1\xi(t)E(t)-2k_2E'(t).\end{aligned} \tag{4.16}$$

We choose $G(t)=\xi(t)F(t)+2k_2E(t)$ that is equivalent to $E(t)$ because of (4.7). Then, from (4.16) we can obtain

$$G'(t)\le -k_0\xi(t)E(t)\le -k\xi(t)G(t),\quad \forall t\ge t_0. \tag{4.17}$$

A simple integration of (4.17), leads to

$$G(t)\le G(t_0)e^{-k\int_{t_0}^t\xi(s)\,ds},\quad \forall t\ge t_0, \tag{4.18}$$

which implies

$$E(t)\le Ke^{-k\int_{t_0}^t\xi(s)\,ds},\quad \forall t\ge t_0. \tag{4.19}$$

4.2 Case: $r>1$

Applying Corollary 4.4, then inequality (4.15) becomes

$$\xi(t)F'(t)\le -k_1\xi(t)E(t)+k_2\left[-E'(t)\right]^{1/(2r-1)},\quad \forall t\ge t_0. \tag{4.20}$$

Multiplying (4.20) by $\xi^\nu(t)E^\nu(t)$ where $\nu=2r-2$, we have

$$\begin{aligned}&\xi^{\nu+1}(t)E^\nu(t)F'(t)\\&\quad\le -k_1\xi^{\nu+1}(t)E^{\nu+1}(t)+k_2\xi^\nu(t)E^\nu(t)\left[-E'(t)\right]^{1/(\nu+1)},\quad \forall t\ge t_0.\end{aligned} \tag{4.21}$$

Using Young's inequality with $q = \nu + 1$ and $q^* = \frac{\nu+1}{\nu}$, yields

$$\begin{aligned} &\xi^{\nu+1}(t)E^{\nu}(t)F'(t) \\ &\quad \leq -k_1\xi^{\nu+1}(t)E^{\nu+1}(t) + k_2\big[\eta\xi^{\nu+1}(t)E^{\nu+1}(t) - C_\eta E'(t)\big] \\ &\quad = -(k_1 - \eta k_2)\xi^{\nu+1}(t)E^{\nu+1}(t) - C_\eta E'(t), \quad \forall t \geq t_0. \end{aligned} \tag{4.22}$$

At this point, we choose $\eta < \frac{k_1}{k_2}$ and recall that $\xi'(t) \leq 0$ and $E'(t) \leq 0$, we obtain

$$\begin{aligned} \big(\xi^{\nu+1}E^{\nu}F\big)'(t) &\leq \xi^{\nu+1}(t)E^{\nu}(t)F'(t) \\ &\leq -k_3\xi^{\nu+1}(t)E^{\nu+1}(t) - k_4E'(t), \quad \forall t \geq t_0, \end{aligned}$$

which implies

$$\big(\xi^{\nu+1}E^{\nu}F + k_4F\big)'(t) \leq -k_3\xi^{\nu+1}(t)E^{\nu+1}(t), \quad \forall t \geq t_0. \tag{4.23}$$

We choose $G(t) = \xi^{\nu+1}(t)E^{\nu}(t)F(t) + k_4E(t)$ that is equivalent to $E(t)$. Then,

$$\begin{aligned} G'(t) &\leq -k_3\xi^{\nu+1}(t)G^{\nu+1}(t) \\ &= -k_3\xi^{2r-1}(t)G^{2r-1}(t), \quad \forall t \geq t_0. \end{aligned} \tag{4.24}$$

A simple integration of (4.24) and using the fact that $G(t) \sim E(t)$, leads to

$$E(t) \leq K\left[\frac{1}{\int_{t_0}^{t}\xi^{2r-1}(s)\,ds + 1}\right]^{1/(2r-2)}, \quad \forall t \geq t_0. \tag{4.25}$$

4.3 Case: $1 < r < 3/2$

To establish (4.4), we note that from simple calculations show that (4.2) and (4.3) yield

$$\int_{t_0}^{\infty} E(t) < \infty.$$

Next, let

$$\sigma(t) = \int_0^t \big\|\nabla v(t) - \nabla v(t-s)\big\|_2^2\,ds,$$

then, we have

$$\begin{aligned} \sigma(t) &\leq c\int_0^t \big[\big\|\nabla v(t)\big\|_2^2 + \big\|\nabla v(t-s)\big\|_2^2\big]\,ds \leq c\int_0^t \big[E(t) + E(t-s)\big]\,ds \leq 2c\int_0^t E(t-s)\,ds \\ &= 2c\int_0^t E(s)\,ds \leq 2c\int_0^{\infty} E(s)\,ds < \infty. \end{aligned}$$

Applying Jensens's inequality for the second term on the right-hand side of (4.15) and using (A_2), we obtain

$$\begin{aligned}
\xi(t)F'(t) &\le -k_1\xi(t)E(t) + k_2\xi(t)(g\circ\nabla v)(t) \\
&= -k_1\xi(t)E(t) + k_2\frac{\sigma(t)}{\sigma(t)}\int_0^t \left[\xi^r(s)g^r(s)\right]^{\frac{1}{r}}\left\|\nabla v(t)-\nabla v(t-s)\right\|_2^2\,ds \\
&\le -k_1\xi(t)E(t) + k_2\sigma(t)\left[\frac{1}{\sigma(t)}\int_0^t \xi^r(s)g^r(s)\left\|\nabla v(t)-\nabla v(t-s)\right\|_2^2\,ds\right]^{\frac{1}{r}} \\
&\le -k_1\xi(t)E(t) + k_2\sigma^{\frac{r-1}{r}}(t)\xi^{r-1}(0)\left[\int_0^t \xi(s)g^r(s)\left\|\nabla v(t)-\nabla v(t-s)\right\|_2^2\,ds\right]^{\frac{1}{r}} \\
&\le -k_1\xi(t)E(t) + k_2\left[\int_0^t -g'(s)\left\|\nabla v(t)-\nabla v(t-s)\right\|_2^2\,ds\right]^{\frac{1}{r}} \\
&\le -k_1\xi(t)E(t) + k_2\left[-E'(t)\right]^{\frac{1}{r}}.
\end{aligned} \tag{4.26}$$

Multiplying (4.26) by $\xi^{\nu}(t)E^{\nu}(t)$, where $\nu = r-1$, we have

$$\xi^{\nu+1}(t)E^{\nu}(t)F'(t) \le -k_1\xi^{\nu+1}(t)E^{\nu+1}(t) + k_2\xi^{\nu}(t)E^{\nu}(t)\left[-E'(t)\right]^{\frac{1}{\nu+1}}, \quad \forall t\ge t_0. \tag{4.27}$$

The remainder of the proof is similar to (4.2). The proof is complete.

5 Blow up

In this section, we state and prove the blow up of the solution to problem (1.1) with negative initial energy.

Let

$$H(t) = -E(t), \tag{5.1}$$

where $E(0) < 0$. From (5.1) and (2.11) we have

$$H'(t) = -E'(t) \ge c_0\left(\|v_t\|_2^2 + \left\|z(1,t)\right\|_2^2\right) \ge 0 \tag{5.2}$$

and $H(t)$ is an increasing function. Using (2.10) and (5.1), we obtain

$$0 < H(0) \le H(t) \le \frac{1}{p}\|v\|_p^p. \tag{5.3}$$

Moreover, similar to the work of Messaoudi [29], we can obtain the following lemma that is needed later.

Lemma 5.1 *Suppose that* (A_1), (A_3), (A_4), (2.4), *and* $E(0) < 0$ *hold. Then, we have, for any* $2 \le s \le p$,

$$\|v\|_p^s \le C\left(-H(t) - \|v_t\|_2^2 - \|\nabla v\|_2^4 - (g\circ\nabla v)(t) - \int_0^1 \left\|z(\rho,t)\right\|_2^2\,d\rho + \|v\|_p^p\right),$$

where C is a positive constant.

Theorem 5.2 *Let the conditions of Lemma 5.1 hold. Then, the solution of problem (1.1) blows up in finite time.*

Proof Set

$$\Gamma(t) = H^{1-\sigma}(t) + \varepsilon \int_\Omega v v_t \, dx + \frac{\alpha}{4}\|\nabla v\|_2^4, \tag{5.4}$$

where $\varepsilon > 0$ is a small constant that will be chosen later, and

$$0 < \sigma \leq \min\left\{\frac{p-2}{2p}, \frac{p-2}{p}\right\}. \tag{5.5}$$

Taking a derivative of (5.4) and using the first equation in (2.8), we have

$$\begin{aligned}\Gamma'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon \int_\Omega v_t^2 \, dx + \varepsilon \int_\Omega v v_{tt} \, dx + \alpha \|\nabla u\|_2^2 \int_\Omega \nabla u \nabla u_t \, dx \\ &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\|v_t\|_2^2 - \varepsilon a\|\nabla v\|_2^2 - \epsilon b\|\nabla v\|_2^4 + \varepsilon\|v\|_p^p \\ &\quad + \varepsilon \int_\Omega \nabla v \int_0^t g(t-s)\nabla v(s) \, ds \, dx - \varepsilon\mu_1 \int_\Omega v v_t \, dx - \varepsilon\mu_2 \int_\Omega z(1,t) v \, dx.\end{aligned} \tag{5.6}$$

Applying Hölder's and Young's inequalities, for $\eta, \delta > 0$, we have

$$\int_\Omega \nabla v \int_0^t g(t-s)\nabla v(s) \, ds \, dx \geq \left(1 - \frac{1}{4\eta}\right)\left(\int_0^t g(s) \, ds\right)\|\nabla v\|_2^2 - \eta(g \circ \nabla v)(t), \tag{5.7}$$

$$\mu_1 \int_\Omega v v_t \, dx \leq \delta\mu_1^2\|v\|_2^2 + \frac{1}{4\delta}\|v_t\|_2^2 \leq \delta\mu_1^2\|v\|_2^2 + \frac{1}{4c_0\delta}H'(t) \tag{5.8}$$

and

$$\mu_2 \int_\Omega z(1,t) v \, dx \leq \delta\mu_2^2\|v\|_2^2 + \frac{1}{4\delta}\left\|z(1,t)\right\|_2^2 \leq \delta\mu_2^2\|v\|_2^2 + \frac{1}{4c_0\delta}H'(t). \tag{5.9}$$

Combining these estimates (5.7)–(5.9) and (5.6), we obtain

$$\begin{aligned}\Gamma'(t) &\geq \left\{(1-\sigma)H^{-\sigma}(t) - \frac{\varepsilon}{2c_0\delta}\right\}H'(t) + \varepsilon\|v_t\|_2^2 - \epsilon b\|\nabla v\|_2^4 + \varepsilon\|v\|_p^p \\ &\quad - \varepsilon\left\{a - \left(1 - \frac{1}{4\eta}\right)\left(\int_0^t g(s) \, ds\right)\right\}\|\nabla v\|_2^2 - \varepsilon\delta\left(\mu_1^2 + \mu_2^2\right)\|v\|_2^2 \\ &\quad - \varepsilon\eta(g \circ \nabla v)(t).\end{aligned} \tag{5.10}$$

Applying (2.10) to the last term $\|v\|_p^p$ on the right-hand side of (5.10) and using (5.1), we see that

$$\begin{aligned}\Gamma'(t) &\geq \left\{(1-\sigma)H^{-\sigma}(t) - \frac{\varepsilon}{2c_0\delta}\right\}H'(t) + \varepsilon\left(\frac{p}{2} + 1\right)\|v_t\|_2^2 + \varepsilon b\left(\frac{p}{4} - 1\right)\|\nabla v\|_2^4 \\ &\quad + \varepsilon\left\{a\left(\frac{p}{2} - 1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\eta}\right)\int_0^t g(s) \, ds\right\}\|\nabla v\|_2^2 + \varepsilon\left(\frac{p}{2} - \eta\right)(g \circ \nabla v)(t)\end{aligned}$$

$$-\varepsilon\delta\big(\mu_1^2+\mu_2^2\big)\|v\|_2^2+\varepsilon\frac{p\zeta}{2}\int_0^1\big\|z(\rho,t)\big\|_2^2\,d\rho+\varepsilon pH(t), \tag{5.11}$$

for some number η with $0<\eta<p/2$. By recalling (2.4), the estimate (5.11) reduces to

$$\begin{aligned}\Gamma'(t)\ge{}&\left\{(1-\sigma)H^{-\sigma}(t)-\frac{\varepsilon}{2c_0\delta}\right\}H'(t)+\varepsilon\left(\frac{p}{2}+1\right)\|v_t\|_2^2+\varepsilon c_1\|\nabla v\|_2^4\\&+\varepsilon c_2\|\nabla v\|_2^2+\varepsilon c_3(g\circ\nabla v)(t)-\varepsilon\delta\big(\mu_1^2+\mu_2^2\big)\|v\|_2^2\\&+\varepsilon\frac{p\zeta}{2}\int_0^1\big\|z(\rho,t)\big\|_2^2\,d\rho+\varepsilon pH(t),\end{aligned} \tag{5.12}$$

where

$$c_1=b\left(\frac{p}{4}-1\right)>0,\quad c_2=a\left(\frac{p}{2}-1\right)-\left(\frac{p}{2}-1+\frac{1}{4\eta}\right)\int_0^t g(s)\,ds>0,\quad c_3=\frac{p}{2}-\eta>0.$$

Therefore, by taking $\delta=H(t)^\sigma/2c_0k$, where $k>0$ is to be specified later, and exploiting (5.3), we se that

$$H(t)^\sigma\|v\|_2^2\le\frac{1}{p^\sigma}\|v\|_p^{\sigma p}\|v\|_2^2\le\frac{c_p^2}{p^\sigma}\|v\|_p^{\sigma p+2}. \tag{5.13}$$

Substituting (5.13) into (5.12), we obtain

$$\begin{aligned}\Gamma'(t)\ge{}&\big\{(1-\sigma)-\varepsilon k\big\}H^{-\sigma}(t)H'(t)+\varepsilon\left(\frac{p}{2}+1\right)\|v_t\|_2^2+\varepsilon c_1\|\nabla v\|_2^4\\&+\varepsilon c_2\|\nabla v\|_2^2+\varepsilon c_3(g\circ\nabla v)(t)+\varepsilon c_4\|v\|_p^p-\varepsilon\frac{c_5}{k}\|v\|_p^{\sigma p+2}\\&+\varepsilon\frac{p\zeta}{2}\int_0^1\big\|z(\rho,t)\big\|_2^2\,d\rho+\varepsilon pH(t),\end{aligned} \tag{5.14}$$

where $c_5=(c_p^2(\mu_1^2+\mu_2^2))/2c_0p^\sigma$. From (5.5) and Lemma 5.1, for $s=\sigma p+2\le p$, we deduce

$$\|v\|_p^{\sigma p+2}\le C\left(-H(t)-\|v_t\|_2^2-\|\nabla v\|_2^4-(g\circ\nabla v)(t)-\int_0^1\big\|z(\rho,t)\big\|_2^2\,d\rho+\|v\|_p^p\right). \tag{5.15}$$

Combining (5.15) with (5.14), we obtain

$$\begin{aligned}\Gamma'(t)\ge{}&\big\{(1-\sigma)-\varepsilon k\big\}H^{-\sigma}(t)H'(t)+\varepsilon\left(\frac{p}{2}+1+\frac{c_5}{k}C\right)\|v_t\|_2^2+\varepsilon c_2\|\nabla v\|_2^2\\&+\varepsilon\left(c_1+\frac{c_5}{k}C\right)\|\nabla v\|_2^4+\varepsilon\left(c_3+\frac{c_5}{k}C\right)(g\circ\nabla v)(t)-\frac{c_5}{k}C\|v\|_p^p\\&+\varepsilon\left(\frac{p\zeta}{2}+\frac{c_5}{k}C\right)\int_0^1\big\|z(\rho,t)\big\|_2^2\,d\rho+\varepsilon\left(p+\frac{c_5}{k}C\right)H(t).\end{aligned} \tag{5.16}$$

Subtracting and adding $\varepsilon\gamma H(t)$ on the right-hand side of (5.16), using (2.10) and (5.1), we deduce

$$\Gamma'(t)\ge\big\{(1-\sigma)-\varepsilon k\big\}H^{-\sigma}(t)H'(t)+\varepsilon\left(\frac{p}{2}-\frac{\gamma}{2}+1+\frac{c_5}{k}C\right)\|v_t\|_2^2$$

$$
\begin{aligned}
&+\varepsilon\left(c_2-a\frac{\gamma}{2}\right)\|\nabla v\|_2^2+\varepsilon\left(c_1-b\frac{\gamma}{4}+\frac{c_5}{k}C\right)\|\nabla v\|_2^4\\
&+\varepsilon\left(c_3-\frac{\gamma}{2}+\frac{c_5}{k}C\right)(g\circ\nabla v)(t)+\left(\frac{\gamma}{p}-\frac{c_5}{k}C\right)\|v\|_p^p\\
&+\varepsilon\left(\frac{p\zeta}{2}-\frac{\gamma\zeta}{2}+\frac{c_5}{k}C\right)\int_0^1\left\|z(\rho,t)\right\|_2^2\,d\rho\\
&+\varepsilon\left(p-\gamma+\frac{c_5}{k}C\right)H(t)+\varepsilon\gamma E_1.
\end{aligned}
\tag{5.17}
$$

First, we fix γ such that

$$
0<\gamma<\min\left\{p,\frac{2c_2}{a},\frac{4c_1}{b},2c_3,\right\}.
$$

Secondly, we take k large enough such that

$$
\frac{\gamma}{p}-\frac{c_5}{k}C>0.
$$

Once k is fixed, we select $\varepsilon>0$ small enough so that

$$
(1-\sigma)-\varepsilon k>0,\quad \text{and}\quad \Gamma(0)=H^{1-\sigma}(0)+\varepsilon\int_\Omega v_0v_1\,dx+\frac{\alpha}{4}\|\nabla v_0\|_2^4>0.
$$

Therefore, we obtain from (5.17) that

$$
\begin{aligned}
\Gamma'(t)\ge\omega\biggl(&\|v_t\|_2^2+\|\nabla v\|_2^2+\|\nabla v\|_2^4+(g\circ\nabla v)(t)\\
&+\int_0^1\left\|z(\rho,t)\right\|_2^2\,d\rho+\|v\|_p^p+H(t)\biggr),
\end{aligned}
\tag{5.18}
$$

where ω is a positive constant.

We now estimate $\Gamma(t)^{\frac{1}{1-\sigma}}$. By Hölder's inequality, we have

$$
\left|\int_\Omega vv_t\,dx\right|\le\|v\|_2\|v_t\|_2\le C_1\|v\|_p\|v_t\|_2,
\tag{5.19}
$$

which implies

$$
\left|\int_\Omega vv_t\,dx\right|^{\frac{1}{1-\sigma}}\le C_1\|v\|_p^{\frac{1}{1-\sigma}}\|v_t\|_2^{\frac{1}{1-\sigma}}.
\tag{5.20}
$$

Young's inequality yields

$$
\left|\int_\Omega vv_t\,dx\right|^{\frac{1}{1-\sigma}}C_1\left(\|v\|_p^{\frac{\mu}{1-\sigma}}+\|v_t\|_2^{\frac{\vartheta}{1-\sigma}}\right),
\tag{5.21}
$$

for $\frac{1}{\mu}+\frac{1}{\vartheta}=1$. To obtain $\frac{\mu}{1-\sigma}=\frac{2}{1-2\sigma}\le p$, by (5.5), we take $\vartheta=2(1-\sigma)$. Therefore, (5.21) becomes

$$
\left|\int_\Omega vv_t\,dx\right|^{\frac{1}{1-\sigma}}C_1\left(\|v\|_p^s+\|v_t\|_2^2\right),
$$

where $s = \frac{2}{1-2\sigma}$. Using Lemma 5.1, we obtain

$$\left|\int_\Omega v v_t \, dx\right|^{\frac{1}{1-\sigma}} \le C_1\biggl(H(t) + \|v_t\|_2^2 + \|\nabla v\|_2^4 + (g \circ \nabla v)(t) + \int_0^1 \bigl\|z(\rho,t)\bigr\|_2^2 \, d\rho + \|v\|_p^p\biggr). \tag{5.22}$$

Combining (5.4) and (5.22), we obtain

$$\begin{aligned}\Gamma^{\frac{1}{1-\sigma}}(t) &= \biggl(H^{1-\sigma}(t) + \varepsilon \int_\Omega v v_t \, dx + \frac{\alpha}{4}\|\nabla v\|_2^4\biggr)^{\frac{1}{1-\sigma}} \\ &\le c_6\biggl(H(t) + \|v_t\|_2^2 + \|\nabla v\|_2^4 + (g \circ \nabla v)(t) \\ &\quad + \int_0^1 \bigl\|z(\rho,t)\bigr\|_2^2 \, d\rho + \|v\|_p^p + \|\nabla v\|_2^{\frac{4}{1-\sigma}}\biggr).\end{aligned} \tag{5.23}$$

We note from (3.8) and (5.3) that

$$\|\nabla v\|_2^{\frac{4}{1-\sigma}} \le \bigl(K_1 E(0)\bigr)^{\frac{2}{1-\sigma}} \le \bigl(K_1 E(0)\bigr)^{\frac{2}{1-\sigma}} \frac{H(t)}{H(0)}. \tag{5.24}$$

It follows from (5.23) and (5.24) that

$$\Gamma^{\frac{1}{1-\sigma}}(t) \le c_7\biggl(H(t) + \|v_t\|_2^2 + \|\nabla v\|_2^4 + (g \circ \nabla v)(t) + \int_0^1 \bigl\|z(\rho,t)\bigr\|_2^2 \, d\rho + \|v\|_p^p\biggr). \tag{5.25}$$

Combining (5.25) with (5.18), we find that

$$\Gamma'(t) \ge \kappa \Gamma^{\frac{1}{1-\sigma}}(t), \quad t \ge 0. \tag{5.26}$$

A simple integration of (5.26) over $(0,t)$ yields

$$\Gamma^{\frac{\sigma}{1-\sigma}}(t) \ge \frac{1}{\Gamma^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\kappa\sigma t}{1-\sigma}}.$$

Consequently, the solution of problem (1.1) blows up in finite time T^* and $T^* \le \frac{1-\sigma}{\kappa\sigma\Gamma^{\frac{\sigma}{1-\sigma}}(0)}$. □

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