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On nonlinear fractional Choquard equation with indefinite potential and general nonlinearity

Fangfang Liao[1], Fulai Chen[1], Shifeng Geng[2] and Dong Liu[3*]

*Correspondence:
liudong@xnu.edu.cn
[3]School of Computer and Artificial Intelligence, Xiangnan University, Chenzhou, 423000 Hunan, China
Full list of author information is available at the end of the article

Abstract

In this paper, we consider a class of fractional Choquard equations with indefinite potential

$$(-\Delta)^{\alpha}u+V(x)u=\left[\int_{\mathbb{R}^N}\frac{M(\epsilon y)G(u)}{|x-y|^{\mu}}\,\mathrm{d}y\right]M(\epsilon x)g(u),\quad x\in\mathbb{R}^N,$$

where $\alpha\in(0,1)$, $N>2\alpha$, $0<\mu<2\alpha$, ϵ is a positive parameter. Here $(-\Delta)^{\alpha}$ stands for the fractional Laplacian, V is a linear potential with periodicity condition, and M is a nonlinear reaction potential with a global condition. We establish the existence and concentration of ground state solutions under general nonlinearity by using variational methods.

Mathematics Subject Classification: 35B32; 35J20; 35J60; 47J15; 58E07

Keywords: Concentration; Ground state solutions; Fractional Choquard equation

1 Introduction and main result

In this paper, we deal with a class of nonlinear fractional Choquard equations with indefinite potential

$$(-\Delta)^{\alpha}u+V(x)u=\left[\int_{\mathbb{R}^N}\frac{M(\epsilon y)G(u)}{|x-y|^{\mu}}\,\mathrm{d}y\right]M(\epsilon x)g(u),\quad x\in\mathbb{R}^N,\tag{1.1}$$

where $\epsilon>0$ is a parameter, $\alpha\in(0,1)$, $N>2\alpha$, $(-\Delta)^{\alpha}$ stands for the fractional Laplacian operator, the nonlinear function G is the primitive function of g with subcritical growth. The operator $(-\Delta)^{\alpha}$ is nonlocal and can be defined by

$$(-\Delta)^{\alpha}u(x)=-\frac{C_{N,\alpha}}{2}\int_{\mathbb{R}^N}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2\alpha}}\,dy,\quad\forall x\in\mathbb{R}^N,$$

where $C_{N,\alpha}$ is a suitable normalization constant. We recall that the problem (1.1) is inspired by the study of standing wave solutions for the time-dependent nonlinear

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Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t}=\hbar^{2\alpha}(-\Delta)^{\alpha}\Psi+\big(V(x)+E\big)\Psi-\big(|x|^{-\mu}*|\Psi|^{q}\big)|\Psi|^{q-2}\Psi,\quad (x,t)\in\mathbb{R}^N\times\mathbb{R},\quad (1.2)$$

where i is the imaginary unit, $\hbar$ is the Planck constant, and Ψ represents the wave function of the state of an electron.

Regarding the applications of equation (1.2), we recall that fractional Laplacian operators are the infinitesimal generators of Lévy stable diffusion processes. They have application in several areas such as anomalous diffusion of plasmas, probability, finance, and population dynamics. For more details on the application background, we refer to Applebaum [4] and the monograph [24] of Molica Bisci–Rădulescu–Servadei.

When $\alpha=1$, problem (1.1) becomes the usual Choquard equation. The early existence and symmetry results were established by Lions [21] and Lieb–Loss [20]. After the celebrated work [20, 21], the existence and qualitative and asymptotic properties of nontrivial solutions for the Choquard equation or its generalized version have been extensively investigated by using various methods of nonlinear analysis (such as the variational method, moving plane method, Lyapunov–Schmidt reduction method, and shooting method). We refer the readers to [2, 5, 7, 14, 25, 26, 29, 30, 35, 43, 44] and the references therein.

For the case $\alpha\in(0,1)$, during the recent years, problem (1.1) has attracted considerable interest, the literature related to this equation is numerous and encompasses several interesting lines of research in nonlinear analysis, including existence, multiplicity, concentration, and qualitative properties of solutions. Let us now briefly recall some related results in this direction.

In [10], d'Avenia–Siciliano–Squassiona studied some results involving existence, regularity, and asymptotic of the solutions for the fractional Choquard equation with constant potential

$$(-\Delta)^s u+\omega u=\left[\frac{1}{|x|^{\mu}}*|u|^p\right]|u|^{p-2}u,\quad x\in\mathbb{R}^N,$$

where $\omega>0$. The analyticity, uniqueness, and radial symmetry of ground state solutions were investigated by Frank–Lenzmann [13]. Later on, under general source terms, Shen–Gao–Yang [32] proved the existence result of ground state solutions for a fractional Choquard equation involving a nonlinearity satisfying Berestycki–Lions-type conditions. Without any symmetry property, Chen–Liu [8] established the existence of positive ground state solutions by using the usual Nehari manifold and concentration compactness principle. We also refer to Zhang–Wu [41] for the existence result of nodal solutions.

Recently, there have been some results for fractional Choquard equations with critical growth; we mention the works of Mukherjee–Sreenadh [27] for an analogous Brezis–Nirenberg-type problem; He–Rădulescu [17] for a small linear perturbation problem; Guan–Rădulescu–Wang [16] for the existence of positive bounded solutions. Moreover, concerning the semiclassical analysis of the singularly perturbed problem

$$\epsilon^{2\alpha}(-\Delta)^{\alpha}u+V(x)u=\epsilon^{\mu-N}\left[\frac{1}{|x|^{\mu}}*G(u)\right]g(u),\quad x\in\mathbb{R}^N,$$

the papers [3, 6, 15, 19, 36, 37] showed the existence or multiplicity of semiclassical solutions which concentrate around the local or global minimum points of the linear potential V. For other related results involving the qualitative and asymptotic analysis of nontrivial solutions to nonlocal elliptic equations, we also refer to the papers [12, 18, 28, 31, 34, 38–40] and the references therein.

We would like to emphasize that, in the works mentioned above, the authors dealt only with the case where the potential V is a constant or positive function, in the sense that the corresponding energy functional is strongly definite, which has the mountain pass geometry structure in general. In the variational framework of strongly definite functional, the classical Nehari manifold method and mountain pass theory are available. However, as far as we know, there are very few works considering the case where the potential $V(x)$ as in problem (1.1) is indefinite (or sign-changing), which motivates the present work to consider this case.

Concerning the indefinite potential case, we would like to mention the recent work done by Fang–Ji [11] in which the authors first considered the fractional Schrödinger equation under the condition (V) and proved that the fractional Schrödinger operator $(-\Delta)^\alpha + V$ has a purely continuous spectrum which is bounded below and consists of closed disjoint intervals, see [11, Theorem 1.1]. So in this framework, we know that the energy functional of problem (1.1) is strongly indefinite, which has a more complicated geometry structure than that of a strongly definite functional. In the sense we can see that zero is no longer a local minimum point of the energy functional, and then the usual Nehari manifold method and mountain pass theorem do not work for this case.

Under the variational framework of strongly indefinite potential, motivated by the work of Alves–Germano [1], Chen–Ji [9] proved the existence and concentration of solutions to fractional Schrödinger equation, which extend the relevant ones in [1] from the classical to fractional Schrödinger equation. Very recently, Zhang–Yuan–Wen [42] investigated the fractional Choquard equation with a pure power nonlinearity, and obtained the existence and concentration properties of ground state solutions. We also mention the recent paper [23] in which the existence and asymptotics of ground states to the fractional Schrödinger equations with indefinite and Hardy potentials are discussed.

Motivated by the above works, in the present paper, we aim to study further the existence and some properties of ground state solutions of the fractional Choquard equation (1.1) under a more general nonlinearity. To be more precise, the interest in the study of this paper is twofold: one is to establish the existence of ground state solutions to problem (1.1); the other is to study the asymptotics of these solutions as $\epsilon \to 0$.

Before stating our results, let us give some suitable conditions about the potentials V, M, and the nonlinearity g. We first assume that V and M satisfy the following conditions:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$ is $\mathbb{Z}^N$-periodic, $0 \notin \sigma((-\Delta)^\alpha + V)$ and $\sigma((-\Delta)^\alpha + V) \cap (-\infty, 0) \neq \emptyset$, where σ denotes the spectrum of Schrödinger operator $(-\Delta)^\alpha + V$;

(M) $M \in C(\mathbb{R}^N, \mathbb{R})$ and $0 < \inf_{x\in\mathbb{R}^N} M(x) \leq M_\infty := \lim_{|x|\to+\infty} M(x) < M(0) = \max_{x\in\mathbb{R}^N} M(x)$.

Meanwhile, we suppose that the nonlinearity g satisfies the following conditions:

(g_1) $g(u) = o(|u|)$ as $|u| \to 0$;

(g_2) There exist $c_0 > 0$ and $q \in (2, \frac{2N-2\mu}{N-2\alpha})$ such that

$$\left|g(u)\right| \leq c_0\left(1 + |u|^{q-1}\right) \quad \text{for all } u \in \mathbb{R};$$

(g_3) $G(u) \geq 0$ for all $u \in \mathbb{R}$ and $\frac{G(u)}{|u|^2} \to +\infty$ as $|u| \to +\infty$;
(g_4) $u \mapsto \frac{g(u)}{|u|}$ is strictly increasing on $(-\infty, 0)$ and on $(0, +\infty)$.
The main result of this paper is the following theorem.

Theorem 1.1 *Assume that* (V), (M) *and* (g_1)–(g_4) *are satisfied, then*
(a) *there exists* $\epsilon_0 > 0$ *such that problem* (1.1) *has a ground state solution* u_ϵ *for each* $\epsilon \in (0, \epsilon_0)$;
(b) *if* $x_\epsilon \in \mathbb{R}^N$ *denotes a global maximum point of* $|u_\epsilon|$, *then*

$$\lim_{\epsilon \to 0} M(\epsilon x_\epsilon) = M(0);$$

(c) $u_\epsilon(x + x_\epsilon) \to u$ *as* $\epsilon \to 0$, *where* u *is a ground state solution of the limit equation*

$$(-\Delta)^\alpha u + V(x)u = M(0)^2 \left[\int_{\mathbb{R}^N} \frac{G(u)}{|x-y|^\mu} \,\mathrm{d}y\right] g(u), \quad x \in \mathbb{R}^N.$$

The features of this paper are the following:

• The problem combines the multiple effects generated by the indefinite potential, reaction potential, and general nonlinearity;

• The strong indefiniteness of the energy functional together with the double nonlocality bring some difficulties in our analysis;

• The lack of compactness due to the unboundedness of the domain leads to the fact that the energy functional does not satisfy the necessary compactness property.

Let us explain shortly the strategies of the proof of Theorem 1.1. Based on the above features, firstly, we intend to make use of the method of generalized Nehari manifold developed by Szulkin–Weth [33] to conquer the difficulty caused by the strong indefiniteness feature. Secondly, we must verify that the energy functional satisfies a necessary compactness condition at some minimax level. This goal will be achieved by doing a finer analysis and using the energy comparison argument to establish some relationships of the ground state energy value between the original problem and certain auxiliary problems. Finally, in order to characterize the concentration property of solutions, we need to draw upon the Moser iteration arguments to show the L^∞-estimate. Summarizing, the results included in the present paper complement several recent contributions to the study of concentration of solutions to the fractional Choquard equation.

2 Variational setting and preliminaries

Throughout this paper, for the sake of simplicity we will use the following notations:

• $L^q(\mathbb{R}^N)$ $(1 \leq q < \infty)$ denotes the Lebesgue space with the norm $\|\cdot\|_q = (\int_{\mathbb{R}^N} |u|^q \,\mathrm{d}x)^{1/q}$;

• $(\cdot,\cdot)_2$ denotes the usual $L^2(\mathbb{R}^N)$ inner product;

• c, c_i, C_i denote positive constants possibly different in different places.

In the following we introduce the variational framework of the fractional Sobolev space and some comprehensive presentations of the space can be found in the book [24].

For any $\alpha \in (0,1)$, the norm of the fractional Sobolev space $D^{\alpha,2}(\mathbb{R}^N)$, which is the completion of $C_0^\infty(\mathbb{R}^N)$, is

$$[u]_{D^{\alpha,2}}^2 = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y.$$

Based on this, we can define the following fractional Sobolev space $H^\alpha(\mathbb{R}^N)$:

$$H^\alpha(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y < +\infty\right\},$$

and the corresponding norm is

$$\|u\|_0 = \left[\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{C_{N,\alpha}}{2} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x\right]^{\frac{1}{2}}.$$

Furthermore, $H^\alpha(\mathbb{R}^N)$ can also be represented as

$$H^\alpha(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : (-\Delta)^{\frac{\alpha}{2}} u \in L^2(\mathbb{R}^N)\right\},$$

with the norm of the form

$$\|u\|_0 = \left[\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{\alpha}{2}} u\right|^2 + |u|^2 \,\mathrm{d}x\right]^{\frac{1}{2}}.$$

Next we define the energy functional associated with problem (1.1), namely

$$\begin{aligned}
\mathcal{J}_\epsilon(u) &= \frac{1}{2} \int_{\mathbb{R}^N} \left[\left|(-\Delta)^{\frac{\alpha}{2}} u\right|^2 + V(x)|u|^2\right] \mathrm{d}x \\
&\quad - \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{M(\epsilon x)G(u(x))M(\epsilon y)G(u(y))}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\left(\left((-\Delta)^\alpha + V(x)\right)u, u\right)_2 - \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{M(\epsilon x)G(u(x))M(\epsilon y)G(u(y))}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{2.1}$$

It is well known that the potential V is bounded in $\mathbb{R}^N$ due to the continuity of V. Let $\mathcal{L} := (-\Delta)^\alpha + V$. From (V), we know that $\mathcal{L}$ is self-adjoint and has a purely continuous spectrum which is bounded below and consists of closed disjoint intervals, see [11, Theorem 1.1]. Furthermore, by (V) again, we get the following orthogonal decomposition:

$$L^2 := L^2(\mathbb{R}^N) = L^- \oplus L^+, \qquad u = u^+ + u^-,$$

in this case, $\mathcal{L}$ is positive definite (resp. negative definite) in L^+ (resp. L^-). Let $|\mathcal{L}|$ denote the absolute value of $\mathcal{L}$, and let $|\mathcal{L}|^{\frac{1}{2}}$ represent the square root of $\mathcal{L}$. We define the working space $E = D(|\mathcal{L}|^{\frac{1}{2}})$. Then E is a Hilbert space, with the inner product of the following form:

$$(u, v) = \left(|\mathcal{L}|^{\frac{1}{2}} u, |\mathcal{L}|^{\frac{1}{2}} v\right)_2 = \int_{\mathbb{R}^N} |\mathcal{L}|^{\frac{1}{2}} u |\mathcal{L}|^{\frac{1}{2}} v \,\mathrm{d}x,$$

and the corresponding norm is $\|u\| = (u,u)^{\frac{1}{2}}$. Obviously, from (V), the two norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Therefore, $E = H^\alpha(\mathbb{R}^N)$. Furthermore, by the decomposition of L^2, we have

$$E = E^- \oplus E^+, \quad \text{where } E^\pm = E \cap L^\pm,$$

which is orthogonal with respect to the two inner products $(\cdot,\cdot)_2$ and $(\cdot,\cdot)$. Moreover, the polar decomposition of $\mathcal{L}$ yields that

$$\mathcal{L}u^- = -|\mathcal{L}|u^-, \qquad \mathcal{L}u^+ = |\mathcal{L}|u^+ \quad \text{for all } u = u^+ + u^- \in E.$$

Define the following bilinear map $A(u,v)$:

$$A(u,v) = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\alpha}{2}} u(-\Delta)^{\frac{\alpha}{2}} v + V(x)uv \,\mathrm{d}x.$$

For every $u \in E$, from the above decomposition, we obtain that

$$A(u,u) = A\bigl(u^+,u^+\bigr) + A\bigl(u^-,u^-\bigr)$$

and

$$A\bigl(u^+,u^+\bigr) = \bigl(u^+,u^+\bigr), \qquad A\bigl(u^-,u^-\bigr) = -\bigl(u^-,u^-\bigr).$$

Therefore, we can rewrite functional (2.1) in the following form:

$$\mathcal{J}_\epsilon(u) = \frac{1}{2}\bigl(\|u^+\|^2 - \|u^-\|^2\bigr) - \Psi_\epsilon(u),$$

where

$$\Psi_\epsilon(u) = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(\epsilon x)G(u(x))M(\epsilon y)G(u(y))}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y.$$

Moreover, according to the conclusion in [24], we have the following embedding property.

Lemma 2.1 *Let $\alpha \in (0,1)$ and $N > 2\alpha$. Then there is a constant $\widehat{c} = \widehat{c}(\alpha,N) > 0$ such that*

$$\|u\|^2_{2^*_\alpha(\mathbb{R}^N)} \le \widehat{c}^{-1}[u]^2_{D^{\alpha,2}}, \quad \forall u \in E,$$

where $2^*_\alpha = 2N/(N-2\alpha)$. The embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for all $p \in [2,2^*_\alpha]$ and $E \hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}^N)$ is compact for all $p \in [2,2^*_\alpha)$.

We also get the following Lion's compactness lemma from the monograph [24].

Lemma 2.2 *Suppose that the sequence $\{u_n\}$ is bounded in E, and for every $r > 0$ there holds*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^2 \,\mathrm{d}x = 0,$$

then $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for all $q \in (2,2^*_\alpha)$.

Since we will treat the nonlocal problem (1.1) with Choquard term, the classical Hardy–Littlewood–Sobolev inequality [22] will be frequently used throughout this paper. Hence we present the following Hardy–Littlewood–Sobolev inequality.

Lemma 2.3 (Hardy–Littlewood–Sobolev inequality [22]) *Let $1 < r,t < +\infty$ and $0 < \mu < N$ be such that $\frac{1}{r} + \frac{1}{t} + \frac{\mu}{N} = 2$. If $\phi \in L^r(\mathbb{R}^N)$ and $\psi \in L^t(\mathbb{R}^N)$, then there exists a sharp constant $C(N,\mu,r,t) > 0$, independent of ϕ and ψ, such that*

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\phi(x)\psi(y)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \le C(N,\mu,r,t)\|\phi\|_r\|\psi\|_t.$$

From (g_1) and (g_2), we can deduce that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\bigl|g(u)\bigr| \le \epsilon|u| + C_\epsilon|u|^{q-1} \quad \text{and} \quad \bigl|G(u)\bigr| \le \frac{\epsilon}{2}|u|^2 + \frac{C_\epsilon}{q}|u|^q. \tag{2.2}$$

Accordingly, we use (2.2), as well as Lemmas 2.1 and 2.3, to obtain the following estimate:

$$\begin{aligned}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(u(x))G(u(y))}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y &\le C(N,\mu,r,t)\bigl\|G(u)\bigr\|_r\bigl\|G(u)\bigr\|_t \\ &\le c_1\biggl[\int_{\mathbb{R}^N}\bigl(\epsilon|u|^2 + c_\epsilon|u|^q\bigr)\,\mathrm{d}x\biggr]^{\frac{2}{r}} \\ &\le \epsilon\|u\|_{2r}^4 + c_2\|u\|_{qr}^{2q}.\end{aligned} \tag{2.3}$$

Since $2 < q < \frac{2N-2\mu}{N-2\alpha} < \frac{2N-\mu}{N-2\alpha}$, we obtain $rq \in (2,2_\alpha^*)$ and $2r \in (2,2_\alpha^*)$. According to Lemma 2.1, we obtain

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(u(x))G(u(y))}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \le \epsilon\|u\|^4 + c_3\|u\|^{2q}. \tag{2.4}$$

Therefore, we get the following relation:

$$\begin{aligned}&\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(\epsilon x)G(u(x))M(\epsilon y)G(u(y))}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad\le M(0)^2\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(u(x))G(u(y))}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad\le \epsilon\|u\|^4 + c_4\|u\|^{2q}.\end{aligned} \tag{2.5}$$

Based on the above discussion, it is easy to see that $\mathcal{J}_\epsilon \in C^1(E,\mathbb{R})$, and the critical points of the functional $\mathcal{J}_\epsilon$ are weak solutions of problem (1.1). Then, for each $u,v \in E$, we have

$$\bigl\langle \mathcal{J}_\epsilon'(u),v\bigr\rangle = \int_{\mathbb{R}^N}(-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}v\,\mathrm{d}x + \int_{\mathbb{R}^N}V(x)uv\,\mathrm{d}x - \bigl\langle\Psi_\epsilon'(u),v\bigr\rangle,$$

where

$$\bigl\langle\Psi_\epsilon'(u),v\bigr\rangle = \iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{M(\epsilon y)G(u)}{|x-y|^\mu}M(\epsilon x)g(u)v\,\mathrm{d}x\,\mathrm{d}y.$$

Using Lemmas 2.1 and 2.3, and combining some standard arguments, we can check the following lemma.

Lemma 2.4 *The functional Ψ_ϵ is weakly sequentially lower semicontinuous and Ψ_ϵ' is weakly sequentially continuous.*

3 The autonomous problem

We will use the limit problem to prove the main results, and next we introduce some important results for the autonomous problem. For any $\pi > 0$, in this section we consider the following autonomous problem:

$$(-\Delta)^{\alpha}u + V(x)u = \pi^2\left[\int_{\mathbb{R}^N} \frac{G(u)}{|x-y|^{\mu}}\,\mathrm{d}y\right]g(u), \quad x \in \mathbb{R}^N, \tag{3.1}$$

where V satisfies the condition in (V). Meanwhile, we define the corresponding functional as follows:

$$\mathcal{J}_{\pi}(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \frac{\pi^2}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(u(x))G(u(y))}{|x-y|^{\mu}}\,\mathrm{d}x\,\mathrm{d}y.$$

Similar to the discussion in Sect. 2, we conclude that $\mathcal{J}_{\pi} \in C^1(E,\mathbb{R}^N)$, and the critical points of functional $\mathcal{J}_{\pi}$ correspond to the weak solutions of the problem (3.1).

In order to establish the existence of ground state solutions for the problem (3.1), we will apply the generalized Nehari manifold method developed by Szulkin and Weth [33]. In the following we introduce the generalized Nehari–Pankov manifold $\mathcal{N}_{\pi}$ of the form

$$\mathcal{N}_{\pi} = \left\{u \in E\backslash E^- : \langle \mathcal{J}_{\pi}'(u), u\rangle = 0 \text{ and } \langle \mathcal{J}_{\pi}'(u), v\rangle = 0, \forall v \in E^-\right\},$$

and set the ground state energy d_{π} of functional $\mathcal{J}_{\pi}$ on $\mathcal{N}_{\pi}$ as follows:

$$d_{\pi} = \inf_{u\in\mathcal{N}_{\pi}} \mathcal{J}_{\pi}(u).$$

Furthermore, for every $u \in E\backslash E^-$, we also define the subspace

$$E(u) = E^- \oplus \mathbb{R}u = E^- \oplus \mathbb{R}u^+,$$

and the convex subset

$$\widehat{E}(u) = E^- \oplus [0,+\infty)u = E^- \oplus [0,+\infty)u^+.$$

Lemma 3.1 *Let $u \in \mathcal{N}_{\pi}$, then for each $v \in \mathcal{X} := \{su + w : s \geq -1, w \in E^-\}$ and $v \neq 0$, we have*

$$\mathcal{J}_{\pi}(u+v) < \mathcal{J}_{\pi}(u).$$

Furthermore, u is a unique global maximum of $\mathcal{J}_{\pi}|_{\widehat{E}(u)}$.

Proof We apply the arguments in the proof of [33, Proposition 2.3] to prove this lemma. First, we notice that for each $u \in \mathcal{N}_{\pi}$, we have

$$0 = \langle \mathcal{J}_{\pi}'(u), \varphi\rangle = A(u,\varphi) - \pi^2\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(u)}{|x-y|^{\mu}} g(u)\varphi\,\mathrm{d}x\,\mathrm{d}y \quad \text{for all } \varphi \in E(u).$$

Let $v = su + w \in \mathcal{X}$, then $u + v = (1 + s)u + w \in \widehat{E}(u)$. Computing directly, we have

$$\begin{aligned}
&\mathcal{J}_\pi(u+v) - \mathcal{J}_\pi(u) \\
&= \frac{1}{2}\big[A(u+v,u+v) - A(u,u)\big] \\
&\quad + \frac{\pi^2}{2}\bigg[\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(u)G(u)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y - \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(u+v)G(u+v)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\bigg] \\
&= -\frac{\|w\|^2}{2} + A\bigg(u, s\Big(\frac{s}{2}+1\Big)u + (1+s)w\bigg) \\
&\quad + \frac{\pi^2}{2}\bigg[\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(u)G(u)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y - \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(u+v)G(u+v)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\bigg] \\
&= -\frac{\|w\|^2}{2} + \pi^2\int_{\mathbb{R}^N}\bigg[\bigg(\int_{\mathbb{R}^N} \frac{G(u(y))}{|x-y|^\mu}\,\mathrm{d}y\bigg) g\big(u(x)\big)\bigg(s\Big(\frac{s}{2}+1\Big)u + (1+s)w\bigg) \\
&\quad + \frac{1}{2}\bigg(\int_{\mathbb{R}^N} \frac{G(u)}{|x-y|^\mu}\,\mathrm{d}y G(u)\bigg) - \frac{1}{2}\bigg(\int_{\mathbb{R}^N} \frac{G(u+v)}{|x-y|^\mu}\,\mathrm{d}y\bigg) G(u+v)\bigg]\,\mathrm{d}x \\
&= -\frac{\|w\|^2}{2} + \pi^2\int_{\mathbb{R}^N} \widehat{g}(s,u,v)\,\mathrm{d}x,
\end{aligned}$$

where

$$\begin{aligned}
\widehat{g}(s,u,v) = {}& \bigg(\int_{\mathbb{R}^N} \frac{G(u)}{|x-y|^\mu}\,\mathrm{d}y\bigg) g(u)\bigg(s\Big(\frac{s}{2}+1\Big)u + (1+s)w\bigg) \\
&+ \frac{1}{2}\bigg(\int_{\mathbb{R}^N} \frac{G(u)}{|x-y|^\mu}\,\mathrm{d}y\bigg) G(u) - \frac{1}{2}\bigg(\int_{\mathbb{R}^N} \frac{G(u+v)}{|x-y|^\mu}\,\mathrm{d}y\bigg) G(u+v).
\end{aligned}$$

According to the argument in [33, Lemma 2.2], we conclude that $\widehat{g}(s,u,v) < 0$, and then we can obtain that $\mathcal{J}_\pi(u+v) < \mathcal{J}_\pi(u)$. Hence, u is a unique global maximum of $\mathcal{J}_\pi|_{\widehat{E}(u)}$. □

Lemma 3.2 *If* $\Omega \subset E^+ \setminus \{0\}$ *is a compact subset, then there exists* $R > 0$ *such that* $\mathcal{J}_\pi < 0$ *on* $E(u) \setminus B_R(0)$ *for each* $u \in \Omega$.

Lemma 3.3 *We have the following conclusions*:

(i) *there exists* $\kappa > 0$ *such that* $d_\pi \ge \inf_{S_\kappa} \mathcal{J}_\pi > 0$, *where* $S_\kappa := \{u \in E^+ : \|u\| = \kappa\}$;

(ii) *for each* $u \in \mathcal{N}_\pi$, $\|u^+\| \ge \max\{\|u^-\|, \sqrt{2d_\pi}\} > 0$.

Proof (i) For each $u \in E^+$, it follows that

$$\mathcal{J}_\pi(u) = \frac{1}{2}\|u\|^2 - \frac{\pi^2}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(u(x))G(u(y))}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.$$

Observe that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(u(x))G(u(y))}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y = o\big(\|u\|^2\big) \quad \text{as } u \to 0.$$

Hence, we find that there exists a small constant κ such that $\inf_{S_\kappa} \mathcal{J}_\pi > 0$ when $\|u\| = \kappa$.

On the other hand, for each $u \in \mathcal{N}_\pi$, there is $s > 0$ such that $s\|u\| = \kappa$, and then $su \in \widehat{E}(u) \cap S_\kappa$. It is easy to check that $\mathcal{J}_\pi(u) = \max_{v \in \widehat{E}(u)} \mathcal{J}_\pi(v) \geq \mathcal{J}_\pi(su)$ according to Lemma 3.1, so $\inf_{\mathcal{N}_\pi} \mathcal{J}_\pi \geq \inf_{S_\kappa} \mathcal{J}_\pi > 0$.

(ii) Let $u \in \mathcal{N}_\pi$, it is easy to see that

$$\begin{aligned} 0 < d_\pi &\leq \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{\pi^2}{2}\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{G(u(x))G(u(y))}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2, \end{aligned}$$

therefore, $\|u^+\| \geq \max\{\|u^-\|, \sqrt{2d_\pi}\} > 0$, finishing the proof. □

Following the idea of the proof of [33, Lemma 2.6], we can establish the uniqueness of a maximum point of $\mathcal{J}_\pi$ restricted to $\widehat{E}(u)$ without proof.

Lemma 3.4 *For any $u \in E \backslash E^-$, the set $\mathcal{N}_\pi \cap \widehat{E}(u)$ has a unique element $\widehat{m}_\pi(u)$, which is the global maximum of $\mathcal{J}_\pi|_{\widehat{E}(u)}$.*

Moreover, employing Lemmas 3.1 and 3.4, we have the following consequence.

Lemma 3.5 *For any $u \in E \backslash E^-$, there exists a unique pair (t, φ) with $t \in (0, +\infty)$ and $\varphi \in E^-$ such that $tu + \varphi \in \mathcal{N}_\pi \cap \widehat{E}(u)$ and*

$$\mathcal{J}_\pi(tu + \varphi) = \max_{w \in \widehat{E}(u)} \mathcal{J}_\pi(w).$$

Lemma 3.6 *The functional $\mathcal{J}_\pi$ is coercive on $\mathcal{N}_\pi$ for each $\pi > 0$, that is, $\mathcal{J}_\pi(u) \to +\infty$ as $\|u\| \to +\infty$.*

Proof Arguing by contradiction, we may assume that there exists a sequence $\{u_n\} \subset \mathcal{N}_\pi$ such that $\mathcal{J}_\pi(u_n) \leq C$ for some $C > 0$ as $\|u_n\| \to +\infty$. Set $w_n := \frac{u_n}{\|u_n\|}$, then using Lemma 3.3(ii), we obtain $\|u_n^+\| \geq \|u_n^-\|$, $\|w_n^+\|^2 \geq \|w_n^-\|^2$, and $\|w_n^+\|^2 \geq \frac{1}{2}$. In the following we show that there exist a sequence $\{y_n\} \subset \mathbb{Z}^N$, $R > 0$, and $\delta > 0$ such that

$$\int_{B_R(y_n)} |w_n^+|^2\,\mathrm{d}x \geq \delta. \tag{3.2}$$

If this is not true, Lemma 2.2 yields that $w_n^+ \to 0$ in $L^q(\mathbb{R}^N)$ for $q \in (2, 2_\alpha^*)$. From Lemmas 2.1 and 2.3, for each $\theta > 0$, we obtain

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{G(\theta w_n^+)G(\theta w_n^+)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \leq \epsilon\theta^4\|w_n^+\|_{2r}^4 + C\theta^{2q}\|w_n^+\|_{qr}^{2q} \to 0.$$

Hence, we derive from the above fact that

$$\begin{aligned} C \geq \mathcal{J}_\pi(\theta w_n^+) &= \frac{1}{2}\theta^2\|w_n^+\|^2 - \frac{\pi^2}{2}\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{g(\theta w_n^+)G(\theta w_n^+)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\ &\geq \frac{\theta^2}{4} - \frac{\pi^2}{2}\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{G(\theta w_n^+)G(\theta w_n^+)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \to \frac{\theta^2}{4}, \end{aligned}$$

which is impossible since θ is arbitrary. Therefore, we get that (3.2) holds.

We define $\tilde{u}_n(x) := u_n(x + y_n)$, and then $\tilde{w}_n(x) := w_n(x + y_n)$. So, we infer from (3.2) that $\tilde{w}_n^+ \rightharpoonup \tilde{w}^+ \neq 0$. Note that $\tilde{u}_n(x) = \tilde{w}_n(x)\|\tilde{u}_n\|$, thus $\tilde{u}_n(x) \to +\infty$ a.e. in $\mathbb{R}^N$ as $\|\tilde{u}_n\| = \|u_n\| \to +\infty$. Taking advantage of Fatou's lemma, we get

$$\begin{aligned}
&\frac{1}{\|u_n\|^2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{G(u_n)G(u_n)}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \frac{1}{\|u_n\|^2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{G(u_n)G(u_n)}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{G(u_n)}{|x-y|^\mu} \frac{G(u_n)}{\|u_n\|^2} \,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{G(\tilde{u}_n)}{|x-y|^\mu} \frac{G(\tilde{u}_n)}{|\tilde{u}_n|^2} |\tilde{w}_n|^2 \,\mathrm{d}x\,\mathrm{d}y \\
&\quad \geq \int_{[\tilde{u}_n \neq 0]} \left[\int_{\mathbb{R}^N} \frac{G(\tilde{u}_n)}{|x-y|^\mu} \,\mathrm{d}y \right] \frac{G(\tilde{u}_n)}{|\tilde{u}_n|^2} |\tilde{w}_n|^2 \,\mathrm{d}x \to +\infty,
\end{aligned}$$

where $[\tilde{u}_n \neq 0]$ denotes the usual Lebesgue measure of the set $\{x \in \mathbb{R}^N : \tilde{u}_n(x) \neq 0\}$. Thus, we have

$$\begin{aligned}
0 &\leq \frac{\mathcal{J}_\pi(u_n)}{\|u_n\|^2} = \frac{1}{2}\|w_n^+\|^2 - \frac{1}{2}\|w_n^-\|^2 - \frac{\pi^2}{2\|u_n\|^2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{G(u_n)G(u_n)}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y \\
&\leq \frac{1}{2} - \frac{\pi^2}{2\|u_n\|^2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{G(u_n)G(u_n)}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y \to -\infty.
\end{aligned}$$

So, we obtain a contradiction. The proof is completed. □

In the following, we introduce the method of generalized Nehari manifold developed by Szulkin and Weth [33]. For this, define the mapping

$$\widehat{m}_\pi : E^+ \backslash \{0\} \to \mathcal{N}_\pi \quad \text{and} \quad m_\pi = \widehat{m}_\pi|_{S^+},$$

with the inverse of m_π being

$$m_\pi^{-1} : \mathcal{N}_\pi \to S^+, \qquad m_\pi^{-1}(u) = \frac{u^+}{\|u^+\|},$$

where $S^+ = \{u \in E^+ : \|u\| = 1\}$. From now on, let us consider the reduction functional $\widehat{I}_\pi : E^+ \backslash \{0\} \to \mathbb{R}$ and the restriction $I_\pi : S^+ \to \mathbb{R}$ given by

$$\widehat{I}_\pi(u) = \mathcal{J}_\pi\big(\widehat{m}_\pi(u)\big) \quad \text{and} \quad I_\pi = \widehat{I}_\pi|_{S^+},$$

which are continuous by Lemma 2.8 in [33]. The following result establishes some significant properties involving the reduced functionals $\widehat{I}_\pi$ and I_π, which play a crucial role in our arguments. And their proofs follow the proofs of [33, Proposition 2.9, Corollary 2.10].

Lemma 3.7 *We have the following important results*:

(a) $\widehat{I}_\pi \in C^1(E^+ \backslash \{0\}, \mathbb{R})$ *and for* $u, v \in E^+$ *and* $u \neq 0$,

$$\langle \widehat{I}_\pi'(u), v \rangle = \frac{\|\widehat{m}_\pi(u)^+\|}{\|u\|} \langle \mathcal{J}_\pi'\big(\widehat{m}_\pi(u)\big), v \rangle.$$

(b) $I_\pi \in C^1(S^+, \mathbb{R})$ *and for each* $u \in S^+$ *and* $v \in T_u(S^+) = \{w \in E^+ : (u,w) = 0\}$,

$$\langle I_\pi'(u), v\rangle = \|\widehat{m}_\pi(u)^+\| \langle \mathcal{J}_\pi'(\widehat{m}_\pi(u)), v\rangle.$$

(c) $\{u_n\}$ *is a* (PS)*-sequence for* I_π *if and only if* $\{\widehat{m}_\pi(u_n)\}$ *is a* (PS)*-sequence for* $\mathcal{J}_\pi$.

(d) $u \in S^+$ *is a critical point of* I_π *if and only if* $\widehat{m}_\pi(u) \in \mathcal{N}_\pi$ *is a critical point of* $\mathcal{J}_\pi$. *Moreover, the corresponding values of* I_π *and* $\mathcal{J}_\pi$ *coincide and*

$$\inf_{S^+} I_\pi = \inf_{\mathcal{N}_\pi} \mathcal{J}_\pi = d_\pi.$$

Furthermore, in view of Lemma 3.5, the ground state energy value d_π has a minimax characterization given by

$$d_\pi = \inf_{\mathcal{N}_\pi} \mathcal{J}_\pi = \inf_{u \in E^+ \setminus \{0\}} \max_{v \in \widehat{E}(u)} \mathcal{J}_\pi(v). \tag{3.3}$$

The existence result of ground state solutions of problem (3.1) is the following:

Lemma 3.8 *Assume that* (V) *and* (g_1)*–*(g_4) *hold. Then problem* (3.1) *has at least one ground state solution.*

Proof We note that Lemma 3.3 shows that $d_\pi > 0$. If $u \in \mathcal{N}_\pi$ with $\mathcal{J}_\pi(u) = d_\pi$, it is easy to see that $m_\pi^{-1}(u) \in S^+$ is a minimizer of functional I_π, and hence it is a critical point of I_π. Then, u is a critical point of the functional $\mathcal{J}_\pi$ according to Lemma 3.7. In the following, we want to prove that there exists a minimizer $\tilde{u} \in \mathcal{N}_\pi$ such that $\mathcal{J}_\pi(\tilde{u}) = d_\pi$. Indeed, applying Ekeland's variational principle, there exists a sequence $\{v_n\} \subset S^+$ such that $I_\pi(v_n) \to d_\pi$ and $I_\pi'(v_n) \to 0$ as $n \to \infty$. Set $u_n = \widehat{m}_\pi(v_n) \in \mathcal{N}_\pi$ for all $n \in \mathbb{N}$, then from Lemma 3.7 we can infer that $\mathcal{J}_\pi(u_n) \to d_\pi$ and $\mathcal{J}_\pi'(u_n) \to 0$. Moreover, Lemma 3.6 shows that $\{u_n\}$ is bounded. Next we claim that

$$\varliminf_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_n|^2 \,\mathrm{d}x > 0.$$

If not, Lemma 2.2 yields that $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for any $q \in (2, 2_\alpha^*)$. Hence, according to Lemma 2.3, we deduce that

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{G(u_n)}{|x-y|^\mu} \big[g(u_n)u_n - G(u_n)\big] \,\mathrm{d}x \,\mathrm{d}y = o_n(1),$$

and we also have

$$\begin{aligned} d_\pi + o_n(1) &= \mathcal{J}_\pi(u_n) - \frac{1}{2}\langle \mathcal{J}_\pi'(u_n), u_n\rangle \\ &= \frac{\pi^2}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{G(u_n)}{|x-y|^\mu} \big[g(u_n)u_n - G(u_n)\big] \,\mathrm{d}x \,\mathrm{d}y \\ &= o_n(1). \end{aligned}$$

Evidently, this is impossible since $d_\pi > 0$. Thus, there exist $\{y_n\} \subset \mathbb{Z}^N$ and $\delta > 0$ such that

$$\int_{B_{1+\sqrt{N}}(y_n)} |u_n|^2 \,\mathrm{d}x \geq \delta.$$

We define $\tilde{u}_n(x) = u_n(x + y_n)$, then it follows that

$$\int_{B_{1+\sqrt{N}}(0)} |\tilde{u}_n|^2 \,\mathrm{d}x \geq \delta. \tag{3.4}$$

According to the periodicity condition, we can conclude that $\|\tilde{u}_n\| = \|u_n\|$ and

$$\mathcal{J}_\pi(\tilde{u}_n) \to d_\pi \quad \text{and} \quad \mathcal{J}'_\pi(\tilde{u}_n) \to 0. \tag{3.5}$$

Passing to a subsequence, we get that $\tilde{u}_n \rightharpoonup \tilde{u}$ in E, $\tilde{u}_n \to \tilde{u}$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $q \in (2, 2^*_\alpha)$, and $\tilde{u}_n(x) \to \tilde{u}(x)$ a.e. on $\mathbb{R}^N$. Hence, combining (3.4) with (3.5), we know that $\tilde{u} \neq 0$ and $\mathcal{J}'_\pi(\tilde{u}) = 0$, which implies that $\tilde{u} \in \mathcal{N}_\pi$ and $\mathcal{J}_\pi(\tilde{u}) \geq d_\pi$.

On the other hand, it follows from (g_4) and Fatou's lemma that

$$\begin{aligned}
d_\pi &= \lim_{n\to\infty}\left[\mathcal{J}_\pi(\tilde{u}_n) - \frac{1}{2}\langle \mathcal{J}'_\pi(\tilde{u}_n), \tilde{u}_n\rangle\right] \\
&= \lim_{n\to\infty} \frac{\pi^2}{2} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(\tilde{u}_n)}{|x-y|^\mu}\left[g(\tilde{u}_n)\tilde{u}_n - G(\tilde{u}_n)\right] \mathrm{d}x\,\mathrm{d}y \\
&\geq \frac{\pi^2}{2} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(\tilde{u})}{|x-y|^\mu}\left[g(\tilde{u})\tilde{u} - G(\tilde{u})\right] \mathrm{d}x\,\mathrm{d}y \\
&= \mathcal{J}_\pi(\tilde{u}) - \frac{1}{2}\langle \mathcal{J}'_\pi(\tilde{u}), \tilde{u}\rangle = \mathcal{J}_\pi(\tilde{u}),
\end{aligned}$$

which shows that $\mathcal{J}_\pi(\tilde{u}) \leq d_\pi$. Thus, $\mathcal{J}_\pi(\tilde{u}) = d_\pi$ and $\tilde{u}$ is a critical point of $\mathcal{J}_\pi$, which implies that $\tilde{u}$ is a ground state solution of problem (3.1), completing the proof of the lemma. □

4 Proof of Theorem 1.1

4.1 Existence of ground state solutions

In the following we will give a proof of the existence of ground state solutions for problem (1.1). As before, we define the associated generalized Nehari manifold

$$\mathcal{N}_\epsilon := \left\{u \in E\backslash E^- : \langle \mathcal{J}'_\epsilon(u), u\rangle = 0 \text{ and } \langle \mathcal{J}'_\epsilon(u), \varphi\rangle = 0, \forall \varphi \in E^-\right\}$$

and the ground state energy value

$$d_\epsilon = \inf_{\mathcal{N}_\epsilon} \mathcal{J}_\epsilon.$$

We also define the mapping

$$\widehat{m}_\epsilon : E^+\backslash\{0\} \to \mathcal{N}_\epsilon \quad \text{and} \quad m_\epsilon = \widehat{m}_\epsilon|_{S^+},$$

with the inverse of m_ϵ being

$$m_\epsilon^{-1} : \mathcal{N}_\epsilon \to S^+, \qquad m_\epsilon^{-1}(u) = \frac{u^+}{\|u^+\|}.$$

Then, the reduction functional $\widehat{I}_\epsilon : E^+\backslash\{0\} \to \mathbb{R}$ and the restriction $I_\epsilon : S^+ \to \mathbb{R}$ are defined by

$$\widehat{I}_\epsilon(u) = \Phi_\epsilon\big(\widehat{m}_\epsilon(u)\big) \quad \text{and} \quad I_\epsilon = \widehat{I}_\epsilon|_{S^+}.$$

Employing the same arguments explored in Sect. 3, we can check that all relevant conclusions in Sect. 3 remain true for $\mathcal{J}_\epsilon$, d_ϵ, $\mathcal{N}_\epsilon$, $\widehat{m}_\epsilon$, m_ϵ, $\widehat{I}_\epsilon$, and I_ϵ, respectively.

Similar to the proof of Lemma 3.5, we can conclude that for every $u \in E\backslash E^-$, there is only one point in $\mathcal{N}_\epsilon \cap \widehat{E}(u)$, and then there exists a unique pair $t \geq 0$ and $\varphi \in E^-$ such that

$$\mathcal{J}_\epsilon(tu + \varphi) = \max_{v\in\widehat{E}(u)} \mathcal{J}_\epsilon(v)$$

and

$$0 < d_\epsilon = \inf_{\mathcal{N}_\epsilon} \mathcal{J}_\epsilon = \inf_{u\in E^+\backslash\{0\}} \max_{v\in\widehat{E}(u)} \mathcal{J}_\epsilon(v).$$

Consider the limit problem

$$(-\Delta)^\alpha u + V(x)u = M(0)^2\left[\int_{\mathbb{R}^N} \frac{G(u)}{|x-y|^\mu}\,dy\right]g(u), \quad x \in \mathbb{R}^N. \tag{4.1}$$

Moreover, for convenience, we denote $\mathcal{J}_0 = \mathcal{J}_{M(0)}$, $d_0 = d_{M(0)}$, and $\mathcal{N}_0 = \mathcal{N}_{M(0)}$.

In the next step we shall establish an important relation between d_ϵ and d_0.

Lemma 4.1 $\lim_{\epsilon\to 0} d_\epsilon = d_0$.

Proof Let $d_{\epsilon_n} = \mathcal{J}_{\epsilon_n}(u_n)$ be the ground state energy of $\mathcal{J}_{\epsilon_n}$ for $u_n \in E$. From Lemma 3.5, we can deduce that there exists a unique pair (t_n, φ_n) with $t_n \in [0, +\infty)$ and $\varphi_n \in E^-$ such that $t_n u_n^+ + \varphi_n \in \mathcal{N}_0$ and

$$\mathcal{J}_0\big(t_n u_n^+ + \varphi_n\big) = \max_{u\in\widehat{E}(u_n)} \mathcal{J}_0(u).$$

In view of the definition of d_0, we conclude that

$$\begin{aligned}
d_0 &\leq \mathcal{J}_0\big(t_n u_n^+ + \varphi_n\big) \\
&= \mathcal{J}_{\epsilon_n}\big(t_n u_n^+ + \varphi_n\big) \\
&\quad + \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(\epsilon_n y)G(t_n u_n^+ + \varphi_n)M(\epsilon_n x)G(t_n u_n^+ + \varphi_n)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(0)G(t_n u_n^+(y) + \varphi_n(y))M(0)G(t_n u_n^+ + \varphi_n)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\
&\leq d_{\epsilon_n} + \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(\epsilon_n y)G(t_n u_n^+ + \varphi_n)M(\epsilon_n x)G(t_n u_n^+ + \varphi_n)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(0)G(t_n u_n^+ + \varphi_n)M(0)G(t_n u_n^+ + \varphi_n)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Letting $\epsilon_n \to 0$ as $n \to \infty$, for each $n \in \mathbb{N}$ we have $M(\epsilon_n x) \le M(0)$, therefore, combining with the above inequality, we obtain $d_0 \le d_{\epsilon_n}$ for any $n \in \mathbb{N}$.

On the other hand, employing Lemma 3.8, we can conclude that problem (4.1) has a ground state solution u_0. According to Lemma 3.5, we know that there exist $t_n \in [0,+\infty)$ and $\varphi_n \in E^-$ such that $t_n u_0^+ + \varphi_n \in \mathcal{N}_{\epsilon_n}$ and

$$\max_{u\in \widehat{E}(u_0)} \mathcal{J}_{\epsilon_n}(u) = \mathcal{J}_{\epsilon_n}\big(t_n u_0^+ + \varphi_n\big) \ge d_{\epsilon_n} \ge d_0 > 0, \quad \forall n \in \mathbb{N}.$$

Moreover, by Lemma 3.2, the sequence $\{t_n u_0^+ + \varphi_n\}$ is bounded. Thus, we can assume that $t_n \to t_0$ and $\varphi_n \rightharpoonup \varphi$ in E^- and

$$d_{\epsilon_n} \le \mathcal{J}_{\epsilon_n}\big(t_n u_0^+ + \varphi_n\big).$$

Therefore, applying Fatou's lemma, we get

$$\begin{aligned}
d_0 &= \liminf_{n\to\infty} d_{\epsilon_n} \le \limsup_{n\to\infty} d_{\epsilon_n} \le \limsup_{n\to\infty} \mathcal{J}_{\epsilon_n}\big(t_n u_0^+ + \varphi_n\big) \\
&\le \limsup_{n\to\infty}\bigg[\frac{1}{2}t_n^2\big\|u_0^+\big\|^2 - \frac{1}{2}\|\varphi_n\|^2 \\
&\quad - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(\epsilon_n y)G(t_n u_0^+ + \varphi_n)M(\epsilon_n x)G(t_n u_0^+ + \varphi_n)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\bigg] \\
&\le \frac{1}{2}t_0^2\big\|u_0^+\big\|^2 - \frac{1}{2}\|\varphi\|^2 \\
&\quad - \frac{M(0)^2}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(t_0 u_0^+ + \varphi(x))G(t_0 u_0^+ + \varphi)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\
&= \mathcal{J}_0\big(t_0 u_0^+ + \varphi\big) \le \mathcal{J}_0(u_0) = d_0,
\end{aligned}$$

which implies that $\lim_{\epsilon\to 0} d_\epsilon = d_0$, ending the proof. □

From the above discussion, we obtain the conclusion $\mathcal{J}_0(t_0 u_0^+ + \varphi) = \mathcal{J}_0(u_0) = d_0$, hence $t_0 u_0^+ + \varphi$ and u_0 are elements of $\mathcal{N}_0 \cap \widehat{E}(u_0)$. Applying Lemma 3.5, we can deduce that there is only one point in $\mathcal{N}_0 \cap \widehat{E}(u_0)$, thus $t_0 u_0^+ + \varphi = u_0$ and $t_n \to t_0 = 1$, $\varphi_n \rightharpoonup \varphi = u_0^-$.

Lemma 4.2 *There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0,\epsilon_0)$, we have $d_\epsilon < d_{M_\infty}$.*

Proof First, we can obtain that $M(0) > M_\infty$ from the assumption (M). So it is easy to see that $d_{M_\infty} > d_0$. Using Lemma 4.1, we find that there is $\epsilon_0 > 0$ such that $d_\epsilon < d_{M_\infty}$ for any $\epsilon \in (0,\epsilon_0)$. □

Now we give the existence result of ground state solutions of problem (1.1) as follows.

Lemma 4.3 *Assume that (V), (M), and (g_1)–(g_4) hold. Then problem (1.1) has a ground state solution for each $\epsilon \in (0,\epsilon_0)$.*

Proof Following the proof of Lemma 3.8 and using Lemma 3.7, we have to prove that there exists $u \in \mathcal{N}_\epsilon$ such that $\mathcal{J}_\epsilon(u) = d_\epsilon$. Observe that, by Lemma 3.7, we know that there exists

$\{u_n\} \subset \mathcal{N}_\epsilon$ such that $\mathcal{J}_\epsilon(u_n) \to d_\epsilon$ and $\mathcal{J}'_\epsilon(u_n) \to 0$, moreover, up to a subsequence, we can assume that $u_n \rightharpoonup u$ in E. Evidently, $\mathcal{J}'_\epsilon(u) = 0$.

In the following we show that $u \neq 0$ and $\mathcal{J}_\epsilon(u) = d_\epsilon$. It follows from Lemma 3.3 that

$$\begin{aligned} o_n(1) &= \langle \mathcal{J}'_\epsilon(u_n), u_n^+ \rangle \\ &= \|u_n^+\|^2 - \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(\epsilon y)G(u_n(y))}{|x-y|^\mu} M(\epsilon x)g(u_n)u_n^+ \,\mathrm{d}x\,\mathrm{d}y \\ &\geq 2d_\epsilon - \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(\epsilon y)G(u_n(y))}{|x-y|^\mu} M(\epsilon x)g(u_n)u_n^+ \,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

which implies that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(\epsilon y)G(u_n(y))}{|x-y|^\mu} M(\epsilon x)g(u_n)u_n^+ \,\mathrm{d}x\,\mathrm{d}y \geq 2d_\epsilon > 0.$$

Therefore, there exist a sequence $\{y_n\} \subset \mathbb{Z}^N$, $R > 0$, and $\delta > 0$ such that

$$\int_{B_R(y_n)} |u_n^+|^2 \,\mathrm{d}x \geq \delta, \quad \forall n \in \mathbb{N}. \tag{4.2}$$

Otherwise, according to Lemma 2.2, we directly get a contradiction.

We claim that the sequence $\{y_n\}$ is bounded in $\mathbb{R}^N$. Arguing by contradiction, we assume that $\{y_n\}$ is unbounded and $|y_n| \to +\infty$ as $n \to \infty$. We set $w_n(x) := u_n(x+y_n)$, then $w_n \rightharpoonup w$, and (4.2) implies that $w \neq 0$. For any $\psi \in C_0^\infty(\mathbb{R}^N)$, computing directly, we have

$$\begin{aligned} o_n(1) &= \langle \mathcal{J}'_\epsilon(u_n), \psi(x-y_n) \rangle \\ &= \int_{\mathbb{R}^N} \big[(-\Delta)^\alpha u_n(x)\psi(x-y_n) + V(x)u_n(x)\psi(x-y_n)\big]\,\mathrm{d}x \\ &\quad - \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(\epsilon y)G(u_n)}{|x-y|^\mu} M(\epsilon x)g(u_n)\psi(x-y_n)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^N} \big[(-\Delta)^\alpha w_n(x)\psi(x) + V(x)w_n(x)\psi(x)\big]\,\mathrm{d}x \\ &\quad - \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(\epsilon y+\epsilon y_n)G(w_n(y))}{|x-y|^\mu} M(\epsilon x+\epsilon y_n)g(w_n)\psi\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.3}$$

Taking the limit $n \to +\infty$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} \big[(-\Delta)^\alpha w\psi + V(x)w\psi\big]\,\mathrm{d}x - \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M_\infty G(w)}{|x-y|^\mu} M_\infty g(w)\psi\,\mathrm{d}x\,\mathrm{d}y \\ &= \langle \mathcal{J}'_\infty(w), \psi \rangle = 0, \quad \forall \psi \in C_0^\infty(\mathbb{R}^N). \end{aligned}$$

From the density of $C_0^\infty(\mathbb{R}^N)$ in E, we derive that

$$\begin{aligned} &\int_{\mathbb{R}^N} \big[(-\Delta)^\alpha w\phi + V(x)w\phi\big]\,\mathrm{d}x - \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M_\infty G(w)}{|x-y|^\mu} M_\infty g(w)\phi\,\mathrm{d}x\,\mathrm{d}y \\ &= \langle \mathcal{J}'_\infty(w), \phi \rangle = 0, \quad \forall \phi \in E, \end{aligned} \tag{4.4}$$

which implies that w is a nontrivial solution of problem (3.1) with $\pi = M_\infty$ and $w \in \mathcal{N}_{M_\infty}$. It follows from Fatou's lemma that

$$
\begin{aligned}
d_{M_\infty} &\leq \mathcal{J}_{M_\infty}(w) = \mathcal{J}_{M_\infty}(w) - \frac{1}{2}\langle \mathcal{J}'_{M_\infty}(w), w\rangle \\
&= \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M_\infty G(w)}{|x-y|^\mu} M_\infty\bigl[g(w)w - G(w)\bigr]\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \liminf_{n\to\infty} \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M_n(\epsilon_n y)G(w_n)}{|x-y|^\mu} M_n(\epsilon_n x)\bigl[g(w_n)w_n - G(w_n)\bigr]\,\mathrm{d}x\,\mathrm{d}y \\
&= \liminf_{n\to\infty} \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(\epsilon y)G(u_n)}{|x-y|^\mu} M(\epsilon x)\bigl[g(u_n)u_n - G(u_n)\bigr]\,\mathrm{d}x\,\mathrm{d}y \\
&= \liminf_{n\to\infty} \left[\mathcal{J}_\epsilon(u_n) - \frac{1}{2}\langle \mathcal{J}'_\epsilon(u_n), u_n\rangle\right] \\
&= d_\epsilon,
\end{aligned}
$$

where $M_n(\epsilon_n x) = M(\epsilon_n x + \epsilon_n y_n)$ and $M_n(\epsilon_n y) = M(\epsilon_n y + \epsilon_n y_n)$. Thus we deduce that $d_{M_\infty} \leq d_\epsilon$ for all $\epsilon > 0$. However, according to Lemma 4.2, we know that $d_\epsilon < d_{M_\infty}$ for $\epsilon < \epsilon_0$, a contradiction. Thus, $\{y_n\}$ is bounded, and then there is $R_0 > 0$ such that $B_{1+\sqrt{N}}(y_n) \subset B_{R_0}(0)$ for all $n \in \mathbb{N}$, so we have

$$
\int_{B_{R_0}(0)} |u_n|^2\,\mathrm{d}x \geq \int_{B_{1+\sqrt{N}}(y_n)} |u_n|^2\,\mathrm{d}x \geq \delta,
$$

which shows that $u_n \rightharpoonup u$ in E and $u \not\equiv 0$. By repeating the arguments leading to (4.3) and (4.4), we know that $u \in \mathcal{N}_\epsilon$ is a nontrivial solution for problem (1.1), thus, $d_\epsilon \leq \mathcal{J}_\epsilon(u)$.

On the other hand, on account of Fatou's lemma, we conclude that

$$
\begin{aligned}
d_\epsilon &= \liminf_{n\to\infty} \left[\mathcal{J}_\epsilon(u_n) - \frac{1}{2}\langle \mathcal{J}'_\epsilon(u_n), u_n\rangle\right] \\
&= \liminf_{n\to\infty} \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(\epsilon y)G(u_n)}{|x-y|^\mu} M(\epsilon x)\bigl[g(u_n)u_n - G(u_n)\bigr]\,\mathrm{d}x\,\mathrm{d}y \\
&\geq \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(\epsilon y)G(u)}{|x-y|^\mu} M(\epsilon x)\bigl[g(u)u - G(u)\bigr]\,\mathrm{d}x\,\mathrm{d}y \\
&= \mathcal{J}_\epsilon(u) - \frac{1}{2}\langle \mathcal{J}'_\epsilon(u), u\rangle \\
&= \mathcal{J}_\epsilon(u).
\end{aligned}
$$

Consequently, $d_\epsilon = \mathcal{J}_\epsilon(u)$, which implies that u is a ground state solution of problem (1.1), ending the proof. □

4.2 Concentration of ground state solutions

We now shall prove the concentration of the maximum points of the ground state solution u_ϵ obtained in Lemma 4.3. Furthermore, the completed proof of Theorem 1.1 will also be given. Our aim is to show that if x_ϵ is a maximum point of $|u_\epsilon|$, then

$$
\lim_{\epsilon\to 0} M(\epsilon x_\epsilon) = M(0).
$$

In other words, we have to show that if $\epsilon_n \to 0$, then, for some subsequence, $\epsilon_n x_{\epsilon_n} \to z$ for some $z \in \mathscr{M}$, where

$$\mathscr{M} = \{x \in \mathbb{R}^N : M(x) = M(0)\}$$

is the set of the maximum points of $M(x)$.

Let $\{\epsilon_n\} \subset (0, \epsilon_0)$ with $\epsilon_n \to 0$ as $n \to \infty$, and we denote $u_n := u_{\epsilon_n}$. Then we get the following relation:

$$\mathcal{J}'_{\epsilon_n}(u_n) = 0 \quad \text{and} \quad \mathcal{J}_{\epsilon_n}(u_n) = d_{\epsilon_n}.$$

Using a standard argument, we can deduce that $\{u_n\}$ is bounded.

Lemma 4.4 *There exist $\{y_n\} \subset \mathbb{Z}^N$ and constants $R > 0$, $\delta > 0$ such that*

$$\int_{B_R(y_n)} |u_n^+|^2 \,\mathrm{d}x \geq \delta.$$

Proof If it is not true, then, using Lemma 2.2, we get $u_n^+ \to 0$ in $L^q(\mathbb{R}^N)$ for $q \in (2, 2_\alpha^*)$. Furthermore, from (2.3), Lemma 2.1, and Hardy–Littlewood–Sobolev inequality, we obtain

$$\begin{aligned}
0 &\leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{M(\epsilon_n y)G(u_n)}{|x-y|^\mu} M(\epsilon_n x) g(u_n) u_n^+ \,\mathrm{d}x\,\mathrm{d}y \\
&\leq M(0)^2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{G(u_n)}{|x-y|^\mu} g(u_n) u_n^+ \,\mathrm{d}x\,\mathrm{d}y \\
&\to 0.
\end{aligned}$$

This, together with the fact that $u_n \in \mathcal{N}_{\epsilon_n}$, leads to $\|u_n^+\| \to 0$. Evidently, this is a contradiction since $\|u_n^+\| \geq \sqrt{2d_0} > 0$, finishing the proof. □

Lemma 4.5 *The sequence $\{\epsilon_n y_n\}$ is bounded in $\mathbb{R}^N$ and $\lim_{n\to\infty} \epsilon_n y_n = z \in \mathscr{M}$.*

Proof Set $v_n(x) := u_n(x + y_n)$, then, up to a subsequence, we have $v_n \rightharpoonup v$ with $v \not\equiv 0$. In the following, we show that the sequence $\{\epsilon_n y_n\}$ is bounded in $\mathbb{R}^N$. Otherwise, we suppose that $|\epsilon_n y_n| \to \infty$ as $n \to \infty$. Observe that u_n is the ground state solution of problem (1.1), and then we obtain the following fact:

$$(-\Delta)^\alpha v_n + V(x) v_n = \left[\int_{\mathbb{R}^N} \frac{M_n(\epsilon_n y) G(v_n)}{|x-y|^\mu} \,\mathrm{d}y\right] M_n(\epsilon_n x) g(v_n), \quad x \in \mathbb{R}^N, \tag{4.5}$$

where $M_n(\epsilon_n x) = M(\epsilon_n x + \epsilon_n y_n)$, and we also have the energy relation

$$\begin{aligned}
\widetilde{E}(v_n) &= \frac{1}{2}\left(\|v_n^+\|^2 - \|v_n^-\|^2\right) - \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{M_n(\epsilon_n x) G(v_n) M_n(\epsilon_n y) G(v_n)}{|x-y|^\mu} \,\mathrm{d}x\,\mathrm{d}y \\
&= \mathcal{J}_{\epsilon_n}(u_n) \\
&= \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{M(\epsilon_n y) G(u_n)}{|x-y|^\mu} M(\epsilon_n y) \left[g(u_n) u_n - G(u_n)\right] \mathrm{d}x\,\mathrm{d}y \\
&= d_{\epsilon_n}.
\end{aligned}$$

Moreover, since $M_n(\epsilon_n x) \to M_\infty$, due to the fact $v_n \rightharpoonup v$, for any $\phi \in C_0^\infty(\mathbb{R}^N)$, we can deduce that

$$\int_{\mathbb{R}^N} \big[(-\Delta)^\alpha v\phi + V(x)v\phi\big]\,\mathrm{d}x - \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M_\infty G(v)}{|x-y|^\mu} M_\infty g(v)\phi\,\mathrm{d}x\,\mathrm{d}y = 0.$$

So we can see that $v \in \mathcal{N}_{M_\infty}$ which is a nontrivial solution of problem (1.1) with $\pi = M_\infty$. According to Fatou's lemma and Lemma 4.1, we obtain

$$\begin{aligned} d_{M_\infty} &\le \mathcal{J}_{M_\infty}(v) = \mathcal{J}_{M_\infty}(v) - \frac{1}{2}\big\langle \mathcal{J}'_{M_\infty}(v), v\big\rangle \\ &= \frac{1}{2}M(0)^2 \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(v(y))}{|x-y|^\mu}\big[g(v)v - G(v)\big]\,\mathrm{d}x\,\mathrm{d}y \\ &\le \liminf_{n\to\infty} \frac{1}{2} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M_n(\epsilon_n y)G(v_n(y))}{|x-y|^\mu} M_n(\epsilon_n x)\big[g(v_n)v_n - G(v_n)\big]\,\mathrm{d}x\,\mathrm{d}y \\ &\le \liminf_{n\to\infty} \frac{1}{2} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(\epsilon_n y)G(u_n(y))}{|x-y|^\mu} M(\epsilon_n x)\big[g(u_n)u_n - G(u_n)\big]\,\mathrm{d}x\,\mathrm{d}y \\ &= \liminf_{n\to\infty}\bigg[\mathcal{J}_{\epsilon_n}(u_n) - \frac{1}{2}\big\langle \mathcal{J}'_{\epsilon_n}(u_n), u_n\big\rangle\bigg] \\ &= \liminf_{n\to\infty} \mathcal{J}_{\epsilon_n}(u_n) = \lim_{n\to\infty} d_{\epsilon_n} = d_0, \end{aligned} \tag{4.6}$$

while $d_0 < d_{M_\infty}$ from Lemma 4.2. So, we get a contradiction. Thus $\{\epsilon_n y_n\}$ is bounded, and, passing to a subsequence, we may assume that $\epsilon_n y_n \to z$. Similar to the above discussion, for any $\psi \in E$, one has

$$\int_{\mathbb{R}^N} \big[(-\Delta)^\alpha v\psi + V(x)v\psi\big]\,\mathrm{d}x - \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M(z)G(v)}{|x-y|^\mu} M(z)g(v)\psi\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Evidently, we know that $v \in \mathcal{N}_{M(z)}$ and it is a ground state solution of the problem

$$(-\Delta)^\alpha v + V(x)v = \bigg[\int_{\mathbb{R}^N} \frac{M(z)G(v(y))}{|x-y|^\mu}\,\mathrm{d}y\bigg] M(z)g(v), \quad x \in \mathbb{R}^N. \tag{4.7}$$

Similarly, we can show that $d_{M(z)} \le d_0$ according to the above argument. Using assumption (M), we know that $M(z) = M(0)$. Hence, we get that

$$\lim_{n\to\infty} \epsilon_n y_n = z \quad \text{and} \quad z \in \mathcal{M},$$

completing the proof. □

Lemma 4.6 *The sequence $\{v_n\}$ converges strongly to v in E, and there exists $C > 0$ such that $\|v_n\|_\infty \le C$ for all $n \in \mathbb{N}$, and $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in n.*

Proof First, following the arguments used in [42], we can derive that $v_n \to v$ in E. Moreover, Lemma 4.5 shows that v_n satisfies the following equation:

$$(-\Delta)^\alpha v_n + V(x)v_n = \bigg[\int_{\mathbb{R}^N} \frac{M_n(\epsilon_n y)G(v_n)}{|x-y|^\mu}\,\mathrm{d}y\bigg] M_n(\epsilon_n x)g(v_n), \quad x \in \mathbb{R}^N.$$

Next we need to claim that there exists $C > 0$ such that

$$\left|\left[\int_{\mathbb{R}^N} \frac{M_n(\epsilon_n y)G(v_n)}{|x-y|^\mu}\,\mathrm{d}y\right]M_n(\epsilon_n x)\right| \le C. \tag{4.8}$$

Employing Hardy–Littlewood–Sobolev inequality, we observe that for any $\gamma > N/\mu$, if $w \in L^{\frac{N\gamma}{(N-\mu)\gamma+N}}$, then

$$\left\|\frac{1}{|x|^\mu} * w\right\|_\gamma \le C(N,\mu,\gamma)\|w\|_{\frac{N\gamma}{(N-\mu)\gamma+N}}.$$

Hence, together with Lemma 2.1, for any $\gamma > N/\mu$, we deduce that

$$\left\|\int_{\mathbb{R}^N} \frac{G(v_n)}{|x-y|^\mu}\,\mathrm{d}y\right\|_\gamma \le C\|v_n\|_{\frac{N\gamma}{(N-\mu)\gamma+N}} \le c_5.$$

Letting $\gamma \to +\infty$, due to the boundedness of M, we can see that (4.8) holds.

For any $L > 0$ and $\beta > 1$, let

$$v_{L,n} = \begin{cases} v_n, & v_n(x) \le L, \\ L, & v_n(x) \ge L. \end{cases}$$

We define the function

$$r(v_n) = r_{L,\beta}(v_n) = v_n v_{L,n}^{2(\beta-1)} \in E.$$

Since r is increasing in $(0,+\infty)$, we obtain

$$(k-l)\big[r(k)-r(l)\big] \ge 0 \quad \text{for any } k,l \in \mathbb{R}^+.$$

Set

$$P(t) = \frac{|t|^2}{2} \quad \text{and} \quad Q(t) = \int_0^t \big(r'(\tau)\big)^{\frac{1}{2}}\,\mathrm{d}\tau.$$

For each $k,l \in \mathbb{R}$, without loss of generality, we may assume that $k > l$, and then Jensen inequality yields that

$$\begin{aligned} P'(k-l)\big[r(k)-r(l)\big] &= (k-l)\big[r(k)-r(l)\big] = (k-l)\int_l^k r'(t)\,\mathrm{d}t \\ &= (k-l)\int_l^k \big(Q'(t)\big)^2\,\mathrm{d}t \ge \left[\int_l^k Q'(t)\,\mathrm{d}t\right]^2. \end{aligned}$$

Similarly, we can conclude that the above inequality is also true for the case $k \le l$. Therefore

$$P'(k-l)\big[r(k)-r(l)\big] \ge \big|Q(k)-Q(l)\big|^2 \quad \text{for any } k,l \in \mathbb{R}. \tag{4.9}$$

Using (4.9), we derive

$$\left|Q(v_n(x)) - Q(v_n(y))\right|^2 \le (v_n(x) - v_n(y))\left[\left(v_n v_{L,n}^{2(\beta-1)}\right)(x) - \left(v_n v_{L,n}^{2(\beta-1)}\right)(y)\right]. \tag{4.10}$$

Combining with (4.5) and (4.10), and taking $r(v_n) = v_n v_{L,n}^{2(\beta-1)}$, we can conclude that

$$\begin{aligned}
&\left[Q(v_n)\right]_{D^{\alpha,2}}^2 + \int_{\mathbb{R}^N} V(x)|v_n|^2 v_{L,n}^{2(\beta-1)}\,\mathrm{d}x \\
&\quad\le \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(v_n(x) - v_n(y))}{|x-y|^{N+2\alpha}}\left[\left(v_n v_{L,n}^{2(\beta-1)}\right)(x) - \left(v_n v_{L,n}^{2(\beta-1)}\right)(y)\right]\mathrm{d}x\,\mathrm{d}y \\
&\qquad + \int_{\mathbb{R}^N} V(x)|v_n|^2 v_{L,n}^{2(\beta-1)}\,\mathrm{d}x \\
&\quad= \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{M_n(\epsilon_n y)G(v_n)}{|x-y|^{\mu}} M_n(\epsilon_n x)g(v_n)v_n v_{L,n}^{2(\beta-1)}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Since

$$L(v_n) \ge \frac{1}{\beta} v_n v_{L,n}^{(\beta-1)},$$

in view of Lemma 2.1, we get

$$\left[Q(v_n)\right]_{D^{\alpha,2}}^2 \ge c_6 \left\|Q(v_n)\right\|_{2_\alpha^*}^2 \ge \left(\frac{1}{\beta}\right)^2 c_7 \left\|v_n v_{L,n}^{(\beta-1)}\right\|_{2_\alpha^*}^2. \tag{4.11}$$

On the other hand, using (2.3) and (4.8), it follows that

$$\left\|v_n v_{L,n}^{(\beta-1)}\right\|_{2_\alpha^*}^2 \le c_8 \beta^2 \int_{\mathbb{R}^N} |v_n|^q v_{L,n}^{2(\beta-1)}\,\mathrm{d}x.$$

Letting $w_{L,n} := v_n v_{L,n}^{(\beta-1)}$, on account of Hölder inequality, we infer that

$$\|w_{L,n}\|_{2_\alpha^*}^2 \le c_9 \beta^2 \left(\int_{\mathbb{R}^N} |v_n|^{2_\alpha^*}\,dx\right)^{\frac{q-2}{2_\alpha^*}} \left(\int_{\mathbb{R}^N} |w_{L,n}|^{\sigma_\alpha^*}\,dx\right)^{\frac{2}{\sigma_\alpha^*}},$$

where $\sigma_\alpha^* := \frac{22_\alpha^*}{2_\alpha^*-(q-2)} \in (2, 2_\alpha^*)$. Moreover, from the boundedness of v_n we derive

$$\|w_{L,n}\|_{2_\alpha^*}^2 \le c_{10}\beta^2 \|w_{L,n}\|_{\sigma_\alpha^*}^2. \tag{4.12}$$

Observe that if $v_n^\beta \in L^{\sigma_\alpha^*}(\mathbb{R}^N)$, using (4.12) and the fact that $v_{L,n} \le v_n$, we obtain

$$\|w_{L,n}\|_{2_\alpha^*}^2 \le c_{11}\beta^2 \left(\int_{\mathbb{R}^N} |v_n|^{\beta\sigma_\alpha^*}\,dx\right)^{\frac{2}{\sigma_\alpha^*}} < \infty. \tag{4.13}$$

Letting $L \to +\infty$ and taking the limit in (4.13), by Fatou's lemma, we have

$$\|v_n\|_{\beta 2_\alpha^*} \le c_{12}^{\frac{1}{\beta}} \beta^{\frac{1}{\beta}} \|v_n\|_{\beta\sigma_\alpha^*}, \tag{4.14}$$

whenever $v_n^{\beta\sigma_\alpha^*} \in L^1(\mathbb{R}^N)$.

We set $\beta := \frac{2^*_\alpha}{\sigma^*_\alpha} > 1$ and note that $v_n \in L^{2^*_\alpha}(\mathbb{R}^N)$, so the above inequality holds for the case of β. Then, observing that $\beta^2\sigma^*_\alpha = \beta 2^*_\alpha$, we know that (4.14) holds with β replaced by β^2. Therefore, we obtain

$$\|v_n\|_{\beta^2 2^*_\alpha} \leq c_{13}^{\frac{1}{\beta^2}} \beta^{\frac{2}{\beta^2}} \|v_n\|_{\beta^2\sigma^*_\alpha} \leq c_{14}^{(\frac{1}{\beta}+\frac{1}{\beta^2})} \beta^{(\frac{1}{\beta}+\frac{2}{\beta^2})} \|v_n\|_{\beta\sigma^*_\alpha}.$$

Using iteration and recalling that $\beta\sigma^*_\alpha := 2^*_\alpha$, we can infer that for each $m \in \mathbb{N}$,

$$\|v_n\|_{\beta^m 2^*_\alpha} \leq c_{15}^{\sum_{i=1}^m \frac{1}{\beta^i}} \beta^{\sum_{i=1}^m \frac{i}{\beta^i}} \|v_n\|_{2^*_\alpha}. \tag{4.15}$$

Letting $m \to +\infty$ and recalling that $\|v_n\|_{2^*_\alpha} \leq \widetilde{K}$, we have

$$\|v_n\|_\infty \leq c_{16}^{\sigma_1} \beta^{\sigma_2} \widetilde{K} < \infty,$$

where

$$\sigma_1 := \sum_{i=1}^\infty \frac{1}{\beta^i} < \infty \quad \text{and} \quad \sigma_2 := \sum_{i=1}^\infty \frac{i}{\beta^i} < \infty.$$

Finally, by using a similar argument as in [3], we can conclude that $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in n. This proves the lemma. □

Lemma 4.7 *There exists $\nu > 0$ such that $\|v_n\|_\infty \geq \nu$ for all $n \in \mathbb{N}$.*

Proof Arguing by contradiction, we assume that $\|v_n\|_\infty \to 0$ as $n \to \infty$. Then according to Lemma 4.6, it is easy to see that $v = 0$, which implies a contradiction, completing the proof. □

Now we are in a position to finish the proof of Theorem 1.1.

Proof of Theorem 1.1 (completion) Assume that p_n is a global maximum point of $|v_n(x)|$ for each $n \in \mathbb{N}$, then

$$|v_n(p_n)| = \max_{x\in\mathbb{R}^N} |v_n(x)|.$$

Since $v_n(x) = u_n(x + y_n)$, we see that $s_n = p_n + y_n$ is a maximum point of $|u_n(x)|$. Lemma 4.7 shows that there exists $\nu > 0$ such that

$$|v_n(p_n)| \geq \nu \quad \text{for all } n \in \mathbb{N},$$

which implies that the sequence $\{p_n\}$ is bounded. So, we conclude from Lemma 4.5 that

$$\epsilon_n s_n = \epsilon_n p_n + \epsilon_n y_n \to z \in \mathscr{M}.$$

Consequently, we have

$$\lim_{n\to\infty} M(\epsilon_n s_n) = M(z), \quad z \in \mathscr{M}.$$

Furthermore, following the proofs of Lemmas 4.5 and 4.6, we know that $u_n(x+s_n)$ converges to a ground state solution v of the following limit equation:

$$(-\Delta)^\alpha v + V(x)v = M(0)^2\left[\int_{\mathbb{R}^N}\frac{G(v)}{|x-y|^\mu}\,dy\right]g(v),\quad x\in\mathbb{R}^N,$$

finishing the proof of all conclusions of Theorem 1.1. □

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Author details
[1]School of Mathematics and Information Science, Xiangnan University, Chenzhou, 423000 Hunan, China. [2]Hunan Key Laboratory for Computation and Simulation in Science and Engineering, School of Mathematics and Computational Science, Xiangtan University, Xiangtan, 411105 Hunan, China. [3]School of Computer and Artificial Intelligence, Xiangnan University, Chenzhou, 423000 Hunan, China.

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