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Existence and nonexistence of solutions for an approximation of the Paneitz problem on spheres

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Abstract

This paper is devoted to studying the nonlinear problem with slightly subcritical and supercritical exponents $(S_{\pm\varepsilon}) : \Delta^2 u - c_n \Delta u + d_n u = K u^{\frac{n+4}{n-4}\pm\varepsilon}$, $u > 0$ on S^n, where $n \geq 5$, ε is a small positive parameter and K is a smooth positive function on S^n. We construct some solutions of $(S_{-\varepsilon})$ that blow up at one critical point of K. However, we prove also a nonexistence result of single-peaked solutions for the supercritical equation $(S_{+\varepsilon})$.

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1 Introduction

Recently, there has been considerable interest in equations involving the biharmonic operator Δ^2. A particular feature of the biharmonic operator is that it is conformally invariant. In this work, we are interested in the generalization of the Paneitz operator [15] to higher dimensions, which was discovered by Branson [7]. More precisely, let (M,g) be a smooth compact Riemannian n-manifol, $n \geq 5$, S_g be the scalar curvature of g, and Ric_g be the Ricci curvature of g. The Paneitz–Branson operator is defined by

$$P_g^n u = \Delta_g^2 u - \operatorname{div}_g(a_n S_g g + b_n Ric_g)\, du + \frac{n-4}{2} Q_g^n u,$$

where

$$a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}, \qquad b_n = \frac{-4}{n-2},$$
$$Q_g^n = -\frac{1}{2(n-1)}\Delta_g S_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2}|Ric_g|^2.$$

If $\widetilde{g} = u^{4/(n-4)} g$ is a metric conformal to g, then for all $\varphi \in C^\infty(M)$ one has

$$P_g^n(u\varphi) = u^{(n+4)/(n-4)} P_{\widetilde{g}}^n(\varphi)$$

and

$$P_g^n(u) = \frac{n-4}{2} Q_{\widetilde{g}}^n u^{(n+4)/(n-4)}. \tag{1.1}$$

Regarding this equation, it is normal to consider the problem of prescribing the Paneitz curvature, that is: given a function $K : M \to \mathbb{R}$ does there exist a metric $\widetilde{g}$ conformally equivalent to g such that $Q_{\widetilde{g}}^n = K$? This problem is equivalent to finding a smooth solution of the following equation,

$$P_g^n(u) = \frac{n-4}{2} K u^{(n+4)/(n-4)}, \quad u > 0 \text{ on } M. \tag{1.2}$$

In the case of the standard sphere (S^n, g), $n \geq 5$. Thus, we are reduced to finding a positive solution u of the problem

$$\mathcal{P}u = \Delta^2 u - c_n \Delta u + d_n u = K u^{\frac{n+4}{n-4}}, \quad u > 0 \text{ on } S^n, \tag{1.3}$$

where $c_n = \frac{1}{2}(n^2 - 2n - 4)$, $d_n = \frac{n-4}{16} n(n^2 - 4)$, and K is a given positive function defined on S^n.

In the last decades, many interesting works have been devoted to study problem (1.3). In [6, 8, 9], the authors treated the lower-dimensional case ($n = 5, 6$). In [10], Felli proved a perturbative theorem and some existence results under the assumptions of symmetry.

The special nature of problem (1.3) appears when we consider it from a variational viewpoint. Indeed, the Euler–Lagrange functional associated to (1.3) does not satisfy the Palais–Smale condition, that is, there exist noncompact sequences along which the functional is bounded and its gradient goes to zero. This fact is due to the presence of the critical exponent. Hence, for the study of problem (1.3), it is interesting to approach it by the following family of subcritical and supercritical problems:

$$(S_{\pm\varepsilon}) \mathcal{P}u = \Delta^2 u - c_n \Delta u + d_n u = K u^{\frac{n+4}{n-4} \pm \varepsilon}, \quad u > 0 \text{ on } S^n,$$

and we need to study the asymptotic behavior of the solutions (u_ε) as $\varepsilon \to 0$ (if they exist). Observe that, for $\varepsilon > 0$, problem $(S_{-\varepsilon})$ always has a positive solution (u_ε). In [13], the author proved some existence and nonexistence results of solutions that blow up at one point for a subcritical and supercritical approximation of a harmonic equation in the unit ball in $\mathbb{R}^n$.

In the present paper, we aim to give sufficient conditions on the function K such that both subcritical and supercritical approximations $(S_{\pm\varepsilon})$ admit or do not admit a positive solution. Our approach follows the ideas introduced first by Bahri, Li, and Rey [4] when they studied an approximation problem of the Yamabe type on domains. This idea has been used by many authors to construct some solutions for different problems. Since our problem is different from the problem studied in [4], we will take account of the new estimates in order to use this method.

To state our main results, we need to introduce some notations.

For $K \equiv 1$, the solutions of (1.3) form a family $\widetilde{\delta}_{(a,\lambda)}$ defined by

$$\widetilde{\delta}_{(a,\lambda)}(x) = \frac{\gamma_n}{2^{\frac{n-4}{2}}} \frac{\lambda^{\frac{n-4}{2}}}{(1+\frac{\lambda^2-1}{2}(1-\cos d(a,x)))^{\frac{n-4}{2}}}, \tag{1.4}$$

where $a \in S^n$, $\lambda > 0$ and $\gamma_n = ((n-4)(n-2)n(n+2))^{(n-4)/8}$.

After performing a stereographic projection π with the point $-a$ as a pole, the function $\widetilde{\delta}_{(a,\lambda)}$ is transformed into

$$\delta_{(0,\lambda)} = \gamma_n \left(\frac{\lambda}{1+\lambda^2|y|^2} \right)^{\frac{n-4}{2}},$$

which is a solution of the problem (see [11])

$$\Delta^2 u = u^{\frac{n+4}{n-4}}, \quad u > 0, \text{on } \mathbb{R}^n.$$

Note that we will use the stereographic projection to collect some technical estimates of the different integral quantities that occur in the paper (see [3]).

The space $H_2^2(S^n)$ is equipped with the norm:

$$\|u\|^2 = \langle u, u \rangle_{\mathcal{P}} = \int_{S^n} \mathcal{P}u.u = \int_{S^n} |\Delta u|^2 + c_n \int_{S^n} |\nabla u|^2 + d_n \int_{S^n} u^2.$$

Our first result deals with construction of single-peaked solutions for the subcritical approximation of the problem $(S_{-\varepsilon})$ with $\varepsilon > 0$. More precisely, we have

Theorem 1.1 *Assume that y is a nondegenerate critical point of K satisfying $\Delta K(y) < 0$. Then, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, the problem $(S_{-\varepsilon})$ has a solution (u_ε) of the form*

$$u_\varepsilon = \alpha_\varepsilon \widetilde{\delta}_{(x_\varepsilon,\lambda_\varepsilon)} + v_\varepsilon, \quad \text{with } v_\varepsilon \in E_{(x_\varepsilon,\lambda_\varepsilon)} \text{ and as } \varepsilon \to 0, \tag{1.5}$$

$$\alpha_\varepsilon \to K(y)^{\frac{4-n}{8}}; \qquad \|v_\varepsilon\| \to 0; x_\varepsilon \to y \text{ and } \lambda_\varepsilon \to +\infty, \tag{1.6}$$

where

$$E_{(x_\varepsilon,\lambda_\varepsilon)} = \left\{ w \in H_2^2(S^n) / \langle w, \varphi \rangle = 0 \ \forall \varphi \in \operatorname{Span}\left\{ \widetilde{\delta}_{(x_\varepsilon,\lambda_\varepsilon)}, \frac{\partial \widetilde{\delta}_{(x_\varepsilon,\lambda_\varepsilon)}}{\partial \lambda}, \frac{\partial \widetilde{\delta}_{(x_\varepsilon,\lambda_\varepsilon)}}{\partial x^j}; j \le n \right\} \right\}.$$

In the second result, we give a sufficient condition on the function K to ensure the nonexistence of single-peaked solutions of $(S_{-\varepsilon})$ with $\varepsilon > 0$.

Theorem 1.2 *Let y be a nondegenerate critical point of K with $\Delta K(y) > 0$. Then, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, problem $(S_{-\varepsilon})$ has no solution (u_ε) of the form* (1.5) *satisfying* (1.6).

In view of the above results, a natural question arises: are equivalent results true for slightly supercritical exponents? The aim of the next result is to answer this question. In [14] and [12], the author proved some nonexistence results of sign-changing solutions for

a biharmonic equation involving slightly supercritical exponents in a bounded domain of $\mathbb{R}^n$. Note that, in the supercritical case, problem $(S_{+\varepsilon})$ becomes more delicate since we lose the Sobolev embedding that is an important point to overcome.

In contrast to the subcritical case, we have the following nonexistence result for the supercritical problem.

Theorem 1.3 *Let y be a nondegenerate critical point of K with $\Delta K(y) < 0$. Then, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, problem $(S_{+\varepsilon})$ has no solution (u_ε) of the form (1.5) satisfying (1.6) and $|u_\varepsilon|_\infty^\varepsilon$ is bounded.*

Remark 1.4 For the subcritical problem $(S_{-\varepsilon})$, the assumption that $|u_\varepsilon|_\infty^\varepsilon$ is bounded is always satisfied. To remove it in the supercritical case we should adapt the proof established in [5] when the author studied a supercritical problem in a bounded domain of $\mathbb{R}^n$, but it is too technical to discuss this in the present paper. Note that this assumption allows us to recover the Sobolev embedding.

The present paper is organized as follows. In Sect. 2, we set up the variational structure and recall some known facts. In Sect. 3, we provide the proof of Theorem 1.1, while Sects. 4 and 5 are devoted to the proofs of Theorem 1.2 and Theorem 1.3, respectively.

2 Variational structure and some known facts

In this section we recall the functional setting and its main features in the subcritical case. For $\varepsilon > 0$, we define the functional

$$\begin{aligned} I_\varepsilon(u) = {} & \frac{1}{2}\int_{S^n} |\Delta u|^2 + c_n \int_{S^n} |\nabla u|^2 \\ & + d_n \int_{S^n} u^2 - \frac{1}{\frac{2n}{n-4} - \varepsilon} \int_{S^n} K|u|^{\frac{2n}{n-4}-\varepsilon}, \quad u \in H_2^2(S^n). \end{aligned} \tag{2.1}$$

The positive critical points of I_ε are solutions of $(S_{-\varepsilon})$.

First, we give the following remark that is established by [16] when S^n is replaced by a bounded domain of $\mathbb{R}^3$.

Remark 2.1 Let $\widetilde{\delta}_{(a,\lambda)}$ be the function defined in (1.4). Assume that $\varepsilon \log \lambda$ is small enough. For $\varepsilon > 0$, we have

$$\widetilde{\delta}_{(a,\lambda)}^{-\varepsilon}(x) = 1 - \varepsilon \log \widetilde{\delta}_{(a,\lambda)} + O\big(\varepsilon^2 \log^2 \lambda\big) \quad \text{in } S^n.$$

Now, we collect some expansions of the gradient of the functional I_ε associated with the problem $(S_{-\varepsilon})$ that will be needed in Sect. 3. Explicit computations, by Remark 2.1, yield the following propositions. For the sake of simplicity, we will write $\widetilde{\delta}$ instead of $\widetilde{\delta}_{(x,\lambda)}$.

Proposition 2.2 *For $u = \alpha\widetilde{\delta}_{(x,\lambda)} + v$ with $v \in E_{(x,\lambda)}$, we have*

$$\big\langle \nabla I_\varepsilon(u), \widetilde{\delta}\big\rangle = \alpha S_n\big(1 - \alpha^{\frac{8}{n-4}-\varepsilon} K(x)\big) + O\left(\varepsilon \log \lambda + \frac{1}{\lambda^2} + \|v\|^2\right),$$

where $S_n = \int_{\mathbb{R}^n} \delta_{(0,1)}^{\frac{2n}{n-4}}$.

Proof We have

$$\langle \nabla I_\varepsilon, h\rangle = \int_{S^n} \mathcal{P}u.h - \int_{S^n} Ku^{p-\varepsilon}h. \tag{2.2}$$

A computation similar to the one performed in [2] shows that

$$\int_{S^n} \mathcal{P}\widetilde{\delta}.\widetilde{\delta} = \int_{S^n} \widetilde{\delta}^{\frac{2n}{n-4}} = \int_{\mathbb{R}^n} \delta^{\frac{2n}{n-4}} = S_n. \tag{2.3}$$

For the integral, we write

$$\int_{S^n} Ku^{\frac{n+4}{n-4}-\varepsilon}\widetilde{\delta} = \int_{S^n} K(\alpha\widetilde{\delta}+v)^{\frac{n+4}{n-4}-\varepsilon}\widetilde{\delta} = \alpha^{\frac{n+4}{n-4}-\varepsilon}\int_{S^n} K\widetilde{\delta}^{\frac{2n}{n-4}-\varepsilon} + O\big(|v|^2\big). \tag{2.4}$$

Expanding of K around x, we obtain

$$\int_{S^n} K\widetilde{\delta}^{\frac{2n}{n-4}-\varepsilon} = \int_{\mathbb{R}^n} K\widetilde{\delta}^{\frac{2n}{n-4}-\varepsilon} = K(x)S_n + O\left(\varepsilon\log\lambda + \frac{1}{\lambda^2}\right). \tag{2.5}$$

Combining (2.2)–(2.5), we easily derive our proposition. □

Proposition 2.3 *For $u = \alpha\widetilde{\delta}_{(x,\lambda)} + v$ with $v \in E_{(x,\lambda)}$, we have the following expansion*:

$$\begin{aligned}\left\langle \nabla I_\varepsilon(u), \lambda\frac{\partial\widetilde{\delta}}{\partial\lambda}\right\rangle &= \alpha^{\frac{n+4}{n-4}-\varepsilon}\left[\frac{\varepsilon S_n K(x)}{n} + \frac{4(n-4)c_2}{n}\frac{\Delta K(x)}{\lambda^2}\right] \\ &\quad + O\left(\varepsilon^2\log\lambda + \frac{\varepsilon\log\lambda}{\lambda^2} + \frac{1}{\lambda^3} + \|v\|^2\right),\end{aligned}$$

where

$$c_2 = \frac{1}{2n}\int_{\mathbb{R}^n} |x|^2\delta_{(0,1)}^{\frac{2n}{n-4}}\,dx.$$

Proof Observe that (see [2])

$$\left\langle \widetilde{\delta}, \lambda\frac{\partial\widetilde{\delta}}{\partial\lambda}\right\rangle = 0 \tag{2.6}$$

and

$$\begin{aligned}\int_{S^n} K\widetilde{\delta}^{\frac{n+4}{n-4}-\varepsilon}\lambda\frac{\partial\widetilde{\delta}}{\partial\lambda} &= \int_{\mathbb{R}^n} K\widetilde{\delta}^{\frac{n+4}{n-4}-\varepsilon}\lambda\frac{\partial\delta}{\partial\lambda} \\ &= -\frac{(n-4)c_2}{n}\frac{4\Delta K(x)}{\lambda^2} - \frac{S_n\varepsilon}{n}K(x) \\ &\quad + O\left(\varepsilon^2\log\lambda + \frac{1}{\lambda^3} + \frac{\varepsilon\log\lambda}{\lambda^2}\right).\end{aligned} \tag{2.7}$$

Combining (2.2), (2.6), and (2.7), we derive our proposition. □

Proposition 2.4 *For $u = \alpha\widetilde{\delta}_{(x,\lambda)} + v$, with $v \in E_{(x,\lambda)}$, we have*

$$\left\langle \nabla I_\varepsilon(u), \frac{1}{\lambda}\frac{\partial\widetilde{\delta}}{\partial x}\right\rangle = -\frac{\alpha^{\frac{n+4}{n-4}-\varepsilon}c_3\nabla K(x)}{\lambda} + O\left(\frac{\varepsilon\log\lambda}{\lambda}\left|\nabla K(x)\right| + \frac{1}{\lambda^2} + \|v\|^2\right).$$

Proof An easy computation shows

$$\left\langle \widetilde{\delta}, \frac{1}{\lambda}\frac{\partial\widetilde{\delta}}{\partial x}\right\rangle = 0 \tag{2.8}$$

and

$$\int_{S^n} K\widetilde{\delta}^{\frac{n+4}{n-4}-\varepsilon}\frac{1}{\lambda}\frac{\partial\widetilde{\delta}}{\partial x} = \int_{\mathbb{R}^n} K\widetilde{\delta}^{\frac{n+4}{n-4}-\varepsilon}\frac{1}{\lambda}\frac{\partial\delta}{\partial x} = c_3\frac{\nabla K(x)}{\lambda} + O\left(\frac{1}{\lambda^2} + \varepsilon^2\log\lambda\right). \tag{2.9}$$

Using (2.2), (2.8), and (2.9), we have our proposition. □

3 Proof of Theorem 1.1

Let

$$M_{\varepsilon,1} = \Big\{m = (\alpha,\lambda,x,v) \in \mathbb{R}\times\mathbb{R}_+^*\times S^n\times H_2^2(S^n) : v \in E_{(x,\lambda)}, \|v\| < \nu_0,$$
$$\left|\alpha^{\frac{8}{n-4}}K(x) - 1\right|\langle \nu_0,\lambda\rangle\frac{1}{\nu_0}, \varepsilon\log\lambda < \nu_0\Big\},$$

where ν_0 is a small positive constant. Let us define the function by

$$\Psi_{\varepsilon,1} : M_{\varepsilon,1} \to \mathbb{R}; \qquad m = (\alpha,\lambda,x,v) \mapsto I_\varepsilon(\alpha\widetilde{\delta}_{(x,\lambda)} + v). \tag{3.1}$$

As in [4], using the Euler–Lagrange coefficients, we easily obtain the following proposition.

Proposition 3.1 *Let $m = (\alpha,\lambda,x,v) \in M_{\varepsilon,1}$. m be a critical point of $\Psi_{\varepsilon,1}$ if and only if $u = \alpha\widetilde{\delta} + v$ is a critical point of I_ε, i.e., if and only if there exists $(A,B,C) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$ such that the following hold*:

$$(E_\alpha) \quad \frac{\partial\Psi_{\varepsilon,1}}{\partial\alpha} = 0, \tag{3.2}$$

$$(E_\lambda) \quad \frac{\partial\Psi_{\varepsilon,1}}{\partial\lambda} = B\left\langle\frac{\partial^2\widetilde{\delta}}{\partial\lambda^2}, v\right\rangle + \sum_{j=1}^n C_j\left\langle\frac{\partial^2\widetilde{\delta}}{\partial x^j\partial\lambda}, v\right\rangle, \tag{3.3}$$

$$(E_x) \quad \frac{\partial\Psi_{\varepsilon,1}}{\partial x} = B\left\langle\frac{\partial^2\widetilde{\delta}}{\partial\lambda\partial x}, v\right\rangle + \sum_{j=1}^n C_j\left\langle\frac{\partial^2\widetilde{\delta}}{\partial x^j\partial x}, v\right\rangle, \tag{3.4}$$

$$(E_v) \quad \frac{\partial\Psi_{\varepsilon,1}}{\partial v} = A\widetilde{\delta} + B\frac{\partial\widetilde{\delta}}{\partial\lambda} + \sum_{j=1}^n C_j\frac{\partial\widetilde{\delta}}{\partial x^j}. \tag{3.5}$$

The results of Theorem 1.1 will be obtained through a careful analysis of (3.2)–(3.5) on $M_{\varepsilon,1}$. As usual in this type of problem, we first deal with the v-part of u, in order to show that it is negligible with respect to the concentration phenomenon. The study of (E_v) yields.

Proposition 3.2 *There exists a smooth map that to any* $(\varepsilon,\alpha,\lambda,x)$ *such that* $(\alpha,\lambda,x,0)$ *in* $M_{\varepsilon,1}$ *associates* $\overline{v}\in E_{(x,\lambda)}$ *such that* $\|v\|<\nu_0$ *and* (E_v) *is satisfied for some* $(A,B,C)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$. *Such a* $\overline{v}$ *is unique, minimizes* $\Psi_{\varepsilon,1}(\alpha,\lambda,x,v)$ *with respect to* v *in* $\{v\in E_{(x,\lambda)}/\|v\|<\nu_0\}$, *and we have the following estimate*

$$\|\overline{v}\| = O\left(\varepsilon+\frac{|\nabla K(x)|}{\lambda}+\frac{1}{\lambda^2}\right). \tag{3.6}$$

Proof Expanding I_ε with respect to $v\in E_{(x,\lambda)}$, we obtain

$$I_\varepsilon(\alpha\widetilde{\delta}+v) = c(\alpha,x,\lambda)+\frac{1}{2}Q(v,v)-f(v)+R(v), \tag{3.7}$$

where $Q(.,.)$ is a quadratic positive-definite form, $f(.)$ is a linear form, and $R(v)$ satisfies $R(v)=o(\|v\|^2)$, $R'(v)=o(\|v\|)$, and $R''(v)=o(1)$.

Since $Q(v,v)$ is positive-definite, we derive that the following problem

$$\min\left\{I_\varepsilon(\alpha\widetilde{\delta}+v), v\in E_{(x,\lambda)} \text{ and } \|v\|<\nu_0\right\} \tag{3.8}$$

is achieved by a unique function $\overline{v}$ that satisfies $\|\overline{v}\|\le c\|f\|$. Now, following [6] we obtain the estimate (3.6). Since $\overline{v}$ is orthogonal to the functions $\{\widetilde{\delta},\partial\widetilde{\delta}/\partial\lambda,\partial\widetilde{\delta}/\partial x^j, j\le n\}$, there exist A, B, and C such that

$$\frac{\partial\Psi_{\varepsilon,1}}{\partial v}(\alpha,\lambda,x,\overline{v}) = \nabla I_\varepsilon(\alpha\widetilde{\delta}+\overline{v}) = A\widetilde{\delta}+B\frac{\partial\widetilde{\delta}}{\partial\lambda}+\sum_{j=1}^{n}C_j\frac{\partial\widetilde{\delta}}{\partial x^j}. \tag{3.9}$$

The proposition follows. □

Proof of Theorem 1.1 Once $\overline{v}$ is defined by Proposition 3.2, we estimate the corresponding numbers A, B, C by taking the scalar product in $H_2^2(S^n)$ of (E_v) with $\widetilde{\delta}$, $\partial\widetilde{\delta}/\partial\lambda$, $\partial\widetilde{\delta}/\partial x$ and $\partial\widetilde{\delta}/\partial x$, respectively. Thus, we obtain a quasidiagonal system whose coefficients are given by

$$\int_{\mathbb{R}^n}|\nabla\delta|^2 = S_n,\qquad \int_{\mathbb{R}^n}\nabla\delta\nabla\frac{\partial\delta}{\partial\lambda}=0,\qquad \int_{\mathbb{R}^n}\left|\nabla\frac{\partial\delta}{\partial\lambda}\right|^2=\frac{\Gamma_1}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right),$$
$$\int_{\mathbb{R}^n}\nabla\frac{\partial\delta}{\partial\lambda}\nabla\frac{\partial\delta}{\partial x}=O\left(\frac{1}{\lambda^3}\right),\qquad \int_{\mathbb{R}^n}\left|\nabla\frac{\partial\delta}{\partial x}\right|^2=\Gamma_2\lambda^2+O\left(\frac{1}{\lambda}\right),\qquad \int_{\mathbb{R}^n}\nabla\delta\nabla\frac{\partial\delta}{\partial x}=0,$$

where Γ_1, Γ_2 are positive constants. The other side is given by

$$\frac{\partial\Psi_{\varepsilon,1}}{\partial\alpha}=\left\langle\frac{\partial\Psi_{\varepsilon,1}}{\partial v},\widetilde{\delta}\right\rangle;\qquad \frac{1}{\alpha}\frac{\partial\Psi_{\varepsilon,1}}{\partial\lambda}=\left\langle\frac{\partial\Psi_{\varepsilon,1}}{\partial v},\frac{\partial\widetilde{\delta}}{\partial\lambda}\right\rangle;\qquad \frac{1}{\alpha}\frac{\partial\Psi_{\varepsilon,1}}{\partial x}=\left\langle\frac{\partial\Psi_{\varepsilon,1}}{\partial v},\frac{\partial\widetilde{\delta}}{\partial x}\right\rangle. \tag{3.10}$$

Using Proposition 2.2, some computations yield

$$\frac{\partial\Psi_{\varepsilon,1}}{\partial\alpha}=-\frac{8}{n-4}S_n\beta+V_\alpha(\varepsilon,\alpha,\lambda,x), \tag{3.11}$$

where $\beta = \alpha - 1/K(y)^{\frac{n-4}{8}}$ and V_α is a smooth function that satisfies

$$V_\alpha = O\left(\beta^2 + \varepsilon \log\lambda + \frac{1}{\lambda^2} + |x-y|^2\right). \tag{3.12}$$

Now, using Proposition 2.3, we obtain

$$\frac{\partial \Psi_{\varepsilon,1}}{\partial \lambda} = \frac{1}{K(y)^{n-4/4}}\left(\frac{\varepsilon S_n}{n\lambda} + \frac{4(n-4)c_2}{n}\frac{\Delta K(x)}{K(x)}\frac{1}{\lambda^3}\right) + V_\lambda(\varepsilon,\alpha,\lambda,x), \tag{3.13}$$

where c_2 and c_3 are defined in Propositions 2.3 and 2.4, and V_λ is a smooth function satisfying

$$V_\lambda = O\left[\frac{1}{\lambda}\left(\frac{1}{\lambda^3} + \frac{|x-y|^2}{\lambda^2} + \varepsilon^2\log\lambda + \frac{\varepsilon\log\lambda}{\lambda^2}\right) + \left(\beta + |x-y|^2\right)\left(\frac{\varepsilon}{\lambda} + \frac{1}{\lambda^3}\right)\right]. \tag{3.14}$$

Lastly, using Proposition 2.4, we have

$$\frac{\partial \Psi_{\varepsilon,1}}{\partial x} = \frac{-c_3}{K(y)^{(n-4)/8}}\nabla K(x) + V_x(\varepsilon,\alpha,\lambda,x), \tag{3.15}$$

where V_x is a smooth function such that

$$V_x = O\left(\frac{1}{\lambda} + \left(\beta + \varepsilon\log\lambda + |x-y|^2\right)|x-y|\right). \tag{3.16}$$

Note that these estimates imply

$$\frac{\partial \Psi_{\varepsilon,1}}{\partial \alpha} = O\left(\beta + \varepsilon\log\lambda + \frac{1}{\lambda^2} + |x-y|^2\right),$$
$$\frac{\partial \Psi_{\varepsilon,1}}{\partial \lambda} = O\left(\frac{1}{\lambda^3} + \frac{\varepsilon}{\lambda}\right), \qquad \frac{\partial \Psi_{\varepsilon,1}}{\partial x} = O\left(|x-y| + \frac{1}{\lambda}\right).$$

The solution of the system in A, B, and C shows that

$$A = O\left(\beta + \varepsilon\log\lambda + \frac{1}{\lambda^2} + |x-y|^2\right), \qquad B = O\left(\frac{1}{\lambda} + \varepsilon\lambda\right), \qquad C = O\left(\frac{|x-y|}{\lambda^2} + \frac{1}{\lambda^3}\right).$$

This allows us to evaluate the right-hand sides in the equations (E_λ) and (E_x), namely

$$B\left\langle \frac{\partial^2\widetilde{\delta}}{\partial\lambda^2}, \overline{v}\right\rangle + \sum_{j=1}^{n} C_j\left\langle \frac{\partial^2\widetilde{\delta}}{\partial x^j\partial\lambda}, \overline{v}\right\rangle = O\left(\left(\frac{1}{\lambda^3} + \frac{\varepsilon}{\lambda} + \frac{|y-x|}{\lambda^2}\right)\|\overline{v}\|\right), \tag{3.17}$$

$$B\left\langle \frac{\partial^2\widetilde{\delta}}{\partial\lambda\partial x}, \overline{v}\right\rangle + \sum_{j=1}^{n} C_j\left\langle \frac{\partial^2\widetilde{\delta}}{\partial x^j\partial x}, \overline{v}\right\rangle = O\left(\left(\frac{1}{\lambda} + \varepsilon\lambda + |x-y|\right)\|\overline{v}\|\right), \tag{3.18}$$

where we have used the following estimates

$$\left\|\frac{\partial^2\widetilde{\delta}}{\partial\lambda^2}\right\| = O\left(\frac{1}{\lambda^2}\right); \qquad \left\|\frac{\partial^2\widetilde{\delta}}{\partial x\partial\lambda}\right\| = O(1); \qquad \left\|\frac{\partial^2\widetilde{\delta}}{\partial x^2}\right\| = O(\lambda^2).$$

Now,we set

$$\frac{1}{\lambda} = \varepsilon^{\frac{1}{2}}\Lambda(1+\zeta); \qquad x = y + \xi,$$

where $\zeta \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ are assumed to be small and $\Lambda = \Lambda(y)$ verifies

$$\frac{S_n}{n} + \frac{4(n-4)c_2}{n}\Lambda^2\frac{\Delta K(y)}{K(y)} = 0.$$

With these changes of variables and using (3.11), (E_α) is equivalent to

$$\beta = V_\alpha(\varepsilon,\beta,\zeta,\xi) = O\bigl(\beta^2 + \varepsilon|\log\varepsilon| + |\xi|^2\bigr). \tag{3.19}$$

Now, using (3.13), we show by an easy computation

$$\begin{aligned}
&\frac{\varepsilon S_n}{n\lambda} + \frac{4(n-4)c_2}{n}\frac{\Delta K(y+\xi)}{K(y+\xi)}\frac{1}{\lambda^3} \\
&\quad = \frac{\varepsilon^{3/2}S_n}{n}\Lambda(1+\zeta) \\
&\qquad + \frac{4(n-4)c_2}{n}\varepsilon^{3/2}\Lambda^3(1+3\zeta)\left(\frac{\Delta K(y)}{K(y)} + \frac{\nabla\Delta K(y)}{K(y)}\xi\right) + O\bigl(\varepsilon^{3/2}\bigl(\zeta^2+\xi^2\bigr)\bigr) \\
&\quad = \varepsilon^{3/2}\left[\frac{8(n-4)c_2\Lambda^3}{n}\frac{\Delta K(y)}{K(y)}\right]\zeta + \varepsilon^{3/2}\left[\frac{4(n-4)c_2\Lambda^3}{n}\frac{\nabla(\Delta K)(y)}{K(y)}\right]\xi \\
&\qquad + O\bigl(\varepsilon^{3/2}\bigl(\zeta^2+\xi^2\bigr)\bigr).
\end{aligned}$$

This implies that (E_λ) is equivalent, on account of (3.14) and (3.17), to

$$\begin{aligned}
&\left[\frac{8(n-4)c_2\Lambda^3}{n}\frac{\Delta K(y)}{K(y)}\right]\zeta + \left[\frac{4(n-4)c_2\Lambda^3}{n}\frac{\nabla(\Delta K)(y)}{K(y)}\right]\xi \\
&\quad = V_\lambda(\varepsilon,\beta,\zeta,\xi) = O\bigl(\beta^2+\zeta^2+\xi^2+\varepsilon^{1/2}\bigr).
\end{aligned} \tag{3.20}$$

Lastly, using (3.15), (3.16), and (3.18), we see that (E_x) is equivalent to

$$D^2K(y)\xi = V_x(\varepsilon,\beta,\zeta,\xi) = O\bigl(\varepsilon^{1/2}+\beta^2+\zeta^2+\xi^2\bigr). \tag{3.21}$$

We remark that V_α, V_λ, and V_x are smooth functions. This system may be written as

$$\begin{cases} \beta = V(\varepsilon,\beta,\zeta,\xi), \\ L(\zeta,\xi) = W(\varepsilon,\beta,\zeta,\xi), \end{cases} \tag{3.22}$$

where L is a fixed linear operator on $\mathbb{R}^{n+1}$ defined by (3.20) and (3.21) and V, W are smooth functions satisfying

$$\begin{cases} V(\varepsilon,\beta,\zeta,\xi) = O(\varepsilon^{1/2}+|\beta|^2+|\xi|^2), \\ W(\varepsilon,\beta,\zeta,\xi) = O(\varepsilon^{1/2}+|\beta|^2+|\zeta|^2+|\xi|^2). \end{cases}$$

Moreover, a simple computation shows that the determinant of L is not equal to zero. Hence, L is invertible, and Brouwer's fixed-point theorem shows that (3.22) has a solution $(\beta_\varepsilon, \zeta_\varepsilon, \xi_\varepsilon)$ for ε small enough, such that

$$|\beta_\varepsilon| = O\big(\varepsilon^{1/2}\big); \qquad |\zeta_\varepsilon| = O\big(\varepsilon^{1/2}\big); \qquad |\xi_\varepsilon| = O\big(\varepsilon^{1/2}\big).$$

Hence, we have constructed $m_\varepsilon = (\alpha_\varepsilon, \lambda_\varepsilon, x_\varepsilon)$ such that $u_\varepsilon := \alpha_\varepsilon \widetilde{\delta}_{(x_\varepsilon,\lambda_\varepsilon)} + \overline{v}_\varepsilon$, satisfies (3.2)–(3.6). Therefore, by Proposition 3.1, u_ε is a critical point of I_ε, i.e., u_ε is a solution of $(S_{-\varepsilon})$. Hence, the proof of Theorem 1.1 is thereby completed. □

4 Proof of Theorem 1.2

The proof of Theorem 1.2 will, by contradiction, suppose that the subcritical problem $(S_{-\varepsilon})$ has a solution of the form (1.5) and satisfying (1.6).

First, we show that $\lambda_\varepsilon^\varepsilon \to 1$ as $\varepsilon \to 0$. For this, multiplying $(S_{-\varepsilon})$ by $\tilde{\delta}_{(x_\varepsilon,\lambda_\varepsilon)}$ and integrating over S^n, we obtain

$$\alpha_\varepsilon \|\tilde{\delta}\|^2 = \int_{S^n} K|\alpha_\varepsilon \tilde{\delta} + v_\varepsilon|^{\frac{n+4}{n-4}-\varepsilon} \tilde{\delta} = \alpha_\varepsilon^{\frac{n+4}{n-4}-\varepsilon} \int_{S^n} K \tilde{\delta}^{\frac{2n}{n-4}-\varepsilon} + O\big(\|v_\varepsilon\|\big). \tag{4.1}$$

From (2.3) and (2.5) we derive

$$\alpha_\varepsilon S_n = \frac{\alpha_\varepsilon^{\frac{n+4}{n-4}-\varepsilon} K(x_\varepsilon) S_n}{\lambda_\varepsilon^{\varepsilon(n-4)}} \big(1 + o(1)\big) + o(1), \tag{4.2}$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Since $\alpha_\varepsilon \to K(y)^{8/(n-4)}$ and $x_\varepsilon \to y$ as $\varepsilon \to 0$, we deduce from (4.2) that $\lambda_\varepsilon^\varepsilon \to 1$ as $\varepsilon \to 0$.

Secondly, we find the estimation of v_ε. Multiplying $(S_{-\varepsilon})$ by v_ε and integrating, we have

$$\begin{aligned}
\|v_\varepsilon\|^2 &= \int_{S^n} K(x)|\alpha_\varepsilon \tilde{\delta} + v_\varepsilon|^{\frac{n+4}{n-4}-\varepsilon} v_\varepsilon = \alpha_\varepsilon^{\frac{n+4}{n-4}-\varepsilon} \int_{S^n} K(x) \tilde{\delta}^{\frac{n+4}{n-4}-\varepsilon} v_\varepsilon \\
&\quad + \left(\frac{n+4}{n-4} - \varepsilon\right) \alpha_\varepsilon^{\frac{8}{n-4}-\varepsilon} \int_{S^n} K(x) \tilde{\delta}^{\frac{8}{n-4}-\varepsilon} v_\varepsilon^2 + O\big(\|v_\varepsilon\|^{\min(3, \frac{2n}{n-4}-\varepsilon)}\big) \\
&= \alpha_\varepsilon^{\frac{n+4}{n-4}-\varepsilon} \int_{S^n} K(x) \tilde{\delta}^{\frac{n+4}{n-4}-\varepsilon} v_\varepsilon + \left(\frac{n+4}{n-4} - \varepsilon\right) \alpha_\varepsilon^{\frac{8}{n-4}-\varepsilon} K(x_\varepsilon) \int_{S^n} \tilde{\delta}^{\frac{8}{n-4}-\varepsilon} v_\varepsilon^2 + o\big(\|v_\varepsilon\|^2\big).
\end{aligned}$$

According to [1], there exists a $\rho > 0$, such that

$$\begin{aligned}
&\|v_\varepsilon\|^2 - \left(\frac{n+4}{n-4} - \varepsilon\right) \alpha_\varepsilon^{\frac{8}{n-4}-\varepsilon} K(x_\varepsilon) \int_{S^n} \tilde{\delta}^{\frac{8}{n-4}-\varepsilon} v_\varepsilon^2 \\
&\quad = \|v_\varepsilon\|^2 - \frac{n+4}{n-4} \int_{S^n} \tilde{\delta}^{\frac{8}{n-4}-\varepsilon} v_\varepsilon^2 + o\big(\|v_\varepsilon\|^2\big) \geq \rho \|v_\varepsilon\|^2.
\end{aligned} \tag{4.3}$$

For the other term, expanding the function K around x_ε and using Holder's inequality, we obtain

$$\left| \alpha_\varepsilon^{\frac{n+4}{n-4}-\varepsilon} \int_{S^n} K(x) \tilde{\delta}^{\frac{n+4}{n-4}-\varepsilon} v_\varepsilon \right| \leq c\|v_\varepsilon\| \left(\varepsilon + \frac{|\nabla K(x_\varepsilon)|}{\lambda_\varepsilon} + \frac{1}{\lambda_\varepsilon^2} \right). \tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$\|v_\varepsilon\| = O\left(\varepsilon + \frac{|\nabla K(x_\varepsilon)|}{\lambda_\varepsilon} + \frac{1}{\lambda_\varepsilon^2}\right).$$

We turn now to the proof of the theorem, multiplying $(S_{-\varepsilon})$ by $\lambda_\varepsilon \partial\tilde{\delta}/\partial\lambda_\varepsilon$ and integrating, we derive

$$\alpha_\varepsilon\left\langle \tilde{\delta}, \lambda_\varepsilon \frac{\partial\tilde{\delta}}{\partial\lambda_\varepsilon}\right\rangle - \int_{S^n} K(x)|\alpha_\varepsilon\tilde{\delta} + v_\varepsilon|^{\frac{n+4}{n-4}}\lambda_\varepsilon \frac{\partial\tilde{\delta}}{\partial\lambda_\varepsilon} = 0. \tag{4.5}$$

Arguing as in the proof of Proposition 2.3, we easily arrive at:

$$\varepsilon S_n K(y) + c_2\frac{\Delta K(y)}{\lambda_\varepsilon^2} + O\left(\varepsilon^2\log\lambda_\varepsilon + \frac{\varepsilon\log\lambda_\varepsilon}{\lambda_\varepsilon^2} + \frac{1}{\lambda_\varepsilon^3}\right) = 0,$$

where we have used the previous estimate of v_ε and the fact that $\lambda_\varepsilon^\varepsilon \to 1$, $\alpha_\varepsilon \to K(y)^{8/(n-2)}$. Thus,

$$\varepsilon S_n K(y) + c_2\frac{\Delta K(y)}{\lambda_\varepsilon^2} = o\left(\varepsilon + \frac{1}{\lambda_\varepsilon^2}\right),$$

which is a contradiction with the assumption of Theorem 1.2.

5 Proof of Theorem 1.3

Arguing by contradiction, suppose that $(S_{+\varepsilon})$ has a solution as stated in Theorem 1.3. We start by showing that $\lambda_\varepsilon^\varepsilon \to 1$ as $\varepsilon \to 0$. Indeed, multiplying $(S_{+\varepsilon})$ by $\tilde{\delta}_{(x_\varepsilon,\lambda_\varepsilon)}$ and integrating over S^n, we obtain

$$\begin{aligned}
\alpha_\varepsilon\|\tilde{\delta}\|^2 &= \int_{S^n} K(x)|\alpha_\varepsilon\tilde{\delta} + v_\varepsilon|^{\frac{n+4}{n-4}+\varepsilon}\tilde{\delta} \\
&= \alpha_\varepsilon^{\frac{n+4}{n-4}+\varepsilon}\int_{S^n} K(x)\tilde{\delta}^{\frac{2n}{n-4}+\varepsilon} + O\left(\int_{S^n}\tilde{\delta}^{\frac{n+4}{n-4}+\varepsilon}|v| + \int_{S^n}\tilde{\delta}|v|^{\frac{n+4}{n-4}+\varepsilon}\right) \\
&= \alpha_\varepsilon^{\frac{n+4}{n-4}+\varepsilon}\int_{S^n} K(x)\tilde{\delta}^{\frac{2n}{n-4}+\varepsilon} + O\left(\lambda_\varepsilon^{\epsilon\frac{n-4}{2}}\int_{S^n}\tilde{\delta}^{\frac{n+4}{n-4}+\varepsilon}|v| + \lambda_\varepsilon^{\varepsilon\frac{n-4}{2}}\int_{S^n}\tilde{\delta}^{1-\varepsilon}|v|^{\frac{n+4}{n-4}+\varepsilon}\right) \\
&= \alpha_\varepsilon^{\frac{n+4}{n-4}+\varepsilon}\int_{S^n} K(x)\tilde{\delta}^{\frac{2n}{n-4}+\varepsilon} + O\left(\lambda_\varepsilon^{\varepsilon\frac{n-4}{2}}\|v\| + \lambda_\varepsilon^{\varepsilon\frac{n-4}{2}}\|v\|^{\frac{n+4}{n-4}+\varepsilon}\right).
\end{aligned} \tag{5.1}$$

As in (2.5), we have

$$\int_{S^n} K(x)\tilde{\delta}^{\frac{2n}{n-4}+\varepsilon} = K(x_\varepsilon)S_n\lambda_\varepsilon^{\varepsilon\frac{n-4}{2}}\big(1 + o(1)\big). \tag{5.2}$$

Consequently, by (2.3) and (5.2), we obtain

$$\alpha_\varepsilon S_n = \alpha_\varepsilon^{\frac{n+4}{n-4}+\varepsilon}K(x_\varepsilon)S_n\lambda_\varepsilon^{\varepsilon(n-4)}\big(1 + o(1)\big) + o(1), \tag{5.3}$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Since $\alpha_\varepsilon \to K(y)^{8/(n-4)}$ and $x_\varepsilon \to y$ as $\varepsilon \to 0$, we deduce from (5.3) that $\lambda_\varepsilon^{\varepsilon(n-4)/2} \to 1$ as $\varepsilon \to 0$.

Remark 5.1 We remark that:

(i) Since $\lambda_\varepsilon^\varepsilon \to 1$ as $\varepsilon \to 0$ it is easy to derive that $\varepsilon \log(1+\lambda_\varepsilon^2|x-x_\varepsilon|^2)$ tends to 0 as $\varepsilon \to 0$ and therefore we obtain:

$$\tilde{\delta}^\varepsilon(x) - c_0^\varepsilon \lambda_\varepsilon^{\varepsilon(n-4)/2} = O\big(\varepsilon \log\big(1+\lambda_\varepsilon^2|x-x_\varepsilon|^2\big)\big) \quad \text{in } S^n.$$

(ii) We also point out that it follows from the assumption that $|u_\varepsilon|_\infty^\varepsilon$ is bounded and $\lambda_\varepsilon^\varepsilon \to 1$ as $\varepsilon \to 0$ that $|v_\varepsilon|_\infty^\varepsilon$ is bounded – a fact that is used in the proof of Lemma 5.2.

Next, we are going to estimate the v_ε-part of u_ε in order to show that it is negligible with respect to the concentration phenomenon. Namely, we have the following estimate.

Lemma 5.2 *The function v_ε defined in* (1.5), *satisfies the following estimate*

$$\|v_\varepsilon\| = O\left(\varepsilon + \frac{|\nabla K(x_\varepsilon)|}{\lambda_\varepsilon} + \frac{1}{\lambda_\varepsilon^2}\right). \tag{5.4}$$

Proof Multiplying $(S_{+\varepsilon})$ by v_ε and integrating, we obtain

$$\begin{aligned}\|v_\varepsilon\|^2 &= \int_{S^n} K|\alpha_\varepsilon\tilde{\delta} + v_\varepsilon|^{\frac{n+4}{n-4}+\varepsilon} v_\varepsilon = \alpha_\varepsilon^{\frac{n+4}{n-4}+\varepsilon} \int_{S^n} K\tilde{\delta}^{\frac{n+4}{n-4}+\varepsilon} v_\varepsilon \\ &\quad + \left(\frac{n+4}{n-4}+\varepsilon\right)\alpha_\varepsilon^{\frac{8}{n-4}+\varepsilon} \int_{S^n} K\tilde{\delta}^{\frac{8}{n-4}+\varepsilon} v_\varepsilon^2 + O\left(\|v_\varepsilon\|^3 + \int_{S^n} |v|_\varepsilon^{\frac{2n}{n-4}+\varepsilon}\right),\end{aligned}$$

therefore

$$\begin{aligned}&\|v_\varepsilon\|^2 - \left(\frac{n+4}{n-4}+\varepsilon\right)\alpha_\varepsilon^{\frac{8}{n-4}+\varepsilon} \int_{S^n} K(y)\tilde{\delta}^{\frac{8}{n-4}+\varepsilon} v_\varepsilon^2 \\ &\quad = \alpha_\varepsilon^{\frac{n+4}{n-4}+\varepsilon} \int_{S^n} K\tilde{\delta}^{\frac{n+4}{n-4}+\varepsilon} v_\varepsilon + O\big(\|v_\varepsilon\|^{\inf(3,\frac{2n}{n-4})}\big).\end{aligned} \tag{5.5}$$

Observe that

$$\begin{aligned}&\|v_\varepsilon\|^2 - \left(\frac{n+4}{n-4}+\varepsilon\right)\alpha_\varepsilon^{\frac{8}{n-4}+\varepsilon} \int_{S^n} K(y)\tilde{\delta}^{\frac{8}{n-4}+\varepsilon} v_\varepsilon^2 \\ &\quad = \|v_\varepsilon\|^2 - \left(\frac{n+4}{n-4}+\varepsilon\right)\alpha_\varepsilon^{\frac{8}{n-4}+\varepsilon} K(x_\varepsilon) \int_{S^n} \tilde{\delta}^{\frac{8}{n-4}+\varepsilon} v_\varepsilon^2 \\ &\quad = \|v_\varepsilon\|^2 - \left(\frac{n+4}{n-4}+\varepsilon\right)\alpha_\varepsilon^{\frac{8}{n-4}+\varepsilon} c_0^\varepsilon \lambda_\varepsilon^{\varepsilon(n-4)/2} K(x_\varepsilon) \int_{S^n} \tilde{\delta}^{\frac{8}{n-4}} v_\varepsilon^2 + o\big(\|v_\varepsilon\|^2\big).\end{aligned} \tag{5.6}$$

Since $\alpha_\varepsilon \to K(y)^{8/(n-4)}$, $x_\varepsilon \to y$, and $\lambda_\varepsilon^{\varepsilon(n-4)/2} \to 1$ as $\varepsilon \to 0$, it follows from [1] that there exists a positive constant $\rho > 0$ independent of ε, such that

$$\|v_\varepsilon\|^2 - \left(\frac{n+4}{n-4}+\varepsilon\right)\alpha_\varepsilon^{\frac{8}{n-4}+\varepsilon} c_0^\varepsilon \lambda_\varepsilon^{\varepsilon(n-4)/2} K(x_\varepsilon) \int_{S^n} \tilde{\delta}^{\frac{8}{n-4}} v_\varepsilon^2 \geq \rho\|v_\varepsilon\|^2. \tag{5.7}$$

Also as in (4.4), we have

$$\alpha_\varepsilon^{\frac{n+4}{n-4}+\varepsilon} \int_{S^n} K(y)\tilde{\delta}^{\frac{n+4}{n-4}+\varepsilon} v_\varepsilon = O\left(\varepsilon + \frac{|\nabla K(x_\varepsilon)|}{\lambda_\varepsilon} + \frac{1}{\lambda_\varepsilon^2}\right)\|v_\varepsilon\|. \tag{5.8}$$

Combining (5.5), (5.7), and (5.8), we obtain the estimate (5.4).

Now, we turn to the proof of Theorem 1.3. Multiplying $(S_{+\varepsilon})$ by $\lambda_\varepsilon \partial\tilde{\delta}/\partial\lambda_\varepsilon$ and integrating over S^n, we derive

$$\alpha_\varepsilon \left\langle \tilde{\delta}, \lambda_\varepsilon \frac{\partial \tilde{\delta}}{\partial \lambda_\varepsilon} \right\rangle - \int_{S^n} K(x) |\alpha_\varepsilon \tilde{\delta} + v_\varepsilon|^{\frac{n+4}{n-4}+\varepsilon} \lambda_\varepsilon \frac{\partial \tilde{\delta}}{\partial \lambda_\varepsilon} = 0. \tag{5.9}$$

Arguing as in the proof of Proposition 2.3, we easily derive:

$$\varepsilon S_n K(y) - c_2 \frac{\Delta K(y)}{\lambda_\varepsilon^2} + O\left(\varepsilon^2 \log \lambda_\varepsilon + \frac{\varepsilon \log \lambda_\varepsilon}{\lambda_\varepsilon^2} + \frac{1}{\lambda_\varepsilon^3} \right) = 0,$$

where we have used the previous estimate of v_ε and the fact that $\lambda_\varepsilon^\varepsilon \to 1$, $\alpha_\varepsilon \to K(y)^{8/(n-4)}$. Thus,

$$\varepsilon S_n K(y) - c_2 \frac{\Delta K(y)}{\lambda_\varepsilon^2} = o\left(\varepsilon + \frac{1}{\lambda_\varepsilon^2} \right),$$

which is a contradiction with the assumption of Theorem 1.3. □

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