Abidin and Marwan *Boundary Value Problems* (2023) 2023:107
https://doi.org/10.1186/s13661-023-01794-3

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On the global existence and analyticity of the mild solution for the fractional Porous medium equation

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Abstract

In this research article we focus on the study of existence of global solution for a three-dimensional fractional Porous medium equation. The main objectives of studying the fractional porous medium equation in the corresponding critical function spaces are to show the existence of unique global mild solution under the condition of small initial data. Applying Fourier transform methods gives an equivalent integral equation of the model equation. The linear and nonlinear terms are then estimated in the corresponding Lei and Lin spaces. Further, the analyticity of solution to the fractional Porous medium equation is also obtained.

Keywords: Global solution; Fractional porous medium equation; Analyticity; Critical spaces

1 Introduction

This paper considers the following 3D fractional porous medium (FPM) equation:

$$\begin{cases} w_t + \alpha \Lambda^s w = \nabla \cdot (w \nabla Q w) \in \mathbb{R}^3 \times \mathbb{R}^+, \\ w(z,0) = w_0 \in \mathbb{R}^3, \end{cases} \tag{FPM}$$

where $w = w(z,t)$ represents the density or concentration. The positive coefficient of dissipation $\alpha > 0$ denotes the viscous property, while $\alpha = 0$ is for the corresponding inviscid property. The symbol Q denotes an abstract operator. The term Λ^s is a fractional Laplacian that is defined by the Fourier transform $\widehat{\Lambda^s w} = |\varphi|^s \widehat{w}$. For the sake of simplicity, α is equal to one.

According to [1], Zhou, Xiao, and Zheng were the first to introduce equation (FPM). In their work, they applied the fractional dissipation term $\alpha \Lambda^s$ to the equation of continuity $wt + \nabla \cdot (wW) = 0$, then Caffarelli and Vázquez [2] established equation (FPM). The potential $W = -\nabla p$ gives the velocity W, and the pressure or velocity potential is related to w via an abstract operator $q = Qw$ [3]. There are plenty of real-world applications for porous media problems. The main purpose of this equation is to provide a mathematical description of fluid's behavior as it permeates porous materials such as biological tissues,

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rocks, or soil [4–6]. The fractional porous medium equation is a fundamental equation to model groundwater flow in deep aquifer, which provides the dynamics, transportation of substances, and the effects of injection or pumping activities of groundwater [7, 8]. In addition to fluid flow, the equation has been adapted to facilitate the modeling of heat and mass transfer phenomena occurring within porous media [9, 10]. We refer the focus of the reader to [11–16] and the references listed within for more detailed research referring to the physical importance of equation (FPM).

In numerous cases, the abstract pressure Qw is a suitable choice. The simplest example is derived from a groundwater filtration model [17, 18]. Zhou, Xiao, and Chen [19] investigated the strong solution for the more general case $\alpha = 0$ and $Qw = (-\Delta)^{-m}w = \Lambda^{-ms}$, $0 < m < 1$, of equation (FPM) in the Besov spaces $B^s_{p,\infty}$. Further, they constructed the existence of local solution for the initial data in the space $B^s_{1,\infty}$. Lin and Zhang [20] obtained the mean field equation by considering the critical case, i.e., $s = 1$. Regarding the existence of solutions and the uniqueness of these equations, the authors suggest the reader refers to the work of Zhou and Biler [1, 21] and their references.

The aggregation equations, which explain aggregation behaviors and collective dynamics in the structure of continuous media [22, 23], is another model that is similar to the one described above. Applications of this equation can be found in domains such as biology, physics, chemistry, and dynamics of populations. The operator Q in the aggregation model may also be expressed in the form of convolution operator with kernel $\mathcal{J}$ as follows: $Qw = \mathcal{J} * w$. The Newton potential $|z|^\gamma$ [24] and the exponential potential $-e^{-|z|}$ [25] are two examples of typical kernels. Further research in this regard can be found in [26, 27] and the related references cited there.

Further, by considering the corresponding initial data, equation (FPM) can be modified to the following form [28]:

$$\begin{aligned} &w_t + \alpha\Lambda^s w + u \cdot \nabla w = -w(\nabla \cdot u); \\ &u = -\nabla Q w. \end{aligned} \tag{1}$$

The turbulent velocity associated with the fractional porous medium equations is not strictly divergence free, and this allows that (1) can be compared to the geostrophic model. Moreover, as the divergence free vector u follows the condition $(\nabla \cdot u = 0)$, equation (1) thus contains the quasi-geostrophic equation [29, 30].

The singularity of an abstract pressure component Qw that ensures the well-posedness or produces the blow-up solutions is one of the most significant challenges related to equation (FPM). The existence of local solution with large initial data belonging to Besov spaces was established by Zhou, Xiao, and Zheng [1], while the global existence of solution was established for small initial data. In addition to that, a blow-up criterion was provided for the solution. The existence of solution and the blow-up condition for equation (FPM) related to its pressure $Qw = \mathcal{J} * w$, where $\mathcal{J}(x) = e^{-|z|}$, in the Sobolev space were obtained by Li and Rodrigo [31]. In addition, Wu and Zhang's work [32] advanced their previous investigation to the case in which $\nabla\mathcal{J} \in W^{1,1}$ and $\mathcal{J}(x) = e^{-|z|}$. For $\nabla\mathcal{J} \in L^1$, Xiao and Zhou [3] showed the existence of local solutions in Fourier–Besov spaces with large initial data, and they obtained the existence of global solutions with small initial data. These results come from the fact that convolution $\mathcal{J} * w$ and its gradient $\nabla\mathcal{J} * w$ can be controlled in Besov spaces. More recently Zhou, Xiao, and Zheng [28] established the existence of local solutions with large initial data and the global existence of solutions when the initial data is small in homogenous Besov spaces $\dot{B}^s_{p,q}$ under the following general condition:

$$\|\nabla P w\|_{\dot{B}^s_{p,q}} \leq C\|w\|_{\dot{B}^{s+\sigma}_{p,q}}.$$

Motivated by the previously mentioned studies and considering the above condition, we achieved the existence of global solution and analyticity to the solution for the three-dimensional fractional porous medium equation in the following critical space:

$$\mathcal{X}^a := \left\{ g \in \mathcal{Z}'(\mathbb{R}^3) \,\middle|\, \int_{\mathbb{R}^3} |\varphi|^a \left|\widehat{g}(\varphi)\right| d\varphi < \infty, a \in \mathbb{R} \right\}.$$

Regarding the function space mentioned before, Lei and Lin [33] established the global existence of mild solution to the classical Navier–Stokes (NS) equation in the critical space $C(\mathbb{R}_+, \mathcal{X}^{-1}) \cap L^1(\mathbb{R}_+, \mathcal{X}^1)$. Bae [34] recently showed the results of Lei and Lin [33] in a little modified way, which is given as follows:

$$\mathcal{L}^\infty_t \mathcal{X}^s := \left\{ g \in \mathcal{Z}'(\mathbb{R}^3 \times \mathbb{R}_+) : \int_{\mathbb{R}^3} \left[\sup_{0 \leq t < +\infty} |\varphi|^s \left|\widehat{g}(\varphi, t)\right| \right] d\varphi < +\infty \right\}$$

with

$$\|g\|_{\mathcal{L}^\infty_t \mathcal{X}^s} = \int_{\mathbb{R}^3} \left[\sup_{0 \leq t < +\infty} |\varphi|^s \left|\widehat{g}(\varphi, t)\right| \right] d\varphi$$

and

$$L^1_t \mathcal{X}^1 := \left\{ g \in \mathcal{Z}'(\mathbb{R}^3 \times \mathbb{R}_+) : \int_{\mathbb{R}^3} \int_0^{+\infty} |\varphi| \left|\widehat{g}(\varphi, \eta)\right| d\eta \, d\varphi < +\infty \right\}$$

with

$$\|g\|_{L^1_t \mathcal{X}^1} = \int_{\mathbb{R}^3} \int_0^t |\varphi| \left|\widehat{g}(\varphi, \eta)\right| d\eta \, d\varphi.$$

In the present investigation, we use the approaches proposed by Lei and Bae [33, 34] to obtain the existence of global solution and analyticity to the solution of equation (FPM). In this paper, $f \lesssim g$ is used to denote $f \leq Cg$, where C represents positive constants (different values may be taken in different places). The symbol $\widehat{f}$ denotes the Fourier transform of f. The following is the arrangement of this research article: In Sect. 2, we give the statements of the two main theorems. In Sect. 3, we show the proof of Theorem 2.1, and the proof of Theorem 2.2 is presented in Sect. 4.

2 Main results

The primary purpose of this research is to determine the existence of global solution to equation (FPM). The following is the related result.

Theorem 2.1 *Let $\frac{1}{2} \leq s \leq 1$ and there exists a constant $\varepsilon_0 > 0$ depending on the value of s such that for all initial data w_0 belongs to $\mathcal{X}^{1-2s}$ satisfies the condition*

$$\|w_0\|_{\mathcal{X}^{1-2s}} < \varepsilon_0,$$

then equation (FPM) *has a unique global in time solution*

$$w \in \mathcal{L}_t^\infty \mathcal{X}^{1-2s} \cap L_t^1 \mathcal{X}^1$$

such that

$$\|w\|_{\mathcal{L}_t^\infty \mathcal{X}^{1-2s}} + \|u\|_{L_t^1 \mathcal{X}^1} \lesssim \|u_0\|_{\mathcal{X}^{1-2s}}.$$

The next objective of this paper is to work on the Gevrey class regularity of solution to equation (FPM). The Gevrey class regularity to the solution for the classical NS equations has been the focus of significant research; for instance, see [35, 36] and the references therein. The specific result is the following.

Theorem 2.2 *Let $\frac{1}{2} \leq s \leq 1$ and then there exists a constant $\kappa_0 > 0$ depending on the value of s such that for all initial data u_0 belongs to $\mathcal{X}^{1-2s}$ satisfying the condition*

$$\|w_0\|_{\mathcal{X}^{1-2s}} \leq \kappa_0.$$

The global solution established in Theorem 2.1 can be analytic in a way that

$$\left\|e^{\sqrt{t}|\delta|^s} w\right\|_{\mathcal{L}_t^\infty \mathcal{X}^{1-2s}} + \left\|e^{\sqrt{t}|\delta|^s} w\right\|_{L_t^1 \mathcal{X}^1} \lesssim \|w_0\|_{\mathcal{X}^{1-2s}},$$

where $e^{\sqrt{t}|\delta|^s}$ is defined as a Fourier multiplier with symbol $e^{\sqrt{t}|\varphi|^s}$.

Throughout this paper, $A \lesssim B$ represents $A \leq CB$ depending on some constant $C > 0$.

3 Proof of Theorem 2.1

The proof of Theorem 2.1 is presented in this section. To prove our key results, we first state the following lemma.

Lemma 3.1 ([37]) *Suppose $\frac{1}{2} \leq s \leq 1$, then we have the following inequality*:

$$|\varphi|^{2(1-s)} \leq 2^{1-2s}\left(|\zeta||\varphi-\zeta|^{1-2s} + |\zeta|^{1-2s}|\varphi-\zeta|\right)$$

for any φ, $\zeta \in \mathbb{R}^3$.

Proof of Theorem 2.1 To get the solution of equation (FPM), we can transform equation (FPM) into the following integral form:

$$w = G_s(t)w_0 + \int_0^t G_s(t-\eta)\nabla \cdot (w\nabla Qw)\, d\eta, \tag{2}$$

where $G_s(t) := e^{-t\Lambda^s}$.

Applying Fourier transform to the above integral form, we get

$$\widehat{w}(\varphi,t) = e^{-t|\cdot|^s}\widehat{w}_0(\varphi) + \int_0^t e^{-(t-\eta))|\cdot|^s} \iota\varphi \cdot \int_{\mathbb{R}^3} \widehat{w}(\varphi-\zeta,\eta)\widehat{\nabla Qw}(\zeta,\eta)\, d\zeta\, d\eta. \tag{3}$$

The multiplication of $|\varphi|^{1-2s}$ to both sides gives us the following:

$$\begin{aligned}|\varphi|^{1-2s}\widehat{w}(\varphi,t) &\lesssim e^{-t|\cdot|^s}|\varphi|^{1-2s}\big|\widehat{w}_0(\varphi)\big| \\ &\quad + \int_0^t e^{-(t-\eta)|\cdot|^s}\int_{\mathbb{R}^3}|\varphi|^{2-2s}\big|\widehat{w}(\varphi-\zeta,\eta)\big|\big|\widehat{w}(\zeta,\eta)\big|\,d\zeta\,d\eta. \end{aligned} \tag{4}$$

By using Lemma 3.1, the nonlinear term can be estimated as follows:

$$\begin{aligned}&\int_0^t\left[\int_{\mathbb{R}^3}|\varphi|^{2-2s}\big|\widehat{u}(\varphi-\zeta,\eta)\big|\big|\widehat{w}(\zeta,\eta)\big|\,d\zeta\right]d\eta \\ &\quad\lesssim 2^{2(1-s)}\left[\int_0^\infty\big|\cdot\|\widehat{w}(\cdot,\eta)\big|\,d\eta\right]*\left[\sup_{0\le t<+\infty}|\cdot|^{1-2s}\big|\widehat{w}(\cdot,t)\big|\right].\end{aligned} \tag{5}$$

Considering equations (4) and (5) and applying Young's inequality, we have

$$\|w\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2s}} \le \|w_0\|_{\mathcal{X}^{1-2s}} + 2^{2(1-2s)}\|w\|_{L_t^1\mathcal{X}^1}\|w\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2s}}. \tag{6}$$

Equation (3) shows that w is estimated in $\mathcal{L}_t^\infty\mathcal{X}^{1-2s}$.

Next we need to estimate w in $L_t^1\mathcal{X}^1$. Multiplying by $|\varphi|$ both sides of equation (3), we have

$$\begin{aligned}|\varphi|\big|\widehat{w}(\varphi,t)\big| &\lesssim |\varphi|^{2s}e^{-t|\varphi|^s}|\xi|^{1-2s}\big|\widehat{u}_0(\varphi)\big| \\ &\quad + \int_0^t|\varphi|^{2s}e^{-(t-\eta)|\varphi|^s}\int_{\mathbb{R}^3}|\varphi|^{2-2s}\big|\widehat{w}(\varphi-\zeta,\eta)\big|\big|\widehat{w}(\zeta,\eta)\big|\,d\zeta\,d\eta.\end{aligned} \tag{7}$$

Applying L_t^1 to inequality (7), utilizing $\int_0^{+\infty}|\varphi|^{2s}e^{-t|\xi|^s}\,dt<\infty$ and Young's inequality, we have

$$\|w\|_{L_t^1\mathcal{X}^1} \lesssim \|w_0\|_{\mathcal{X}^{1-2s}} + 2^{2(1-2s)}\|w\|_{L_t^1\mathcal{X}^1}\|w\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2s}}. \tag{8}$$

Denote

$$S := \mathcal{L}_t^\infty\mathcal{X}^{1-2s}\cap L_t^1\mathcal{X}^1$$

and

$$H := \|u\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2s}} + \|w\|_{L_t^1\mathcal{X}^1},$$

then from equations (3) and (8), we have

$$\|w\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2s}} + \|w\|_{L_t^1\mathcal{X}^1} \lesssim \|u_0\|_{\mathcal{X}^{1-2s}} + 2^{2(1-2s)}\big(\|w\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2s}} + \|w\|_{L_t^1\mathcal{X}^1}\big)^2,$$

that is,

$$H \lesssim 2^{2(1-2s)}H^2 + \|w_0\|_{\mathcal{X}^{1-2s}}.$$

By choosing

$$\psi = 1 - 4 \cdot 2^{2(1-2s)} \|w_0\|_{\mathcal{X}^{1-2s}}$$

and applying the Banach fixed point principle, it is easy to achieve the existence of global solution in S for small initial data belonging to $\mathcal{X}^{1-2s}$. □

4 Proof of Theorem 2.2

The lemma stated below is helpful in constructing the proof of Theorem 2.2.

Lemma 4.1 [38] *Let* $0 < \pi \le t < \infty$ *and* $0 \le s \le 1$, *then there holds the following inequality*:

$$t|m|^s - \frac{1}{2}\left(t^2 - \pi^2\right)|m|^{2s} - \pi|m-n|^s - \pi|n|^s \le \frac{1}{2}$$

for any $m, n \in \mathbb{R}^3$.

The construction of the proof of Theorem 2.2 follows the idea of Lemarié-Rieusset [36], where he showed the analyticity for the solution, i.e.,

$$\sup_{0<t<\infty} \sup_{\varphi \in \mathbb{R}^3} e^{\sqrt{t}|\varphi|^s} |\varphi|^{2s} \left|\widehat{w}(\varphi, t)\right| < \infty.$$

Suppose $f(z,t) = e^{\sqrt{t}|\delta|^s} w(z,t)$, considering the integral form (2), we can write

$$f(z,t) = e^{\sqrt{t}|\delta|^s} G_s(t) w_0 - \int_0^t e^{\sqrt{t}|\delta|^s} G_s(t-\eta) \nabla \cdot (w \nabla Q w)\, d\eta.$$

We can easily get that

$$\begin{aligned}\left|\widehat{f}(\varphi,t)\right| &\lesssim e^{\sqrt{t}|\varphi|^s - t|\varphi|^{2s}} \left|\widehat{w}_0(\varphi)\right| \\ &\quad + \int_0^t e^{\sqrt{t}|\varphi|^s - (t-\eta)|\varphi|^{2s} - \sqrt{\eta}(|\varphi-\zeta|^s + |\zeta|^s)} |\varphi| \left(\int_{\mathbb{R}^3} \left|\widehat{w}(\varphi-\zeta,\eta)\right| \left|\widehat{w}(\zeta,\eta)\right| d\zeta \right) d\eta.\end{aligned}$$

Noticing that

$$e^{\sqrt{t}|\varphi|^s - \frac{1}{2}t|\varphi|^{2s}} = e^{-\frac{1}{2}(\sqrt{t}|\varphi|^s - 1)^2 + \frac{1}{2}} \le e^{\frac{1}{2}} \tag{9}$$

and Lemma 4.1, we have

$$e^{\sqrt{t}|\varphi|^s - \frac{1}{2}(t-\eta)|\varphi|^{2s} - \sqrt{\eta}(|\varphi-\zeta|^s + |\zeta|^s)} \le e^{\frac{1}{2}}. \tag{10}$$

Utilizing inequalities (9), (10), we have

$$\begin{aligned}\left|\widehat{f}(\varphi,t)\right| &\lesssim e^{-\frac{1}{2}t|\varphi|^s} \left|\widehat{w}_0(\varphi)\right| \\ &\quad + \int_0^t e^{-\frac{1}{2}(t-\eta)|\varphi|^{2s}} |\varphi| \left(\int_{\mathbb{R}^3} \left|\widehat{w}(\varphi-\zeta,\eta)\right| \left|\widehat{w}(\zeta,\eta)\right| d\zeta \right) d\eta.\end{aligned} \tag{11}$$

The rest of the theorem follows similar steps as in the proof of Theorem 2.1. That's why the remaining details are skipped here.

5 Conclusion

In this paper we considered the fractional porous medium equation and established the existence of global solution in the corresponding critical function spaces for small initial data belonging to these spaces. These spaces were previously considered related to the existence of solution for the classical case of Navier–Stokes equations [33]. The existence of solution for equation (FPM) was previously studied in various function spaces, for instance, [19, 28]. This paper extended the study of equation (FPM) to Lei and Lin spaces and achieved the existence of global solution for small initial data. Moreover, this paper also provided the analyticity of the solution of equation (FPM).

Funding
This research has no external funding.

Availability of data and materials
Not applicable

Declarations

Ethics approval and consent to participate
Not applicable

Competing interests
The authors declare no competing interests.

Author contributions
MZA worked on the problem and wrote the original draft. MM revised the mathematical calculations, made corrections and did several improvements. All authors have read and approved the final version of the manuscript.

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Received: 19 August 2023 Accepted: 25 October 2023 Published online: 14 November 2023

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