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# Existence and multiplicity of solutions for the Cauchy problem of a fractional Lorentz force equation 

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#### Abstract

This paper aims to deal with the Cauchy problem of a fractional Lorentz force equation. By the methods of reducing and topological degree in cone, the existence and multiplicity of solutions to the problem were obtained, which extend and enrich some previous results.

Mathematics Subject Classification: 26A33; 34G20; 34B15 Keywords: Fractional Lorentz force equation; Cauchy problem; Fixed point; Existence; Multiplicity


## 1 Introduction

In this paper, we consider the Cauchy problem of a fractional Lorentz force equation as follows:

$$
\left\{\begin{array}{l}
{ }_{0} D_{t}^{\beta}\left(\frac{{ }_{0} D_{t}^{\alpha} u(t)}{\sqrt{1-\left.l_{0}^{\alpha} D_{t}^{\alpha} u(t)\right|^{2}}}\right)=E(t, u(t))+{ }_{0} D_{t}^{\alpha} u(t) \otimes B(t, u(t)), \quad t \in(0, T),  \tag{1.1}\\
u(0)={ }_{0} D_{t}^{\alpha} u(0)=\mathbf{0},
\end{array}\right.
$$

where $|\cdot|$ is the usual Euclidean norm in $\mathbb{R}^{3},{ }_{0} D_{t}^{\beta}$ and ${ }_{0} D_{t}^{\alpha}$ are the left Riemann-Liouville fractional derivatives with orders $\alpha, \beta \in(0,1], \otimes$ is the vector product, $\mathbf{0}$ represents the zero vector, $E, B \in C\left([0, T] \times \mathbb{R}^{3} ; \mathbb{R}^{3}\right)$ stand for the electric and magnetic fields, respectively. Let $\phi$ stand for a relativistic acceleration operator defined by

$$
\phi(v)=\frac{v}{\sqrt{1-|v|^{2}}}, \quad v \in \mathcal{B}(1)
$$

$\mathcal{B}(\delta)$ means the open ball of center 0 and radius $\delta$.
In recent years, the qualitative theoretical analysis of the following relativistic oscillator equation has attracted the attention of many scholars, which comes from the classical

[^0]theory of relativity (see [1-3]):
\[

$$
\begin{equation*}
\left(\frac{m_{0} u^{\prime}}{\sqrt{1-\frac{u^{\prime 2}}{c^{2}}}}\right)^{\prime}=F \tag{1.2}
\end{equation*}
$$

\]

where $F$ means the restoring force, $m_{0}$ stands for the particle's rest mass, and $c$ represents the speed of light in a vacuum. Since the relativistic acceleration operator is a singular operator, it brings many difficulties during the course of analysis. For example, Bereanu, Jebelean, and Mawhin [4] considered the existence and multiplicity of radial solutions to the following Neumann boundary problem by critical-point theory in Minkowski space:

$$
\left\{\begin{array}{l}
\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla v|^{2}}}\right)=g(|x|, u) \quad \text { on } \mathcal{A}  \tag{1.3}\\
\frac{\partial u}{\partial \nu}=0 \quad \text { on } \partial \mathcal{A}
\end{array}\right.
$$

where $0 \leq R_{1}<R_{2}, \mathcal{A}=\left\{x \in \mathbb{R}^{N}: R_{1} \leq|x| \leq R_{2}\right\}, g:\left[R_{1}, R_{2}\right] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Mawhin [5] made a further study on the multiplicity of radial solutions of a Neumann boundary condition with periodic nonlinearity by Hamiltonian techniques. Moreover, Coelho et al. [6] employed the reducing method that converts the singular problem to an equivalent nonsingular problem to investigate some Dirichlet boundary value problems with parameters and obtained the existence and multiplicity of solutions by variational methods. Furthemore, Jebelean, Mawhin, and Şerban [7] considered the multiplicity of periodic solutions to the $N$-dimensional relativistic pendulum equation with periodic nonlinearity by a geometric method in critical-point theory. For more papers, we refer the reader to [8-11] and references therein.
Recently, the famous Lorentz force equation has attracted the attention of many scholars, which is an important equation in the field of mathematical physics and can be used to describe the effect of an electromagnetic field on the trajectory of a slowly accelerated charged particle in $\mathbb{R}^{3}$. One of the important questions is the existence of periodic motion to charged particles. From different perspectives, by different functional methods, Bereanu and Mawhin [12] and Arcoya, Bereanu, and Torres [13] established sufficient conditions for the existence of circular motion when the electric field is nonsingular. For the singular case like a Coulomb electric potential or the magnetic dipole, Garzón and Torres [14] gave a positive answer by the topological degree method.
Inspired by the above literature, an interesting question naturally arises in the mind. Can we consider the existence and multiplicity of solutions to a fractional Lorentz force equation? It should be mentioned that compared to the above paper, fractional derivatives lack some basic properties such as monotonicity, convexity-concavity, and so on. This brings many difficulties such as the estimation of inequality and prior bounds. Moreover, as far as we know, there are few papers investigating the existence and multiplicity of solutions to a fractional Lorentz force equation. Furthermore, if $\alpha=\beta=1$, the operator ${ }_{0} D_{t}^{\beta}\left(\phi\left({ }_{0} D_{t}^{\alpha} u(t)\right)\right)$ reduces to $\left(\phi\left(u^{\prime}\right)\right)^{\prime}$, so the fractional-order model is more general than the integer-order model. Based on the above reasons, the Cauchy problem of a fractional Lorentz force equation was considered. By the methods of reducing and topological degree in cone, the existence and multiplicity of solutions to the problem (1.1) were obtained. Also, for the topics on initial value problems or boundary value problems of fractional differential models, one can refer to [15-17] and references therein.

The rest of this paper is organized as follows. To begin with, the basic space, the definitions and properties of left Riemann-Liouville fractional integrals and derivatives, and some necessary lemmas are given in Sect. 2. Moreover, based on the methods of reducing and topological degree in cone, the existence and multiplicity of solutions to the problem (1.1) are proved in Sect. 3.

## 2 Preliminaries

Let $\mathbf{c}$ represent a vector and $\mathbb{R}_{+}=[0,+\infty)$. Setting $C:=C\left([0, T], \mathbb{R}^{3}\right)$ with the norm $\|u\|_{\infty}=$ $\max _{t \in[0, T]}|u(t)|$, define

$$
C^{\alpha}=\left\{u:[0, T] \rightarrow \mathbb{R}^{3} \mid u \in C \text { and }{ }_{0} D_{t}^{\alpha} u \in C\right\},
$$

whose norm is $\|u\|=\max \left\{\|u\|_{\infty},\left\|_{0} D_{t}^{\alpha} u\right\|_{\infty}\right\}$.

Definition $2.1([15,16])$ Let $u$ be a function defined on $[0, T]$.
(i) The left Riemann-Liouville fractional integral of order $\alpha>0$ for a function $u$ is defined by

$$
{ }_{0} I_{t}^{\alpha} u(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} u(s) d s, \quad t \in[0, T]
$$

provided the right-hand side is pointwise defined on $[0, T]$, where $\Gamma(\alpha)$ is the standard gamma function.
(ii) If $\alpha=n, n \in \mathbb{N}$, it reduces to the usual definitions

$$
{ }_{0} I_{t}^{n} u(t)=\frac{1}{\Gamma(n)} \int_{0}^{t}(t-s)^{n-1} u(s) d s, \quad t \in[0, T] .
$$

Definition $2.2([15,16])$ Let $u$ be a function defined on $[0, T]$.
(i) The left Riemann-Liouville fractional derivatives of order $\alpha>0$ for a function $u$ is defined by

$$
{ }_{0} D_{t}^{\alpha} u(t)=\frac{d^{n}}{d t^{n}} I_{t}^{n-\alpha} u(t), \quad t \in[0, T],
$$

where $n-1 \leq \alpha<n$ and $n \in \mathbb{N}_{+}$.
(ii) If $\alpha=n-1, n \in \mathbb{N}_{+}$, it reduces to the usual definition

$$
{ }_{0} D_{t}^{n-1} u(t)=u^{n-1}(t) .
$$

Lemma 2.3 ([17]) Assume that $u \in C(0, T) \cap L^{1}(0, T)$ with a fractional derivative of order $\alpha>0$ that belongs to $C(0, T) \cap L^{1}(0, T)$. Then,

$$
{ }_{0} I_{t}^{\alpha}{ }_{0} D_{t}^{\alpha} u(t)=u(t)+c_{1} t^{\alpha-1}+c_{2} t^{\alpha-2}+\cdots+c_{N} t^{\alpha-N},
$$

for some $c_{i} \in \mathbb{R}, i=1,2, \ldots, N$, where $N=[\alpha]+1$.
Define $P_{c}=\{x \in P \mid\|x\| \leq c\}$, where $P$ is a cone of Banach space $E$ and $c$ is a positive constant.

Lemma 2.4 ([18]) Let $P$ be a cone of Banach space $E$ and $\Phi: P_{c} \rightarrow P_{c}$ be a completely continuous map. There exists a nonnegative continuous concave functional $\theta$ such that $\theta(x) \leq\|x\|$ for $x \in P$ and numbers $0<a<b<d \leq c$ satisfying the following conditions:
(i) $\{x \in P(\theta, b, d) \mid \theta(x)>b\} \neq \emptyset$ and $\theta(\Phi x)>b$ for $x \in P(\theta, b, d)$, where $P(\theta, b, d)=$ $\{x \in P \mid \theta(x) \geq b$ and $\|x\| \leq d\}$;
(ii) $\|\Phi x\|<a$ for $x \in P_{a}$;
(iii) $\theta(\Phi x)>b$ for $x \in P(\theta, b, c)$ with $\|\Phi x\|>d$.

Then, $\Phi$ has at least three fixed points $x_{1}, x_{2}, x_{3}$ in $P_{c}$.

In order to obtain a priori bounds, the following assumption is presented:
(H1) There exist functions $\lambda_{i} \in C\left([0, T], \mathbb{R}_{+}\right), i=1,2,3,4$ such that for $t \in[0, T], u \in \mathbb{R}$

$$
\begin{array}{ll}
|E(t, u)| \leq \lambda_{1}(t)+\lambda_{2}(t)|u|^{\mu}, & \mu \in(0,1], \\
|B(t, u)| \leq \lambda_{3}(t)+\lambda_{4}(t)|u|^{\nu}, & v \in(0,1]
\end{array}
$$

where $\left\|\lambda_{2}\right\|_{\infty}+\left\|\lambda_{4}\right\|_{\infty}<\frac{\Gamma(\alpha+1) \Gamma(\beta+1)}{\sqrt{3} T^{\alpha+\beta}}$.
If $u(t)$ is a solution of (1.1), by applying the operator ${ }_{0} I_{t}^{\beta}$ on both sides of the equation, we have

$$
{ }_{0} D_{t}^{\alpha} u(t)=\phi^{-1}\left({ }_{0} I_{t}^{\beta}\left(E(t, u(t))+{ }_{0} D_{t}^{\alpha} u(t) \otimes B(t, u(t))\right)+\mathbf{c}_{1} t^{\beta-1}\right),
$$

which together with ${ }_{0} D_{t}^{\alpha} u(0)=\mathbf{0}$ imply that $\mathbf{c}_{1}=\mathbf{0}$ and

$$
u(t)={ }_{0} I_{t}^{\alpha}\left(\phi^{-1}\left({ }_{0} I_{t}^{\beta}\left(E(t, u(t))+{ }_{0} D_{t}^{\alpha} u(t) \otimes B(t, u(t))\right)\right)\right)+\mathbf{c}_{2} t^{\alpha-1}
$$

which together with $u(0)=\mathbf{0}$ yield that $c_{2}=\mathbf{0}$ and

$$
u(t)={ }_{0} I_{t}^{\alpha}\left(\phi^{-1}\left({ }_{0} I_{t}^{\beta}\left(E(t, u(t))+{ }_{0} D_{t}^{\alpha} u(t) \otimes B(t, u(t))\right)\right)\right)
$$

Since $\left|{ }_{0} D_{t}^{\alpha} u(t)\right|<1$, it follows that

$$
\begin{aligned}
& |u(t)| \leq \sqrt{3}_{0} I_{t}^{\alpha}\left(\phi^{-1}\left(\frac{T^{\beta}}{\Gamma(\beta+1)}\left(\left\|\lambda_{1}\right\|_{\infty}+\left\|\lambda_{3}\right\|_{\infty}+\left\|\lambda_{2}\right\|_{\infty}\|u\|_{\infty}^{\mu}+\left\|\lambda_{4}\right\|_{\infty}\|u\|_{\infty}^{v}\right)\right)\right) \\
& =\sqrt{3}{ }_{0} I_{t}^{\alpha}\left(\frac{\frac{T^{\beta}}{\Gamma(\beta+1)}\left(\left\|\lambda_{1}\right\|_{\infty}+\left\|\lambda_{3}\right\|_{\infty}+\left\|\lambda_{2}\right\|_{\infty}\|u\|_{\infty}^{\mu}+\left\|\lambda_{4}\right\|_{\infty}\|u\|_{\infty}^{\nu}\right)}{\sqrt{1+\frac{T^{2}}{(\Gamma(\beta+1))^{2}}}\left(\left\|a_{1}\right\|_{\infty}+\left\|a_{3}\right\|_{\infty}+\left\|a_{2}\right\|_{\infty}\|u\|_{\infty}^{\mu}+\left\|a_{4}\right\|_{\infty}\|u\|_{\infty}^{\nu}\right)^{2}}\right) \\
& \leq \sqrt{3} I_{0}^{\alpha}\left(\frac{T^{\beta}}{\Gamma(\beta+1)}\left(\left\|\lambda_{1}\right\|_{\infty}+\left\|\lambda_{3}\right\|_{\infty}+\left\|\lambda_{2}\right\|_{\infty}\|u\|_{\infty}^{\mu}+\left\|\lambda_{4}\right\|_{\infty}\|u\|_{\infty}^{\nu}\right)\right) \\
& \leq \frac{\sqrt{3} T^{\alpha+\beta}}{\Gamma(\alpha+1) \Gamma(\beta+1)}\left(\left\|\lambda_{1}\right\|_{\infty}+\left\|\lambda_{3}\right\|_{\infty}+\left\|\lambda_{2}\right\|_{\infty}\|u\|_{\infty}^{\mu}+\left\|\lambda_{4}\right\|_{\infty}\|u\|_{\infty}^{\nu}\right),
\end{aligned}
$$

which together with $\left\|\lambda_{2}\right\|_{\infty}+\left\|\lambda_{4}\right\|_{\infty}<\frac{\Gamma(\alpha+1) \Gamma(\beta+1)}{\sqrt{3} T^{\alpha+\beta}}$ imply that there exists a positive constant $r>1$ such that $|u(t)|<r$. Thus, the solutions of (1.1) must belong to $\mathcal{B}(r)$.

Let $\Lambda:=\phi^{-1}(\overline{\mathcal{B}}(\omega)) \subset \mathcal{B}(1)$, where $\omega=\frac{\sqrt{3} T^{\beta}}{\Gamma(\beta+1)}\left(\left\|\lambda_{1}\right\|_{\infty}+\left\|\lambda_{3}\right\|_{\infty}+\left\|\lambda_{2}\right\|_{\infty} r^{\mu}+\left\|\lambda_{4}\right\|_{\infty} r^{\nu}\right)$.
Moreover, choose $\sigma \in(0,1)$ such that

$$
\begin{equation*}
\frac{\sigma}{\sqrt{1-\sigma^{2}}} \geq \omega, \quad \Lambda \subset \overline{\mathcal{B}}(\sigma) \tag{2.1}
\end{equation*}
$$

Denote $\psi: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ :

$$
\psi(x)= \begin{cases}\frac{x}{\sqrt{1-|x|^{2}}}, & |x| \leq \sigma \\ \frac{x}{\sqrt{1-|\sigma|^{2}}}, & |x| \geq \sigma\end{cases}
$$

Let $\bar{U}=\left\{u \in C \mid\|u\|_{\infty} \leq r\right\}$. By the method of [7], we can obtain the following lemma.
Lemma 2.5 A function $u \in C^{\alpha} \cap \bar{U}$ is a solution of problem (1.1) if and only if it is a solution of the following system:

$$
\left\{\begin{array}{l}
{ }_{0} D_{t}^{\beta}\left(\psi\left({ }_{0} D_{t}^{\alpha} u(t)\right)\right)=E(t, u(t))+{ }_{0} D_{t}^{\alpha} u(t) \otimes B(t, u(t)), \quad t \in(0, T),  \tag{2.2}\\
u(0)={ }_{0} D_{t}^{\alpha} u(0)=\mathbf{0}
\end{array}\right.
$$

Proof On the one hand, if $u \in C^{\alpha} \cap \bar{U}$ is a solution of problem (1.1), by ${ }_{0} D_{t}^{\alpha} u(0)=\mathbf{0}$, we can obtain that

$$
\phi\left({ }_{0} D_{t}^{\alpha} u(t)\right)={ }_{0} I_{t}^{\beta}\left(E(t, u(t))+{ }_{0} D_{t}^{\alpha} u(t) \otimes B(t, u(t))\right)
$$

which together with ${ }_{0} D_{t}^{\alpha} u(t) \mid<1$ yield that

$$
\left|\phi\left({ }_{0} D_{t}^{\alpha} u(t)\right)\right| \leq \frac{\sqrt{3} T^{\beta}}{\Gamma(\beta+1)}\left(\left\|\lambda_{1}\right\|_{\infty}+\left\|\lambda_{3}\right\|_{\infty}+\left\|\lambda_{2}\right\|_{\infty} r^{\mu}+\left\|\lambda_{4}\right\|_{\infty} r^{\nu}\right)=\omega
$$

Thus, for any $t \in[0, T],{ }_{0} D_{t}^{\alpha} u(t) \in \Lambda$ and ${ }_{0} D_{t}^{\alpha} u(t) \mid \leq \sigma$, which implies that $\phi\left({ }_{0} D_{t}^{\alpha} u(t)\right)=$ $\psi\left({ }_{0} D_{t}^{\alpha} u(t)\right)$.

On the other hand, if $u \in C^{\alpha} \cap \bar{U}$ is a solution of problem (2.2), we just need to show that ${ }_{0} D_{t}^{\alpha} u(t) \mid \leq \sigma$ for any $t \in[0, T]$. If not, we assume that there exists a $t_{*} \in[0, T]$ such that $\left|{ }_{0} D_{t}^{\alpha} u\left(t_{*}\right)\right|>\sigma$. Since

$$
\begin{equation*}
{ }_{0} I_{t}^{\beta}\left({ }_{0} D_{t}^{\beta}\left(\psi\left({ }_{0} D_{t}^{\alpha} u(t)\right)\right)\right)=\psi\left({ }_{0} D_{t}^{\alpha} u(t)\right)-\mathbf{c}_{3} t^{\beta-1}, \tag{2.3}
\end{equation*}
$$

which together with ${ }_{0} D_{t}^{\alpha} u(0)=\mathbf{0}$ yield that $\mathbf{c}_{3}=\mathbf{0}$ and

$$
\begin{equation*}
\omega \geq\left|{ }_{0} I_{t}^{\beta}\left({ }_{0} D_{t}^{\beta}\left(\psi\left({ }_{0} D_{t}^{\alpha} u(t)\right)\right)\right)\right|_{t=t_{*}}\left|=\left|\psi\left({ }_{0} D_{t}^{\alpha} u(t)\right)\right|_{t=t_{*}}\right|>\frac{\sigma}{\sqrt{1-|\sigma|^{2}}}, \tag{2.4}
\end{equation*}
$$

which contradicts the definition of $\sigma$. Thus, the proof is complete.

## 3 Main results

### 3.1 Existence

Define the operator $\Phi: C^{\alpha} \rightarrow C^{\alpha}$ by

$$
\begin{equation*}
\Phi u(t)={ }_{0} I_{t}^{\alpha}\left(\psi^{-1}\left({ }_{0} I_{t}^{\beta}\left(E(t, u(t))+{ }_{0} D_{t}^{\alpha} u(t) \otimes B(t, u(t))\right)\right)\right) \tag{3.1}
\end{equation*}
$$

where

$$
\psi^{-1}(x)= \begin{cases}\frac{x}{\sqrt{1+|x|^{2}}}, & |x| \leq \frac{\sigma}{\sqrt{1-\sigma^{2}}} \\ x \sqrt{1-|\sigma|^{2}}, & |x| \geq \frac{\sigma}{\sqrt{1-\sigma^{2}}}\end{cases}
$$

Letting

$$
\Pi=\left\{u \in C^{\alpha} \cap \bar{u} \mid u=\Phi u \text { and }\left|{ }_{0} D_{t}^{\alpha} u\right| \leq \sigma\right\}
$$

Thus, a function $u \in \Pi$ is a solution of the problem (1.1). Define

$$
\bar{\Omega}=\left\{u \in C^{\alpha} \mid\|u\| \leq \rho\right\},
$$

where $\rho=\max \left\{\sigma, \frac{\sigma T^{\alpha}}{\Gamma(\alpha+1)}\right\}$. It follows that $\bar{\Omega}$ is a nonempty, convex, closed set. Moreover, if $\frac{\sigma T^{\alpha}}{\Gamma(\alpha+1)}<1$, we have $\bar{\Omega} \subset C^{\alpha} \cap \bar{U}$.

Remark 3.1 If $T^{\alpha} \leq \Gamma(\alpha+1)$, it follows that $\rho=\sigma$.

Lemma 3.2 Assuming that the condition (H1) is satisfied, $\Phi: \bar{\Omega} \rightarrow \bar{\Omega}$ is completely continuous, provided that $\frac{\sigma T^{\alpha}}{\Gamma(\alpha+1)}<1$.

Proof It is not difficult to obtain that $\Phi$ is continuous by the continuity of $E$ and $B$. Based on (H1), for $(t, u) \in[0,1] \times \bar{\Omega}$, we have

$$
\begin{aligned}
& \left|{ }_{0} I_{t}^{\beta}\left(E(t, u(t))+{ }_{0} D_{t}^{\alpha} u(t) \otimes B(t, u(t))\right)\right| \\
& \quad \leq \frac{\sqrt{3} T^{\beta}}{\Gamma(\beta+1)}\left(\left\|\lambda_{1}\right\|_{\infty}+\left\|\lambda_{3}\right\|_{\infty}+\left\|\lambda_{2}\right\|_{\infty} \rho^{\mu}+\left\|\lambda_{4}\right\|_{\infty} \rho^{\nu}\right) \\
& \quad \leq \omega,
\end{aligned}
$$

which together with $\omega \leq \frac{\sigma}{\sqrt{1-\sigma^{2}}}$ yield that

$$
\left|{ }_{0} D_{t}^{\alpha}(\Phi u(t))\right| \leq \frac{\omega}{\sqrt{1+\omega^{2}}} \leq \sigma \leq \rho
$$

and

$$
|\Phi u(t)| \leq \frac{\omega T^{\alpha}}{\Gamma(\alpha+1) \sqrt{1+\omega^{2}}} \leq \frac{\sigma T^{\alpha}}{\Gamma(\alpha+1)} \leq \rho .
$$

Thus, we have $\|\Phi u\| \leq \rho$, which implies $\Phi: \bar{\Omega} \rightarrow \bar{\Omega}$ is uniformly bounded. For simplicity, let $f(t)=E(t, u(t))+{ }_{0} D_{t}^{\alpha} u(t) \otimes B(t, u(t))$. Hence, $\forall t_{1}, t_{2} \in[0,1]$, assuming that $t_{1} \leq t_{2}$, for any $u \in \bar{\Omega}$, it follows that

$$
\begin{aligned}
&{ }_{0} I_{t_{2}}^{\beta} f(t)-{ }_{0} I_{t_{1}}^{\beta} f(t) \mid \\
&=\left|\frac{1}{\Gamma(\beta)} \int_{0}^{t_{1}}\left(\left(t_{2}-s\right)^{\beta-1}-\left(t_{1}-s\right)^{\beta-1}\right) f(s) d s+\frac{1}{\Gamma(\beta)} \int_{t_{1}}^{t_{2}}\left(t_{2}-s\right)^{\beta-1} f(s) d s\right| \\
& \leq \frac{\sqrt{3}\left(\left\|\lambda_{1}\right\|_{\infty}+\left\|\lambda_{3}\right\|_{\infty}+\left\|\lambda_{2}\right\|_{\infty} \rho^{\mu}+\left\|\lambda_{4}\right\|_{\infty} \rho^{\nu}\right)}{\Gamma(\beta)}\left(\int_{0}^{t_{1}}\left(\left(t_{1}-s\right)^{\beta-1}-\left(t_{2}-s\right)^{\beta-1}\right) d s\right. \\
&\left.+\int_{t_{1}}^{t_{2}}\left(t_{2}-s\right)^{\beta-1} d s\right) \\
&= \frac{\sqrt{3}\left(\left\|a_{1}\right\|_{\infty}+\left\|a_{3}\right\|_{\infty}+\left\|a_{2}\right\|_{\infty} \rho^{\mu}+\left\|a_{4}\right\|_{\infty} \rho^{\nu}\right)}{\Gamma(\beta+1)}\left(2\left(t_{2}-t_{1}\right)^{\beta}+t_{1}^{\beta}-t_{2}^{\beta}\right),
\end{aligned}
$$

which implies that

$$
\left|{ }_{0} I_{t_{2}^{\beta}}^{\beta} f(t)-{ }_{0} I_{t_{1}}^{\beta} f(t)\right| \rightarrow 0 \quad \text { uniformly as } t_{1} \rightarrow t_{2} .
$$

Moreover, we have

$$
\begin{aligned}
&{ }_{0} D_{t}^{\alpha}\left(\Phi u\left(t_{2}\right)\right)-{ }_{0} D_{t}^{\alpha}\left(\Phi u\left(t_{1}\right)\right) \mid \\
&=\left|\psi^{-1}\left({ }_{0} I_{t_{2}}^{\beta} f(t)\right)-\psi^{-1}\left({ }_{0} I_{t_{1}}^{\beta} f(t)\right)\right| \\
&=\left|\frac{\left(\sqrt{\left.1+{ }_{0} I_{t_{1}}^{\beta} f(t)\right)^{2}}-\sqrt{\left.1+{ }_{0} I_{t_{2}}^{\beta} f(t)\right)^{2}}{ }_{0} I_{t_{2}}^{\beta} f(t)+\sqrt{\left.1+{ }_{0} I_{t_{2}}^{\beta} f(t)\right)^{2}}{ }_{0} I_{t_{2}}^{\beta} f(t)-{ }_{0} I_{t_{1}}^{\beta} f(t)\right)}{\sqrt{1+\left({ }_{0} I_{t_{1}}^{\beta} f(t)\right)^{2}} \sqrt{\left.1+{ }_{0} I_{t_{2}}^{\beta} f(t)\right)^{2}}}\right| \\
&= \left\lvert\, \frac{{ }_{0} I_{t_{2}}^{\beta} f(t)\left({ }_{0} I_{t_{1}}^{\beta} f(t)+{ }_{0} I_{t_{2}}^{\beta} f(t)\right){ }_{0} I_{t_{1}}^{\beta} f(t)-{ }_{\left.{ }_{0} I_{t_{2}}^{\beta} f(t)\right)}^{\left(\sqrt{\left.1+{ }_{0} I_{t_{1}}^{\beta} f(t)\right)^{2}}+\sqrt{\left.1+{ }_{0} I_{t_{2}}^{\beta} f(t)\right)^{2}}\right) \sqrt{\left.1+{ }_{0} I_{t_{1}}^{\beta} f(t)\right)^{2}} \sqrt{\left.1+{ }_{0} I_{t_{2}}^{\beta} f(t)\right)^{2}}}}{}\right. \\
& \left.+\frac{\left.\sqrt{\left.1+{ }_{0} I_{t_{2}}^{\beta} f(t)\right)^{2}}{ }_{0} I_{t_{2}}^{\beta} f(t)-{ }_{0} I_{t_{1}}^{\beta} f(t)\right)}{\sqrt{\left.1+{ }_{0} I_{t_{1}}^{\beta} f(t)\right)^{2}} \sqrt{\left.1+{ }_{0} I_{t_{2}}^{\beta} f(t)\right)^{2}}} \right\rvert\, \rightarrow 0 \quad \text { uniformly as } t_{1} \rightarrow t_{2} .
\end{aligned}
$$

Similarly, we can obtain

$$
\left|\Phi u\left(t_{2}\right)-\Phi u\left(t_{1}\right)\right| \leq \frac{\sigma}{\Gamma(\alpha+1)}\left(2\left(t_{2}-t_{1}\right)^{\alpha}+t_{1}^{\alpha}-t_{2}^{\alpha}\right),
$$

which yields that

$$
\left|\Phi u\left(t_{2}\right)-\Phi u\left(t_{1}\right)\right| \rightarrow 0 \quad \text { uniformly as } t_{1} \rightarrow t_{2} .
$$

Thus, $\Phi$ is equicontinuous on $\bar{\Omega}$. By the Arzelà-Ascoli theorem, it follows that $\Phi: \bar{\Omega} \rightarrow \bar{\Omega}$ is completely continuous.

Based on Lemma 3.2, we can obtain the following theorem immediately by Schauder's fixed-point theorem.

Theorem 3.3 If the assumption (H1) is satisfied, there exists a fixed point $u \in \bar{\Omega}$ such that $\Phi u=u$, provided that $T^{\alpha} \leq \Gamma(\alpha+1)$.

Remark 3.4 By Theorem 3.3, we known that the fixed point $u \in \bar{\Omega} \subset \Pi$, which tells us that the problem (1.1) has at least one solution on $\bar{\Omega}$.

### 3.2 Multiplicity

For any $u, v \in C^{\alpha}, u \geq v$ means $u_{i} \geq v_{i}$, where $u_{i}$ and $v_{i}$ are components of $u$ and $v$, respectively, $i=1,2,3$. Let $P$ be a cone of $C^{\alpha}$, where $P=\left\{u \in C^{\alpha} \mid u(t) \geq \mathbf{0}\right\}$. Define a nonnegative continuous functional $\theta$ on $P$ by

$$
\theta(\Phi u)=\min _{t \in[\tau, T-\tau]} \Phi_{1} u(t)+\min _{t \in[\tau, T-\tau]} \Phi_{2} u(t)+\min _{t \in[\tau, T-\tau]} \Phi_{3} u(t),
$$

where $\left(\frac{\sqrt{3}}{3}\right)^{\frac{1}{\alpha}} T<\tau<T, \Phi_{i} u$ stand for the component of $\Phi u, i=1,2,3$. Since, for any $u, v \in$ $C^{\alpha}$ and $s \in[0,1]$, we have

$$
\begin{aligned}
\theta(s \Phi u+(1-s) \Phi v)= & \min _{t \in[\tau, T-\tau]}\left\{s \Phi_{1} u(t)+(1-s) \Phi_{1} v(t)\right\} \\
& +\min _{t \in[\tau, T-\tau]}\left\{s \Phi_{2} u(t)+(1-s) \Phi_{2} v(t)\right\} \\
& +\min _{t \in[\tau, T-\tau]}\left\{s \Phi_{3} u(t)+(1-s) \Phi_{3} v(t)\right\} \\
\geq & s\left(\min _{t \in[\tau, T-\tau]} \Phi_{1} u(t)+\min _{t \in[\tau, T-\tau]} \Phi_{2} u(t)+\min _{t \in[\tau, T-\tau]} \Phi_{3} u(t)\right) \\
& \times(1-s)\left(\min _{t \in[\tau, T-\tau]} \Phi_{1} v(t)+\min _{t \in[\tau, T-\tau]} \Phi_{2} v(t)+\min _{t \in[\tau, T-\tau]} \Phi_{3} v(t)\right) \\
= & s \theta(\Phi u)+(1-s) \theta(\Phi v),
\end{aligned}
$$

which implies that the functional $\theta$ is concave. Choose $0<a<b<d \leq c \leq \sigma<1$ satisfying

$$
\begin{aligned}
& \omega_{1}<\frac{a}{\sqrt{1-a^{2}}} \\
& \omega_{2} \leq \frac{d}{\sqrt{1-d^{2}}} \\
& \omega_{3} \leq \frac{c}{\sqrt{1-c^{2}}}
\end{aligned}
$$

where

$$
\begin{aligned}
& \omega_{1}=\frac{\sqrt{3} T^{\beta}}{\Gamma(\beta+1)}\left(\left\|\lambda_{1}\right\|_{\infty}+\left\|\lambda_{3}\right\|_{\infty}+\left\|\lambda_{2}\right\|_{\infty} a^{\mu}+\left\|\lambda_{4}\right\|_{\infty} a^{\nu}\right), \\
& \omega_{2}=\frac{\sqrt{3} T^{\beta}}{\Gamma(\beta+1)}\left(\left\|\lambda_{1}\right\|_{\infty}+\left\|\lambda_{3}\right\|_{\infty}+\left\|\lambda_{2}\right\|_{\infty} d^{\mu}+\left\|\lambda_{4}\right\|_{\infty} d^{\nu}\right), \\
& \omega_{3}=\frac{\sqrt{3} T^{\beta}}{\Gamma(\beta+1)}\left(\left\|\lambda_{1}\right\|_{\infty}+\left\|\lambda_{3}\right\|_{\infty}+\left\|\lambda_{2}\right\|_{\infty} c^{\mu}+\left\|\lambda_{4}\right\|_{\infty} c^{\nu}\right) .
\end{aligned}
$$

Moreover, it follows that

$$
\frac{a}{\sqrt{1-a^{2}}}<\frac{d}{\sqrt{1-d^{2}}} \leq \frac{c}{\sqrt{1-c^{2}}} \leq \frac{\sigma}{\sqrt{1-\sigma^{2}}}
$$

In order to study the positive solution of the initial value problem in the cone, the following assumptions are naturally required.
(H2) $E(t, u), B(t, u):[0, T] \times \mathbb{R}_{+}^{3} \rightarrow \mathbb{R}_{+}^{3}$ are continuous and ${ }_{0} D_{t}^{\alpha} u \otimes B(t, u) \geq \mathbf{0}$.
(H3) $E_{i}(t, u)>\Delta b$ for $(t, u) \in[0, T] \times[0, d], i=1,2,3$,
where $E_{i}(t, u)$ stand for the component of $E(t, u)$,

$$
\Delta=\frac{\Gamma(\alpha+\beta+1) \sqrt{1+\omega_{3}^{2}}}{3 \tau^{\alpha+\beta}} .
$$

Lemma 3.5 Assuming that the conditions (H1) and (H2) are satisfied, $\Phi: P_{c} \rightarrow P_{c}$ is completely continuous, provided that $T^{\alpha} \leq \Gamma(\alpha+1)$.

Proof To begin with, by (H2), we can obtain that $\Phi u \geq \mathbf{0}$ for any $u \in P_{c}$ and $\Phi$ is continuous. Moreover, for any $u \in P_{c}$, in the same way as Lemma 3.2, one can obtain

$$
\left|{ }_{0} D_{t}^{\alpha}(\Phi u(t))\right| \leq \frac{\omega_{3}}{\sqrt{1+\omega_{3}^{3}}} \leq c \leq \sigma
$$

and

$$
|\Phi u(t)| \leq \frac{c T^{\alpha}}{\Gamma(\alpha+1)} \leq c \leq \sigma
$$

because of $\omega_{3} \leq \frac{c}{\sqrt{1-c^{2}}}$. Thus, $\Phi: P_{c} \rightarrow P_{c}$ is uniformly bounded. Similarly, we can also obtain that $\Phi$ is equicontinuous on $P_{c}$. In view of the Arzelà-Ascoli theorem, it follows that $\Phi: P_{c} \rightarrow P_{c}$ is completely continuous.

Theorem 3.6 Assuming that the conditions (H1), (H2), and (H3) are satisfied, if $d=c$, there exist at least three fixed points $u_{1}, u_{2}, u_{3}$ in $P_{c}$ meeting $\Phi u=u$, provided that $T^{\alpha}=$ $\Gamma(\alpha+1)$.

Proof By Lemma 3.5, we know that $\Phi: P_{c} \rightarrow P_{c}$ is completely continuous. From (H1), in the same way as Lemma 3.2, it follows that $\|\Phi u\|<a$ for $u \in P_{a}$, which yields that the condition (ii) of Lemma 2.4 is satisfied. Letting

$$
u_{0}(t)=\left(u_{0,1}(t), u_{0,2}(t), u_{0,3}(t)\right)=\left(\frac{b+d}{2 \sqrt{3} T^{\alpha}} t^{\alpha}, \frac{b+d}{2 \sqrt{3} T^{\alpha}} t^{\alpha}, \frac{b+d}{2 \sqrt{3} T^{\alpha}} t^{\alpha}\right),
$$

we have $\left\|u_{0}\right\|_{\infty}=\frac{b+d}{2} \leq d$. Moreover, since $0<\alpha \leq 1$, one can obtain that for any $i \in$ $\{1,2,3\}$,

$$
\begin{aligned}
{ }_{0} D_{t}^{\alpha} u_{0, i}(t) & =\frac{d}{d t} I_{t}^{1-\alpha} u_{0, i}(t) \\
& =\frac{b+d}{2 \sqrt{3} T^{\alpha}} \frac{d}{d t}\left(\frac{1}{\Gamma(1-\alpha)} \int_{0}^{t}(t-s)^{-\alpha} s^{\alpha} d s\right) \\
& \stackrel{\varsigma=\frac{s}{t}}{=} \frac{b+d}{2 \sqrt{3} T^{\alpha}} \frac{d}{d t}\left(\frac{t}{\Gamma(1-\alpha)} \int_{0}^{1}(1-\varsigma)^{-\alpha} \varsigma^{\alpha} d \varsigma\right) \\
& =\frac{(b+d) \Gamma(1+\alpha)}{2 \sqrt{3} T^{\alpha}}=\frac{b+d}{2 \sqrt{3}},
\end{aligned}
$$

which yields that $\left\|_{0} D_{t}^{\alpha} u_{0}\right\|_{\infty} \leq \frac{b+d}{2} \leq d$. Thus, $\left\|u_{0}\right\| \leq d$. Since $\left(\frac{\sqrt{3}}{3}\right)^{\frac{1}{\alpha}} T<\tau<T$, we have

$$
\theta\left(u_{0}\right)=\frac{\sqrt{3}(b+d)}{2 T^{\alpha}} \tau^{\alpha}>b
$$

Hence,

$$
\{u \in P(\theta, b, d) \mid \theta(u)>b\} \neq \emptyset .
$$

Thus, by (H2) and (H3), for $(t, u) \in[0, T] \times[0, d]$, one has

$$
\theta(\Phi u)=\min _{t \in[\tau, T-\tau]} \Phi_{1} u(t)+\min _{t \in[\tau, T-\tau]} \Phi_{2} u(t)+\min _{t \in[\tau, T-\tau]} \Phi_{3} u(t)
$$

$$
\begin{aligned}
\geq & \frac{1}{\Gamma(\alpha)} \min _{t \in[\tau, T-\tau]} \int_{0}^{t}(t-s)^{\alpha-1} \frac{{ }_{0} I_{s}^{\beta} E_{1}(s, u(s))}{\sqrt{1+\left.{ }_{0} I_{s}^{\beta}\left(E(s, u(s))+{ }_{0} D_{t}^{\alpha} u(s) \otimes B(s, u(s))\right)\right|^{2}}} d s \\
& +\frac{1}{\Gamma(\alpha)} \min _{t \in[\tau, T-\tau]} \int_{0}^{t}(t-s)^{\alpha-1} \frac{{ }_{0} I_{s}^{\beta} E_{2}(s, u(s))}{\sqrt{1+\left.{ }_{0} I_{s}^{\beta}\left(E(s, u(s))+{ }_{0} D_{t}^{\alpha} u(s) \otimes B(s, u(s))\right)\right|^{2}}} d s \\
& +\frac{1}{\Gamma(\alpha)} \min _{t \in[\tau, T-\tau]} \int_{0}^{t}(t-s)^{\alpha-1} \frac{{ }_{0} I_{s}^{\beta} E_{3}(s, u(s))}{\sqrt{1+\left.{ }_{0} I_{s}^{\beta}\left(E(s, u(s))+{ }_{0} D_{t}^{\alpha} u(s) \otimes B(s, u(s))\right)\right|^{2}}} d s \\
> & \frac{3 \Delta b}{\Gamma(\beta+1) \Gamma(\alpha) \sqrt{1+\omega_{3}^{2}}} \min _{t \in[\tau, T-\tau]} \int_{0}^{t}(t-s)^{\alpha-1} s^{\beta} d s \\
= & \frac{3 \Delta b}{\Gamma(\beta+1) \Gamma(\alpha) \sqrt{1+\omega_{3}^{2}}} \min _{t \in[\tau, T-\tau]} \frac{\Gamma(\alpha) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta} \\
= & \frac{3 \Delta b \tau^{\alpha+\beta}}{\Gamma(\alpha+\beta+1) \sqrt{1+\omega_{3}^{2}}}=b .
\end{aligned}
$$

Thus, the condition (i) of Lemma 2.4 holds. If $d=c$, the condition (i) implies (iii) in Lemma 2.4. Then, there exist at least three fixed points $u_{1}, u_{2}, u_{3} \in P_{c}$ meeting $\Phi u=u$.

Remark 3.7 By Theorem 3.6, it follows that the fixed points $u_{1}, u_{2}, u_{3} \in P_{c} \subset \Pi$, which yields that the problem (1.1) has at least three solutions on $P_{c}$.

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The authors declare no competing interests.

## Author contributions

X. Shen edited the original draft and checked the validation; T. Ye checked the validation and conducted analysis; T. Shen wrote the main manuscript text. All authors reviewed the manuscript.

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