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COMMENT

A generalized Darbo's fixed point theorem and its applications to different types of hybrid differential equations

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Abstract

In this article, a generalization of Darbo's fixed point theorem using a new contraction operator is obtained to solve our proposed hybrid differential and fractional hybrid differential equations in a Banach space. The applicability of our results with the help of a suitable example has also been shown.

Mathematics Subject Classification: 26A33; 34L30; 45G10; 47H10

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1 Introduction

Fractional integrals and fractional derivatives have a history as old as calculus. Fractional differential equations involving fractional-order derivatives have gained much importance in recent years because of their variety of practical applications in different fields of science and engineering. Fixed point theory and measures of noncompactness (*m.n.c.*) are widely used in solving different types of differential and integral equations. *m.n.c.* was first introduced by Kuratowski [17] in 1930. After that, Darbo [6] generalized Schauder's fixed point theorem with the help of Kuratowski *m.n.c.* Presently, a huge number of new research works related to different types of integral and differential equations have been done by several mathematicians. Arab et al. [2] studied the solvability of fractional functional-integral equations using *m.n.c.* In [8], the authors established some new fixed point theorems involving *m.n.c.* and control functions. Furthermore, by employing this *m.n.c.*, they discussed the existence of solutions to integral equations in a Banach space. In [7], the authors presented the existence of solutions to the nonlinear functional integral equations in two variables with the help of fixed point theory. In [15], the authors discussed the existence of solutions to functional integral equations by using *m.n.c.* and operator-type contractions. Das et al. [10] analyzed the problem of the existence of solutions to a generalized proportional fractional integral equation via new fixed point theorems.

In [5], the authors considered the applications of *m.n.c.* in the study of asymptotic stability. Deb et al. [11] discussed new fixed point theorems via *m.n.c.* and their applications on fractional equations involving an operator with iterative relations. In [4], the authors discussed the existence of solutions for infinite systems of differential equations in tempered sequence spaces. In [20], the authors worked on the solvability of nonlinear functional integral equations of two variables in a Banach algebra. In [19], the authors first introduced a new tempered sequence space and introduced an *m.n.c.* in this space. Furthermore, by using *m.n.c.* and generalizing Darbo's fixed point theorem, they discussed the existence of solutions to an infinite system of fractional differential equations. In [12], the authors presented a generalization of Darbo's fixed point theorem, and they used it to investigate the solvability of an implicit fractional order integral equation in ℓ_p $(1 \le p < \infty)$ spaces. In [13], the authors established a new tempered sequence space, i.e., the tempered sequence space ℓ_p^α $(p \ge 1)$ and obtained an *m.n.c.* in this space. Also, they studied the solvability of an infinite system of Langevin fractional differential equations in this space by using *m.n.c.* with Darbo's fixed point theorem. In [18], the authors discussed the solvability of an infinite system of fractional differential equations in a new tempered sequence space. In [14], the authors studied the existence of solutions for an infinite system of Hilfer fractional boundary value problems in tempered sequence spaces by using Meir-Keeler condensing operators with some numerical examples. Das et al. investigated the solvability of generalized fractional integral equations of two variables in [9]. They developed a fixed point theorem that broadened Darbo's fixed point theorem (DFPT) by employing the measure of noncompactness and a contraction operator. Additionally, they discovered the associated coupled fixed point theorem, used this generalized DFPT to solve generalized fractional integral equations of two variables, and presented an example to explain their conclusions.

In this investigation, we study the following hybrid differential equation

$$\frac{d}{d\varrho}\left[\frac{Ł(\varrho)}{\mathfrak{F}(\varrho, Ł(\varrho))}\right] = \Lambda\big(\varrho, Ł(\varrho)\big), \quad \varrho \in [0,b] = J \text{ and } Ł(0) = 0$$

and the following fractional hybrid differential equation

$$D^q\left[\frac{Ł(\varrho)}{\mathfrak{F}(\varrho, Ł(\varrho))}\right] = \Lambda\big(\varrho, Ł(\varrho)\big), \quad 0 < q < 1, \varrho \in [0,b] = J \text{ and } Ł(0) = 0,$$

where

$$D^q f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{(n)} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\, ds, \quad n = [\alpha] + 1$$

and $[\alpha]$ denotes the integer part of number α.

2 Preliminaries

First, we recall the definition of an *m.n.c.* (see [3]).

Assume that $\|\cdot\|_H$ is a norm on a real Banach space H and $\mathcal{W}(\pi, d_0) = \{\theta \in H : \|\theta - \pi\|_H \le d_0\}$.

Let,

- The collection of all non-empty and bounded subsets of H be denoted by M_H and the collection of all non-empty relatively compact subsets of H be denoted by N_H,
- The closure and the convex closure of $\mathcal{V} \subseteq H$ be denoted by $\bar{\mathcal{V}}$ and $\operatorname{Conv} \mathcal{V}$, respectively,

- $\mathcal{R} = (-\infty, \infty)$, and $\mathcal{R}^+ = [0, \infty)$.

Definition 2.1 [3] An *m.n.c.* in H is a function $\mathcal{G} : M_H \to \mathcal{R}^+$ that satisfies the given conditions:

(i) $\forall \mathcal{V} \in M_H$, $\mathcal{G}(\mathcal{V}) = 0$ gives $\mathcal{V}$ is relatively compact,
(ii) $\ker \mathcal{G} = \{\mathcal{V} \in M_H : \mathcal{G}(\mathcal{V}) = 0\} \neq \emptyset$,
(iii) $\mathcal{V} \subseteq \mathcal{V}_1 \implies \mathcal{G}(\mathcal{V}) \leq \mathcal{G}(\mathcal{V}_1)$,
(iv) $\mathcal{G}(\bar{\mathcal{V}}) = \mathcal{G}(\mathcal{V})$,
(v) $\mathcal{G}(\mathrm{Conv}\,\mathcal{V}) = \mathcal{G}(\mathcal{V})$,
(vi) $\mathcal{G}(\kappa\mathcal{V} + (1-\kappa)\mathcal{V}_1) \leq \kappa\mathcal{G}(\mathcal{V}) + (1-\kappa)\mathcal{G}(\mathcal{V}_1)$ for all $\kappa \in [0,1]$,
(vii) if $\mathcal{V}_p \in M_H$, $\mathcal{V}_p = \bar{\mathcal{V}}_p$, $\mathcal{V}_{p+1} \subset \mathcal{V}_p$ for all $p = 1, 2, 3, 4, \dots$ and $\lim_{p\to\infty} \mathcal{G}(\mathcal{V}_p) = 0$ then $\bigcap_{p=1}^{\infty} \mathcal{V}_p \neq \emptyset$.

The subfamily $\ker\mathcal{G}$, defined by (ii), represents the kernel of measure $\mathcal{G}$, and since $\mathcal{G}(\mathcal{V}_\infty) \leq \mathcal{G}(\mathcal{V}_p)$ for any p, we can say that $\mathcal{G}(\mathcal{V}_\infty) = 0$. Then, $\mathcal{V}_\infty = \bigcap_{p=1}^{\infty} \mathcal{V}_p \in \ker\mathcal{G}$.

2.1 Useful theorems and definitions

First, we state some important theorems:

Theorem 2.2 (Schauder [1]) *Assume that $\mathcal{Q}$ is a non-empty, bounded, closed and convex subset (n.b.c.c.s.) of a Banach Space H. Then every compact continuous mapping $\mathfrak{J} : \mathcal{Q} \to \mathcal{Q}$ has at least one fixed point.*

Theorem 2.3 (Darbo [6]) *Assume that $\mathcal{Q}$ is an n.b.c.c.s. of a Banach Space H, and let $\mathfrak{J} : \mathcal{Q} \to \mathcal{Q}$ be a continuous mapping. Let us have a constant $\chi \in [0,1)$ such that*

$$\mathcal{G}(\mathfrak{J}\omega) \leq \chi \cdot \mathcal{G}(\omega), \quad \omega \subset \mathcal{Q}.$$

Then, there exists at least one fixed point for $\mathfrak{J}$ in $\mathcal{Q}$.

Now, we define some functions that are important for generalization of the Darbo's fixed point theorem ($\mathcal{DF}$PT).

Definition 2.4 A pair of operators (ψ, ϕ), where $\psi, \phi : (0,\infty) \to (0,\infty)$, is a pair of generalized altering distance functions if subsequent postulates hold:

(1) ψ is nondecreasing and continuous;
(2) $\lim_{n\to\infty} \phi(t_n) = 0 \implies \lim_{n\to\infty} t_n = 0$.

Definition 2.5 [16] A simulation function is an operator $\Theta : [0,\infty)^2 \to [0,\infty)$ satisfying the following conditions:

(1) $\Theta(0,0) = 0$;
(2) $\Theta(a,b) < b - a$, $a, b > 0$;
(3) if $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,\infty)$ so that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n > 0$, then $\limsup\limits_{n\to\infty} \Theta(a_n, b_n) < 0$.

3 Fixed point theorems

Theorem 3.1 *Assume that $\mathbb{E}$ is an n.b.c.c.s. of a Banach space H. Also, let $\mathcal{K}:\mathbb{E}\to\mathbb{E}$ be a continuous mapping with*

$$\psi\big(\mathcal{G}(\mathcal{K}\mathfrak{P})\big)\leq\alpha\big(\mathcal{G}(\mathfrak{P})\big)\psi\big(\mathcal{G}(\mathfrak{P})\big)-\beta\big(\mathcal{G}(\mathfrak{P})\big)\phi\big(\mathcal{G}(\mathfrak{P})\big), \tag{3.1}$$

where $\alpha,\beta:[0,\infty)\to[0,1)$ are continuous, (ψ,ϕ) is a pair of generalized altering distance functions, $\mathfrak{P}\subset\mathbb{E}$ and $\mathcal{G}$ is an arbitrary m.n.c. Then, $\mathcal{K}$ has at least one fixed point in $\mathbb{E}$.

Proof First, we consider the sequence $\{\mathbb{E}_r\}_{r=1}^{\infty}$ with $\mathbb{E}_1=\mathbb{E}$ and $\mathbb{E}_{r+1}=\overline{\mathrm{Conv}}(\mathcal{K}\mathbb{E}_r)$ for all $r\in\mathbb{N}$ (the set of natural numbers). So, $\mathcal{K}\mathbb{E}_1=\mathcal{K}\mathbb{E}\subseteq\mathbb{E}=\mathbb{E}_1$, $\mathbb{E}_2=\overline{\mathrm{Conv}}(\mathcal{K}\mathbb{E}_1)\subseteq\mathbb{E}=\mathbb{E}_1$. By proceeding in the same manner, we get $\mathbb{E}_1\supseteq\mathbb{E}_2\supseteq\mathbb{E}_3\supseteq\cdots\supseteq\mathbb{E}_r\supseteq\mathbb{E}_{r+1}\supseteq\cdots$.

If $\mathcal{G}(\mathbb{E}_{r_0})=0$ for some $r_0\in\mathbb{N}$, then $\mathbb{E}_{r_0}$ is a compact set. Then, using Theorem 2.2, we can say that $\mathcal{K}$ has a fixed point in $\mathbb{E}$.

So, let $\mathcal{G}(\mathbb{E}_r)>0$ for all $r\in\mathbb{N}$.

Now,

$$\begin{aligned}&\psi\big(\mathcal{G}(\mathbb{E}_{r+1})\big)\\&\quad=\psi\big(\mathcal{G}\big(\overline{\mathrm{Conv}}(\mathcal{K}\mathbb{E}_r)\big)\big)\\&\quad=\psi\big(\mathcal{G}(\mathcal{K}\mathbb{E}_r)\big)\\&\quad\leq\alpha\big(\mathcal{G}(\mathbb{E}_r)\big)\psi\big(\mathcal{G}(\mathbb{E}_r)\big)-\beta\big(\mathcal{G}(\mathbb{E}_r)\big)\phi\big(\mathcal{G}(\mathbb{E}_r)\big)\\&\quad\leq\alpha\big(\mathcal{G}(\mathbb{E}_r)\big)\psi\big(\mathcal{G}(\mathbb{E}_r)\big).\end{aligned}$$

Since $\{\mathcal{G}(\mathbb{E}_r)\}_{r=1}^{\infty}$ is a nonnegative decreasing sequence; therefore, it is convergent to some nonnegative number a (say), i.e., $\lim_{r\to\infty}\mathcal{G}(\mathbb{E}_r)=a$. Let $a\neq0$. So, as $r\to\infty$, we get,

$$\psi(a)\leq\alpha(a)\psi(a),$$

which implies

$$\alpha(a)\geq1,$$

which contradicts our assumption. Hence, $a=0$, which gives

$$\lim_{r\to\infty}\mathcal{G}(\mathbb{E}_r)=0.$$

As we have $\mathbb{E}_r\supseteq\mathbb{E}_{r+1}$, from part (vii) of Definition 2.1, we get $\mathbb{E}_\infty=\bigcap_{r=1}^{\infty}\mathbb{E}_r$ is a nonempty, closed, convex set, which is invariant under $\mathcal{K}$ and belongs to $\ker\mathcal{G}$. Thus, by Schauder's theorem (Theorem 2.2), $\mathcal{K}$ has at least one fixed point in $\mathbb{E}$. □

Corollary 3.2 *Assume that $\mathbb{E}$ is an n.b.c.c.s. of H. Also, let $\mathcal{K}:\mathbb{E}\to\mathbb{E}$ be a continuous mapping with*

$$\psi\big(\mathcal{G}(\mathcal{K}\mathfrak{P})\big)\leq\chi\psi\big(\mathcal{G}(\mathfrak{P})\big), \tag{3.2}$$

where $\chi \in [0,1)$, $\mathfrak{P} \subset \mathbb{E}$ *and* $\mathcal{G}$ *is an arbitrary m.n.c. Then,* $\mathcal{K}$ *has at least one fixed point in* $\mathbb{E}$.

Proof Putting $\alpha \equiv \chi \in [0,1)$ and $\beta \equiv 0$ in equation (3.1) of Theorem 3.1, we can get the above result. □

Remark 3.3 If we chose $\psi(t) = t$ for all $t \in [0,\infty)$ in equation (3.2), then we get

$$\mathcal{G}(\mathcal{K}\mathfrak{P}) \le \chi \mathcal{G}(\mathfrak{P}).$$

Hence, we conclude that our fixed point theorem is a generalization of $\mathcal{DF}$PT.

Theorem 3.4 *Assume that* $\mathbb{E}$ *is an n.b.c.c.s. of* H*. Also, let* $\mathcal{K} : \mathbb{E} \to \mathbb{E}$ *be a continuous mapping with*

$$\big(\mathcal{G}(\mathcal{K}\mathfrak{P}) + l\big)^{\mathcal{G}(\mathcal{K}\mathfrak{P})} \le l^{\beta(\mathcal{G}(\mathfrak{P}))\mathcal{G}(\mathfrak{P})}, \quad l > 1, \tag{3.3}$$

where $\beta : [0,\infty) \to [0,1)$ *is continuous,* $\mathfrak{P} \subset \mathbb{E}$ *and* $\mathcal{G}$ *is an arbitrary m.n.c. Then,* $\mathcal{K}$ *has at least one fixed point in* $\mathbb{E}$.

Proof First, we consider the sequence $\{\mathbb{E}_r\}_{r=1}^{\infty}$ with $\mathbb{E}_1 = \mathbb{E}$ and $\mathbb{E}_{r+1} = \overline{\mathrm{Conv}}(\mathcal{K}\mathbb{E}_r)$ for all $r \in \mathbb{N}$ (the set of natural numbers). Also, $\mathcal{K}\mathbb{E}_1 = \mathcal{K}\mathbb{E} \subseteq \mathbb{E} = \mathbb{E}_1$, and $\mathbb{E}_2 = \overline{\mathrm{Conv}}(\mathcal{K}\mathbb{E}_1) \subseteq \mathbb{E} = \mathbb{E}_1$. Proceeding in the same manner, we get $\mathbb{E}_1 \supseteq \mathbb{E}_2 \supseteq \mathbb{E}_3 \supseteq \cdots \supseteq \mathbb{E}_r \supseteq \mathbb{E}_{r+1} \supseteq \cdots$.

Let $\mathcal{G}(\mathbb{E}_{r_0}) = 0$ for some $r_0 \in \mathbb{N}$. So, $\mathbb{E}_{r_0}$ is a compact set. Then, using Theorem 2.2, we conclude that $\mathcal{K}$ has a fixed point in $\mathbb{E}$.

Let $\mathcal{G}(\mathbb{E}_r) > 0$ for all $r \in \mathbb{N}$.

Now,

$$\begin{aligned}
&\big(\mathcal{G}(\mathbb{E}_{r+1}) + l\big)^{\mathcal{G}(\mathbb{E}_{r+1})} \\
&\quad = \big(\mathcal{G}\big(\overline{\mathrm{Conv}}\mathcal{K}(\mathbb{E}_r)\big) + l\big)^{\mathcal{G}(\overline{\mathrm{Conv}}\mathcal{K}(\mathbb{E}_r))} \\
&\quad = \big(\mathcal{G}\big(\mathcal{K}(\mathbb{E}_r)\big) + l\big)^{\mathcal{G}(\mathcal{K}(\mathbb{E}_r))} \\
&\quad \le l^{\beta(\mathcal{G}(\mathbb{E}_r))\mathcal{G}(\mathbb{E}_r)}.
\end{aligned}$$

Since $\mathcal{G}(\mathbb{E}_{r+1}) + l > l$, $\mathcal{G}(\mathbb{E}_{r+1}) \le \beta(\mathcal{G}(\mathbb{E}_r))\mathcal{G}(\mathbb{E}_r)$. Also, as $\beta(\mathcal{G}(\mathbb{E}_r)) \in [0,1)$, so $\mathcal{G}(\mathbb{E}_{r+1}) < \mathcal{G}(\mathbb{E}_r)$.

Since $\{\mathcal{G}(\mathbb{E}_r)\}_{r=1}^{\infty}$ is a nonnegative decreasing sequence, it is convergent to some nonnegative number a (say), i.e., $\lim_{r\to\infty} \mathcal{G}(\mathbb{E}_r) = a$. So, as $r \to \infty$, we get

$$a \le \beta(a)a.$$

If $a > 0$, then

$$\beta(a) \ge 1,$$

which contradicts our assumption. Hence, $a = 0$, which gives

$$\lim_{r\to\infty} \mathcal{G}(\mathbb{E}_r) = 0.$$

Thus, we have $\mathbb{E}_r \supseteq \mathbb{E}_{r+1}$. Now, from part (vii) of Definition 2.1, $\mathbb{E}_\infty = \bigcap_{r=1}^{\infty} \mathbb{E}_r$ is a non-empty, closed, convex set, which is invariant under $\mathcal{K}$ and belongs to $\ker \mathcal{G}$. Thus, by Schauder's theorem (Theorem 2.2), $\mathcal{K}$ has at least one fixed point in $\mathbb{E}$. □

Theorem 3.5 *Assume that $\mathbb{E}$ is an n.b.c.c.s. of a Banach space H. Also, let $\mathcal{K} : \mathbb{E} \to \mathbb{E}$ be a continuous mapping with*

$$\Theta\left\{\left(\mathcal{G}(\mathcal{K}\mathfrak{P}) + l\right)^{\mathcal{G}(\mathcal{K}\mathfrak{P})}, l^{\beta(\mathcal{G}(\mathfrak{P}))\mathcal{G}(\mathfrak{P})}\right\} \geq 0, \quad l > 1, \tag{3.4}$$

where Θ is a simulation function, $\beta : [0,\infty) \to [0,1)$ is continuous, $\mathfrak{P} \subset \mathbb{E}$ and $\mathcal{G}$ is an arbitrary m.n.c. Then, $\mathcal{K}$ has at least one fixed point in $\mathbb{E}$.

Proof First, we consider the sequence $\{\mathbb{E}_r\}_{r=1}^{\infty}$ with $\mathbb{E}_1 = \mathbb{E}$ and $\mathbb{E}_{r+1} = \overline{\mathrm{Conv}}(\mathcal{K}\mathbb{E}_r)$ for all $r \in \mathbb{N}$ (the set of natural numbers). It is obvious that $\mathcal{K}\mathbb{E}_1 = \mathcal{K}\mathbb{E} \subseteq \mathbb{E} = \mathbb{E}_1$ and $\mathbb{E}_2 = \overline{\mathrm{Conv}}(\mathcal{K}\mathbb{E}_1) \subseteq \mathbb{E} = \mathbb{E}_1$. By proceeding in the same manner, we get $\mathbb{E}_1 \supseteq \mathbb{E}_2 \supseteq \mathbb{E}_3 \supseteq \cdots \supseteq \mathbb{E}_r \supseteq \mathbb{E}_{r+1} \supseteq \cdots$.

Let $\mathcal{G}(\mathbb{E}_{r_0}) = 0$ for some $r_0 \in \mathbb{N}$. So, $\mathbb{E}_{r_0}$ is a compact set. Then, using Theorem 2.2, we conclude that $\mathcal{K}$ has a fixed point in $\mathbb{E}$.

Let $\mathcal{G}(\mathbb{E}_r) > 0$ for all $r \in \mathbb{N}$.

If we apply the properties of simulation functions, then

$$\Theta\left\{\left(\mathcal{G}(\mathcal{K}\mathfrak{P}) + l\right)^{\mathcal{G}(\mathcal{K}\mathfrak{P})}, l^{\beta(\mathcal{G}(\mathfrak{P}))\mathcal{G}(\mathfrak{P})}\right\} \geq 0,$$

implies

$$\left(\mathcal{G}(\mathcal{K}\mathfrak{P}) + l\right)^{\mathcal{G}(\mathcal{K}\mathfrak{P})} \leq l^{\beta(\mathcal{G}(\mathfrak{P}))\mathcal{G}(\mathfrak{P})}.$$

Now, we can apply the steps of the above theorem to get the desired result. □

4 Measure of noncompactness on $\mathcal{C}([0,b])$

Let us consider the set of all real continuous functions on $J = [0,b]$, which is denoted by $H = \mathcal{C}(J)$. Then, H is a Banach space with the norm

$$\|\mathfrak{X}\| = \sup\left\{\left|\mathfrak{X}(q)\right| : q \in J\right\}, \quad \mathfrak{X} \in H.$$

Assume that $\Omega (\neq \emptyset) \subseteq H$ is bounded. For $\mathfrak{X} \in \Omega$ and for a $\sigma > 0$, $\mu(\mathfrak{X},\sigma)$ is the modulus of the continuity of $\mathfrak{X}$ that is written as

$$\mu(\mathfrak{X},\sigma) = \sup\left\{\left|\mathfrak{X}(q_1) - \mathfrak{X}(q_2)\right| : q_1, q_2 \in J, |q_2 - q_1| \leq \sigma\right\}.$$

We also define

$$\mu(\Omega,\sigma) = \sup\left\{\mu(\mathfrak{X},\sigma) : \mathfrak{X} \in \Omega\right\}$$

and

$$\mu_0(\Omega) = \lim_{\sigma \to 0} \mu(\Omega,\sigma),$$

where the *m.n.c.* in H is denoted by the function μ_0, and the Hausdorff *m.n.c.* is denoted by ζ and defined by $\zeta(\Omega) = \frac{1}{2}\mu_0(\Omega)$ (see [3]).

5 Solvability of a hybrid differential equation

In this portion, we will check the existence of a solution to the following hybrid differential equation in the Banach space H:

$$\frac{d}{d\varrho}\left[\frac{Ł(\varrho)}{\mathfrak{F}(\varrho, Ł(\varrho))}\right] = \Lambda\big(\varrho, Ł(\varrho)\big), \quad \varrho \in [0, b] = J \text{ and } Ł(0) = 0. \tag{5.1}$$

Equation (5.1) is equivalent to the hybrid integral equation (5.2) given as

$$Ł(\varrho) = \mathfrak{F}\big(\varrho, Ł(\varrho)\big) \int_0^{\varrho} \Lambda\big(u, Ł(u)\big)\, du. \tag{5.2}$$

Let

$$\mathbb{Y}_g = \big\{Ł \in H : \|Ł\| \leq g\big\}.$$

To establish the existence of a solution to (5.1), we need the following assumptions:

(I) $\mathfrak{F} : J \times \mathbb{R} \to \mathbb{R}$ is continuous, and there is a constant $£_1 > 0$ satisfying

$$\big|\mathfrak{F}\big(\varrho, Ł_1(x)\big) - \mathfrak{F}\big(\varrho, Ł_2(y)\big)\big| \leq £_1 \big|Ł_1(x) - Ł_2(y)\big|,$$

for all $\varrho, x, y \in J$ and $Ł_1, Ł_2 \in H$.
Also, for all $\varrho \in J$,

$$\mathfrak{F}(\varrho, 0) = z_0 > 0.$$

(II) $\Lambda : J \times \mathbb{R} \to \mathbb{R}$ is continuous, and there is a constant $\Lambda_1 > 0$ satisfying

$$\big|\Lambda\big(\varrho, Ł_1(x)\big) - \Lambda\big(\varrho, Ł_2(y)\big)\big| \leq \Lambda_1 \big|Ł_1(x) - Ł_2(y)\big|,$$

for all $\varrho, x, y \in J$ and $Ł_1, Ł_2 \in H$.
Also, for all $\varrho \in J$,

$$\Lambda(\varrho, 0) = 0.$$

(III) There is a positive number g_0 such that

$$b(£_1 g_0 + z_0)\Lambda_1 < 1.$$

Theorem 5.1 *Under assumptions* (*I*)-(*III*), *the equation* (5.2) *has at least one solution in* H.

Proof We consider the operator $\mathcal{S} : H \to H$ defined as

$$(\mathcal{S}Ł)(\varrho) = \mathfrak{F}\big(\varrho, Ł(\varrho)\big) \int_0^{\varrho} \Lambda\big(u, Ł(u)\big)\, du.$$

Step (1): In this step, we will prove that the operator $\mathcal{S}$ maps $\mathbb{Y}_{g_0}$ into $\mathbb{Y}_{g_0}$. Let $Ł \in \mathbb{Y}_{g_0}$. Now, we have

$$\begin{aligned}
&\left|(\mathcal{S}Ł)(\varrho)\right| \\
&\quad\leq \left|\mathfrak{F}\big(\varrho, Ł(\varrho)\big)\right| \int_0^{\varrho} \left|\Lambda\big(u, Ł(u)\big)\right| du \\
&\quad\leq \left[\left|\mathfrak{F}\big(\varrho, Ł(\varrho)\big) - \mathfrak{F}(\varrho, 0)\right| + \left|\mathfrak{F}(\varrho, 0)\right|\right] \int_0^{\varrho} \left[\left|\Lambda\big(\varrho, Ł(\varrho)\big) - \Lambda(\varrho, 0)\right| + \left|\Lambda(\varrho, 0)\right|\right] du \\
&\quad\leq \big(£_1 \|Ł\| + z_0\big) \int_0^{\varrho} \Lambda_1 \|Ł\| \, du \\
&\quad\leq \big(£_1 \|Ł\| + z_0\big) \Lambda_1 \|Ł\| b \\
&\quad\leq (£_1 g_0 + z_0) \Lambda_1 g_0 b.
\end{aligned}$$

Hence, $\|Ł\| \leq g_0$ gives

$$\|\mathcal{S}\| < g_0,$$

due to assumption (III). So, $\mathcal{S}$ maps $\mathbb{Y}_{g_0}$ to $\mathbb{Y}_{g_0}$.

Step (2): In this section, the continuity of $\mathcal{S}$ on $\mathbb{Y}_{g_0}$ will be established. Let $\sigma > 0$ and $Ł, Ł' \in \mathbb{Y}_{g_0}$ such that $\|Ł - Ł'\| < \sigma$. Then,

$$\begin{aligned}
&\left|(\mathcal{S}Ł)(\varrho) - \big(\mathcal{S}Ł'\big)(\varrho)\right| \\
&\quad= \left|\mathfrak{F}\big(\varrho, Ł(\varrho)\big) \int_0^{\varrho} \Lambda\big(u, Ł(u)\big) du - \mathfrak{F}\big(\varrho, Ł'(\varrho)\big) \int_0^{\varrho} \Lambda\big(u, Ł'(u)\big) du\right| \\
&\quad\leq \left|\left[\mathfrak{F}\big(\varrho, Ł(\varrho)\big) - \mathfrak{F}\big(\varrho, Ł'(\varrho)\big)\right] \int_0^{\varrho} \Lambda\big(u, Ł(u)\big) du\right| \\
&\qquad+ \left|\mathfrak{F}\big(\varrho, Ł'(\varrho)\big)\right| \left|\int_0^{\varrho} \left[\Lambda\big(u, Ł(u)\big) - \Lambda\big(u, Ł'(u)\big)\right] du\right| \\
&\quad\leq \left|\mathfrak{F}\big(\varrho, Ł(\varrho)\big) - \mathfrak{F}\big(\varrho, Ł'(\varrho)\big)\right| \int_0^{\varrho} \left|\Lambda\big(u, Ł(u)\big)\right| du \\
&\qquad+ \left|\mathfrak{F}\big(\varrho, Ł'(\varrho)\big)\right| \int_0^{\varrho} \left|\Lambda\big(u, Ł(u)\big) - \Lambda\big(u, Ł'(u)\big)\right| du \\
&\quad\leq £_1 \left\|Ł - Ł'\right\| b \Lambda_1 \|Ł\| + \big(£_1 \left\|Ł'\right\| + z_0\big) \left\|Ł - Ł'\right\| b \Lambda_1 \\
&\quad< £_1 \sigma b \Lambda_1 g_0 + (£_1 g_0 + z_0) \sigma b \Lambda_1.
\end{aligned}$$

If $\sigma \to 0$, then $|(\mathcal{S}Ł)(\varrho) - (\mathcal{S}Ł')(\varrho)| \to 0$, i.e., $\|\mathcal{S}Ł - \mathcal{S}Ł'\| \to 0$. Hence, $\mathcal{S}$ is continuous on $\mathbb{Y}_{g_0}$.

Step (3): Now, assume that $\Delta_Ł \ (\neq \emptyset) \subseteq \mathbb{Y}_{g_0}$. Let $\sigma > 0$ be arbitrary and choose $Ł \in \Delta_Ł$ and $\varrho_1, \varrho_2 \in J$ such that $|\varrho_2 - \varrho_1| \leq \sigma$ with $\varrho_2 \geq \varrho_1$.

We have,

$$\begin{aligned}
&\left|(\mathcal{S}Ł)(\varrho_1) - (\mathcal{S}Ł)(\varrho_2)\right| \\
&\quad= \left|\mathfrak{F}\big(\varrho_1, Ł(\varrho_1)\big) \int_0^{\varrho_1} \Lambda\big(u, Ł(u)\big) du - \mathfrak{F}\big(\varrho_2, Ł(\varrho_2)\big) \int_0^{\varrho_2} \Lambda\big(u, Ł(u)\big) du\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\mathfrak{F}(\varrho_1,Ł(\varrho_1)) \int_0^{\varrho_1} \Lambda(u,Ł(u))\,du - \mathfrak{F}(\varrho_1,Ł(\varrho_2)) \int_0^{\varrho_1} \Lambda(u,Ł(u))\,du\right| \\
&\quad + \left|\mathfrak{F}(\varrho_1,Ł(\varrho_2)) \int_0^{\varrho_1} \Lambda(u,Ł(u))\,du - \mathfrak{F}(\varrho_1,Ł(\varrho_2)) \int_0^{\varrho_2} \Lambda(u,Ł(u))\,du\right| \\
&\quad + \left|\mathfrak{F}(\varrho_2,Ł(\varrho_2)) - \mathfrak{F}(\varrho_1,Ł(\varrho_2))\right| \int_0^{\varrho_2} \left|\Lambda(u,Ł(u))\right| du \\
&\leq \left|\mathfrak{F}(\varrho_1,Ł(\varrho_1)) - \mathfrak{F}(\varrho_1,Ł(\varrho_2))\right| \int_0^{\varrho_1} \left|\Lambda(u,Ł(u))\right| du \\
&\quad + \left|\mathfrak{F}(\varrho_1,Ł(\varrho_2))\right| \int_{\varrho_1}^{\varrho_2} \left|\Lambda(u,Ł(u))\right| du \\
&\quad + \left|\mathfrak{F}(\varrho_2,Ł(\varrho_2)) - \mathfrak{F}(\varrho_1,Ł(\varrho_2))\right| \int_0^{\varrho_2} \left|\Lambda(u,Ł(u))\right| du \\
&\leq £_1\Lambda_1 b g_0 \left|Ł(\varrho_1) - Ł(\varrho_2)\right| \\
&\quad + \left|\mathfrak{F}(\varrho_1,Ł(\varrho_2))\right| \Lambda_1 g_0 |\varrho_2 - \varrho_1| \\
&\quad + \left|\mathfrak{F}(\varrho_2,Ł(\varrho_2)) - \mathfrak{F}(\varrho_1,Ł(\varrho_2))\right| \Lambda_1 g_0 b \\
&\leq £_1\Lambda_1 b g_0 \mu(Ł,\sigma) + \left|\mathfrak{F}(\varrho_1,Ł(\varrho_2))\right| \Lambda_1 g_0 |\varrho_2 - \varrho_1| \\
&\quad + \left|\mathfrak{F}(\varrho_2,Ł(\varrho_2)) - \mathfrak{F}(\varrho_1,Ł(\varrho_2))\right| \Lambda_1 g_0 b,
\end{aligned}
$$

where

$$\mu(Ł,\sigma) = \sup\left\{\left|Ł(\varrho_2) - Ł(\varrho_1)\right|; |\varrho_2 - \varrho_1| \leq \sigma; \varrho_1,\varrho_2 \in J\right\}.$$

Since $\mathfrak{F}$ is continuous, $|\mathfrak{F}(\varrho_2,Ł(\varrho_2)) - \mathfrak{F}(\varrho_1,Ł(\varrho_2))| \to 0$ as $\sigma \to 0$.
Therefore,

$$
\begin{aligned}
\mu(\mathcal{S}Ł,\sigma) \leq{}& £_1\Lambda_1 b g_0 \mu(Ł,\sigma) \\
&+ \sup_{|\varrho_2-\varrho_1|\leq\sigma} \left|\mathfrak{F}(\varrho_1,Ł(\varrho_2))\right| \Lambda_1 g_0 \sigma \\
&+ \sup_{|\varrho_2-\varrho_1|\leq\sigma} \left|\mathfrak{F}(\varrho_2,Ł(\varrho_2)) - \mathfrak{F}(\varrho_1,Ł(\varrho_2))\right| \Lambda_1 g_0 b.
\end{aligned}
$$

As $\sigma \to 0$, via taking the $\sup_{Ł\in\Delta_Ł}$, we get

$$\mu_0(\mathcal{S}\Delta_Ł) \leq £_1\Lambda_1 b g_0 \mu_0(\Delta_Ł).$$

Thus, by Corollary 3.2, $\mathcal{S}$ has a fixed point in $\Delta_Ł \subseteq \mathbb{Y}_{g_0}$, i.e., equation (5.1) has a solution in $H = \mathcal{C}(J)$. □

Now, we consider an example to demonstrate Theorem 5.1.

Example 5.2 Let us take the following $\mathcal{HDE}$:

$$\frac{d}{d\rho}\left[\frac{Ł(\varrho)(1+\varrho^2)}{2+Ł(\varrho)}\right] = \frac{\varrho^3 Ł(\varrho)}{3+2\varrho^2}, \tag{5.3}$$

where $\varrho \in [0,1] = J$ and $Ł(0) = 0$.

Here,

$$\mathfrak{F}(\varrho, Ł) = \frac{2 + Ł}{1 + \varrho^2},$$

$$b = 1,$$

and

$$\Lambda(\varrho, Ł) = \frac{\varrho^3 Ł}{3 + 2\varrho^2}.$$

It is also trivial that $\mathfrak{F}$ is continuous and

$$\left|\mathfrak{F}\big(\varrho, Ł_1(x)\big) - \mathfrak{F}\big(\varrho, Ł_2(y)\big)\right| \leq £_1 \left|Ł_1(x) - Ł_2(y)\right|,$$

for all $\varrho, x, y \in J$ and $Ł_1, Ł_2 \in H$. Therefore, $£_1 = 1$ and $z_0 = \mathfrak{F}(\varrho, 0) = \frac{2}{1+\varrho^2} \leq 2$. Λ is also continuous such that

$$\left|\Lambda\big(\varrho, Ł_1(x)\big) - \Lambda\big(\varrho, Ł_2(y)\big)\right| \leq \frac{1}{3}\left|Ł_1(x) - Ł_2(y)\right|,$$

for all $\varrho, x, y \in J$ and $Ł_1, Ł_2 \in H$.

Therefore, $\Lambda(\varrho, 0) = 0$ and $\Lambda_1 = \frac{1}{3}$.

Now, from the inequality of assumption (III), we have $\frac{g_0+2}{3} < 1$ and $g_0 < 1$. Then, we can say that assumption (III) is also fulfilled for the value

$$g_0 = \frac{1}{2}.$$

Thus, all the presumptions from (I) to (III) of Theorem 5.1 are fulfilled. According to Theorem 5.1, we can say that there exists a solution for the equation (5.3) in $H = \mathcal{C}(J)$.

6 Solvability of fractional hybrid differential equation

In this portion, the existence of a solution to a hybrid fractional differential equation in a Banach space H will be established.

Consider the following fractional hybrid differential equation:

$$D^q\left[\frac{Ł(\varrho)}{\mathfrak{F}(\varrho, Ł(\varrho))}\right] = \Lambda\big(\varrho, Ł(\varrho)\big), \quad 0 < q < 1, \varrho \in [0, b] = J \text{ and } Ł(0) = 0. \tag{6.1}$$

Equation (6.1) is equivalent to the following hybrid integral equation:

$$Ł(\varrho) = \frac{\mathfrak{F}(\varrho, Ł(\varrho))}{\Gamma(q)} \int_0^{\varrho} (\varrho - u)^{q-1} \Lambda\big(u, Ł(u)\big)\, du. \tag{6.2}$$

Let

$$\mathbb{Y}_g = \big\{Ł \in H : \|Ł\| \leq g\big\}.$$

To establish the existence of a solution of (6.2), we need the following assumptions:

(I) $\mathfrak{F} : J \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exists a constant $£_1 > 0$ satisfying

$$\big|\mathfrak{F}\big(\varrho, Ł_1(x)\big) - \mathfrak{F}\big(\varrho, Ł_2(y)\big)\big| \leq £_1\big|Ł_1(x) - Ł_2(y)\big|,$$

for all $\varrho, x, y \in J$ and $Ł_1, Ł_2 \in H$.

Also, let, for all $\varrho \in J$,

$$\mathfrak{F}(\varrho, 0) = z_0 > 0.$$

(II) $\Lambda : J \times \mathbb{R} \to \mathbb{R}$ is continuous, and there is a constant $\Lambda_1 > 0$ such that

$$\big|\Lambda\big(\varrho, Ł_1(x)\big) - \Lambda\big(\varrho, Ł_2(y)\big)\big| \leq \Lambda_1\big|Ł_1(x) - Ł_2(y)\big|,$$

for all $\varrho, x, y \in J$ and $Ł_1, Ł_2 \in H$.

Also, for all $\varrho \in J$,

$$\Lambda(\varrho, 0) = 0.$$

(III) There is a positive number g_0 such that

$$\frac{b^q(£_1 g_0 + z_0)\Lambda_1}{\Gamma(q+1)} \leq 1.$$

Theorem 6.1 *Under assumptions* (*I*)-(*III*), *equation* (6.2) *has at least one solution in* $H = \mathcal{C}(J)$.

Proof We consider the operator $\mathcal{S} : H \to H$ defined as

$$(\mathcal{S}Ł)(\varrho) = \frac{\mathfrak{F}(\varrho, Ł(\varrho))}{\Gamma(q)} \int_0^{\varrho} (\varrho - u)^{q-1} \Lambda\big(u, Ł(u)\big)\, du.$$

Step (1): In this step, we will prove that the operator $\mathcal{S}$ maps $\mathbb{Y}_{g_0}$ into $\mathbb{Y}_{g_0}$. Let $Ł \in \mathbb{Y}_{g_0}$. Now, we have

$$\begin{aligned}
&\big|(\mathcal{S}Ł)(\varrho)\big| \\
&\quad \leq \frac{|\mathfrak{F}(\varrho, Ł(\varrho))|}{\Gamma(q)} \int_0^{\varrho} (\varrho - u)^{q-1} \big|\Lambda\big(u, Ł(u)\big)\big|\, du \\
&\quad \leq \frac{1}{\Gamma(q)} \big[\big|\mathfrak{F}\big(\varrho, Ł(\varrho)\big) - \mathfrak{F}(\varrho, 0)\big| + \big|\mathfrak{F}(\varrho, 0)\big|\big] \\
&\qquad \times \int_0^{\varrho} (\varrho - u)^{q-1} \big[\big|\Lambda\big(u, Ł(u)\big) - \Lambda(u, 0)\big| + \big|\Lambda(u, 0)\big|\big]\, du \\
&\quad \leq \frac{1}{\Gamma(q)} \big(£_1 \|Ł\| + z_0\big) \int_0^{\varrho} (\varrho - u)^{q-1} \Lambda_1 \|Ł\|\, du \\
&\quad \leq \frac{1}{\Gamma(q)} \big(£_1 \|Ł\| + z_0\big) \Lambda_1 \|Ł\| \frac{b^q}{q} \\
&\quad \leq \frac{(£_1 g_0 + z_0)\Lambda_1 g_0 b^q}{\Gamma(q+1)}.
\end{aligned}$$

Hence, $\|Ł\| \le g_0$ gives

$$\|\mathcal{S}\| \le g_0.$$

Due to assumption (III), $\mathcal{S}$ maps $\mathbb{Y}_{g_0}$ to $\mathbb{Y}_{g_0}$.

Step (2): In this section, continuity of $\mathcal{S}$ on $\mathbb{Y}_{g_0}$ will be established. Let $\sigma > 0$ and $Ł, Ł' \in \mathbb{Y}_{g_0}$ such that $\|Ł - Ł'\| < \sigma$. Then,

$$\begin{aligned}
&\bigl|(\mathcal{S}Ł)(\varrho) - (\mathcal{S}Ł')(\varrho)\bigr| \\
&\quad= \frac{1}{\Gamma(q)}\biggl|\mathfrak{F}\bigl(\varrho, Ł(\varrho)\bigr)\int_0^{\varrho}(\varrho-u)^{q-1}\Lambda\bigl(u, Ł(u)\bigr)\,du \\
&\qquad- \mathfrak{F}\bigl(\varrho, Ł'(\varrho)\bigr)\int_0^{\varrho}(\varrho-u)^{q-1}\Lambda\bigl(u, Ł'(u)\bigr)\,du\biggr| \\
&\quad\le \frac{1}{\Gamma(q)}\biggl|\bigl[\mathfrak{F}\bigl(\varrho, Ł(\varrho)\bigr) - \mathfrak{F}\bigl(\varrho, Ł'(\varrho)\bigr)\bigr]\int_0^{\varrho}(\varrho-u)^{q-1}\Lambda\bigl(u, Ł(u)\bigr)\,du\biggr| \\
&\qquad+ \frac{1}{\Gamma(q)}\bigl|\mathfrak{F}\bigl(\varrho, Ł'(\varrho)\bigr)\bigr|\biggl|\int_0^{\varrho}(\varrho-u)^{q-1}\bigl[\Lambda\bigl(u, Ł(u)\bigr) - \Lambda\bigl(u, Ł'(u)\bigr)\bigr]\,du\biggr| \\
&\quad\le \frac{1}{\Gamma(q)}\bigl|\mathfrak{F}\bigl(\varrho, Ł(\varrho)\bigr) - \mathfrak{F}\bigl(\varrho, Ł'(\varrho)\bigr)\bigr|\int_0^{\varrho}(\varrho-u)^{q-1}\bigl|\Lambda\bigl(u, Ł(u)\bigr)\bigr|\,du \\
&\qquad+ \frac{1}{\Gamma(q)}\bigl|\mathfrak{F}\bigl(\varrho, Ł'(\varrho)\bigr)\bigr|\int_0^{\varrho}(\varrho-u)^{q-1}\bigl|\Lambda\bigl(u, Ł(u)\bigr) - \Lambda\bigl(u, Ł'(u)\bigr)\bigr|\,du \\
&\quad\le \frac{1}{\Gamma(q+1)}£_1\bigl\|Ł - Ł'\bigr\|b^q\Lambda_1\|Ł\| + \frac{1}{\Gamma(q+1)}\bigl(£_1\bigl\|Ł'\bigr\| + z_0\bigr)\bigl\|Ł - Ł'\bigr\|b^q\Lambda_1 \\
&\quad< \frac{1}{\Gamma(q+1)}\bigl\{£_1\sigma b^q\Lambda_1 g_0 + (£_1 g_0 + z_0)\sigma b^q\Lambda_1\bigr\}.
\end{aligned}$$

If $\sigma \to 0$, then $|(\mathcal{S}Ł)(\varrho) - (\mathcal{S}Ł')(\varrho)| \to 0$, i.e., $\|\mathcal{S}Ł - \mathcal{S}Ł'\| \to 0$. Hence, $\mathcal{S}$ is continuous on $\mathbb{Y}_{g_0}$.

Step (3): Now, assume that $\Delta_{Ł}(\neq \emptyset) \subseteq \mathbb{Y}_{g_0}$. Let $\sigma > 0$ be arbitrary and choose $Ł \in \Delta_{Ł}$ and $\varrho_1, \varrho_2 \in J$ such as $|\varrho_2 - \varrho_1| \le \sigma$ with $\varrho_2 \ge \varrho_1$.

Now,

$$\begin{aligned}
&\bigl|(\mathcal{S}Ł)(\varrho_1) - (\mathcal{S}Ł)(\varrho_2)\bigr| \\
&\quad= \frac{1}{\Gamma(q)}\biggl|\mathfrak{F}\bigl(\varrho_1, Ł(\varrho_1)\bigr)\int_0^{\varrho_1}(\varrho-u)^{q-1}\Lambda\bigl(u, Ł(u)\bigr)\,du \\
&\qquad- \mathfrak{F}\bigl(\varrho_2, Ł(\varrho_2)\bigr)\int_0^{\varrho_2}(\varrho-u)^{q-1}\Lambda\bigl(u, Ł(u)\bigr)\,du\biggr| \\
&\quad\le \frac{1}{\Gamma(q)}\biggl|\mathfrak{F}\bigl(\varrho_1, Ł(\varrho_1)\bigr)\int_0^{\varrho_1}(\varrho-u)^{q-1}\Lambda\bigl(u, Ł(u)\bigr)\,du \\
&\qquad- \mathfrak{F}\bigl(\varrho_1, Ł(\varrho_2)\bigr)\int_0^{\varrho_1}(\varrho-u)^{q-1}\Lambda\bigl(u, Ł(u)\bigr)\,du\biggr| \\
&\qquad+ \frac{1}{\Gamma(q)}\biggl|\mathfrak{F}\bigl(\varrho_1, Ł(\varrho_2)\bigr)\int_0^{\varrho_1}(\varrho-u)^{q-1}\Lambda\bigl(u, Ł(u)\bigr)\,du \\
&\qquad- \mathfrak{F}\bigl(\varrho_1, Ł(\varrho_2)\bigr)\int_0^{\varrho_2}(\varrho-u)^{q-1}\Lambda\bigl(u, Ł(u)\bigr)\,du\biggr|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(q)}\big|\mathfrak{F}\big(\varrho_2,Ł(\varrho_2)\big)-\mathfrak{F}\big(\varrho_1,Ł(\varrho_2)\big)\big|\int_0^{\varrho_2}(\varrho-u)^{q-1}\big|\Lambda\big(u,Ł(u)\big)\big|\,du\\
\le{}&\frac{1}{\Gamma(q)}\big|\mathfrak{F}\big(\varrho_1,Ł(\varrho_1)\big)-\mathfrak{F}\big(\varrho_1,Ł(\varrho_2)\big)\big|\int_0^{\varrho_1}(\varrho-u)^{q-1}\big|\Lambda\big(u,Ł(u)\big)\big|\,du\\
&+\frac{1}{\Gamma(q)}\big|\mathfrak{F}\big(\varrho_1,Ł(\varrho_2)\big)\big|\int_{\varrho_1}^{\varrho_2}(\varrho-u)^{q-1}\big|\Lambda\big(u,Ł(u)\big)\big|\,du\\
&+\frac{1}{\Gamma(q)}\big|\mathfrak{F}\big(\varrho_2,Ł(\varrho_2)\big)-\mathfrak{F}\big(\varrho_1,Ł(\varrho_2)\big)\big|\int_0^{\varrho_2}(\varrho-u)^{q-1}\big|\Lambda\big(u,Ł(u)\big)\big|\,du\\
\le{}&\frac{1}{\Gamma(q+1)}£_1\Lambda_1 b^q g_0\big|Ł(\varrho_1)-Ł(\varrho_2)\big|\\
&+\frac{1}{\Gamma(q+1)}\big|\mathfrak{F}\big(\varrho_1,Ł(\varrho_2)\big)\big|\Lambda_1 g_0\big\{-(\varrho_1-\varrho_2)^q-(\varrho_2-\varrho_1)^q\big\}\\
&+\frac{1}{\Gamma(q)}\big|\mathfrak{F}\big(\varrho_2,Ł(\varrho_2)\big)-\mathfrak{F}\big(\varrho_1,Ł(\varrho_2)\big)\big|\Lambda_1 g_0 b^q\\
\le{}&\frac{1}{\Gamma(q+1)}£_1\Lambda_1 b^q g_0\mu(Ł,\sigma)\\
&+\frac{1}{\Gamma(q+1)}\big|\mathfrak{F}\big(\varrho_1,Ł(\varrho_2)\big)\big|\Lambda_1 g_0\big\{-(\varrho_1-\varrho_2)^q-(\varrho_2-\varrho_1)^q\big\}\\
&+\frac{1}{\Gamma(q)}\big|\mathfrak{F}\big(\varrho_2,Ł(\varrho_2)\big)-\mathfrak{F}\big(\varrho_1,Ł(\varrho_2)\big)\big|\Lambda_1 g_0 b^q,
\end{aligned}
$$

where

$$\mu(Ł,\sigma)=\sup\big\{\big|Ł(\varrho_2)-Ł(\varrho_1)\big|;|\varrho_2-\varrho_1|\le\sigma;\varrho_1,\varrho_2\in J\big\}.$$

Since $\mathfrak{F}$ is continuous, $|\mathfrak{F}(\varrho_2,Ł(\varrho_2))-\mathfrak{F}(\varrho_1,Ł(\varrho_2))|\to 0$ as $\sigma\to 0$.
As $\sigma\to 0$, taking $\sup_{Ł\in\Delta_Ł}$, we get

$$\mu_0(\mathcal{S}\Delta_Ł)\le\frac{1}{\Gamma(q+1)}£_1\Lambda_1 b^q g_0\mu_0(\Delta_Ł).$$

Thus, by Corollary 3.2, $\mathcal{S}$ has a fixed point in $\Delta_Ł\subseteq\mathbb{Y}_{g_0}$, i.e., the equation (6.1) has a solution in $H=\mathcal{C}(J)$. □

Now, with the help of the following example, Theorem 6.1 will be verified.

Example 6.2 Consider the following $F\mathcal{HDE}$:

$$D^{\frac{1}{2}}\left[\frac{Ł(\varrho)(1+\varrho^2)}{2+Ł(\varrho)}\right]=\frac{\varrho^3Ł(\varrho)}{3+2\varrho^2}\tag{6.3}$$

for $\varrho\in[0,1]=J$ and $Ł(0)=0$.

Here,

$$\mathfrak{F}(\varrho,Ł)=\frac{2+Ł}{1+\varrho^2},$$

$$b=1,q=\frac{1}{2},$$

and

$$\Lambda(\varrho, Ł) = \frac{\varrho^3 Ł}{3 + 2\varrho^2}.$$

It is also trivial that $\mathfrak{F}$ is continuous and

$$\left|\mathfrak{F}\big(\varrho, Ł(x)\big) - \mathfrak{F}\big(\varrho, Ł'(y)\big)\right| \leq \left|Ł(x) - Ł'(y)\right|.$$

Therefore, $£_1 = 1$ and $z_0 = \mathfrak{F}(\varrho, 0) = \frac{2}{1+\varrho^2} \leq 2$. Λ is also continuous such that

$$\left|\Lambda\big(\varrho, Ł(x)\big) - \Lambda\big(\varrho, Ł'(y)\big)\right| \leq \frac{1}{3}\left|Ł(x) - Ł'(y)\right|.$$

Therefore, $\Lambda(\varrho, 0) = 0$ and $\Lambda_1 = \frac{1}{3}$.

Now, from the inequality in assumption (III), we have

$$\frac{g_0 + 2}{3\Gamma(\frac{3}{2})} < 1 \quad \Longrightarrow \quad g_0 < 3\Gamma\left(\frac{3}{2}\right) - 2 \approx 0.658.$$

Then, we can say that assumption (III) is also fulfilled for the value $g_0 = \frac{1}{2}$. Thus, all the presumptions of Theorem 6.1 are fulfilled. According to Theorem 6.1, we can say that there exists a solution for equation (6.3) in $H = \mathcal{C}(J)$.

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