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Remarks on a fractional nonlinear partial integro-differential equation via the new generalized multivariate Mittag-Leffler function

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Abstract
Introducing a new generalized multivariate Mittag-Leffler function which is a generalization of the multivariate Mittag-Leffler function, we derive a sufficient condition for the uniqueness of solutions to a brand new boundary value problem of the fractional nonlinear partial integro-differential equation using Banach's fixed point theorem and Babenko's technique. This has many potential applications since uniqueness is an important topic in many scientific areas, and the method used clearly opens directions for studying other types of equations and corresponding initial or boundary value problems. In addition, we use Python which is a high-level programming language efficiently dealing with the summation of multi-indices to compute approximate values of the generalized Mittag-Leffler function (it seems impossible to do so by any existing integral representation of the Mittag-Leffler function), and provide an example showing applications of key results derived.

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1 Introduction

Partial differential equations have played an important role in various scientific areas, such as physics and engineering [1–8]. There are many interesting studies on uniqueness and existence of solutions, based on the theory of fixed points, for fractional nonlinear PDEs and corresponding initial or boundary value problems, as well as for integral equations [9, 10]. Ouyang and Zhu et al. [11–13] studied the time fractional PDEs given below:

$$\begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t,x) - a(t)\frac{\partial^2}{\partial x^2} u(t,x) = v(t, u(\tau_1(t),x), \dots, u(\tau_l(t),x)), & t \in [0, T_0], \\ u(t,x) = 0, \quad (t,x) \in [0,T_0] \times \partial\Omega, \\ u(0,x) = \psi(x), \quad x \in \Omega, \end{cases}$$

where $0 < \alpha \leq 1$, the function $a(t)$ is a diffusion coefficient, l is a positive integer, $\Omega \subset R^l$ is a bounded domain with a smooth boundary $\partial\Omega$, $\psi \in L^2(\Omega)$, and the function $\nu : [0, T_0] \times R^l \to R$ satisfies certain conditions. Ouyang [11] investigated the existence of the local solutions using Leray–Schauder's fixed point theorem. Additionally, Zhu et al. [12, 13] converted the above time fractional partial differential equations into a form of the time fractional differential equations in the Banach space $L^2(\Omega)$, and, using Banach's fixed point theorem and strict contraction principle, derived results on the existence and uniqueness.

Let $a(x) \in C[0, T]$, $g : [0, T] \times R \to R$, and $f : C[0, T] \to R$. Very recently, Li [14] studied the uniqueness of solutions for the following nonlinear integro-differential equation with a nonlocal boundary condition and variable coefficients for $l < \alpha \leq l + 1$:

$$\begin{cases} {}_CD^\alpha u(x) + a(x)I^\beta u(x) = g(x, u(x)), & x \in [0, T], \\ u(0) = -f(u), \qquad u''(0) = \cdots = u^{(l)}(0) = 0, \\ \int_0^T u(x)\,dx = \lambda, \end{cases} \tag{1.1}$$

where λ is a constant. In particular for $l = 1$, equation (1.1) turns out to be

$$\begin{cases} {}_CD^\alpha u(x) + a(x)I^\beta u(x) = g(x, u(x)), & x \in [0, T], \\ u(0) = -f(u), \qquad \int_0^T u(x)\,dx = \lambda. \end{cases}$$

This paper aims to study the uniqueness of solutions for the following new equation with $0 < \alpha \leq 1$ and $m = 1, 2, \ldots$, in the space $S([0, 1]^2)$:

$$\begin{cases} \frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t, x) + \sum_{i=1}^m \lambda_i I_t^{\gamma_i} I_x^{\beta_i} u(t, x) = \nu(t, x, u(t, x)), \quad \gamma_i \geq 0, \beta_i \geq 0, \\ u(0, x) + u(1, x) - \psi(x) = 0, \quad (t, x) \in [0, 1]^2, \end{cases} \tag{1.2}$$

where all λ_i are constants, $\psi(x)$ is a continuous function on $[0, 1]$, and $\nu : [0, 1]^2 \times R \to R$ is a function which satisfies conditions to be given. Equation (1.2) with its initial condition is new and, to the best of our knowledge, has never been investigated earlier.

The remainder of the paper is organized in the following manor. Section 3 studies the uniqueness of solutions for equation (1.2) by the newly introduced generalized multivariate Mittag-Leffler function and Banach's fixed point theory. Section 4 presents a demonstrative example which illuminates applications of the key results based on the value of a generalized multivariate Mittag-Leffler function calculated by our Python code. Finally, in Sect. 5, we provide a summary of the work.

2 Preliminaries

We define I_t^α as the partial Riemann–Liouville fractional integral of order $\alpha > 0$ [15, 16] given by

$$\big(I_t^\alpha u\big)(t, x) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} u(s, x)\,ds,$$

and $\frac{{}_c\partial^\alpha}{\partial t^\alpha}$ as the partial Liouville–Caputo fractional derivative of order $\alpha > 0$ [15] by

$$\left(\frac{{}_c\partial^\alpha}{\partial t^\alpha} u\right)(t, x) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} u_s'(s, x)\,ds, \quad 0 < \alpha \leq 1.$$

From [17, 18], we have for $0 < \alpha \le 1$,

$$\begin{aligned}&\left(I_t^0 u\right)(t,x) = u(t,x),\\&I_t^\alpha\left(\frac{{}_c\partial^\alpha}{\partial t^\alpha}u\right)(t,x) = u(t,x) - u(0,x).\end{aligned}$$

The set $S([0,1]^2)$ is a Banach space equipped with the following norm:

$$\|u\| = \sup_{t\in[0,1],x\in[0,1]} \left|u(t,x)\right| \quad \text{for } u \in S\left([0,1]^2\right),$$

where u is continuous on $[0,1]^2$.

Definition 1 *A generalized multivariate Mittag-Leffler function is defined by the following series*:

$$\begin{aligned}&E_{(\alpha_1,\dots,\alpha_m),\epsilon}^{(\beta_1,\dots,\beta_m),\delta}(\zeta_1,\dots,\zeta_m)\\&\quad = \sum_{l=0}^{\infty}\sum_{\substack{l_1+\cdots+l_m=l\\ l_1\ge 0,\dots,l_m\ge 0}}\binom{l}{l_1,\dots,l_m}\frac{\zeta_1^{l_1}\cdots\zeta_m^{l_m}}{\Gamma(\alpha_1 l_1+\cdots+\alpha_m l_m+\epsilon)\Gamma(\beta_1 l_1+\cdots+\beta_m l_m+\delta)},\end{aligned}$$

where $\alpha_i, \epsilon, \delta > 0, \beta_i \ge 0, \zeta_j \in \mathcal{C}$ for $1 \le j \le m$ and

$$\binom{l}{l_1,\dots,l_m} = \frac{l!}{l_1!\cdots l_m!}.$$

In particular,

$$\begin{aligned}E_{(\alpha_1,\dots,\alpha_m),\epsilon}^{(0,\dots,0),1}(\zeta_1,\dots,\zeta_m) &= E_{(\alpha_1,\dots,\alpha_m),\epsilon}(\zeta_1,\dots,\zeta_m)\\&= \sum_{l=0}^{\infty}\sum_{\substack{l_1+\cdots+l_m=l\\ l_1\ge 0,\dots,l_m\ge 0}}\binom{l}{l_1,\dots,l_m}\frac{\zeta_1^{l_1}\cdots\zeta_m^{l_m}}{\Gamma(\alpha_1 l_1+\cdots+\alpha_m l_m+\epsilon)},\end{aligned}$$

which is the multivariate Mittag-Leffler function given in [19] *since* $\Gamma(1) = 1$. *Moreover,*

$$E_{\alpha,\epsilon}^{0,1}(\zeta) = E_{\alpha,\epsilon}(\zeta) = \sum_{l=0}^{\infty}\frac{\zeta^l}{\Gamma(\alpha l+\epsilon)}, \quad \zeta \in \mathcal{C},$$

which is the well-known two-parameter Mittag-Leffler function.

Babenko's approach [20] is a highly effective method that can be employed to solve various integral and differential equations [9, 17] by treating a bounded integral operator as a "normal" variable and using the inverse operator to deduce solutions. The method itself is similar to the Laplace transform while working on differential and integral equations with constant coefficients, but it can be applied to equations with continuous and bounded variable coefficients. To show this approach, we will consider the following equation in the space $C[0,1]$ (the space of all continuous functions on $[0,1]$) for constants a and b:

$$\begin{cases} {}_cD_0^{\beta}u(t) + a\,{}_cD_0^{\beta_1}u(t) + bI_0^{\alpha}u(t) = t^2, \quad 0 < \beta_1 < \beta \leq 1, \alpha > 0, \\ u(0) = 0, \end{cases} \tag{2.1}$$

where

$${}_cD_0^{\beta}u(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} u'(s)\,ds$$

and

$$I_0^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,ds.$$

Obviously,

$$I_0^{\beta}\big({}_cD_0^{\beta}u(t)\big) = u(t) - u(0) = u(t).$$

Applying I_0^{α} to equation (2.1), we have

$$\big(1 + aI_0^{\beta-\beta_1} + bI_0^{\beta+\alpha}\big)u(t) = I_0^{\beta}t^2 = \frac{2}{\Gamma(\beta+3)}t^{\beta+2}.$$

Considering the inverse operator of $(1 + aI_0^{\beta-\beta_1} + bI_0^{\beta+\alpha})$, we informally get by Babenko's technique

$$\begin{aligned} u(t) &= \frac{2}{\Gamma(\beta+3)}\big(1 + aI_0^{\beta-\beta_1} + bI_0^{\beta+\alpha}\big)^{-1}t^{\beta+2} \\ &= \frac{2}{\Gamma(\beta+3)}\sum_{n=0}^{\infty}(-1)^n\big(aI_0^{\beta-\beta_1} + bI_0^{\beta+\alpha}\big)^n t^{\beta+2} \\ &= \frac{2}{\Gamma(\beta+3)}\sum_{n=0}^{\infty}(-1)^n \sum_{n_1+n_2=n}\binom{n}{n_1, n_2} a^{n_1} I_0^{(\beta-\beta_1)n_1} b^{n_2} I_0^{(\beta+\alpha)n_2} t^{\beta+2} \\ &= 2\sum_{n=0}^{\infty}(-1)^n \sum_{n_1+n_2=n}\binom{n}{n_1, n_2} a^{n_1} b^{n_2} \frac{t^{n_1(\beta-\beta_1)+n_2(\beta+\alpha)+\beta+2}}{\Gamma((\beta-\beta_1)n_1 + (\beta+\alpha)n_2 + \beta + 3)} \\ &= 2t^{\beta+2}\sum_{n=0}^{\infty}(-1)^n \sum_{n_1+n_2=n}\binom{n}{n_1, n_2} a^{n_1} b^{n_2} \frac{t^{(\beta-\beta_1)n_1+(\beta+\alpha)n_2}}{\Gamma((\beta-\beta_1)n_1 + (\beta+\alpha)n_2 + \beta + 3)}, \end{aligned}$$

using

$$I_0^{(\beta-\beta_1)n_1+(\beta+\alpha)n_2} t^{\beta+2} = \frac{\Gamma(\beta+3)t^{(\beta-\beta_1)n_1+(\beta+\alpha)n_2+\beta+2}}{\Gamma((\beta-\beta_1)n_1 + (\beta+\alpha)n_2 + \beta + 3)}.$$

This implies that

$$\|u\| \leq 2\sum_{n=0}^{\infty} \sum_{n_1+n_2=n}\binom{n}{n_1, n_2} \frac{|a|^{n_1}|b|^{n_2}}{\Gamma((\beta-\beta_1)n_1 + (\beta+\alpha)n_2 + \beta + 3)}$$

$$= 2E_{(\beta-\beta_1,\beta+\alpha),\beta+3}\big(|a|,|b|\big) < +\infty,$$

which gives that the series solution

$$u(t) = 2t^{\beta+2}\sum_{n=0}^{\infty}(-1)^n\sum_{n_1+n_2=n}\binom{n}{n_1,n_2}a^{n_1}b^{n_2}\frac{t^{(\beta-\beta_1)n_1+(\beta+\alpha)n_2}}{\Gamma((\beta-\beta_1)n_1+(\beta+\alpha)n_2+\beta+3)}$$

is an element in $C[0,1]$.

3 Uniqueness of solutions

Theorem 2 *Let $\psi \in C[0,1]$, λ_i be real constants, $\beta_i,\gamma_i \ge 0$ for all $i = 1,2,\dots,m$, and $\nu : [0,1]^2 \times R \to R$ be a continuous and bounded function. In addition, we assume that $0 < \alpha \le 1$ and*

$$q = 1 - \frac{1}{2}\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\gamma_i+\alpha+1)\Gamma(\beta_i+1)}E^{(\beta_1,\dots,\beta_m),1}_{(\gamma_1+\alpha,\dots,\gamma_m+\alpha),1}\big(|\lambda_1|,\dots,|\lambda_m|\big) > 0.$$

Then $u(t,x)$ is a solution to equation (1.2) if and only if it satisfies the following integral equation in the space $S([0,1]^2)$:

$$\begin{aligned}
u(t,x) = {} & \sum_{l=0}^{\infty}(-1)^l\sum_{\substack{l_1+\cdots+l_m=l\\ l_1\ge 0,\dots,l_m\ge 0}}\binom{l}{l_1,\dots,l_m}\lambda_1^{l_1}\cdots\lambda_m^{l_m} \\
& \times I_t^{l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)+\alpha}I_x^{\beta_1 l_1+\cdots+\beta_m l_m}\nu(t,x,u) \\
& - \frac{1}{2}\sum_{l=0}^{\infty}(-1)^l\sum_{\substack{l_1+\cdots+l_m=l\\ l_1\ge 0,\dots,l_m\ge 0}}\binom{l}{l_1,\dots,l_m}\lambda_1^{l_1}\cdots\lambda_m^{l_m} \\
& \times I_t^{l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)}I_{t=1}^{\alpha}I_x^{\beta_1 l_1+\cdots+\beta_m l_m}\nu(t,x,u) \\
& + \frac{1}{2}\sum_{l=0}^{\infty}(-1)^l\sum_{\substack{l_1+\cdots+l_m=l\\ l_1\ge 0,\dots,l_m\ge 0}}\binom{l}{l_1,\dots,l_m}\lambda_1^{l_1}\cdots\lambda_m^{l_m} \\
& \times I_t^{l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)}I_x^{\beta_1 l_1+\cdots+\beta_m l_m}\psi(x) \\
& + \frac{1}{2}\sum_{i=1}^{m}\lambda_i\sum_{l=0}^{\infty}(-1)^l\sum_{\substack{l_1+\cdots+l_m=l\\ l_1\ge 0,\dots,l_m\ge 0}}\binom{l}{l_1,\dots,l_m}\lambda_1^{l_1}\cdots\lambda_m^{l_m} \\
& \times I_t^{l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)}I_x^{\beta_1 l_1+\cdots+\beta_m l_m}I_{t=1}^{\gamma_i+\alpha}I_x^{\beta_i}u.
\end{aligned}\tag{3.1}$$

Furthermore,

$$\begin{aligned}
\|u\| \le {} & \frac{1}{q}E^{(\beta_1,\dots,\beta_m),1}_{(\gamma_1+\alpha,\dots,\gamma_m+\alpha),\alpha+1}\big(|\lambda_1|,\dots,|\lambda_m|\big)\sup_{(t,x)\in[0,1]\times[0,1]u\in R}\big|\nu(t,x,u)\big| \\
& + \frac{1}{2q}\bigg(\frac{1}{\Gamma(\alpha+1)}+1\bigg)E^{(\beta_1,\dots,\beta_m),1}_{(\gamma_1+\alpha,\dots,\gamma_m+\alpha),1}\big(|\lambda_1|,\dots,|\lambda_m|\big) \\
& \times\Big(\sup_{(t,x)\in[0,1]\times[0,1]u\in R}\big|\nu(t,x,u)\big| + \max_{x\in[0,1]}\big|\psi(x)\big|\Big) < +\infty.
\end{aligned}$$

Proof Applying I_t^α to equation (1.2), we get

$$I_t^\alpha \frac{{}_c\partial^\alpha}{\partial t^\alpha} u(t,x) + \sum_{i=1}^m \lambda_i I_t^{\alpha+\gamma_i} I_x^{\beta_i} u(t,x) = I_t^\alpha v\bigl(t,x,u(t,x)\bigr).$$

This implies that

$$u(t,x) - u(0,x) + \sum_{i=1}^m \lambda_i I_t^{\alpha+\gamma_i} I_x^{\beta_i} u(t,x) = I_t^\alpha v\bigl(t,x,u(t,x)\bigr), \quad \text{and}$$
$$u(1,x) - u(0,x) + \sum_{i=1}^m \lambda_i I_{t=1}^{\alpha+\gamma_i} I_x^{\beta_i} u(t,x) = I_{t=1}^\alpha v\bigl(t,x,u(t,x)\bigr).$$

Using

$$-u(0,x) - u(1,x) = -\psi(x),$$

we get

$$u(0,x) = \frac{1}{2}\psi(x) + \frac{1}{2}\sum_{i=1}^m \lambda_i I_{t=1}^{\alpha+\gamma_i} I_x^{\beta_i} u(t,x) - \frac{1}{2} I_{t=1}^\alpha v\bigl(t,x,u(t,x)\bigr).$$

This further implies that

$$\begin{aligned}&\left(1 + \sum_{i=1}^m \lambda_i I_t^{\alpha+\gamma_i} I_x^{\beta_i}\right) u(t,x) \\ &\quad = I_t^\alpha v\bigl(t,x,u(t,x)\bigr) + \frac{1}{2}\psi(x) + \frac{1}{2}\sum_{i=1}^m \lambda_i I_{t=1}^{\alpha+\gamma_i} I_x^{\beta_i} u(t,x) - \frac{1}{2} I_{t=1}^\alpha v\bigl(t,x,u(t,x)\bigr).\end{aligned}$$

Using Babenko's method, we deduce that

$$\begin{aligned}&u(t,x) \\ &\quad = \left(1 + \sum_{i=1}^m \lambda_i I_t^{\alpha+\gamma_i} I_x^{\beta_i}\right)^{-1} \\ &\qquad \times \left(I_t^\alpha v\bigl(t,x,u(t,x)\bigr) + \frac{1}{2}\psi(x) + \frac{1}{2}\sum_{i=1}^m \lambda_i I_{t=1}^{\alpha+\gamma_i} I_x^{\beta_i} u(t,x) - \frac{1}{2} I_{t=1}^\alpha v\bigl(t,x,u(t,x)\bigr)\right) \\ &\quad = \sum_{l=0}^\infty (-1)^l \left(\sum_{i=1}^m \lambda_i I_t^{\alpha+\gamma_i} I_x^{\beta_i}\right)^l \\ &\qquad \times \left(I_t^\alpha v\bigl(t,x,u(t,x)\bigr) + \frac{1}{2}\psi(x) + \frac{1}{2}\sum_{i=1}^m \lambda_i I_{t=1}^{\alpha+\gamma_i} I_x^{\beta_i} u(t,x) - \frac{1}{2} I_{t=1}^\alpha v\bigl(t,x,u(t,x)\bigr)\right) \\ &\quad = \sum_{l=0}^\infty (-1)^l \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\ge 0,\ldots,l_m\ge 0}} \binom{l}{l_1,\ldots,l_m} \lambda_1^{l_1}\cdots\lambda_m^{l_m} I_t^{(\alpha+\gamma_1)l_1+\cdots+(\alpha+\gamma_m)l_m} I_x^{\beta_1 l_1+\cdots+\beta_m l_m}\end{aligned}$$

$$
\begin{aligned}
&\times \left(I_t^{\alpha} v\big(t,x,u(t,x)\big) + \frac{1}{2}\psi(x) + \frac{1}{2}\sum_{i=1}^{m} \lambda_i I_{t=1}^{\alpha+\gamma_i} I_x^{\beta_i} u(t,x) - \frac{1}{2} I_{t=1}^{\alpha} v\big(t,x,u(t,x)\big)\right) \\
= &\sum_{l=0}^{\infty} (-1)^l \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\geq 0,\ldots,l_m\geq 0}} \binom{l}{l_1,\ldots,l_m} \lambda_1^{l_1}\cdots\lambda_m^{l_m} \\
&\times I_t^{l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)+\alpha} I_x^{\beta_1 l_1+\cdots+\beta_m l_m} v(t,x,u) \\
&- \frac{1}{2}\sum_{l=0}^{\infty} (-1)^l \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\geq 0,\ldots,l_m\geq 0}} \binom{l}{l_1,\ldots,l_m} \lambda_1^{l_1}\cdots\lambda_m^{l_m} \\
&\times I_t^{l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)} I_{t=1}^{\alpha} I_x^{\beta_1 l_1+\cdots+\beta_m l_m} v(t,x,u) \\
&+ \frac{1}{2}\sum_{l=0}^{\infty} (-1)^l \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\geq 0,\ldots,l_m\geq 0}} \binom{l}{l_1,\ldots,l_m} \lambda_1^{l_1}\cdots\lambda_m^{l_m} \\
&\times I_t^{l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)} I_x^{\beta_1 l_1+\cdots+\beta_m l_m} \psi(x) \\
&+ \frac{1}{2}\sum_{i=1}^{m} \lambda_i \sum_{l=0}^{\infty} (-1)^l \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\geq 0,\ldots,l_m\geq 0}} \binom{l}{l_1,\ldots,l_m} \lambda_1^{l_1}\cdots\lambda_m^{l_m} \\
&\times I_t^{l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)} I_x^{\beta_1 l_1+\cdots+\beta_m l_m} I_{t=1}^{\gamma_i+\alpha} I_x^{\beta_i} u,
\end{aligned}
$$

by the multinomial theorem. We will now show that $u \in S([0,1]^2)$. Indeed,

$$
\begin{aligned}
\|u\| \leq &\sum_{l=0}^{\infty} \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\geq 0,\ldots,l_m\geq 0}} \binom{l}{l_1,\ldots,l_m} \frac{|\lambda_1|^{l_1}\cdots|\lambda_m|^{l_m}}{\Gamma(l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)+\alpha+1)} \\
&\times \frac{1}{\Gamma(l_1\beta_1+\cdots+l_m\beta_m+1)} \sup_{(t,x)\in[0,1]\times[0,1],u\in R} \big|v(t,x,u)\big| \\
&+ \frac{1}{2\Gamma(\alpha+1)} \sum_{l=0}^{\infty} \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\geq 0,\ldots,l_m\geq 0}} \binom{l}{l_1,\ldots,l_m} \frac{|\lambda_1|^{l_1}\cdots|\lambda_m|^{l_m}}{\Gamma(l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)+1)} \\
&\times \frac{1}{\Gamma(l_1\beta_1+\cdots+l_m\beta_m+1)} \sup_{(t,x)\in[0,1]\times[0,1],u\in R} \big|v(t,x,u)\big| \\
&+ \frac{1}{2}\sum_{l=0}^{\infty} \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\geq 0,\ldots,l_m\geq 0}} \binom{l}{l_1,\ldots,l_m} \\
&\times \frac{|\lambda_1|^{l_1}\cdots|\lambda_m|^{l_m}}{\Gamma(l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)+1)\Gamma(l_1\beta_1+\cdots+l_m\beta_m+1)} \max_{x\in[0,1]} \big|\psi(x)\big| \\
&+ \frac{1}{2}\sum_{i=1}^{m} \frac{|\lambda_i|}{\Gamma(\gamma_i+\alpha+1)\Gamma(\beta_i+1)} \sum_{l=0}^{\infty} \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\geq 0,\ldots,l_m\geq 0}} \binom{l}{l_1,\ldots,l_m} \\
&\times \frac{|\lambda_1|^{l_1}\cdots|\lambda_m|^{l_m}}{\Gamma(l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)+1)\Gamma(l_1\beta_1+\cdots+l_m\beta_m+1)} \|u\|.
\end{aligned}
$$

Since

$$q = 1 - \frac{1}{2}\sum_{i=1}^{m} \frac{|\lambda_i|}{\Gamma(\gamma_i + \alpha + 1)\Gamma(\beta_i + 1)} E^{(\beta_1,\dots,\beta_m),1}_{(\gamma_1+\alpha,\dots,\gamma_m+\alpha),1}\big(|\lambda_1|, \dots, |\lambda_m|\big) > 0,$$

we come to

$$\begin{aligned}
\|u\| &\le \frac{1}{q} E^{(\beta_1,\dots,\beta_m),1}_{(\gamma_1+\alpha,\dots,\gamma_m+\alpha),\alpha+1}\big(|\lambda_1|, \dots, |\lambda_m|\big) \sup_{(t,x)\in[0,1]\times[0,1] u\in R} \big|v(t,x,u)\big| \\
&\quad + \frac{1}{2q}\left(\frac{1}{\Gamma(\alpha+1)} + 1\right) E^{(\beta_1,\dots,\beta_m),1}_{(\gamma_1+\alpha,\dots,\gamma_m+\alpha),1}\big(|\lambda_1|, \dots, |\lambda_m|\big) \\
&\quad \times \Big(\sup_{(t,x)\in[0,1]\times[0,1] u\in R} \big|v(t,x,u)\big| + \max_{x\in[0,1]} \big|\psi(x)\big| \Big) < +\infty,
\end{aligned}$$

since v is bounded. Hence $u \in S([0,1]^2)$. This marks the completion of the proof. □

Theorem 3 *Let $\psi \in C[0,1]$, λ_i be real constants, $\gamma_i, \beta_i \ge 0$ for all $i = 1,2,\dots,m$, and $v : [0,1]^2 \times R \to R$ be a bounded and continuous function that satisfies the following Lipschitz condition for $\mathcal{M} > 0$:*

$$\big|v(t,x,u_1) - v(t,x,u_2)\big| \le \mathcal{M}|u_1 - u_2|, \quad u_1, u_2 \in R.$$

Furthermore, we suppose that $0 < \alpha \le 1$ and

$$\begin{aligned}
W &= \mathcal{M} E^{(\beta_1,\dots,\beta_m),1}_{(\gamma_1+\alpha,\dots,\gamma_m+\alpha),\alpha+1}\big(|\lambda_1|, \dots, |\lambda_m|\big) \\
&\quad + \frac{1}{2}\left(\frac{\mathcal{M}}{\Gamma(\alpha+1)} + \sum_{i=1}^{m} \frac{|\lambda_i|}{\Gamma(\gamma_i+\alpha+1)\Gamma(\beta_i+1)}\right) E^{(\beta_1,\dots,\beta_m),1}_{(\gamma_1+\alpha,\dots,\gamma_m+\alpha),1}\big(|\lambda_1|, \dots, |\lambda_m|\big) < 1.
\end{aligned}$$

Then there is a unique solution in the space $S([0,1]^2)$ to equation (1.2).

Proof Let $\mathcal{T}$ be the mapping defined on the space $S([0,1]^2)$ by

$$\begin{aligned}
(\mathcal{T}u)(t,x) &= \sum_{l=0}^{\infty} (-1)^l \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\ge 0,\dots,l_m\ge 0}} \binom{l}{l_1,\dots,l_m} \lambda_1^{l_1}\cdots\lambda_m^{l_m} \\
&\quad \times I_t^{l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)+\alpha} I_x^{\beta_1 l_1+\cdots+\beta_m l_m} v(t,x,u) \\
&\quad - \frac{1}{2}\sum_{l=0}^{\infty} (-1)^l \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\ge 0,\dots,l_m\ge 0}} \binom{l}{l_1,\dots,l_m} \lambda_1^{l_1}\cdots\lambda_m^{l_m} \\
&\quad \times I_t^{l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)} I_{t=1}^{\alpha} I_x^{\beta_1 l_1+\cdots+\beta_m l_m} v(t,x,u) \\
&\quad + \frac{1}{2}\sum_{l=0}^{\infty} (-1)^l \sum_{\substack{l_1+\cdots+l_m=l \\ l_1\ge 0,\dots,l_m\ge 0}} \binom{l}{l_1,\dots,l_m} \lambda_1^{l_1}\cdots\lambda_m^{l_m} \\
&\quad \times I_t^{l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)} I_x^{\beta_1 l_1+\cdots+\beta_m l_m} \psi(x)
\end{aligned}$$

$$+\frac{1}{2}\sum_{i=1}^{m}\lambda_i\sum_{l=0}^{\infty}(-1)^l\sum_{\substack{l_1+\cdots+l_m=l\\ l_1\geq 0,\ldots,l_m\geq 0}}\binom{l}{l_1,\ldots,l_m}\lambda_1^{l_1}\cdots\lambda_m^{l_m}$$
$$\times I_t^{l_1(\alpha+\gamma_1)+\cdots+l_m(\alpha+\gamma_m)}I_x^{\beta_1 l_1+\cdots+\beta_m l_m}I_{t=1}^{\gamma_i+\alpha}I_x^{\beta_i}u.$$

From the proof of Theorem 2, we claim that $\mathcal{T}u\in S([0,1]^2)$. We will prove that $\mathcal{T}$ is contractive. In fact, for $u_1,u_2\in S([0,1]^2)$, we get from Theorem 2 that

$$\begin{aligned}\|\mathcal{T}u_1-\mathcal{T}u_2\| &\leq \mathcal{M}E_{(\gamma_1+\alpha,\ldots,\gamma_m+\alpha),\alpha+1}^{(\beta_1,\ldots,\beta_m),1}\big(|\lambda_1|,\ldots,|\lambda_m|\big)\|u_1-u_2\|\\ &\quad+\frac{\mathcal{M}}{2\Gamma(\alpha+1)}E_{(\gamma_1+\alpha,\ldots,\gamma_m+\alpha),1}^{(\beta_1,\ldots,\beta_m),1}\big(|\lambda_1|,\ldots,|\lambda_m|\big)\|u_1-u_2\|\\ &\quad+\frac{1}{2}\sum_{i=1}^{m}\frac{|\lambda_i|}{\Gamma(\gamma_i+\alpha+1)\Gamma(\beta_i+1)}E_{(\gamma_1+\alpha,\ldots,\gamma_m+\alpha),1}^{(\beta_1,\ldots,\beta_m),1}\big(|\lambda_1|,\ldots,|\lambda_m|\big)\|u_1-u_2\|\\ &=W\|u_1-u_2\|,\end{aligned}$$

by noting that

$$\big|v(t,x,u_1)-v(t,x,u_2)\big|\leq\mathcal{M}|u_1-u_2|.$$

Since $W<1$, there is a unique solution to equation (1.2) in the space $S([0,1]^2)$ by Banach's fixed point theorem. Hence Theorem 3 follows. □

4 Example

Example 4 Consider the following equation with a boundary condition:

$$\begin{cases}\frac{{}_c\partial^{0.5}}{\partial t^{0.5}}u(t,x)+\frac{1}{15}I_t^{1.5}I_x^{0.5}u(t,x)+\frac{1}{21}I_t^{2.5}I_x^{1.1}u(t,x)\\ \quad=\frac{1}{18}\cos(txu)+t^2+\sin x,\\ u(0,x)+u(1,x)=x^2+1,\quad (t,x)\in[0,1]\times[0,1].\end{cases}\tag{4.1}$$

Then there is a unique solution in the space $S([0,1]^2)$ to equation (4.1).

Proof Let

$$v(t,x,u)=\frac{1}{18}\cos(txu)+t^2+\sin x.$$

Obviously,

$$\big|v(t,x,u_1)-v(t,x,u_2)\big|\leq\frac{1}{18}\big|\cos(txu_1)-\cos(txu_2)\big|\leq\frac{1}{18}|u_1-u_2|,$$

for all $u_1,u_2\in R$, by noting that $(t,x)\in[0,1]\times[0,1]$. Therefore $\mathcal{M}=1/18$, and

$$\beta_1=0.5,\qquad\beta_2=1.1,$$
$$\alpha=0.5,\qquad\gamma_1=1.5,\qquad\gamma_2=2.5,$$
$$\lambda_1=\frac{1}{15},\qquad\lambda_2=\frac{1}{21},$$

from equation (4.1). We evaluate the following W given in Theorem 3 via Python language to get

$$\begin{aligned} W &= \frac{1}{18} E^{(0.5,1.1),1}_{(1.5+0.5,2.5+0.5),0.5+1}\left(\frac{1}{15}, \frac{1}{21}\right) \\ &\quad + \frac{1}{2}\left(\frac{1}{18\Gamma(0.5+1)} + \frac{1}{15\Gamma(1.5+0.5+1)\Gamma(0.5+1)}\right. \\ &\quad \left. + \frac{1}{21\Gamma(2.5+0.5+1)\Gamma(1.1+1)}\right) \\ &\quad \times E^{(0.5,1.1),1}_{(1.5+0.5,2.5+0.5),1}\left(\frac{1}{15}, \frac{1}{21}\right) \approx 0.120560441333871 < 1. \end{aligned}$$

By Theorem 3, the result follows. □

Remark 5 The Python language is quite useful when computing the values of the multivariate Mittag-Leffler function or the newly introduced generalized multivariate Mittag-Leffler function. These functions appear often in many fields and play an important role in studying integral or differential equations with various conditions, as well as in finding approximate solutions, such as for equation (2.1) as an example.

5 Conclusion

We have obtained a sufficient condition for uniqueness of solution to the new boundary value problem (1.2) involving double integral operators by using the new generalized multivariate Mittag-Leffler function, Babenko's approach, as well as by applying Banach's fixed point theorem. Moreover, we made use of the Python language to aid in finding the approximate value of a generalized Mittag-Leffler function, which currently seems unfeasible to do so by any existing integral representations of the Mittag-Leffler function. Finally, we presented an example that applies the results of the key theorems derived. The technique used certainly works for different types of PDE and corresponding initial or boundary value problems.

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Nomenclature
I^{α}_{t}, the partial Riemann–Liouville fractional integral of order $\alpha > 0$ with respect to t; $\frac{{}_c\partial^{\alpha}}{\partial t^{\alpha}}$„ the partial Liouville–Caputo fractional derivative of order $\alpha > 0$ with respect to t; $S([0,1,^2)]$ the Banach space equipped with the following norm: $\|u\| = \sup_{t\in[0,1],x\in[0,1]} |u(t,x)|$ for $u \in S([0,1]^2)$, where u is continuous on $[0,1]^2$; $E^{(\beta_1,\dots,\beta_m),\delta}_{(\alpha_1,\dots,\alpha_m),\epsilon}(\zeta_1,\dots,\zeta_m)$, is the generalized multivariate Mittag-Leffler function.

Availability of data and materials
No data were used to support this study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
CL, methodology, writing–original draft preparation. RS, supervision, and project administration. JB, editing, and writing–original draft preparation. AH, editing, and writing–original draft preparation. All authors read and approved the final manuscript.

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