# Determination of rigid inclusions immersed in an isotropic elastic body from boundary measurement 

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#### Abstract

We study the determination of some rigid inclusions immersed in an isotropic elastic medium from overdetermined boundary data. We propose an accurate approach based on the topological sensitivity technique and the reciprocity gap concept. We derive a higher-order asymptotic formula, connecting the known boundary data and the unknown inclusion parameters. The obtained formula is interesting and useful tool for developing accurate and robust numerical algorithms in geometric inverse problems.


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## 1 Introduction

Let $\Omega$ be a regular domain in $\mathbb{R}^{3}$ occupied by a homogeneous isotropic linear elastic materials. The elastic displacement vector $w$ in $\Omega$ satisfies the following linear elasticity system:

$$
\begin{cases}-\operatorname{div} \sigma(w)=\mathcal{F} & \text { in } \Omega,  \tag{1.1}\\ w=\mathcal{U}_{d} & \text { on } \Gamma_{a} \\ w=0 & \text { on } \Gamma_{i}\end{cases}
$$

where

- $\sigma(w)=\left(\sigma_{i j}(w)\right)_{1 \leq i, j \leq d}$ is the stress tensor,
- $\mathcal{F}$ is the gravitational force,
- $\mathcal{U}_{d}$ is a given boundary data measured or imposed on the accessible part $\Gamma_{a}$ of the boundary $\partial \Omega$,
- $\Gamma_{i}$ is a non-accessible part of $\partial \Omega$, such that $\partial \Omega=\Gamma_{a} \cup \Gamma_{i}, \Gamma_{a} \cap \Gamma_{i}=\emptyset$ and $\operatorname{meas}\left(\Gamma_{a}\right) \neq 0$.

The stress tensor is given by the Hooke law:

$$
\sigma_{i j}(w)=2 \mu e_{i j}(w)+\lambda \delta_{i j} \sum_{k=1}^{d} e_{k k}(w), \quad 1 \leq i, j \leq d
$$

where

- $\delta_{i j}$ is the Kronecker symbol,
$-e(w)=\left(e_{i j}(w)\right)_{1 \leq i, j \leq d}$ is the strain tensor given by

$$
e_{i j}(w)=\frac{1}{2}\left(\frac{\partial w_{i}}{\partial x_{j}}+\frac{\partial w_{j}}{\partial x_{i}}\right), \quad 1 \leq i, j \leq d
$$

- $\mu$ and $\lambda$ are the Lamé coefficients given by

$$
\mu=\frac{E}{2(1+v)} \quad \text { and } \quad \lambda=\frac{E v}{(1+v)(1-2 v)},
$$

with $E$ is the Young modulus, and $v$ is the Poisson ratio.
We suppose that the elastic medium $\Omega$ contains a finite number of well-separated rigid inclusions $\mathcal{I}_{i}, i=1, \ldots, m$, not close to the boundary $\partial \Omega$. In this work, we assume that each inclusion $\mathcal{I}_{i}$ is characterized by its center $\xi_{i} \in \Omega$, size $\rho_{i}>0$, and its shape $\mathcal{I}_{i}$ with $\mathcal{I}_{i} \subset \mathbb{R}^{3}$, which are fixed bounded and smooth domains containing the origin. In other word, the inclusion $\mathcal{I}_{i}$ can be defined as $\mathcal{I}_{i}=\xi_{i}+\rho_{i} \mathcal{I}_{i}, i=1, \ldots, m$.
The problem that we consider can be formulated as follows:

- Given two boundaries data on the boundary $\Gamma_{a}$ : a measured displacement $\mathcal{U}_{d}$ and an imposed force $g$.
- Find the unknown inclusion $\mathcal{I}=\bigcup_{i=1}^{m} \mathcal{I}_{i} \subset \subset \Omega$, such that the displacement field $w_{\mathcal{I}}$ in the presence of inclusion satisfies the following overdetermined elasticity problem [1]

$$
\begin{cases}-\operatorname{div} \sigma\left(w_{\mathcal{I}}\right)=\mathcal{F} & \text { in } \Omega \backslash \overline{\mathcal{I}},  \tag{1.2}\\ w_{\mathcal{I}}=\mathcal{U}_{d} & \text { on } \Gamma_{a} \\ \sigma\left(w_{\mathcal{I}}\right) \mathbf{n}=g & \text { on } \Gamma_{a} \\ w_{\mathcal{I}}=0 & \text { on } \Gamma_{i} \\ w_{\mathcal{I}}=0 & \text { on } \partial \mathcal{I}\end{cases}
$$

where $\mathbf{n}$ denotes the outward normal to the boundary $\Gamma_{a}$.
In this formulation, the elastic domain $\Omega \backslash \overline{\mathcal{I}}$ is unknown since the inclusion geometry $\mathcal{I}$ is unknown. It is well known that this kind of problem is ill-posed in the sense of Hadamard [2]. The majority of investigation focusing on this type of problems fall into the category of shape optimization and utilize the shape derivation techniques $[3,4]$.

In this work, we suggest a new formulation for solving the above inverse problem based on the reciprocity gap concept [5-7] and the topological sensitivity analysis method [815].

More precisely, let $\mathcal{I}_{\xi, \rho}=\xi+\rho \mathcal{I}$ be an unknown inclusion, strictly embedded inside the elastic medium $\Omega$. The reciprocity gap functional is a scalar quantity describing the elastic material response to an imposed force on the boundary $\partial \Omega$. Related to the presence of the
inclusion $\mathcal{I}_{\xi, \rho}$, this function is defined by

$$
\begin{align*}
& \mathcal{R}_{\xi, \rho}: H^{1}(\Omega) \longrightarrow \mathbb{R} \\
& \mathcal{R}_{\xi, \rho}(u)=\int_{\partial \Omega} \sigma(u) \mathbf{n} w_{\rho} d s-\int_{\partial \Omega} \sigma\left(w_{\rho}\right) \mathbf{n} u d s \tag{1.3}
\end{align*}
$$

where $w_{\rho}$ is the solution of the elasticity system in the perforated domain $\Omega \backslash \overline{\mathcal{I}_{\xi, \rho}}$ :

$$
\begin{cases}-\operatorname{div} \sigma\left(w_{\rho}\right)=\mathcal{F} & \text { in } \Omega \backslash \overline{\mathcal{I}_{\xi, \rho}},  \tag{1.4}\\ w_{\rho}=\mathcal{U}_{d} & \text { on } \Gamma_{a}, \\ w_{\rho}=0 & \text { on } \Gamma_{i}, \\ w_{\rho}=0 & \text { on } \partial \mathcal{I}_{\xi, \rho} .\end{cases}
$$

In the absence of any inclusions, the reciprocity gap functional is denoted by $\mathcal{R}_{0}$ and defined by:

$$
\mathcal{R}_{0}(u)=\int_{\partial \Omega} \sigma(u) \mathbf{n} w_{0} d s-\int_{\partial \Omega} \sigma\left(w_{0}\right) \mathbf{n} u d s,
$$

where $w_{0}$ is the solution to the elasticity problem in the entire domain

$$
\begin{cases}-\operatorname{div} \sigma\left(w_{0}\right)=\mathcal{F} & \text { in } \Omega,  \tag{1.5}\\ w_{0}=\mathcal{U}_{d} & \text { on } \Gamma_{a}, \\ w_{0}=0 & \text { on } \Gamma_{i} .\end{cases}
$$

Our goal is to establish a relation between the boundary data and the unknown parameters characterizing the inclusion $\mathcal{I}_{\xi, \rho}$. To this end, we will develop a higher-order asymptotic formula connecting the known boundary data and the unknown inclusion parameters; the location $\xi$, the size $\rho$, and the shape $\mathcal{I}$ [16].

This article is organized as follows: Sect. 2 deals with some technical results. A preliminary estimate describing the variation of the reciprocity gap functional with respect to the presence of an inclusion $\mathcal{I}=\xi+\rho \mathcal{I}$ inside the domain $\Omega$ is presented in Proposition 2.1. To derive the expected formula, we start our analysis by studying the influence of the presence of the inclusion on the elastic displacement vector. In Sect. 3, we calculate a higher-order asymptotic expansion of the perturbed displacement vector $w_{\rho}$ with respect to the inclusion size $\rho$. In Sect. 4, we derive a boundary asymptotic formula of high order for the reciprocity gap functional.

## 2 Preliminary results

We begin this study by the presentation of some technical results.

### 2.1 Variation of $\mathcal{R}_{\xi, \rho}$

We consider the subspace

$$
\mathcal{V}=\left\{u \in H^{1}(\Omega) ; \operatorname{div} \sigma(u)=0 \text { in } \Omega\right\} .
$$

The restriction of the reciprocity gap functional $\mathcal{R}_{\xi, \rho}$ to the subspace $\mathcal{V}$ leads to the following estimate.

Proposition 2.1 On $\mathcal{V}$, the variation of $\mathcal{R}$ reads

$$
\begin{equation*}
\mathcal{R}_{\xi, \rho}(u)-\mathcal{R}_{0}(u)=\int_{\partial \mathcal{I}_{\xi, \rho}} \sigma\left(w_{\rho}-w_{0}\right) \mathbf{n} u d s-\int_{\mathcal{I}_{\xi, \rho}} \sigma(u): e\left(w_{0}\right) d x . \tag{2.1}
\end{equation*}
$$

Proof Since $\operatorname{div} \sigma(u)=0$ in $\Omega \backslash \overline{\mathcal{I}_{\xi, \rho}}$, and $w_{\rho}=0$ on $\partial \mathcal{I}_{\xi, \rho}$. By Green's formula, one can check

$$
\int_{\Omega \backslash \overline{\mathcal{I}_{\xi, \rho}}} \sigma(u): e\left(w_{\rho}\right) d x=\int_{\partial \Omega} \sigma(u) \mathbf{n} w_{\rho} d s
$$

The weak variational formulation of (1.4) implies

$$
\int_{\Omega \backslash \overline{\mathcal{I}_{\xi, \rho}}} \sigma\left(w_{\rho}\right): e(u) d x=\int_{\partial \Omega} \sigma\left(w_{\rho}\right) \mathbf{n} u d s+\int_{\partial \mathcal{I}_{\xi, \rho}} \sigma\left(w_{\rho}\right) \mathbf{n} u d s+\int_{\Omega \backslash \overline{\mathcal{I}_{\xi, \rho}}} \mathcal{F} u .
$$

From the fact $\int_{\Omega \backslash \overline{\mathcal{I}_{\xi, \rho}}} \sigma\left(w_{\rho}\right): e(u) d x=\int_{\Omega \backslash \overline{\mathcal{I}_{\xi, \rho}}} \sigma(u): e\left(w_{\rho}\right) d x$, it follows

$$
\begin{equation*}
\mathcal{R}_{\xi, \rho}(u)=\int_{\partial \mathcal{I}_{\xi, \rho}} \sigma\left(w_{\rho}\right) \mathbf{n} u d s+\int_{\Omega \backslash \overline{\mathcal{I}_{\xi, \rho}}} \mathcal{F} u d x \quad \forall u \in \mathcal{V} \tag{2.2}
\end{equation*}
$$

Similarly,

$$
\begin{aligned}
& \int_{\Omega} \sigma(u): e\left(w_{0}\right) d x=\int_{\partial \Omega} \sigma(u) \mathbf{n} w_{0} d s, \quad \forall u \in \mathcal{V}, \\
& \int_{\Omega} \sigma\left(w_{0}\right): e(u) d x=\int_{\partial \Omega} \sigma\left(w_{0}\right) \mathbf{n} u d s+\int_{\Omega} \mathcal{F} u d x .
\end{aligned}
$$

Then,

$$
\begin{equation*}
\mathcal{R}_{0}(u)=\int_{\Omega} \mathcal{F} u d x, \quad \forall u \in \mathcal{V} . \tag{2.3}
\end{equation*}
$$

From (2.2) and (2.3), we deduce

$$
\mathcal{R}_{\xi, \rho}(u)-\mathcal{R}_{0}(u)=\int_{\partial \mathcal{I}_{\xi, \rho}} \sigma\left(w_{\rho}\right) \mathbf{n} u d s-\int_{\mathcal{I}_{\xi, \rho}} \mathcal{F} u d x .
$$

Using the fact that $-\operatorname{div} \sigma\left(w_{0}\right)=\mathcal{F}$ in $\mathcal{I}_{\xi, \rho}$, one can derive

$$
\mathcal{R}_{\xi, \rho}(u)-\mathcal{R}_{0}(u)=\int_{\partial \mathcal{I}_{\xi, \rho}} \sigma\left(w_{\rho}-w_{0}\right) \mathbf{n} u d s-\int_{\mathcal{I}_{\xi, \rho}} \sigma\left(w_{0}\right): e(u) d x, \quad \forall u \in \mathcal{V}
$$

### 2.2 Green's function and related sub-space

Let $G$ be the fundamental solution of the linear elasticity equation in $\mathbb{R}^{3}$, which is given by:

$$
\begin{equation*}
G(y)=\frac{1}{r}\left(\beta I+\gamma e_{r} e_{r}^{T}\right) \tag{2.4}
\end{equation*}
$$

where $I$ is the $3 \times 3$ identity matrix, $r=\|y\|, e_{r}=y / r, e_{r}{ }^{T}$ is the transposed vector of $e_{r}$, and

$$
\beta=\frac{\lambda+3 \mu}{8 \pi \mu(\lambda+2 \mu)}, \quad \gamma=\frac{\lambda+\mu}{8 \pi \mu(\lambda+2 \mu)} .
$$

The $j$ th column $G_{j}$ of $G$ satisfies the equation

$$
-\operatorname{div} \sigma\left(G_{j}\right)=\delta e_{j} \quad \text { in } \mathbb{R}^{3},
$$

with $\left(e_{j}\right)_{j=1}^{3}$ is the canonical basis of $\mathbb{R}^{3}$, and $\delta$ is the Dirac distribution.
To make relation (2.1) more explicit, we consider the following sub-space of $\mathcal{V}$, defined by the restriction of the functions $G_{\eta}^{j}(x)=G^{j}(x-\eta), \eta \in \mathbb{R}^{3} \backslash \bar{\Omega}$ to the domain $\Omega$ :

$$
\mathcal{V}^{j}=\left\{x \mapsto G^{j}(x-\eta)_{\mid \Omega}, \eta \in \mathbb{R}^{3} \backslash \bar{\Omega}\right\} .
$$

For each $1 \leq j \leq 3$, we denote by $\mathcal{R}_{\xi, \rho}^{j}$, the reciprocity gap function associated with the sub-space $\mathcal{V}^{j}$. Identifying each function $x \mapsto G^{j}(x-\eta)_{\mid \Omega}$ with its parameter $\eta$, then $\mathcal{R}_{\xi, \rho}^{j}$ can be represented as:

$$
\mathcal{R}_{\xi, \rho}^{j}(\eta)=\int_{\partial \Omega} \sigma\left(G^{j}(x-\eta)\right) \mathbf{n} w_{\rho} d s(x)-\int_{\partial \Omega} \sigma\left(w_{\rho}\right) \mathbf{n} G^{j}(x-\eta) d s(x), \quad \forall \eta \in \mathbb{R}^{3} \backslash \bar{\Omega} .
$$

From Proposition 2.1, one can deduce the following corollary.
Corollary 2.2 For each $1 \leq j \leq 3$, the function $\mathcal{R}_{\xi, \rho}^{j}$ verifies

$$
\begin{aligned}
& \mathcal{R}_{\xi, \rho}^{j}(\eta)-\mathcal{R}_{0}^{j}(\eta) \\
& \quad=\int_{\partial \mathcal{I}_{\xi, \rho}} \sigma\left(w_{\rho}-w_{0}\right) \mathbf{n} G^{j}(x-\eta) d s-\int_{\mathcal{I}_{\xi, \rho}} \sigma_{x}\left(G^{j}(x-\eta)\right): e\left(w_{0}\right) d x, \quad \forall \eta \in \mathbb{R}^{3} \backslash \bar{\Omega} .
\end{aligned}
$$

Next, we will derive a higher-order asymptotic formula, connecting the known boundary data and the unknown inclusion parameters. The proposed approach is based on a topological sensitivity analysis for the elasticity operator with respect to the presence of geometric perturbations. To this end, we need some technical results.

### 2.3 Technical results

Let $\rho>0$, for a function $u$ defined on a given bounded open domain $\omega \subset \mathbb{R}^{3}$, we define the function $\tilde{u}$ on $\tilde{\omega}:=\omega / \rho$ by:

$$
\tilde{u}(y)=u(x), \quad y=x / \rho .
$$

We have the following relations:

$$
\begin{equation*}
|u|_{1, \omega}=\rho^{1 / 2}|\tilde{u}|_{1, \tilde{\omega}}, \quad\|u\|_{0, \omega}=\rho^{3 / 2}\|\tilde{u}\|_{0, \tilde{\omega}} . \tag{2.5}
\end{equation*}
$$

Let $r>0$ be such that the closed ball $\overline{B(\xi, r)}$ is included in $\Omega$, and $\overline{\mathcal{I}_{\xi, \rho}} \subset B(\xi, r)$. We denote by $\Gamma_{r}$ the boundary of $B(\xi, r)$, and we define the domains

$$
\Omega_{r}=\Omega \backslash \overline{B(\xi, r)}, \quad D_{\xi, \rho}=B(\xi, r) \backslash \overline{\mathcal{I}_{\xi, \rho}} \quad \text { and } \quad \Omega_{\xi, \rho}=\Omega \backslash \overline{\mathcal{I}_{\xi, \rho}} .
$$

Lemma 2.3 Let $h \in H^{1 / 2}(\partial \mathcal{I})^{3}$ and $w$ be the solution to the following elasticity exterior problem:

$$
\begin{cases}-\operatorname{div} \sigma(w)=0 & \text { in } \mathbb{R}^{3} \backslash \overline{\mathcal{I}} \\ w \rightarrow 0 & \text { at } \infty \\ w=h & \text { on } \partial \mathcal{I}\end{cases}
$$

Then, there exists a constant $c>0$, independent of $h$ and $\rho$, such that

$$
\begin{aligned}
& \|w\|_{0, D_{\xi, \rho} / \rho} \leq c \rho^{-1 / 2}\|h\|_{1 / 2, \partial \mathcal{I}}, \\
& \|w\|_{0, \Omega_{r} / \rho} \leq c \rho^{-1 / 2}\|h\|_{1 / 2, \partial \mathcal{I}}, \\
& |w|_{1, D_{\xi, \rho} / \rho} \leq c\|h\|_{1 / 2, \partial \mathcal{I}}, \\
& |w|_{1, \Omega_{r} / \rho} \leq c \rho^{1 / 2}\|h\|_{1 / 2, \partial \mathcal{I}} .
\end{aligned}
$$

Lemma 2.4 Let $\rho>0$ such that $\overline{\mathcal{I}_{\xi, \rho}} \subset B(\xi, r)$. For a given $g \in H^{1 / 2}(\Gamma)^{3}$, and $h \in$ $H^{1}(B(\xi, r))^{3}$, let $u_{\rho}$ be the solution to the elasticity system

$$
\begin{cases}-\operatorname{div} \sigma\left(u_{\rho}\right)=0 & \text { in } \Omega_{\xi, \rho} \\ u_{\rho}=g & \text { on } \Gamma \\ u_{\rho}=h & \text { on } \partial \mathcal{I}_{\xi, \rho}\end{cases}
$$

Then, there exists a constant $c>0$ (independent of $g, h$, and $\rho$ ), and $\rho_{1}>0$ such that for all $0<\rho<\rho_{1}$, we have:

$$
\begin{aligned}
& \left|u_{\rho}\right|_{1, \Omega_{r}} \leq c\left(\|g\|_{1 / 2, \Gamma}+\rho\|h(\xi+\rho y)\|_{1 / 2, \partial \mathcal{I}}\right) \\
& \left\|u_{\rho}\right\|_{0, D_{\xi, \rho}} \leq c\left(\|g\|_{1 / 2, \Gamma}+\rho\|h(\xi+\rho y)\|_{1 / 2, \partial \mathcal{I}}\right) \\
& \left|u_{\rho}\right|_{1, D_{\xi, \rho}} \leq c\left(\|g\|_{1 / 2, \Gamma}+\rho^{1 / 2}\|h(\xi+\rho y)\|_{1 / 2, \partial \mathcal{I}}\right)
\end{aligned}
$$

Remark 2.5 For the proofs of Lemma 2.3 and Lemma 2.4, one can consult [14], where similar results have been proved for the Stokes problem [17]. The well-posedness of the exterior elasticity problem and the integral representation of its solution are discussed in [18, 19].

## 3 Asymptotic expansion

We derive an asymptotic formula for the elastic displacement vector with respect to the presence of an inclusion $\mathcal{I}_{\xi, \rho}$ inside the domain $\Omega$.

### 3.1 First-order estimate

We derive a preliminary estimate describing the influence of the created inclusion $\mathcal{I}_{\xi, \rho}$ on the displacement field $w_{\rho}$.

Proposition 3.1 Let $\mathcal{I}_{\xi, \rho}$ be an inclusion of size $\rho$ strictly embedded into $\Omega$. Then, the perturbed elastic displacement vector $w_{\rho}$ satisfies that there exists a constant $c>0$, inde-
pendent of $\rho$, such that

$$
\left.\| w_{\rho}(x)-w_{0}(x)-E_{0}((x-\xi) / \rho)\right\}_{H^{1}(\Omega)} \overline{\left.\overline{\xi_{\xi, \rho}}\right)} \leq c \rho,
$$

where $E_{0}$ is the leading term of the displacement field variation $w_{\rho}-w_{0}$, defined as the solution to the elasticity exterior problem:

$$
\begin{cases}-\operatorname{div} \sigma\left(E_{0}\right)=0 & \text { in } \mathbb{R}^{3} \backslash \overline{\mathcal{I}},  \tag{3.1}\\ E_{0} \rightarrow 0 & \text { at } \infty \\ E_{0}=-w_{0}(\xi) & \text { on } \partial \mathcal{I} .\end{cases}
$$

Proof Setting

$$
Z_{\xi, \rho}(x)=w_{\rho}(x)-w_{0}(x)-E_{0}((x-\xi) / \rho), \quad \forall x \in \Omega \backslash \overline{\mathcal{I}_{\xi, \rho}} .
$$

From (1.4), (1.5), and (3.1), one can check that $z_{\rho}$ satisfies

$$
\begin{cases}-\operatorname{div} \sigma\left(Z_{\xi, \rho}\right)=0 & \text { in } \Omega \backslash \overline{\mathcal{I}_{\xi, \rho}}, \\ Z_{\xi, \rho}=-E_{0}((x-\xi) / \rho) & \text { on } \Gamma, \\ Z_{\xi, \rho}=-w_{0}+w_{0}(\xi) & \text { on } \partial \mathcal{I}_{\xi, \rho} .\end{cases}
$$

Using Lemma 2.4 , one can justify that there exists a constant $c>0$, independent of $\rho$, such that

$$
\begin{equation*}
\left|Z_{\xi, \rho}\right|_{1, \Omega \backslash \overline{I_{\xi, \rho}}} \leq c\left\{\left\|E_{0}((x-\xi) / \rho)\right\|_{1 / 2, \Gamma}+\rho^{1 / 2}\left\|w_{0}(\xi+\rho y)-w_{0}(\xi)\right\|_{1 / 2, \partial \mathcal{I}}\right\} \tag{3.2}
\end{equation*}
$$

Since $\operatorname{div} \sigma\left(E_{0}((x-\xi) / \rho)\right)=0$ in $\Omega_{r}$, by trace theorem [20]

$$
\left\|E_{0}((x-\xi) / \rho)\right\|_{1 / 2, \Gamma} \leq\left\|E_{0}((x-\xi) / \rho)\right\|_{1, \Omega_{r}} .
$$

Using the change of variable: $x=\xi+\rho y$, and (2.5):

$$
\begin{aligned}
\left\|E_{0}((x-\xi) / \rho)\right\|_{1, \Omega_{r}} & \leq\left\|E_{0}((x-\xi) / \rho)\right\|_{0, \Omega_{r}}+\left|E_{0}((x-\xi) / \rho)\right|_{1, \Omega_{2}}, \\
& \leq \rho^{3 / 2}\left\|E_{0}(y)\right\|_{0, \Omega_{r} / \rho}+\rho^{1 / 2}\left|E_{0}(y)\right|_{1, \Omega_{r} / \rho} .
\end{aligned}
$$

Then, by Lemma 2.3, we obtain:

$$
\left\|E_{0}((x-\xi) / \rho)\right\|_{1 / 2, \Gamma} \leq c \rho .
$$

Expanding $w_{0}(\xi+\rho y)=w_{0}(\xi)+\rho \nabla w_{0}\left(x_{\xi}\right), x_{\xi} \in \mathcal{I}_{\xi, \rho}$, and using the fact that $\nabla w_{0}$ is uniformly bounded, the third term in (3.2) may be approximated as:

$$
\left\|w_{0}(\xi+\rho y)-w_{0}(\xi)\right\|_{1 / 2, \partial \mathcal{I}} \leq c \rho .
$$

Finally, the combination of the above approximations implies the desired estimate

$$
\left.\left.\| w_{\rho}(x)-w_{0}(x)-E_{0}((x-\xi) / \rho)\right\}_{H^{1}(\Omega)} \overline{\mathcal{I}_{\xi_{, j}}}\right) \leq c \rho .
$$

Corollary 3.2 The perturbed displacement vector $w_{\rho}$ admits the estimate

$$
w_{\rho}(x)=w_{0}(x)+E_{0}((x-\xi) / \rho)+O(\rho) \quad \text { in } \Omega \backslash \overline{\mathcal{I}_{\xi, \rho}}
$$

### 3.2 Higher-order expansion

Here, we will give a generalization of the previous estimate to the higher-order case. The obtained asymptotic behavior is illustrated by the following theorem.

Theorem 3.3 Let $\mathcal{I}_{\xi, \rho}=\xi+\rho \mathcal{I}$ be a given inclusion, strictly embedded in the elastic domain $\Omega$. Then, the perturbed displacement vector $w_{\rho}$ admits the following asymptotic behavior

$$
\begin{equation*}
\left.w_{\rho}(x)=\sum_{n=0}^{K} \rho^{n}\left[C_{n}(x)+E_{n}((x-\xi) / \rho)\right)\right]+O\left(\rho^{K+1}\right) \quad \text { in } \Omega \backslash \overline{\mathcal{I}_{\xi, \rho}}, \tag{3.3}
\end{equation*}
$$

where:

- $K \in \mathbb{N}^{*}$ is an arbitrary chosen integer, denoting the asymptotic order,
- $\left\{C_{0}, C_{1}, \ldots, C_{K}\right\}$ are smooth vector functions, representing the corrected terms. Each vector function $C_{i}$ satisfies an auxiliary elasticity problem in $\Omega$,
- $\left\{E_{0}, E_{1}, \ldots, E_{K}\right\}$ are smooth functions, solutions to a sequence of exterior elasticity problems in $\mathbb{R}^{3} \backslash \overline{\mathcal{I}}$.

Proof To construct the terms $\left(C_{n}\right)_{0 \leq n \leq K}$ and $\left(E_{n}\right)_{0 \leq n \leq K}$ of the expected asymptotic expansion, we use an iterative process: Starting from the fact that $C_{0}=w_{0}$ (see (1.5)), and $E_{0}$ is the solution to (3.1).

- The terms $C_{1}$ and $E_{1}$ :

With the help of a single-layer potential on the boundary $\partial \mathcal{I}$, the solution $E_{0}$ can be defined as (see [18]):

$$
E_{0}(y)=\int_{\partial \mathcal{I}} G(y-z) S_{0}(z) d s(z), \quad \forall y \in \mathbb{R}^{3} \backslash \overline{\mathcal{I}}
$$

where:

- $G$ is the elasticity fundamental solution, defined in (2.4),
- the function $S_{0} \in H^{-1 / 2}(\partial \mathcal{I})$ is the associated density, defined as the solution to the following boundary integral equation

$$
\int_{\partial \mathcal{I}} G(y-z) S_{0}(z) d s(z)=-w_{0}(\xi), \quad \forall y \in \partial \mathcal{I}
$$

As one can observe, for each $x \in \mathbb{R}^{3} \backslash \overline{\mathcal{I}_{\xi, \rho}}$, we have

$$
\begin{aligned}
E_{0}((x-\xi) / \rho) & =\int_{\partial \mathcal{I}} G((x-\xi) / \rho-z) S_{0}(z) d s(z) \\
& =\rho \int_{\partial \mathcal{I}} G((x-\xi)-\rho z) S_{0}(z) d s(z)
\end{aligned}
$$

Since the inclusion $\mathcal{I}_{\xi, \rho}$ is not close to the boundary $\partial \Omega$, one can check that for all $z \in \partial \mathcal{I}$, the function

$$
\Phi_{x-\xi, z}: \rho \mapsto \Phi_{x-\xi, z}(\rho)=\rho G((x-\xi)-\rho z)
$$

is smooth with respect to $\rho$ and admits the following asymptotic expansion

$$
\Phi_{x-\xi, z}(\rho)=\sum_{m=1}^{K} \frac{\rho^{m}}{m!} \Phi_{x-\xi, z}^{(m)}(0)+o\left(\rho^{K}\right)
$$

where $\Phi_{x-\xi, z}^{(m)}(0)$ is the $m$ th derivative of $\Phi_{x-\xi, z}$ at $\rho=0$. It depends on the $m$ th derivative of Green's function $G$ at the point $x-\xi$.

Consequently, the function $x \mapsto E_{0}((x-z) / \varepsilon)$ satisfies the following asymptotic behavior:

$$
\begin{equation*}
E_{0}((x-z) / \rho)=\sum_{m=1}^{K} \rho^{m} E_{0}^{(m)}(x-\xi)+o\left(\rho^{K}\right) \tag{3.4}
\end{equation*}
$$

with $E_{0}^{(m)}$ is the smooth function defined in $\mathbb{R}^{3} \backslash \overline{\mathcal{I}}$ by:

$$
\begin{equation*}
E_{0}^{(m)}(x-\xi)=\frac{1}{m!} \int_{\partial \mathcal{I}} \Phi_{x-\xi, z}^{(m)}(0) S_{0}(z) d s(z), \quad \forall x \in \mathbb{R}^{3} \backslash \overline{\mathcal{I}} . \tag{3.5}
\end{equation*}
$$

Exploiting the developed asymptotic analysis for the function $E_{0}$, we choose the terms $C_{1}$ and $E_{1}$ as follows:

- $C_{1}$ depends on $E_{0}$, defined as the solution to the following auxiliary elasticity problem:

$$
\begin{cases}-\operatorname{div} \sigma\left(C_{1}\right)=0 & \text { in } \Omega \\ C_{1}=-E_{0}^{(1)}(x-\xi) & \text { on } \Gamma\end{cases}
$$

with $E_{0}^{(1)}$ is defined by (3.5), when $m=1$.

- $E_{1}$ depends on $C_{0}$ and $C_{1}$. It is chosen as the solution to the following exterior elasticity problem:

$$
\begin{cases}-\operatorname{div} \sigma\left(E_{1}\right)=0 & \text { in } \mathbb{R}^{3} \backslash \overline{\mathcal{I}}, \\ E_{1} \rightarrow 0 & \text { at } \infty, \\ E_{1}=-C_{1}(\xi)-D C_{0}(\xi)(y) & \text { on } \partial \mathcal{I},\end{cases}
$$

where $D C_{0}(z)$ is the derivative of the function $C_{0}$ at the point $\xi$.

- The terms $C_{n}$, and $E_{n}, n \geq 2$ :

Assume that we have already obtained the terms $\left\{C_{i}, E_{i}\right.$, for all $\left.1 \leq i \leq n-1\right\}$, and we want to construct the terms $C_{n}$ and $E_{n}$.
Due to a single-layer potential [18], for each $i \in\{1,2, \ldots, n-1\}$, the term $E_{i}$ can be expressed as

$$
E_{i}(y)=\int_{\partial \mathcal{I}} G(y-z) S_{i}(z) d s(z), \quad \forall y \in \mathbb{R}^{3} \backslash \overline{\mathcal{I}}
$$

with $S_{i} \in H^{-1 / 2}(\partial \mathcal{I})$ is the corresponding density, defined as the solution to a given boundary integral equation.

Using (3.4), the function $x \mapsto E_{i}((x-z) / \varepsilon)$ reads

$$
\begin{equation*}
E_{i}((x-z) / \rho)=\sum_{m=1}^{K} \rho^{m} E_{i}^{(m)}(x-\xi)+o\left(\rho^{K}\right) \tag{3.6}
\end{equation*}
$$

with $E_{i}^{(m)}$ is given by

$$
\begin{equation*}
E_{i}^{(m)}(x-\xi)=\frac{1}{m!} \int_{\partial \mathcal{I}} \Phi_{x-\xi, z}^{(m)}(0) S_{i}(z) d s(z), \quad \forall x \in \mathbb{R}^{3} \backslash \overline{\mathcal{I}} \tag{3.7}
\end{equation*}
$$

Generalizing the technique that we used for deriving the terms $C_{1}$ and $E_{1}$, the terms $C_{n}$ and $E_{n}$ are obtained as follows:

- The tern $C_{n}$ is constructed with the help of the terms $E_{0}, E_{1}, \ldots$, and $E_{n-1}$. It is chosen as the solution to the following auxiliary elasticity problem:

$$
\begin{cases}-\operatorname{div} \sigma\left(C_{n}\right)=0 & \text { in } \Omega  \tag{3.8}\\ C_{n}=-\sum_{m=1}^{n} E_{n-m}^{(m)}(x-\xi) & \text { on } \Gamma\end{cases}
$$

with $E_{i}^{(m)}$ is given by (3.7).

- The term $E_{n}$ is constructed with the help of the corrected terms $C_{0}, C_{1}, \ldots$, and $C_{n}$. It is defined as the solution to the following exterior elasticity problem:

$$
\begin{cases}-\operatorname{div} \sigma\left(E_{n}\right)=0 & \text { in } \mathbb{R}^{3} \backslash \overline{\mathcal{I}}  \tag{3.9}\\ E_{n} \rightarrow 0 & \text { at } \infty \\ E_{n}=-C_{n}(z)-\sum_{m=1}^{n} \frac{1}{m!} D^{m} C_{n-m}(\xi)\left(y^{m}\right) & \text { on } \partial \mathcal{I}\end{cases}
$$

Here, $D^{m} C_{i}(\xi)$ denotes the derivation of order $m$ of the function $C_{i}$ at the point $\xi$ and $y^{m}=(y, \ldots, y) \in\left(\mathbb{R}^{3}\right)^{m}$.

- Justification of the developed expansion:

The last step of this proof is devoted to justifying that the constructed terms $\left\{C_{i}, E_{i}, 1 \leq\right.$ $i \leq K\}$ gives the expected asymptotic behavior. Posing:

$$
\begin{equation*}
\left.Z_{\xi, \rho}^{K}(x)=\sum_{n=0}^{K} \rho^{n}\left[C_{n}(x)+E_{n}((x-\xi) / \rho)\right)\right]-w_{\rho}(x), \quad x \in \Omega \backslash \overline{\mathcal{I}_{\xi, \rho}} . \tag{3.10}
\end{equation*}
$$

Our aim is to prove that

$$
Z_{\xi, \rho}^{K}(x)=O\left(\rho^{K+1}\right)
$$

From (1.4), (1.5), (3.9), and (3.8), it follows that

$$
-\operatorname{div} \sigma\left(Z_{\xi, \rho}^{K}\right)=0 \quad \text { in } \Omega \backslash \overline{\mathcal{I}_{\xi, \rho}} .
$$

The boundary conditions satisfied by the vector function $Z_{\xi, \rho}^{K}$ are defined as follows:

- On the boundary $\partial \mathcal{I}_{\xi, \rho}$ : Exploiting the equations (3.8)-(3.9), the multi-linearity of $D^{m} C_{n-m}(\xi)$, Taylor's Theorem, and the fact that $\|x-\xi\|=O(\rho)$ on $\partial \mathcal{I}_{\xi, \rho}$, one can establish

$$
Z_{\xi, \rho}^{K}(x)=\sum_{n=0}^{K} \rho^{n}\left[C_{n}(x)-\sum_{m=0}^{K-n} \frac{1}{m!} D^{m} C_{n}(\xi)\left((x-\xi)^{m}\right)\right]=O\left(\rho^{K+1}\right)
$$

- On the boundary $\Gamma$ : Making use of the prescribed Dirichlet condition in (3.8), the system (3.9), and the approximation (3.4), one can derive

$$
\begin{aligned}
Z_{\xi, \rho}^{K}(x) & =\rho^{K} E_{K}((x-\xi) / \rho)+\sum_{n=0}^{K-1} \rho^{n}\left[E_{n}((x-\xi) / \rho)-\sum_{m=1}^{K-n} \rho^{m} E_{n}^{(m)}(x-\xi)\right] \\
& =O\left(\rho^{K+1}\right)
\end{aligned}
$$

## 4 Boundary formula for the reciprocity gap functional

This section deals with a boundary asymptotic formula representing the reciprocity gap functional variation with respect to the perforation of an inclusion $\mathcal{I}_{\xi, \rho}$ inside the elastic medium $\Omega$.

It is established in Corollary 2.2 that for all $1 \leq j \leq 3$,

$$
\begin{aligned}
& \mathcal{R}_{\xi, \rho}^{j}(\eta)-\mathcal{R}_{0}^{j}(\eta) \\
& \quad=\int_{\partial \mathcal{I}_{\xi, \rho}} \sigma\left(w_{\rho}-w_{0}\right) \mathbf{n} G^{j}(x-\eta) d s-\int_{\mathcal{I}_{\xi, \rho}} \sigma_{x}\left(G^{j}(x-\eta)\right): e\left(w_{0}\right) d x, \quad \forall \eta \in \mathbb{R}^{3} \backslash \bar{\Omega} .
\end{aligned}
$$

Exploiting the derived higher-order expansion in Theorem 3.3, the variation $w_{\rho}-w_{0}$ reads:

$$
\left.w_{\rho}(x)-w_{0}(x)=\sum_{n=0}^{K} \rho^{n} E_{n}((x-\xi) / \rho)\right)+\sum_{n=1}^{K} \rho^{n} C_{n}(x)+o\left(\rho^{K}\right) \quad \text { in } \Omega \backslash \overline{\mathcal{I}_{\xi, \rho}} .
$$

Then, $\mathcal{R}_{\xi, \rho}^{j}(\eta)-\mathcal{R}_{0}^{j}(\eta)$ can be rewritten as:

$$
\begin{align*}
\mathcal{R}_{\xi, \rho}^{j}(\eta)-\mathcal{R}_{0}^{j}(\eta)= & \left.\sum_{n=0}^{K} \rho^{n} \int_{\partial \mathcal{I}_{\xi, \rho}} \sigma_{x}\left(E_{n}((x-\xi) / \rho)\right)\right) \mathbf{n} G^{j}(x-\eta) d s(x) \\
& +\sum_{n=1}^{K} \rho^{n} \int_{\partial \mathcal{I}_{\xi, \rho}} \sigma\left(C_{n}\right) \mathbf{n} G^{j}(x-\eta) d s  \tag{4.1}\\
& -\int_{\mathcal{I}_{\xi, \rho}} \sigma_{x}\left(G^{j}(x-\eta)\right): e\left(w_{0}\right) d x+o\left(\rho^{K}\right)
\end{align*}
$$

Aiming to derive the expected formula for the reciprocity gap functional $\mathcal{R}_{\xi, \rho}^{j}$, we start our analysis by estimating the integral terms in (4.1).

### 4.1 Preliminary calculus

We will establish an estimate for each term in the right-hand side of (4.1). The following lemma treat the first integral term.

Lemma 4.1 The first term in (4.1) satisfies the expansion:

$$
\left.\sum_{n=0}^{K} \rho^{n} \int_{\partial \mathcal{I}_{\xi, \rho}} \sigma_{x}\left(E_{n}((x-\xi) / \rho)\right)\right) \mathbf{n} G^{j}(x-\eta) d s(x)=\sum_{n=0}^{K-1} \rho^{n+1} Y_{\eta, \mathcal{I}}^{n, j}(\xi)+o\left(\rho^{K}\right)
$$

where $\xi \mapsto \mathcal{Y}_{\eta, \mathcal{I}}^{n, j}(\xi), 1 \leq n \leq K$ are defined by:

$$
\mathcal{Y}_{\eta, \mathcal{I}}^{n, j}(\xi)=\sum_{m=0}^{n} \frac{1}{m!} \int_{\partial \mathcal{I}}\left[\sigma_{y}\left(E_{n-m}\right)(y) \mathbf{n}(y)\right] \cdot\left[\nabla^{(m)} G^{j}(\xi-\eta)\left(y^{m}\right)\right] d s(y), \quad \forall \xi \in \Omega
$$

with $\nabla^{(m)} G^{j}(z)$ is the mth derivation of $G^{j}$ at the point $z$.
Proof By the change of variable $x=\xi+\rho y$,

$$
\left.\int_{\partial \mathcal{I}_{\xi, \rho}} \sigma_{x}\left(E_{n}((x-\xi) / \rho)\right)\right) \mathbf{n} G^{j}(x-\eta) d s(x)=\rho \int_{\partial \mathcal{I}} \sigma_{y}\left(E_{n}\right)(y) \mathbf{n} G^{j}(\xi-\eta+\rho y) d s(y)
$$

Since $\eta \in \mathbb{R}^{3} \backslash \bar{\Omega}$ and $\overline{\mathcal{I}_{\xi, \rho}} \subset \Omega$, the function $y \mapsto G^{j}(\xi-\eta+\rho y)$ is $C^{\infty}$ in the neighborhood of $\xi-\eta$. One can derive:

$$
\begin{aligned}
G^{j}(\xi-\eta+\rho y) & =G^{j}(\xi-\eta)+\sum_{m=1}^{K-1} \frac{\rho^{m}}{m!} \nabla^{(m)} G^{j}(\xi-\eta)\left(y^{m}\right)+o\left(\rho^{K}\right) \\
& =\sum_{m=0}^{K-1} \frac{\rho^{m}}{m!} \nabla^{(m)} G^{j}(\xi-\eta)\left(y^{m}\right)+o\left(\rho^{K}\right)
\end{aligned}
$$

It follows:

$$
\begin{aligned}
& \left.\int_{\partial \mathcal{I}_{\xi, \rho}} \sigma_{x}\left(E_{n}((x-\xi) / \rho)\right)\right) \mathbf{n}(x) G^{j}(x-\eta) d s(x) \\
& \quad=\sum_{m=0}^{K-2} \frac{\rho^{m+1}}{m!} \int_{\partial \mathcal{I}} \sigma_{y}\left(E_{n}\right)(y) \mathbf{n}(y)\left[\nabla^{(m)} G^{j}(\xi-\eta)\left(y^{m}\right)\right] d s(y)+o\left(\rho^{K}\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \left.\sum_{n=0}^{K} \rho^{n} \int_{\partial \mathcal{I}_{\xi, \rho}} \sigma_{x}\left(E_{n}((x-\xi) / \rho)\right)\right) \mathbf{n} G^{j}(x-\eta) d s(x) \\
& \quad=\sum_{n=0}^{K-1} \rho^{n+1} \sum_{m=0}^{n} \frac{1}{m!} \int_{\partial \mathcal{I}} \sigma_{y}\left(E_{n-m}\right)(y) \mathbf{n}(y)\left[\nabla^{(m)} G^{j}(\xi-\eta)\left(y^{m}\right)\right] d s(y)+o\left(\rho^{K}\right)
\end{aligned}
$$

Lemma 4.2 The second term in (4.1) verifies the expansion:

$$
\sum_{n=1}^{K} \rho^{n} \int_{\partial \mathcal{I}_{\xi, \rho}} \sigma\left(C_{n}\right) \mathbf{n} G^{j}(x-\eta) d s(x)=\sum_{n=0}^{K-3} \rho^{n+3} \mathcal{Z}_{\eta, \mathcal{I}}^{n, j}(\xi)+o\left(\rho^{K}\right)
$$

where the leading terms $\xi \mapsto \mathcal{Z}_{\eta, \mathcal{I}}^{n, j}(\xi), 1 \leq n \leq K$ are defined by:

$$
\mathcal{Z}_{\eta, \mathcal{I}}^{n, j}(\xi)=\sum_{q=0}^{n} \sum_{m=0}^{q} \frac{1}{m!(q-m)!} \int_{\partial \mathcal{I}} \mathcal{M}_{n+1-q}^{(m)}(\xi)(y) \mathbf{n} \cdot\left[\nabla^{(q-m)} G^{j}(\xi-\eta)\left(y^{q-m}\right)\right] d s(y)
$$

with $\mathcal{M}_{p}^{(m)}(\xi)(y)$ is the matrix $\left[\mathcal{M}_{p}^{(m)}(\xi)(y)\right]=\left\{\nabla^{(m)}\left[\sigma\left(C_{p}\right)\right]_{i, l}(\xi)\left(y^{m}\right)\right\}_{1 \leq i, l \leq 3}$.

Proof Changing the variable $x$ by $\xi+\rho y$,

$$
\int_{\partial \mathcal{I}_{\xi, \rho}} \sigma_{x}\left(C_{n}\right) \mathbf{n} G^{j}(x-\eta) d s(x)=\rho^{2} \int_{\partial \mathcal{I}} \sigma_{x}\left(C_{n}\right)(\xi+\rho y) \mathbf{n} G^{j}(\xi-\eta+\rho y) d s(x)
$$

Using the fact that $x \mapsto G^{j}(\xi-\eta+\rho y)$ is regular in the neighborhood of $\xi-\eta$,

$$
G^{j}(\xi-\eta+\rho y)=\sum_{m=0}^{K} \frac{\rho^{m}}{m!} \nabla^{(m)} G^{j}(\xi-\eta)\left(y^{m}\right)+o\left(\rho^{K}\right) .
$$

Similarly, for each $1 \leq i, l \leq 3$,

$$
\left[\sigma_{x}\left(C_{n}\right)\right]_{i, l}(\xi+\rho y)=\sum_{m=0}^{K} \frac{\rho^{m}}{m!} \nabla^{(m)}\left[\sigma\left(C_{n}\right)\right]_{i, l}(\xi)\left(y^{m}\right)+o\left(\rho^{K}\right)
$$

where $\left[\sigma_{x}\left(C_{n}\right)\right]_{i, l}, 1 \leq i, l \leq 3$ denote the coefficients of the matrix $\sigma_{x}\left(C_{n}\right)$.
Hence, we deduce:

$$
\begin{aligned}
& \int_{\partial \mathcal{I}_{\xi, \rho}} \sigma_{x}\left(C_{n}\right) \mathbf{n} G^{j}(x-\eta) d s(x) \\
& \quad=\sum_{q=0}^{K-2} \rho^{q+2} \sum_{m=0}^{q} \frac{1}{m!(q-m)!} \\
& \quad \times \int_{\partial \mathcal{O}} \mathcal{M}_{n}^{(m)}(\xi)(y) \mathbf{n}(y) . \nabla^{(q-m)} G^{j}(\xi-\eta)\left(y^{(q-m)}\right) d s(y)+o\left(\rho^{K}\right)
\end{aligned}
$$

where $\mathcal{M}_{n}^{(m)}(\xi)(y)$ is the matrix $\left[\mathcal{M}_{n}^{(m)}(\xi)(y)\right]=\left\{\nabla^{(m)}\left[\sigma\left(C_{n}\right)\right]_{i, l}(\xi)\left(y^{m}\right)\right\}_{1 \leq i, l \leq 3}$. Then, we obtain:

$$
\begin{aligned}
& \sum_{n=1}^{K} \rho^{n} \int_{\partial \mathcal{I}_{\xi, \rho}} \sigma\left(C_{n}\right) \mathbf{n} G^{j}(x-\eta) d s(x) \\
& \quad=\sum_{n=0}^{K} \rho^{n} \sum_{q=0}^{K-2} \rho^{q+3} \sum_{m=0}^{q} \frac{1}{m!(q-m)!} \int_{\partial \mathcal{O}} \mathcal{M}_{n+1}^{(m)}(\xi)(y) \mathbf{n}(y) . \nabla^{(q-m)} G^{j}(\xi-\eta)\left(y^{(q-m)}\right) d s(y) \\
& \quad+o\left(\rho^{K}\right) .
\end{aligned}
$$

Using the Cauchy formula for the product of two polynomials yields:

$$
\begin{aligned}
& \sum_{n=1}^{K} \rho^{n} \int_{\partial \mathcal{I}_{\xi, \rho}} \sigma\left(C_{n}\right) \mathbf{n} G^{j}(x-\eta) d s(x) \\
& \quad=\sum_{n=0}^{K} \rho^{n+3} \sum_{q=0}^{n} \sum_{m=0}^{q} \frac{1}{m!(q-m)!} \int_{\partial \mathcal{I}} \mathcal{M}_{n+1-q}^{(m)}(\xi)(y) \mathbf{n} \cdot \nabla^{(q-m)} G^{j}(\xi-\eta)\left(y^{q-m}\right) d s \\
& \quad+o\left(\rho^{K}\right) .
\end{aligned}
$$

Lemma 4.3 The third term in (4.1) admits the expansion:

$$
\int_{\mathcal{I}_{\xi, \rho}} \sigma_{x}\left(G^{j}\right)(x-\eta): e\left(w_{0}\right) d x=\sum_{n=0}^{K-3} \rho^{n+3} \mathcal{X}_{\eta, \mathcal{I}}^{n, j}(\xi)+o\left(\rho^{K}\right)
$$

Here, the leading terms $\xi \mapsto \mathcal{X}_{\eta, \mathcal{I}}^{n, j}(\xi), 1 \leq n \leq K$ are given by:

$$
\mathcal{X}_{\eta, \mathcal{I}}^{n, j}(\xi)=\sum_{m=0}^{n} \frac{1}{m!(n-m)!} \int_{\mathcal{I}} \nabla^{(m)} \sigma_{x}\left(G^{j}\right)(\xi-\eta)\left(y^{m}\right): \nabla^{(n-m)} e\left(w_{0}\right)(\xi)\left(y^{n-m}\right) d y
$$

Proof Recall that $I_{\xi, \rho}=\xi+\rho \mathcal{I}$. The replacement of the variable $x$ by $\xi+\rho y$ implies:

$$
\int_{\mathcal{I}_{\xi, \rho}} \sigma_{x}\left(G^{j}\right)(x-\eta): e\left(w_{0}\right) d x=\rho^{3} \int_{\mathcal{I}} \sigma_{x}\left(G^{j}\right)(\xi-\eta+\rho y): e\left(w_{0}\right)(\xi+\rho y) d y
$$

Since $\eta \in \mathbb{R}^{3} \backslash \bar{\Omega}$, and $I_{\xi, \rho}$ is strictly embedded in $\Omega, y \mapsto G^{j}(\xi-\eta+\rho y)$, and $y \mapsto w_{0}(\xi+\rho y)$ are sufficiently smooth in $\mathcal{I}$. By the Taylor-Young formula, one can deduce for all $1 \leq i, l \leq$ 3:

$$
\begin{aligned}
& e\left(w_{0}\right)_{i, l}(\xi+\rho y)=e\left(w_{0}\right)_{i, l}(\xi)+\sum_{m=1}^{K} \frac{\rho^{m}}{m!} \nabla^{(m)}\left[e\left(w_{0}\right)_{i, l}\right](\xi)\left(y^{m}\right)+o\left(\rho^{K}\right), \\
& {\left[\sigma_{x}\left(G^{j}\right)\right]_{i, l}(\xi-\eta+\rho y)=\sum_{m=0}^{K} \frac{\rho^{m}}{m!} \nabla^{(m)}\left[\sigma_{x}\left(G^{j}\right)\right]_{i, l}(\xi-\eta)\left(y^{m}\right)+o\left(\rho^{K}\right) .}
\end{aligned}
$$

Multiplying the two previous polynomials, one can derive:

$$
\int_{\mathcal{I}_{\xi, \rho}} \sigma_{x}\left(G^{j}\right)(x-\eta): e\left(w_{0}\right) d x=\sum_{n=0}^{K-3} \rho^{n+3} \mathcal{X}_{\eta, \mathcal{I}}^{n, j}(\xi)+o\left(\rho^{K}\right)
$$

with

$$
\mathcal{X}_{\eta, \mathcal{I}}^{n, j}(\xi)=\sum_{m=0}^{n} \frac{1}{m!(n-m)!} \int_{\mathcal{I}} \nabla^{(m)} \sigma_{x}\left(G^{j}\right)(\xi-\eta)\left(y^{m}\right): \nabla^{(n-m)} e\left(w_{0}\right)(\xi)\left(y^{n-m}\right) d y
$$

### 4.2 Asymptotic formula for the reciprocity gap function

In this section, we present a higher-order asymptotic formula, representing the variation of the reciprocity gap functional with respect to the presence of an inclusion $\mathcal{I}_{\xi, \rho}=\xi+\rho \mathcal{I}$. The obtained result is summarized in the following theorem. It follows immediately by application of Lemmas 4.1, 4.2, and 4.3.

Theorem 4.4 Assume that the elastic medium contains an unknown inclusion of the form $\mathcal{I}_{\xi, \rho}=\xi+\rho \mathcal{I}$. Then, for each $1 \leq j \leq 3$, the reciprocity gap functional variation $\mathcal{R}_{\xi, \rho}^{j}-\mathcal{R}_{0}^{j}$ admits the following asymptotic formula:

$$
\begin{equation*}
\mathcal{R}_{\xi, \rho}^{j}(\eta)-\mathcal{R}_{0}^{j}(\eta)=\sum_{n=1}^{K} \rho^{n} \mathcal{W}_{\eta, \mathcal{I}}^{n, j}(\xi)+o\left(\rho^{K}\right), \quad \forall \eta \in \mathbb{R}^{3} \backslash \bar{\Omega} \tag{4.2}
\end{equation*}
$$

where the leading terms $\mathcal{W}_{\eta, \mathcal{I}}^{n, j}(\xi), 1 \leq j \leq 3,1 \leq n \leq K$ are defined as

$$
\mathcal{W}_{\eta, \mathcal{I}}^{n, j}(\xi)= \begin{cases}Y_{\eta, \mathcal{I}}^{n-1, j}(\xi) & \text { if } 1 \leq n \leq 2 \\ \mathcal{Z}_{\eta, \mathcal{I}}^{n-3, j}(\xi)+Y_{\eta, \mathcal{I}}^{n-1, j}(\xi)-\mathcal{X}_{\eta, \mathcal{I}}^{n-3, j} & \text { if } 3 \leq n \leq K\end{cases}
$$

### 4.3 Concluding remarks and forthcoming works

The asymptotic formula established in Theorem 4.4 can be exploited as the basis for developing numerical algorithms and identifying unknown inclusions of the form $\mathcal{I}_{\xi, \rho}$ from boundary measured data. In fact:

- Problem (1.4) cannot be solved in practice since the domain $\Omega \backslash \overline{\mathcal{I}_{\xi, \rho}}$ is unknown, but the elastic displacement vector $w_{\rho}$ and the tracking force $\sigma\left(w_{\rho}\right) n$ can be imposed or measured on the boundary $\partial \Omega$.
- Problem (1.5) can be solved, and the solution $w_{0}$ can be approximated in the safe elastic domain $\Omega$.
- The elasticity fundamental solution $G^{j}$ is known and can be calculated explicitly.

Then, the variation

$$
\begin{aligned}
B^{j}(\eta) & =\mathcal{R}_{\xi, \rho}^{j}(\eta)-\mathcal{R}_{0}^{j}(\eta) \\
& =\int_{\partial \Omega} \sigma\left(G^{j}(x-\eta)\right) \mathbf{n}\left(w_{\rho}-w_{0}\right) d s(x)-\int_{\partial \Omega} \sigma\left(w_{\rho}-w_{0}\right) \mathbf{n} G^{j}(x-\eta) d s(x)
\end{aligned}
$$

can be used as a measured datum on $\partial \Omega$ for all $\eta \in \mathbb{R}^{3} \backslash \bar{\Omega}$.
Neglecting the term $o\left(\rho^{K}\right)$, Theorem 4.4 provides us a nonlinear system satisfied by the unknown inclusion parameters: the location $\xi$, the size $\rho$, and the shape $\mathcal{I}$ :

$$
\sum_{n=1}^{K} \rho^{n} \mathcal{W}_{\eta, \mathcal{I}}^{n, j}(\xi)=B^{j}(\eta) \quad \forall 1 \leq j \leq 3, \forall \eta \in \mathbb{R}^{3} \backslash \bar{\Omega}
$$

Solving this problem in its general form is not an easy task but first, one can develop a numerical algorithm for reconstructing the location $\xi$ and the size $\rho$ of the unknown inclusion. The shape $\mathcal{I}$ can be numerically approximated after specifying some geometric inclusions form. This attractive issue will be discussed further in a forthcoming paper.

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Not applicable.

## Declarations

## Competing interests

The authors declare no competing interests.

## Author contributions

Authors Mohamed Abdelwahed, Nejmeddine Chorfi and Maatoug Hassine declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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## References

1. Pasquale, C., Leszek, G., Roberto, L., Junior Joao, S.: Multiplicity of positive solutions for a degenerate nonlocal problem with p-laplacian. Adv. Nonlinear Anal. 11(1), 357-368 (2022)
2. Blatt, S., Hopper, C., Vorderobermeier, N.: A regularized gradient flow for the p-elastic energy. Adv. Nonlinear Anal. 11, 1383-1411 (2022)
3. Muvasharkhan, J., Murat, R., Madi, Y.: On the numerical solution of one inverse problem for a linearized two-dimensional system of navier-stokes equations. Opusc. Math. 42(5), 709-725 (2022)
4. Garcke, H., Huttl, P., Knopf, P.: Shape and topology optimization involving the eigenvalues of an elastic structure: a multi-phase-field approach. Adv. Nonlinear Anal. 11, 159-197 (2022)
5. Abdelwahed, M., Chorfi, N., Hassine, M.: Asymptotic formulas for the identification of small inhonogeneities in a fluid midium. Electron. J. Differ. Equ. 186, 1 (2015)
6. Andrieux, S., Ben Abda, A.: Identification of planar cracks by complete overdetermined data: inversion formulae. Inverse Probl. 12, 553-563 (1996)
7. Alves, C., Silvestre, A.L.: On the determination of point-forces on a stokes system. Math. Comput. Simul. 66, 385-397 (2004)
8. Abdelwahed, M., Hassine, M.: Topological optimization method for a geometric control problem in stokes flow. Appl. Numer. Math. 59, 1823-1838 (2009)
9. Abdelwahed, M., Hassine, M., Masmoudi, M.: Optimal shape design for fluid flow using topological perturbation technique. J. Math. Anal. Appl. 356, 548-563 (2009)
10. Badra, M., Caubet, F., Dambrine, M.: Detecting an obstacle immersed in a fluid by shape optimization methods. Math Models Methods Appl. Sci. 21, 2069-2101 (2011)
11. Garreau, S., Guillaume, P., Masmoudi, M.: The topological asymptotic for pde systems: The elastic case. SIAM J. Control Optim. 39, 1756-1778 (2011)
12. Guillaume, P., Sid Idris, K.: Topological sensitivity and shape optimization for the stokes equations. SIAM J. Control Optim. 43, 1-31 (2004)
13. Hassine, M., Khelif, K.: On the high-order topological asymptotic expansion for shape functions. Electron. J. Differ. Equ. 110, 1 (2016)
14. Hassine, M., Masmoudi, M.: The topological asymptotic expansion for the quasi-stokes problem. ESAIM Control Optim. Calc. Var. 10, 478-504 (2004)
15. Sokolowski, J., Zochowski, A.: On the topological derivative in shape optimization. SIAM J. Control Optim. 37, 1251-1272 (1999)
16. Astashova, I., Bartusek, M., Dosla, Z., Marini, M.: Asymptotic proximity to higher order nonlinear differential equations. Adv. Nonlinear Anal. 11, 1598-1613 (2022)
17. Wang, Y., Wu, W.: Initial boundary value problems for the three-dimensional compressible elastic navier-stokes-poisson equations. Adv. Nonlinear Anal. 10, 1356-1383 (2021)
18. Dautray, R., Lions, J.: In: Analyse Mathémathique et Calcul Numérique Pour les Sciences et les Techniques, MASSON, Paris (1987)
19. Watanabe, K.: Stability of stationary solutions to the three-dimensional navier-stokes equations with surface tension. Adv. Nonlinear Anal. 12, 279-284 (2023)
20. Simsen, J., Simsen, M., Wittbold, P.: Reaction-diffusion coupled inclusions with variable exponents and large diffusion. Opusc. Math. 41, 539-570 (2021)

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