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Decay of the 3D Lüst model

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Abstract

In this paper, we consider the time-decay rate of the strong solution to the Cauchy problem for the three-dimensional Lüst model. In particular, the optimal decay rates of the higher-order spatial derivatives of the solution are obtained. The $\dot{H}^{-s}$ $(0 \le s < \frac{3}{2})$ negative Sobolev norms are shown to be preserved along time evolution and enhance the decay rates.

Mathematics Subject Classification: 35Q35; 35D35; 76D05

Keywords: Lüst model; Decay; Negative Sobolev estimates

1 Introduction

In this paper, we consider the decay estimates for the Cauchy problem of the Lüst model [4]

$$\begin{cases} u_t + u \cdot \nabla u - B \cdot \nabla B + \nabla p - \mu \Delta u = -J \cdot \nabla J, \\ B_t - \Delta B_t + u \cdot \nabla B - B \cdot \nabla u - \nu \Delta B = \nabla \times [J \cdot \nabla J - J \times B - u \cdot \nabla J - J \cdot \nabla u], \\ \nabla \cdot u = 0, \qquad \nabla \cdot B = 0, \\ u(x,0) = u_0(x), \qquad B(x,0) = B_0(x), \end{cases} \tag{1.1}$$

where u, B, and p denote the plasma velocity field, the magnetic field, and the pressure, $J = \nabla \times B$ is the electric current density, and $J \times B$ is called the Lorentz force. The parameters μ and ν are the viscosity and the resistivity constants, respectively. In this paper, for simplicity, we assume $\nu = \mu$.

In [1], Bae and Shin studied the well-posedness and blow-up criterion to the Lüst model with initial data in $H^3(\mathbb{R}^3)$. The authors proved that this solution is defined globally-in-time and decays algebraically when $\mathcal{E}_0 := \|u_0\|_{H^3} + \|B_0\|_{H^3} + \|J_0\|_{H^3}$ is sufficiently small. More precisely, we show the main results of [1] as follows:

Lemma 1.1 ([4]) *There exists T depending on $\mathcal{E}_0$ such that there exists a unique solution of* (1.1) *on* $[0,T)$ *satisfying*

$$\|u(t)\|_{H^3}^2 + \|B(t)\|_{H^3}^2 + \|J(t)\|_{H^3}^2 + 2\mu \int_0^t \left(\|\nabla u(s)\|_{H^3}^2 + \|\nabla B(s)\|_{H^3}^2\right) ds \le C\mathcal{E}_0.$$

Moreover, $T > 0$ is the maximal existence time of the solution if and only if*

$$\limsup_{t \nearrow T*} \int_0^t \left(\|\nabla \times u\|_{BMO} + \|\nabla J\|_{BMO} \right) dt = \infty.$$

Lemma 1.2 ([4]) *If $\mathcal{E}_0$ is sufficiently small, $T* = \infty$ in Lemma 1.1. Moreover, (u, B, J) decays in time as follows*:

$$\left\| \Lambda^k (u, B, J) \right\|_{L^2} \leq C_0 (1+t)^{-\frac{k}{2}}, \quad k = 1, 2, 3,$$

where $\Lambda = \sqrt{-\Delta}$ and C_0 depends only on $\mathcal{E}_0$.

The main purpose of this paper is to improve Bae and Shin's results on the global well-posedness and decay estimates for system (1.1).

First, we show the following theorem on global well-posedness. It is worth pointing out that system (1.1) consists of the 3D incompressible Navier–Stokes equations coupled with Maxwell equations. Thanks to the convective term, the existence of a global strong solution without any additional initial conditions is also one of the big challenges in mathematical research. Hence, in order to obtain the global well-posedness for system (1.1), we also need to provide some assumptions on the initial data. Here, we adopt the pure energy method [2, 5] and standard continuity argument, assuming the H^3-norm of initial data is sufficiently small, to prove the global well-posedness of strong solutions. More precisely, we establish the following theorem:

Theorem 1.3 (Global well-posedness) *Assume that $(u_0, B_0, J_0) \in H^N(\mathbb{R}^3)$ with $N \geq 3$. Then, there exists a constant $\varepsilon_0 > 0$ such that if*

$$\|u_0\|_{H^3} + \|B_0\|_{H^3} + \|J_0\|_{H^3} \leq \varepsilon_0, \tag{1.2}$$

then there exists a unique global solution (u, B, J) satisfying that for all $t \geq 0$,

$$\begin{aligned} &\left\| u(t) \right\|_{H^N}^2 + \left\| B(t) \right\|_{H^N}^2 + \left\| J(t) \right\|_{H^N}^2 + 2\mu \int_0^t \left(\left\| \nabla u(s) \right\|_{H^N}^2 + \left\| \nabla B(s) \right\|_{H^N}^2 \right) ds \\ &\quad \leq C(\|u_0\|_{H^N}^2 + \|B_0\|_{H^N}^2 + \|J_0\|_{H^N}^2. \end{aligned} \tag{1.3}$$

The time-decay rate of solutions is also an interesting topic in the study of the Cauchy problem of dissipative equations. Here, we establish the negative Sobolev norm estimates and show that the solutions of system (1.1) satisfy some negative algebraic decay estimates.

Theorem 1.4 (Decay) *Let $N \geq 3$ and (1.2) hold. Assume that $(u_0, B_0, J_0) \in H^N(\mathbb{R}^3) \cap \dot{H}^{-s}(\mathbb{R}^3)$ for some $s \in [0, \frac{3}{2})$. Then, for all $t \geq 0$, we have*

$$\left\| \Lambda^{-s} u(t) \right\|_{L^2} + \left\| \Lambda^{-s} B(t) \right\|_{L^2} + \left\| \Lambda^{-s} J(t) \right\|_{L^2} \leq C \tag{1.4}$$

and

$$\left\| \nabla^l u \right\|_{H^{N-l}} + \left\| \nabla^l B \right\|_{H^{N-l}} + \left\| \nabla^l J \right\|_{H^{N-l}} \leq C(1+t)^{-\frac{l+s}{2}}, \quad \text{for } l = 0, 1, \ldots, N. \tag{1.5}$$

Applying the Hardy–Littlewood–Sobolev theorem [5], we easily obtain that for $p \in [1,2)$, $L^p(\mathbb{R}^3) \subset \dot{H}^{-s}(\mathbb{R}^3)$ with $s = 3(\frac{1}{p} - \frac{1}{2}) \in [0, \frac{3}{2})$. Therefore, Theorem 1.4 implies that:

Corollary 1.5 *Under the assumptions of Theorem 1.4, if we replace the $\dot{H}^{-s}(\mathbb{R}^3)$ assumption by $(u_0, B_0, J_0) \in L^p(\mathbb{R}^3)$ $(1 \le p \le 2)$, then, for $l = 0, 1, \ldots, N$, the following decay estimate holds*:

$$\left\|\nabla^l u\right\|_{H^{N-l}} + \left\|\nabla^l B\right\|_{H^{N-l}} + \left\|\nabla^l J\right\|_{H^{N-l}} \le C(1+t)^{-[\frac{3}{2}(\frac{1}{p}-\frac{1}{2})+\frac{l}{2}]}. \tag{1.6}$$

Remark 1.6 The decay estimate (1.6) is optimal because it is equivalent to the decay rate of the heat equation.

2 Main results

2.1 Proof of Theorem 1.3

First, we give the following equality implying conservation of the energy:

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|B\|_{L^2}^2 + \|J\|_{L^2}^2\right) + \mu\left(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right) = 0. \tag{2.1}$$

In addition, for $k \ge 0$, Bae and Shin [1] established the following inequality:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\Lambda^k u\right\|_{L^2}^2 + \left\|\Lambda^k B\right\|_{L^2}^2 + \left\|\Lambda^k J\right\|_{L^2}^2\right) + \mu\left(\left\|\Lambda^{k+1} u\right\|_{L^2}^2 + \left\|\Lambda^{k+1} B\right\|_{L^2}^2\right) \\
&\quad \le C\left(\|\nabla u\|_{L^\infty} + \|\nabla B\|_{L^\infty} + \|\nabla J\|_{L^\infty}\right)\left(\left\|\Lambda^k u\right\|_{L^2}^2 + \left\|\Lambda^k B\right\|_{L^2}^2 + \left\|\Lambda^k J\right\|_{L^2}^2\right).
\end{aligned} \tag{2.2}$$

By using the embedding $H^2(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ and Lemma 1.1, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\Lambda^k u\right\|_{L^2}^2 + \left\|\Lambda^k B\right\|_{L^2}^2 + \left\|\Lambda^k J\right\|_{L^2}^2\right) + \mu\left(\left\|\Lambda^{k+1} u\right\|_{L^2}^2 + \left\|\Lambda^{k+1} B\right\|_{L^2}^2\right) \\
&\quad \le C\left(\|\nabla u\|_{H^2} + \|\nabla B\|_{H^2} + \|\nabla J\|_{H^2}\right)\left(\left\|\Lambda^k u\right\|_{L^2}^2 + \left\|\Lambda^k B\right\|_{L^2}^2 + \left\|\Lambda^k J\right\|_{L^2}^2\right) \\
&\quad \le C\mathcal{E}_0\left(\left\|\Lambda^k u\right\|_{L^2}^2 + \left\|\Lambda^k B\right\|_{L^2}^2 + \left\|\Lambda^k J\right\|_{L^2}^2\right).
\end{aligned} \tag{2.3}$$

Summing (2.3) over $k = 1, 2, 3, \ldots, N$, and adding (2.1) to the resulting inequality, yields that

$$\begin{aligned}
&\frac{d}{dt}\left(\|u\|_{H^N}^2 + \|B\|_{H^N}^2 + \|J\|_{H^N}^2\right) + 2\mu\left(\|\nabla u\|_{H^N}^2 + \|\nabla B\|_{H^N}^2\right) \\
&\quad \le C_0\mathcal{E}_0\left(\|\nabla u\|_{H^N}^2 + \|\nabla B\|_{H^N}^2\right).
\end{aligned} \tag{2.4}$$

Thus, (u, B, J) exists globally-in-time when $C_0\mathcal{E}_0 < 2\mu$, and the proof of Theorem 1.3 complete.

2.2 Proof of Theorem 1.4

We will derive the evolution of the negative Sobolev norms of the solution to system (1.1). In order to estimate the nonlinear terms, we need to restrict ourselves to that $s \in [0, \frac{1}{2}]$ and $s \in (\frac{1}{2}, \frac{3}{2})$, respectively.

First, taking Λ^{-s} to $(1.1)_1$, by taking the inner product of them with $\Lambda^{-s}u$ and $\Lambda^{-s}B$, respectively, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\Lambda^{-s}u\|_{L^2}^2+\|\Lambda^{-s}B\|_{L^2}^2+\|\Lambda^{-s}J\|_{L^2}^2\big)+\mu\big(\|\Lambda^{-s}\nabla u\|_{L^2}^2+\|\Lambda^{-s}\nabla B\|_{L^2}^2\big)\\
&\quad=-\int_{\mathbb{R}^3}\Lambda^{-s}(u\cdot\nabla u)\cdot\Lambda^{-s}u\,dx+\int_{\mathbb{R}^3}\Lambda^{-s}(B\cdot\nabla B)\cdot\Lambda^{-s}u\,dx\\
&\qquad-\int_{\mathbb{R}^3}\Lambda^{-s}(J\cdot\nabla J)\cdot\Lambda^{-s}u\,dx-\int_{\mathbb{R}^3}\Lambda^{-s}(u\cdot\nabla B)\cdot\Lambda^{-s}B\,dx\\
&\qquad+\int_{\mathbb{R}^3}\Lambda^{-s}(B\cdot\nabla u)\cdot\Lambda^{-s}B\,dx-\int_{\mathbb{R}^3}\Lambda^{-s}(J\times B)\cdot\Lambda^{-s}B\,dx\\
&\qquad-\int_{\mathbb{R}^3}\Lambda^{-s}(u\cdot\nabla J)\cdot\Lambda^{-s}B\,dx-\int_{\mathbb{R}^3}\Lambda^{-s}(J\cdot\nabla u)\cdot\Lambda^{-s}B\,dx\\
&\qquad+\int_{\mathbb{R}^3}\Lambda^{-s}(J\cdot\nabla J)\cdot\Lambda^{-s}B\,dx\\
&\quad=:I_1+I_2+\cdots+I_9.
\end{aligned}\tag{2.5}$$

The main tool to estimate the nonlinear terms on the right-hand side of (2.5) is the Sobolev interpolation inequality. This forces us to require that $s\in(0,\frac{3}{2})$. If $s\in(0,\frac{1}{2}]$, we have $\frac{1}{2}+\frac{s}{3}<1$ and $\frac{3}{s}\geq 6$. Hence, applying the Kato–Ponce inequality [3], the Sobolev embedding theorem, Hölder's inequality, and Young's inequality, we obtain

$$\begin{aligned}
I_1&\leq\|\Lambda^{-s}u\|_{L^2}\|\Lambda^{-s}(u\cdot\nabla u)\|_{L^2}\leq C\|\Lambda^{-s}u\|_{L^2}\|u\cdot\nabla u\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}}\\
&\leq C\|\Lambda^{-s}u\|_{L^2}\|u\|_{L^{\frac{3}{s}}}\|\nabla u\|_{L^2}\leq C\|\Lambda^{-s}u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}+s}\|\Delta u\|_{L^2}^{\frac{1}{2}-s}\|\nabla u\|_{L^2}\\
&\leq C\|\nabla u\|_{H^1}^2\|\Lambda^{-s}u\|_{L^2},
\end{aligned}\tag{2.6}$$

$$\begin{aligned}
I_2&\leq\|\Lambda^{-s}u\|_{L^2}\|\Lambda^{-s}(B\cdot\nabla B)\|_{L^2}\leq C\|\Lambda^{-s}u\|_{L^2}\|B\cdot\nabla B\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}}\\
&\leq C\|\Lambda^{-s}u\|_{L^2}\|B\|_{L^{\frac{3}{s}}}\|\nabla B\|_{L^2}\leq C\|\Lambda^{-s}u\|_{L^2}\|\nabla B\|_{L^2}^{\frac{1}{2}+s}\|\Delta u\|_{L^2}^{\frac{1}{2}-s}\|\nabla B\|_{L^2}\\
&\leq C\|\nabla J\|_{H^1}^2\|\Lambda^{-s}u\|_{L^2},
\end{aligned}\tag{2.7}$$

$$\begin{aligned}
I_3&\leq\|\Lambda^{-s}u\|_{L^2}\|\Lambda^{-s}(J\cdot\nabla J)\|_{L^2}\leq C\|\Lambda^{-s}u\|_{L^2}\|J\cdot\nabla J\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}}\\
&\leq C\|\Lambda^{-s}u\|_{L^2}\|J\|_{L^{\frac{3}{s}}}\|\nabla J\|_{L^2}\leq C\|\Lambda^{-s}u\|_{L^2}\|\nabla J\|_{L^2}^{\frac{1}{2}+s}\|\Delta J\|_{L^2}^{\frac{1}{2}-s}\|\nabla J\|_{L^2}\\
&\leq C\|\nabla J\|_{H^1}^2\|\Lambda^{-s}u\|_{L^2},
\end{aligned}\tag{2.8}$$

$$\begin{aligned}
I_4&\leq\|\Lambda^{-s}B\|_{L^2}\|\Lambda^{-s}(u\cdot\nabla B)\|_{L^2}\leq C\|\Lambda^{-s}B\|_{L^2}\|u\cdot\nabla B\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}}\\
&\leq C\|\Lambda^{-s}B\|_{L^2}\|u\|_{L^{\frac{3}{s}}}\|\nabla B\|_{L^2}\leq C\|\Lambda^{-s}B\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}+s}\|\Delta u\|_{L^2}^{\frac{1}{2}-s}\|\nabla B\|_{L^2}\\
&\leq C\big(\|\nabla u\|_{H^1}^2+\|\nabla B\|_{L^2}^2\big)\|\Lambda^{-s}B\|_{L^2},
\end{aligned}\tag{2.9}$$

$$\begin{aligned}
I_5&\leq\|\Lambda^{-s}B\|_{L^2}\|\Lambda^{-s}(B\cdot\nabla u)\|_{L^2}\leq C\|\Lambda^{-s}B\|_{L^2}\|B\cdot\nabla u\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}}\\
&\leq C\|\Lambda^{-s}B\|_{L^2}\|B\|_{L^{\frac{3}{s}}}\|\nabla u\|_{L^2}\leq C\|\Lambda^{-s}B\|_{L^2}\|\nabla B\|_{L^2}^{\frac{1}{2}+s}\|\Delta B\|_{L^2}^{\frac{1}{2}-s}\|\nabla u\|_{L^2}\\
&\leq C\big(\|\nabla B\|_{H^1}^2+\|\nabla u\|_{L^2}^2\big)\|\Lambda^{-s}B\|_{L^2},
\end{aligned}\tag{2.10}$$

$$\begin{aligned}
I_6 &\le \|\Lambda^{-s}B\|_{L^2}\|\Lambda^{-s}(J\times B)\|_{L^2} \le C\|\Lambda^{-s}B\|_{L^2}\|J\times B\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}} \\
&\le C\|\Lambda^{-s}B\|_{L^2}\|B\|_{L^{\frac{3}{s}}}\|J\|_{L^2} \le C\|\Lambda^{-s}B\|_{L^2}\|\nabla B\|_{L^2}^{\frac{1}{2}+s}\|\Delta B\|_{L^2}^{\frac{1}{2}-s}\|J\|_{L^2} \\
&\le C\|\nabla B\|_{H^1}^2\|\Lambda^{-s}B\|_{L^2},
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
I_7 &\le \|\Lambda^{-s}B\|_{L^2}\|\Lambda^{-s}(u\cdot\nabla J)\|_{L^2} \le C\|\Lambda^{-s}B\|_{L^2}\|u\cdot\nabla J\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}} \\
&\le C\|\Lambda^{-s}B\|_{L^2}\|u\|_{L^{\frac{3}{s}}}\|\nabla J\|_{L^2} \le C\|\Lambda^{-s}B\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}+s}\|\Delta u\|_{L^2}^{\frac{1}{2}-s}\|\nabla J\|_{L^2} \\
&\le C\big(\|\nabla B\|_{H^1}^2+\|\nabla u\|_{H^1}^2\big)\|\Lambda^{-s}B\|_{L^2},
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
I_8 &\le \|\Lambda^{-s}B\|_{L^2}\|\Lambda^{-s}(J\cdot\nabla u)\|_{L^2} \le C\|\Lambda^{-s}B\|_{L^2}\|J\cdot\nabla u\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}} \\
&\le C\|\Lambda^{-s}B\|_{L^2}\|\nabla u\|_{L^{\frac{3}{s}}}\|J\|_{L^2} \le C\|\Lambda^{-s}B\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}+s}\|\nabla\Delta u\|_{L^2}^{\frac{1}{2}-s}\|J\|_{L^2} \\
&\le C\big(\|\nabla B\|_{L^2}^2+\|\Delta u\|_{H^1}^2\big)\|\Lambda^{-s}B\|_{L^2}
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
I_9 &\le \|\Lambda^{-s}B\|_{L^2}\|\Lambda^{-s}(J\cdot\nabla J)\|_{L^2} \le C\|\Lambda^{-s}B\|_{L^2}\|J\cdot\nabla J\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}} \\
&\le C\|\Lambda^{-s}B\|_{L^2}\|J\|_{L^{\frac{3}{s}}}\|\nabla J\|_{L^2} \le C\|\Lambda^{-s}B\|_{L^2}\|\nabla J\|_{L^2}^{\frac{1}{2}+s}\|\Delta J\|_{L^2}^{\frac{1}{2}-s}\|\nabla J\|_{L^2} \\
&\le C\|\nabla B\|_{H^2}^2\|\Lambda^{-s}B\|_{L^2}.
\end{aligned} \tag{2.14}$$

Summing (2.5)–(2.14) gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\Lambda^{-s}u\|_{L^2}^2+\|\Lambda^{-s}B\|_{L^2}^2+\|\Lambda^{-s}J\|_{L^2}^2\big)+\mu\big(\|\Lambda^{-s}\nabla u\|_{L^2}^2+\|\Lambda^{-s}\nabla B\|_{L^2}^2\big) \\
&\quad\le C\big(\|\nabla B\|_{H^2}^2+\|\nabla u\|_{H^2}^2\big)\big(\|\Lambda^{-s}u\|_{L^2}+\|\Lambda^{-s}B\|_{L^2}\big).
\end{aligned} \tag{2.15}$$

If $s\in(\frac{1}{2},\frac{3}{2})$, we will estimate the right-hand sides of (2.5) and obtain the negative Sobolev norm estimates in another way. Since $s\in(\frac{1}{2},\frac{3}{2})$, we easily obtain $\frac{1}{2}+\frac{s}{3}<1$ and $\frac{3}{s}\in(2,6)$. Therefore, using the Kato–Ponce inequality and Sobolev's embedding theorem, we arrive at

$$I_1 \le C\|\Lambda^{-s}u\|_{L^2}\|u\|_{L^{\frac{3}{s}}}\|\nabla u\|_{L^2} \le C\|\Lambda^{-s}u\|_{L^2}\|u\|_{L^2}^{s-\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}-s}\|\nabla u\|_{L^2}, \tag{2.16}$$

$$I_2 \le C\|\Lambda^{-s}u\|_{L^2}\|B\|_{L^{\frac{3}{s}}}\|\nabla B\|_{L^2} \le C\|\Lambda^{-s}u\|_{L^2}\|B\|_{L^2}^{s-\frac{1}{2}}\|\nabla B\|_{L^2}^{\frac{3}{2}-s}\|\nabla B\|_{L^2}, \tag{2.17}$$

$$\begin{aligned}
I_3 &\le C\|\Lambda^{-s}u\|_{L^2}\|J\|_{L^{\frac{3}{s}}}\|\nabla J\|_{L^2} \le C\|\Lambda^{-s}u\|_{L^2}\|J\|_{L^2}^{s-\frac{1}{2}}\|\nabla J\|_{L^2}^{\frac{3}{2}-s}\|\nabla J\|_{L^2}, \\
&\le C\|\nabla B\|_{H^2}^2\|\Lambda^{-s}u\|_{L^2},
\end{aligned} \tag{2.18}$$

$$I_4 \le C\|\Lambda^{-s}B\|_{L^2}\|u\|_{L^{\frac{3}{s}}}\|\nabla B\|_{L^2} \le C\|\Lambda^{-s}B\|_{L^2}\|u\|_{L^2}^{s-\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}-s}\|\nabla B\|_{L^2}, \tag{2.19}$$

$$I_5 \le C\|\Lambda^{-s}B\|_{L^2}\|B\|_{L^{\frac{3}{s}}}\|\nabla u\|_{L^2} \le C\|\Lambda^{-s}B\|_{L^2}\|B\|_{L^2}^{s-\frac{1}{2}}\|\nabla B\|_{L^2}^{\frac{3}{2}-s}\|\nabla u\|_{L^2}, \tag{2.20}$$

$$I_6 \le C\|\Lambda^{-s}B\|_{L^2}\|B\|_{L^{\frac{3}{s}}}\|J\|_{L^2} \le C\|\Lambda^{-s}B\|_{L^2}\|B\|_{L^2}^{s-\frac{1}{2}}\|\nabla B\|_{L^2}^{\frac{3}{2}-s}\|\nabla B\|_{L^2}, \tag{2.21}$$

$$I_7 \le C\left\|\Lambda^{-s}B\right\|_{L^2}\|u\|_{L^{\frac{3}{s}}}\|\nabla J\|_{L^2} \le C\left\|\Lambda^{-s}B\right\|_{L^2}\|u\|_{L^2}^{s-\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}-s}\|\nabla J\|_{L^2}, \tag{2.22}$$

$$\begin{aligned} I_8 &\le C\left\|\Lambda^{-s}B\right\|_{L^2}\|\nabla u\|_{L^{\frac{3}{s}}}\|J\|_{L^2} \le C\left\|\Lambda^{-s}B\right\|_{L^2}\|\nabla u\|_{L^2}^{s-\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{3}{2}-s}\|J\|_{L^2} \\ &\le C\left(\|\nabla u\|_{H^1}^2 + \|\nabla B\|_{L^2}^2\right)\left\|\Lambda^{-s}B\right\|_{L^2} \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} I_9 &\le C\left\|\Lambda^{-s}u\right\|_{L^2}\|J\|_{L^{\frac{3}{s}}}\|\nabla J\|_{L^2} \le C\left\|\Lambda^{-s}B\right\|_{L^2}\|J\|_{L^2}^{s-\frac{1}{2}}\|\nabla J\|_{L^2}^{\frac{3}{2}-s}\|\nabla J\|_{L^2} \\ &\le C\|\nabla B\|_{H^1}^2\left\|\Lambda^{-s}B\right\|_{L^2}. \end{aligned} \tag{2.24}$$

Summing (2.5) and (2.16)–(2.24) gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\left\|\Lambda^{-s}u\right\|_{L^2}^2 + \left\|\Lambda^{-s}B\right\|_{L^2}^2 + \left\|\Lambda^{-s}J\right\|_{L^2}^2\right) + \mu\left(\left\|\Lambda^{-s}\nabla u\right\|_{L^2}^2 + \left\|\Lambda^{-s}\nabla B\right\|_{L^2}^2\right) \\ &\quad\le C\left(\left\|\Lambda^{-s}u\right\|_{L^2} + \left\|\Lambda^{-s}B\right\|_{L^2}\right) \\ &\qquad\times\Big[\|\nabla B\|_{H^2}^2 + \|\nabla u\|_{H^1}^2 + \left(\|u\|_{L^2} + \|B\|_{L^2}\right)^{s-\frac{1}{2}} \\ &\qquad\times\left(\|\nabla u\|_{L^2} + \|\nabla B\|_{L^2}\right)^{\frac{3}{2}-s}(\|\nabla u\|_{L^2} + \|\nabla B\|_{H^1}\Big]. \end{aligned} \tag{2.25}$$

Next, by using the negative Sobolev norm estimates (2.15) and (2.25), we establish the decay estimates for the solution of system (1.1).

First, one considers the case $s \in (0, \frac{1}{2}]$. Define

$$\mathcal{E}_l(t) = \left\|\nabla^l u\right\|_{L^2}^2 + \left\|\nabla^l B\right\|_{L^2}^2 + \left\|\nabla^l J\right\|_{L^2}^2$$

and

$$\mathcal{E}_{-s}(t) = \left\|\Lambda^{-s}u\right\|_{L^2}^2 + \left\|\Lambda^{-s}B\right\|_{L^2}^2 + \left\|\Lambda^{-s}J\right\|_{L^2}^2.$$

Integrating in time (2.15), and applying (1.3), we obtain

$$\begin{aligned} \mathcal{E}_{-s}(t) &\le \mathcal{E}_{-s}(0) + C\int_0^t\left(\|\nabla B\|_{H^2}^2 + \|\nabla u\|_{H^2}^2\right)\sqrt{\mathcal{E}_{-s}(\tau)}d\tau \\ &\le C_0\Big(1 + \sup_{0\le\tau\le t}\sqrt{\mathcal{E}_{-s}(\tau)}\,d\tau\Big), \end{aligned}$$

which implies (1.4) for $s \in [0, \frac{1}{2}]$, that is

$$\left\|\Lambda^{-s}u(t)\right\|_{L^2} + \left\|\Lambda^{-s}B(t)\right\|_{L^2} + \left\|\Lambda^{-s}J(t)\right\|_{L^2} \le C_0. \tag{2.26}$$

In addition, if $l = 1, 2, \dots, N$, by the Sobolev interpolation inequality, we deduce that

$$\left\|\nabla^{l+1}v\right\|_{L^2} \ge C\left\|\nabla^l v\right\|_{L^2}^{1+\frac{1}{l+s}}\left\|\Lambda^{-s}v\right\|_{L^2}^{-\frac{1}{l+s}}.$$

By this fact and (2.26), we derive that

$$\left\|\nabla^{l+1}(u,B,J)\right\|_{L^2}^2 \ge C_0\left\|\nabla^l(u,B,J)\right\|_{L^2}^2)^{1+\frac{1}{l+s}}.$$

Hence, by using (2.4), one has

$$\frac{d}{dt}\mathcal{E}_l + C_0(\mathcal{E}_l)^{1+\frac{1}{l+s}} \le 0, \quad \text{for } l = 1, 2, \ldots, N,$$

that is

$$\mathcal{E}_l(t) \le C_0(1+t)^{-l-s}, \quad \text{for } l = 1, 2, \ldots, N,$$

which implies that (1.5) holds for the case $s \in [0, \frac{1}{2}]$.

In addition, the arguments for $s \in [0, \frac{1}{2}]$ cannot be applied to $s \in (\frac{1}{2}, \frac{3}{2})$. However, observing that $u_0, B_0, J_0 \in \dot{H}^{-\frac{1}{2}}$ hold since $\dot{H}^{-s} \cap L^2 \subset \dot{H}^{-s'}$ for any $s' \in [0, s]$, we can deduce from what we have proved for (1.4) and (1.5) with $s = \frac{1}{2}$ that

$$\left\|\nabla^l u\right\|_{L^2}^2 + \left\|\nabla^l B\right\|_{L^2}^2 + \left\|\nabla^l J\right\|_{L^2}^2 \le C_0(1+t)^{-\frac{1}{2}-l}, \quad \text{for } l = 0, 1, \ldots, N. \tag{2.27}$$

By (2.25), we have

$$\begin{aligned}
\mathcal{E}_{-s}(t) &\le \mathcal{E}_{-s}(0) + C\int_0^t \|u\|_{L^2}^{s-\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{3}{2}-s} \sqrt{\mathcal{E}_{-s}(\tau)}\, d\tau \\
&\quad + C\int_0^t \left(\|\nabla u\|_{H^1}^2 + \|\varrho\|_{H^3}^2\right)\sqrt{\mathcal{E}_{-s}(\tau)}\, d\tau \\
&\le C + C\int_0^t (1+\tau)^{-\frac{7}{4}-\frac{s}{2}}\, d\tau \sup_{\tau\in[0,t]} \sqrt{\mathcal{E}_{-s}(\tau)} + C \sup_{\tau\in[0,t]} \sqrt{\mathcal{E}_{-s}(\tau)} \\
&\le C + C \sup_{\tau\in[0,t]} \sqrt{\mathcal{E}_{-s}(\tau)}, \quad \text{for } s \in \left(\frac{1}{2}, \frac{3}{2}\right),
\end{aligned}$$

which means (1.4) holds for $s \in (\frac{1}{2}, \frac{3}{2})$. Moreover, we can repeat the arguments leading to (1.5) for $s \in [0, \frac{1}{2}]$ to prove that they also hold for $s \in (\frac{1}{2}, \frac{3}{2})$. Hence, we complete the proof.

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Availability of data and materials
The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
Sheng wrote the main manuscript text and reviewed the manuscript.

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