Boundary Value Problems
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Boundary value problems of quaternion-valued differential equations: solvability and Green's function

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Abstract

This paper is associated with Sturm–Liouville type boundary value problems and periodic boundary value problems for quaternion-valued differential equations (QDEs). Employing the theory of quaternionic matrices, we prove the conditions for the solvability of the linear boundary valued problem and find Green's function.

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1 Introduction

In 1843, Irish mathematician William Rowan Hamilton [1] introduced the concept of quaternion on the basis of complex numbers. In a quaternion set $\mathbb{H}$, the quaternion q is denoted by

$$q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k},$$

where q_0, q_1, q_2, q_3 are real numbers, and i, j, k satisfy the following operations:

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1, \qquad \mathrm{ij} = -\mathrm{ji} = \mathrm{k}, \qquad \mathrm{ki} = -\mathrm{ik} = \mathrm{j}, \qquad \mathrm{jk} = -\mathrm{kj} = \mathrm{i}.$$

Similar to complex numbers, the quaternion q can be regarded as a four-dimensional real vector $q = (q_0, q_1, q_2, q_3)^T \in \mathbb{R}^4$. With quaternion vectors, rotations in three and four dimensions can be algebraically processed. Thus quaternions show more advantages than real-valued vectors in physics and engineering applications. But even more important, quaternions are 4-vectors whose multiplication rules are controlled by a simple noncommutative division algebra. In other words, quaternions are not exchanged regarding multiplication operations.

To the best of our knowledge, the theory of ordinary differential equations (ODEs) has been relatively systematic and complete [2–4]. But quaternion-valued differential equations (QDEs) are a new type of differential equations. Due to the noncommutativity of multiplication, many properties in the fields of real or complex numbers cannot be applied to the field of quaternions directly, which brings a great challenge to the development of QDEs.

♣ *Qualitative properties of QDEs*

In 2006, Campos and Mawhin [5] considered the existence of periodic solutions of first-order QDE

$$q' = a(t)q + c(t,q)$$

by using topological degree methods. In 2009, Wilczyński [6] continued to study the quaternion Riccati equations

$$q' = q^2 + f(t)$$

by means of isolating segments and the Brouwer fixed point theorem. In the same year, Gasull, Llibre, and Zhang [7] studied the first-order homogeneous QDE

$$q' = aq^n,$$

in which they described the phase portraits of the homogeneous QDEs and discussed the periodic orbits, homoclinic loops, and invariant tori at $n = 2, 3$. In 2011, Zhang [8] studied the global structure of the quaternion Bernoulli equation

$$q' = aq + aq^n.$$

By using the Liouvillian theorem of integrability and the topological characterization of 2-dimensional torus, they proved that the quaternion Bernoulli equations may have invariant tori that possess a full Lebesgue measure subset of $\mathbb{H}$. In 2018, Cai and Kou [9] transformed the process of solving QDEs to an algebraic quaternion problem by Laplace transformation, which provides a new approach to study the linear QDEs.

Most of the above mentioned works focused on one-dimensional QDEs, and they mainly discussed the qualitative properties of QDEs (e.g., the existence of periodic orbits, homoclinic loops, and invariant tori, integrability, the existence of periodic solutions, and so on). They did not provide the algorithm to compute the exact solutions to linear QDEs. In 2021, Xia, Kou, and Liu [10] gave a systematic framework for the theory of linear QDEs. They proved that the set of all the solutions to the linear homogenous QDEs is actually a right-free module, not a linear vector space. On the noncommutativity of the quaternion algebra, many concepts and properties for the ODEs cannot be used. They should be redefined accordingly. A definition of Wronskian is introduced under the framework of quaternions, which is different from the standard one in ODEs. Liouville formula for QDEs is given. Recently, Xia [11] developed the classical method of constant variation and gave a method for solving linear inhomogeneous quaternion-valued differential equations. It is worth noting that their research does not involve boundary value problems of QDEs

$$\begin{cases} q^{(n)} = f(t, q, q', \dots, q^{(n-1)}), & t \in J, \\ U(q) = \mathbf{B}, \end{cases} \tag{1.1}$$

where $J \subset \mathbb{R}$ is a real interval, $\mathbf{B} \in \mathbb{H}^n$, $f \in C(J \times \mathbb{H}^n, \mathbb{H})$, $U : C^{(n-1)}(J, \mathbb{H}) \to \mathbb{H}^n$.

♣ *Boundary value problems of QDEs*

In this paper, we continue to study the general theory of linear QDEs and pay more attention to boundary value problems (BVPs) of second-order linear QDEs under linear boundary conditions

$$\begin{cases} q'' + a_1(t)q' + a_2(t)q = f(t), & t \in J, \\ U_1(q) = B_1, & U_2(q) = B_2, \end{cases} \tag{1.2}$$

where $J = [a,b] \subset \mathbb{R}$, $a_1(t), a_2(t) \in C(J,\mathbb{H})$, and $f(t), U_i(q) \in C(J,\mathbb{H})$, U_1, U_2 are linear with respect to q, that is, $U(q)$ satisfies a linear relationship

$$U(\alpha q) = \alpha U(q), \qquad U(q_1 + q_2) = U(q_1) + U(q_2).$$

When

$$f(t) \equiv 0, \quad t \in J, \quad \text{and} \quad B_1 = B_2 = 0 \tag{1.3}$$

hold, (1.2) is called the *quaternion homogeneous linear BVP*; when

$$f(t) \equiv 0, \quad t \in J, \quad \text{or} \quad B_1 = B_2 = 0$$

holds, (1.2) is called the *quaternion semi-homogeneous linear BVP*; when (1.3) are not satisfied, (1.2) is called *inhomogeneous quaternion linear BVP*.

By the application of quaternion matrix theory, we prove the solvability of the boundary value problem in both the *resonant* ($\operatorname{ddet} Q(q) = 0$) and *nonresonant* ($\operatorname{ddet} Q(q) \neq 0$) cases, where the definition of double determinant $\operatorname{ddet} Q$ can be found in Sect. 3.2.

We also give the Green's function of homogeneous and inhomogeneous boundary value problems, and then we verify the properties of Green's function, which are similar to ODEs, such as the solution of BVPs can be uniquely expressed as an integral equation. We will discuss this problem in *Sturm–Liouville type boundary value conditions*

$$q(a) + q'(a) = B_1, \qquad q(b) + q'(b) = B_2, \tag{1.4}$$

and *periodic boundary value conditions*

$$q(a) - q(b) = 0, \qquad q'(a) - q'(b) = 0. \tag{1.5}$$

This paper is organized as follows. In Sect. 2, we give some basic results on quaternion and quaternion matrix including but not limited to determinant, rank, right (left) eigenvalue, and Cramer's rule of linear quaternion algebraic equation. In Sect. 3, we discuss the solvability of BVPs for second-order linear QDEs in resonant and nonresonant cases. You will also see some computational examples in this section. In Sects. 4 and 5, we calculate the Green's function of second-order BVPs under Sturm–Liouville type boundary value conditions and periodic boundary value conditions, respectively.

2 Preliminary

In this section, we give some definitions of quaternion matrix such as the definition of determinant [12], double determinant [13], inverse matrix [13, 14], and left or right eigenvalues [15–17]. Moreover, we give Cramer's rule [12, 18, 19] of linear quaternion algebraic equation, which is used to discuss the solvability of BVPs for second-order linear QDEs.

2.1 Quaternion and quaternion matrix

First of all, we define for the quaternion $q \in \mathbb{H}$ with

$$q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k},$$

where q_0, q_1, q_2, q_3 are real numbers and $\mathbb{H}$ is the set of quaternions. In addition,

$$\mathcal{R}q = q_0 \quad \text{and} \quad \Im q = q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k}$$

denote the *real part* and the *imaginary part* of q respectively. And the *conjugate* is

$$\bar{q} = q_0 - q_1\mathrm{i} - q_2\mathrm{j} - q_3\mathrm{k} = \mathcal{R}q - \Im q.$$

For any q and $h = h_0 + h_1\mathrm{i} + h_2\mathrm{j} + h_3\mathrm{k}$, it is easy to check that

$$\overline{qh} = \bar{h}\bar{q}, \qquad \mathcal{R}\{q\bar{h}\} = \mathcal{R}\{\bar{q}h\},$$

and

$$\begin{aligned} qh &= q_0h_0 - q_1h_1 - q_2h_2 - q_3h_3 + (q_0h_1 + q_1h_0 + q_2h_3 - q_3h_2)\mathrm{i} \\ &\quad + (q_0h_2 - q_1h_3 + q_2h_0 + q_3h_1)\mathrm{j} + (q_0h_3 + q_1h_2 - q_2h_1 + q_3h_0)\mathrm{k}. \end{aligned}$$

For given $q, h \in \mathbb{H}$, we introduce the *inner product*

$$\langle q, h\rangle = q_0h_0 + q_1h_1 + q_2h_2 + q_3h_3,$$

and the *modulus*

$$\|q\| = \sqrt{\langle q, q\rangle} = \left(q_0^2 + q_1^2 + q_2^2 + q_3^2\right)^{\frac{1}{2}}.$$

After simple calculation, it can be concluded that

$$\|q\|^2 = q\bar{q}, \qquad \|qh\| = \|q\|\|h\|.$$

Then

$$q^{-1} = \frac{\bar{q}}{\|q\|^2} \qquad \text{when } q \neq 0.$$

At last, we should note that for any $q = q_0 + q_1\mathrm{i} + q_2\mathrm{j} + q_3\mathrm{k} \in \mathbb{H}$, it can be rewritten as

$$q = (q_0 + q_1\mathrm{i}) + (q_2 + q_3\mathrm{i})\mathrm{j},$$

that is, for any $q \in \mathbb{H}$, there exist $Q_1, Q_2 \in \mathbb{C}$ such that $q = Q_1 + Q_2\mathrm{j}$. Similarly, for any $A \in \mathbb{H}^{n\times n}$, there exist $A_1, A_2 \in \mathbb{C}^{n\times n}$ such that $A = A_1 + A_2\mathrm{j}$; for any $\nu \in \mathbb{H}^n$, there exist $\nu_1, \nu_2 \in \mathbb{C}^n$ such that $\nu = \nu_1 + \nu_2\mathrm{j}$.

2.2 Determinant and double determinant

For any $A = (a_{ij})_{2\times 2} \in \mathbb{H}^{2\times 2}$, the determinant based on expanding along the first row is

$$r\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

and the determined based on permutation is defined as follows:

$$\det_p\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \varepsilon(\sigma_1)a_{22}a_{11} + \varepsilon(\sigma_2)a_{21}a_{12} = a_{22}a_{11} - a_{21}a_{12}, \tag{2.1}$$

where $\sigma_1 = (2)(1)$, $\sigma_2 = (21)$, and

$$\varepsilon(\sigma_1) = (-1)^{(1-1)+(1-1)} = 1, \qquad \varepsilon(\sigma_2) = (-1)^{(2-1)} = -1.$$

For

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

A^+ is the conjugate transpose of A, that is,

$$A^+ = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where a_{ij} is the quaternion. As a result,

$$r\det(AA^+) = \|a_{11}\|^2\|a_{22}\|^2 + \|a_{12}\|^2\|a_{21}\|^2 - a_{12}\bar{a}_{22}a_{21}\bar{a}_{11} - a_{11}\bar{a}_{21}a_{22}\bar{a}_{12}, \tag{2.2}$$

$$\det_P(A^+A) = \|a_{11}\|^2\|a_{22}\|^2 + \|a_{12}\|^2\|a_{21}\|^2 - \bar{a}_{12}a_{11}\bar{a}_{21}a_{22} - \bar{a}_{22}a_{21}\bar{a}_{11}a_{12}. \tag{2.3}$$

Remark 2.1 To our knowledge,

$$\overline{a_{12}\bar{a}_{22}a_{21}\bar{a}_{11}} = a_{11}\bar{a}_{21}a_{22}\bar{a}_{12}, \qquad \overline{\bar{a}_{12}a_{11}\bar{a}_{21}a_{22}} = \bar{a}_{22}a_{21}\bar{a}_{11}a_{12}, \tag{2.4}$$

that is to say,

$$r\det(AA^+) = \det_P(A^+A). \tag{2.5}$$

Proof Obviously, (2.4) is true, which implies that

$$r\det(AA^+) = \|a_{11}\|^2\|a_{22}\|^2 + \|a_{12}\|^2\|a_{21}\|^2 - 2\mathcal{R}\{a_{12}\bar{a}_{22}a_{21}\bar{a}_{11}\}$$

and

$$\det_P(A^+A) = \|a_{11}\|^2\|a_{22}\|^2 + \|a_{12}\|^2\|a_{21}\|^2 - 2\mathcal{R}\{\bar{a}_{12}a_{11}\bar{a}_{21}a_{22}\}$$

are both real numbers. It is worth noticing that

$$\mathcal{R}\{a_{12}\bar{a}_{22}a_{21}\bar{a}_{11}\} = \mathcal{R}\{\bar{a}_{12}a_{11}\bar{a}_{21}a_{22}\}$$

holds under the circumstance of $\mathcal{R}\{q\bar{h}\} = \mathcal{R}\{\bar{q}h\}$. For simplicity, we may take

$$r\det(AA^+) = \det_P(A^+A) = \|a_{11}\|^2\|a_{22}\|^2 + \|a_{12}\|^2\|a_{21}\|^2 - 2\mathcal{R}\{a_{12}\bar{a}_{22}a_{21}\bar{a}_{11}\}. \tag{2.6}$$

In conclusion, our proof is complete. □

Definition 2.1 For any $A = (a_{ij})_{2\times 2} \in \mathbb{H}^{2\times 2}$, the real number

$$\operatorname{ddet} A := r\det(AA^+) = \det_P(A^+A) \tag{2.7}$$

is called *double determinant*.

Due to the noncommutativity algebra of quaternion, Cramer's rule for linear systems of equations over the real domain no longer applies to quaternion field. Now we introduce Cramer's rule for systems of linear equations with quaternion.

Lemma 2.1 (*see* [12]) *For the right linear equations*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n, \end{cases} \tag{2.8}$$

where $a_{ij}, b_i \in \mathbb{H}$, $i,j = 1,2,\ldots,n$. *Denote* $\alpha_j = (a_{1j}, a_{2j}, \ldots a_{nj})^T$, $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, *and* $\beta = (b_1, b_2, \ldots, b_n)^T$. *If* $\operatorname{ddet} A \neq 0$, *then there exists a unique solution*

$$x_j = \frac{1}{\operatorname{ddet} A}\bar{D}_j, \tag{2.9}$$

where

$$D_j = \det_p\left[\begin{pmatrix} \alpha_1^+ \\ \vdots \\ \alpha_{j-1}^+ \\ \alpha_n^+ \\ \alpha_{j+1}^+ \\ \vdots \\ \alpha_{n-1}^+ \\ \beta^+ \end{pmatrix}\left(\alpha_1, \ldots, \alpha_{j-1}, \alpha_n, \alpha_{j+1}, \ldots, \alpha_{n-1}, \alpha_j\right)\right], \quad j = 1,2,\ldots,n. \tag{2.10}$$

In fact, invertible matrix in the matrix analysis of real and complex matrices is introduced in the same way for quaternion matrix.

Definition 2.2 (see [14]) A matrix $A \in \mathbb{H}^{n\times n}$ is called *invertible* when there exists $B \in \mathbb{H}^{n\times n}$ such that $AB = BA = I_n$, where I_n is the identity matrix. Denote the unique B by A^{-1}.

Lemma 2.2 *(see [13]) The matrix $A = (a_{ij})_{n\times n} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{H}^{n\times n}$ is invertible if and only if* $\operatorname{ddet} A \neq 0$. *Moreover, $A^{-1} = (b_{jk})_{n\times n}$, where*

$$b_{jk} = \frac{1}{\operatorname{ddet} A}\bar{\omega}_{kj}, \quad k, j = 1, 2, \dots, n, \tag{2.11}$$

$$\omega_{kj} = \det_p \left[\begin{pmatrix} \alpha_1^+ \\ \vdots \\ \alpha_{j-1}^+ \\ \alpha_n^+ \\ \alpha_{j+1}^+ \\ \vdots \\ \alpha_{n-1}^+ \\ \mathrm{e}_k^+ \end{pmatrix} \Big(\alpha_1 \cdots \alpha_{j-1}\alpha_n\alpha_{j+1} \cdots \alpha_{n-1}\alpha_j\Big) \right]. \tag{2.12}$$

2.3 Right (left) eigenvalue of quaternionic matrix

Due to the lack of commutativity in multiplication, quaternion matrices also have left and right eigenvalues.

Definition 2.3 For a matrix $A \in \mathbb{H}^{2\times 2}$, if

$$Ax = x\lambda \quad (\text{or } Ax = \lambda x) \tag{2.13}$$

and x is the nonzero quaternion column vector, then the quaternion λ is called the *right (left) eigenvalue*.

Definition 2.4 For $A \in \mathbb{H}^{n\times n}$, suppose $A = A_1 + A_2\mathrm{j}$ with $A_1, A_2 \in \mathbb{C}^{n\times n}$. Define the *complex adjoint matrix* of A

$$\phi(A) = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix}. \tag{2.14}$$

For $\nu \in \mathbb{H}^n$, suppose $\nu = \nu_1 + \nu_2\mathrm{j}$ with $\nu_1, \nu_2 \in \mathbb{C}^n$, define the *complex adjoint vector* of ν

$$\varphi(\nu) = \begin{pmatrix} \nu_1 \\ -\overline{\nu_2} \end{pmatrix},$$

then denote

$$\varphi(\nu)^* = \begin{pmatrix} \nu_2 \\ \overline{\nu_1} \end{pmatrix}.$$

Lemma 2.3 (*see* [16]) *For* $A \in \mathbb{H}^{n\times n}$, $\nu, \mu \in \mathbb{H}^n$, *and* $\lambda \in \mathbb{C}$, $A\nu = \mu + \nu\lambda$ *holds if and only if* $\phi(A)\varphi(\nu) = \varphi(\mu) + \varphi(\nu)\lambda$ *holds.*

Lemma 2.4 (*see* [17]) *For* $A \in \mathbb{H}^{n\times n}$, $\nu \in \mathbb{H}^n$, *and* $\lambda \in \mathbb{C}$, $A\nu = \nu\lambda$ *holds if and only if* $\phi(A)\varphi(\nu) = \varphi(\nu)\lambda$ *holds.*

2.4 Rank of quaternionic matrix

Next, we give some results on the rank of quaternionic matrix, the reader will find these in [20] and [21].

Definition 2.5 Let A be any nonzero $m \times n$ matrix on the ring R. Define the order of the subsquare with the largest nonzero factor in A as the *rank* of A, and simplify it as rank A.

Definition 2.6 The number of rows of A that constitute the largest possible sub-block (not necessarily a square) with non-right zero factors is called the *left rank of the rows* of A, the column number of the largest possible sub-block of non-left zero factors formed by the columns of A is called the *column-right rank* of A.

Remark 2.2 Base on the definition of rank, it is easy to check that:

(1) Both the row-left rank and column-right rank of A are equal to the rank of A;

(2) For a zero matrix, all the above rank numbers are defined as 0;

(3) The above definition naturally applies to the skew field, and the rank of the matrix A on the skew field is the order of the largest nonsingular sub-block in A.

Proposition 2.5 (*see* [20]) *On the skew field, the sufficient and necessary condition for* $AX = \beta$ *to have a solution is that* $\operatorname{rank}(A, \beta) = \operatorname{rank} A$.

For convenience, we give the following more general conclusion and proof.

Lemma 2.6 (*see* [21]) *Suppose that* F *is a skew field and* $A \in F^{m\times n}$. *A sufficient and necessary condition for matrix equation*

$$A_{m\times n}X_{n\times s} = B_{m\times s} \tag{2.15}$$

to have a solution on F *is that*

$$\operatorname{rank} A = \operatorname{rank}(A, B) = r,$$

where $0 \le r \le \min\{m, n\}$.

Proof Suppose $\operatorname{rank} A = r$, then there exist invertible matrices P, Q such that

$$PAQ = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$$

and

$$\operatorname{rank} A = \operatorname{rank}(PA) = \operatorname{rank}(AQ) = \operatorname{rank}(PAQ).$$

Let

$$Y = Q^{-1}X, \qquad PB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where B_1 is the $r \times s$ matrix and B_2 is the $(m-r) \times s$ matrix, then (2.15) has a solution, that is,

$$\begin{pmatrix} I_r & O \\ O & O \end{pmatrix} Y = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

has a solution if and only if $B_2 = O_{(m-r)\times s}$. Obviously,

$$\begin{aligned} \operatorname{rank}(A,B) &= \operatorname{rank}\left(P(A,B)\begin{pmatrix} Q & O \\ O & I_s \end{pmatrix}\right) \\ &= \operatorname{rank}(PAQ, PB) \\ &= \operatorname{rank}\begin{pmatrix} I_r & O & B_1 \\ O & O & B_2 \end{pmatrix}. \end{aligned}$$

From this, we can obtain $B_2 = O_{(m-r)\times s}$ if and only if $\operatorname{rank}(A,B) = r = \operatorname{rank} A$. □

3 Solvability of BVPs for second-order linear QDEs

In this section, we consider the solvability of boundary value problems (BVPs) of second-order linear QDEs under the linear boundary conditions

$$\begin{cases} q'' + a_1(t)q' + a_2(t)q = f(t), & t \in J, \\ U_1(q) = B_1, \qquad U_2(q) = B_2, \end{cases}$$

where $J = [a,b] \subset \mathbb{R}$, $a_1(t), a_2(t) \in C(J,\mathbb{H})$, and $f(t) \in C(J,\mathbb{H})$. For convenience, we give some basic theory of linear QDEs, which can be found in [10].

3.1 Some lemmas

Consider the linear system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A(t)\mathbf{x}(t), \tag{3.1}$$

where $\mathbf{x}(t) \in \mathbb{H}^n$, $A(t)$ is an $n \times n$ continuous quaternion-valued function matrix on the interval $I = [a,b]$, $I \subseteq \mathbb{R}$.

Suppose that $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)$ are solutions of linear system (3.1) if

$$\mathbf{x}_1(t)r_1 + \mathbf{x}_2(t)r_2 + \cdots + \mathbf{x}_n(t)r_n = 0, \quad r_i \in \mathbb{H}$$

implies that

$$r_1 = \cdots = r_k = 0, \quad t \in I,$$

then $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)$ is said to be *right independent*. Otherwise, $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)$ is said to be *right dependent*. Denote

$$M(t) = \bigl(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)\bigr) = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \dots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \dots & x_{2n}(t) \\ \vdots & \dots & \vdots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \dots & x_{nn}(t) \end{pmatrix}$$

is a solution matrix of (3.1), if $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)$ are right independent, it is said to be a fundamental matrix of (3.1).

Definition 3.1 The *Wronskian* of solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)$ is defined by

$$W_{QDE}(t) = \operatorname{ddet} M(t) = \det_P\bigl(M^+(t)M(t)\bigr).$$

Remark 3.1 When $n = 2$, we can obtain

$$W_{QDE}(t) = \operatorname{ddet} M(t) := r\det\bigl(M(t)M^+(t)\bigr) = \det_P\bigl(M^+(t)M(t)\bigr).$$

Lemma 3.1 (*see* [15]) *Suppose that* $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)$ *are solutions of linear system* (3.1), *then* $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)$ *are right independent on interval* I *if and only if* $W_{QDE}(t) \neq 0$.

Lemma 3.2 (*see* [15]) *If* $q(t)$ *is differentiable and if* $q(t)q'(t) = q'(t)q(t)$, *it follows from that*

$$\bigl[\exp q(t)\bigr]' = \bigl[\exp q(t)\bigr]q'(t) = q'(t)\bigl[\exp q(t)\bigr], \tag{3.2}$$

where $\exp q(t) = \sum_{n=0}^{\infty} \frac{q^{n(t)}}{n!}$ *is the exponential function of* $q(t)$.

Lemma 3.3 (*see* [17]) *Let* $\Phi(t)$ *be a fundamental matrix of homogenous equations* (3.1). *Any solution* $\psi^H(t)$ *of linear homogenous equations*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A(t)\mathbf{x}(t) \tag{3.3}$$

can be represented by

$$\psi^H(t) = \Phi(t)q,$$

where q *is a constant quaternionic vector.*

Lemma 3.4 (*see* [11])(*Variation of constants formula*) *The general solution of nonhomogenous equation*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A(t)\mathbf{x}(t) + f(t) \tag{3.4}$$

is given by

$$\psi^{NH}(t) = \Phi(t)q + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)f(s)\,ds, \tag{3.5}$$

where $t_0 \in [a,b]$, q is a constant quaternionic vector.

3.2 Solvability of BVPs

From now on, we analyze the homogeneous equation

$$q'' + a_1(t)q' + a_2(t)q = 0 \tag{3.6}$$

and the inhomogeneous equation

$$q'' + a_1(t)q' + a_2(t)q = f(t). \tag{3.7}$$

By the transformation

$$\begin{cases} q_{11} = q, \\ q_{12} = q', \end{cases}$$

equations (3.6) and (3.7) can be converted into

$$\begin{pmatrix} q_{11} \\ q_{12} \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -a_2(t) & -a_1(t) \end{pmatrix} \begin{pmatrix} q_{11} \\ q_{12} \end{pmatrix}, \tag{3.8}$$

$$\begin{pmatrix} q_{11} \\ q_{12} \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -a_2(t) & -a_1(t) \end{pmatrix} \begin{pmatrix} q_{11} \\ q_{12} \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \tag{3.9}$$

respectively. Simultaneously,

$$A(t) = \begin{pmatrix} 0 & 1 \\ -a_2(t) & -a_1(t) \end{pmatrix}, \qquad \mathbf{F}(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

Because $q_1(t), q_2(t)$ are linearly independent solutions of equation (3.6), define

$$\Phi(t) = \begin{pmatrix} q_1(t) & q_2(t) \\ q_1'(t) & q_2'(t) \end{pmatrix} = \big(\Phi_1(t), \Phi_2(t)\big). \tag{3.10}$$

Proposition 3.5 $\Phi(t)$ *is a fundamental matrix of equation* (3.8).

Proof It is easy to verify that $\Phi_1'(t) = A(t)\Phi_1(t)$, $\Phi_2'(t) = A(t)\Phi_2(t)$. Therefore, $\Phi(t) = (\Phi_1(t), \Phi_2(t))$ is a solution matrix of equation (3.8). And more importantly,

$$\begin{aligned}
W_{QDE}(t) = {} & \operatorname{ddet}\Phi(t) \\
& = r\det\Phi(t)\Phi^+(t) \\
& = r\det\begin{pmatrix} q_1(t) & q_2(t) \\ q_1'(t) & q_2'(t)\end{pmatrix}\begin{pmatrix} \bar{q}_1(t) & \bar{q}_1'(t) \\ \bar{q}_2(t) & \bar{q}_2'(t)\end{pmatrix} \\
& = \|q_1(t)\|^2\|q_2'(t)\|^2 + \|q_1'(t)\|^2\|q_2(t)\|^2 \\
& \quad - q_2(t)\bar{q}_2'(t)q_1'(t)\bar{q}_1(t) - q_1(t)\bar{q}_1'(t)q_2'(t)\bar{q}_2(t) \\
& = \|q_1(t)\|^2\|q_2'(t)\|^2 + \|q_1'(t)\|^2\|q_2(t)\|^2 - 2\mathcal{R}\{q_1(t)\bar{q}_1'(t)q_2'(t)\bar{q}_2(t)\}.
\end{aligned} \tag{3.11}$$

For convenience, we can assume that

$$\begin{aligned}
q_1(t) &= a_0(t) + a_1(t)\mathrm{i} + a_2(t)\mathrm{j} + a_3(t)\mathrm{k}, \\
q_2(t) &= b_0(t) + b_1(t)\mathrm{i} + b_2(t)\mathrm{j} + b_3(t)\mathrm{k}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
q_2(t)\bar{q}_2'(t) &= \big(b_0(t) + b_1(t)\mathrm{i} + b_2(t)\mathrm{j} + b_3(t)\mathrm{k}\big)\big(b_0'(t) - b_1'(t)\mathrm{i} - b_2'(t)\mathrm{j} - b_3'(t)\mathrm{k}\big) \\
&= A_0(t) + A_1(t)\mathrm{i} + A_2(t)\mathrm{j} + A_3(t)\mathrm{k}, \\
q_2(t)\bar{q}_2'(t)q_1'(t) &= \big(A_0(t) + A_1(t)\mathrm{i} + A_2(t)\mathrm{j} + A_3(t)\mathrm{k}\big)\big(a_0'(t) + a_1'(t)\mathrm{i} + a_2'(t)\mathrm{j} + a_3'(t)\mathrm{k}\big) \\
&= B_0(t) + B_1(t)\mathrm{i} + B_2(t)\mathrm{j} + B_3(t)\mathrm{k}, \\
q_2(t)\bar{q}_2'(t)q_1'(t)\bar{q}_1(t) &= \big(B_0(t) + B_1(t)\mathrm{i} + B_2(t)\mathrm{j} + B_3(t)\mathrm{k}\big)\big(a_0(t) - a_1(t)\mathrm{i} - a_2(t)\mathrm{j} - a_3(t)\mathrm{k}\big) \\
&= \big(B_0(t)a_0(t) + B_1(t)a_1(t) + B_2(t)a_2(t) + B_3(t)a_3(t)\big) \\
&\quad + \big(-B_0(t)a_1(t) + B_1(t)a_0(t) - B_2(t)a_3(t) + B_3(t)a_2(t)\big)\mathrm{i} \\
&\quad + \big(-B_0(t)a_2(t) + B_1(t)a_3(t) + B_2(t)a_0(t) - B_3(t)a_1(t)\big)\mathrm{j} \\
&\quad + \big(-B_0(t)a_3(t) - B_1(t)a_2(t) + B_2(t)a_1(t) + B_3(t)a_0(t)\big)\mathrm{k}
\end{aligned}$$

with

$$\begin{aligned}
A_0(t) &= b_0(t)b_0'(t) + b_1(t)b_1'(t) + b_2(t)b_2'(t) + b_3(t)b_3'(t), \\
A_1(t) &= -b_0(t)b_1'(t) + b_1(t)b_0'(t) - b_2(t)b_3'(t) + b_3(t)b_2'(t), \\
A_2(t) &= -b_0(t)b_2'(t) + b_1(t)b_3'(t) + b_2(t)b_0'(t) - b_3(t)b_1'(t), \\
A_3(t) &= -b_0(t)b_3'(t) - b_1(t)b_2'(t) + b_2(t)b_1'(t) + b_3(t)b_0'(t), \\
B_0(t) &= A_0(t)a_0'(t) - A_1(t)a_1'(t) - A_2(t)a_2'(t) - A_3(t)a_3'(t), \\
B_1(t) &= A_0(t)a_1'(t) + A_1(t)a_0'(t) + A_2(t)a_3'(t) - A_3(t)a_2'(t), \\
B_2(t) &= A_0(t)a_2'(t) - A_1(t)a_3'(t) + A_2(t)a_0'(t) + A_3(t)a_1'(t), \\
B_3(t) &= A_0(t)a_3'(t) - A_1(t)a_2'(t) - A_2(t)a_1'(t) + A_3(t)a_0'(t).
\end{aligned}$$

Through a series of calculations, we are able to obtain $W_{QDE}(t) \neq 0$, which implies that $\Phi(t)$ is a fundamental matrix of equation (3.8). □

Proposition 3.6 *The inhomogeneous equation* (3.7) *has a solution*

$$z(t) = \frac{1}{\operatorname{ddet}\Phi(t)} \int_a^t \big(q_1(t)\bar{\omega}_{21}(s) + q_2(t)\bar{\omega}_{22}(s)\big) f(s)\,\mathrm{d}s. \tag{3.12}$$

Proof From Lemma 2.1, Lemma 2.2, and Lemma 3.4, it easily follows that

$$z(t) = \Phi(t) \int_a^t \Phi^{-1}(s)\mathbf{F}(s)\,\mathrm{d}s \tag{3.13}$$

is a solution to the inhomogeneous equation (3.9). It should be noted that

$$\Phi^{-1}(t) = \frac{1}{\operatorname{ddet}\Phi(t)} \begin{pmatrix} \bar{\omega}_{11}(t) & \bar{\omega}_{21}(t) \\ \bar{\omega}_{12}(t) & \bar{\omega}_{22}(t) \end{pmatrix},$$

then

$$\begin{aligned}
\Phi(t) \int_a^t \Phi^{-1}(s)\mathbf{F}(s)\,\mathrm{d}s &= \frac{1}{\operatorname{ddet}\Phi(t)} \int_a^t \Phi(t) \begin{pmatrix} \bar{\omega}_{11}(s) & \bar{\omega}_{21}(s) \\ \bar{\omega}_{12}(s) & \bar{\omega}_{22}(s) \end{pmatrix} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} \mathrm{d}s \\
&= \frac{1}{\operatorname{ddet}\Phi(t)} \int_a^t \begin{pmatrix} q_1(t) & q_2(t) \\ q_1'(t) & q_2'(t) \end{pmatrix} \begin{pmatrix} \bar{\omega}_{11}(s) & \bar{\omega}_{21}(s) \\ \bar{\omega}_{12}(s) & \bar{\omega}_{22}(s) \end{pmatrix} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} \mathrm{d}s \\
&= \frac{1}{\operatorname{ddet}\Phi(t)} \int_a^t \begin{pmatrix} (q_1(t)\bar{\omega}_{21}(s) + q_2(t)\bar{\omega}_{22}(s))f(s) \\ (q_1'(t)\bar{\omega}_{21}(s) + q_2'(t)\bar{\omega}_{22}(s))f(s) \end{pmatrix} \mathrm{d}s.
\end{aligned}$$

The first component of the above equation gives the solution to equation (3.9), that is,

$$\frac{1}{\operatorname{ddet}\Phi(t)} \int_a^t \big(q_1(t)\bar{\omega}_{21}(s) + q_2(t)\bar{\omega}_{22}(s)\big) f(s)\,\mathrm{d}s, \tag{3.14}$$

where

$$\begin{aligned}
\omega_{21}(t) &= \det_P \begin{pmatrix} \Phi_2^+(t) \\ e_2^+ \end{pmatrix} \begin{pmatrix} \Phi_2(t) & \Phi_1(t) \end{pmatrix} \\
&= \det_P \begin{pmatrix} \bar{q}_2(t) & \bar{q}_2'(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_2(t) & q_1(t) \\ q_2'(t) & q_1'(t) \end{pmatrix} \\
&= \det_P \begin{pmatrix} \|q_2(t)\|^2 + \|q_2'(t)\|^2 & \bar{q}_2(t)q_1(t) + \bar{q}_2'(t)q_1'(t) \\ q_2'(t) & q_1'(t) \end{pmatrix} \\
&= \big\|q_2(t)\big\|^2 q_1'(t) - q_2'(t)\bar{q}_2(t)q_1(t),
\end{aligned} \tag{3.15}$$

$$\begin{aligned}\omega_{22}(t) &= \det_P \begin{pmatrix} \Phi_1^+(t) \\ e_2^+ \end{pmatrix} \begin{pmatrix} \Phi_1(t) & \Phi_2(t) \end{pmatrix} \\ &= \det_P \begin{pmatrix} \bar{q}_1(t) & \bar{q}_1'(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_1(t) & q_2(t) \\ q_1'(t) & q_2'(t) \end{pmatrix} \\ &= \det_P \begin{pmatrix} \|q_1(t)\|^2 + \|\bar{q}_1'(t)\|^2 & \bar{q}_1(t)q_2(t) + \bar{q}_1'(t)q_2'(t) \\ q_1'(t) & q_2'(t) \end{pmatrix} \\ &= \left\|q_1(t)\right\|^2 q_2'(t) - q_1'(t)\bar{q}_1(t)q_2(t). \end{aligned} \tag{3.16}$$

The special solution of the inhomogeneous equation (3.7) can be obtained by taking the conjugate of equations (3.15) and (3.16) into equation (3.14). □

Substituting $q_1(t)$, $q_2(t)$ separately into $U_1(q)$, $U_2(q)$, we get $U_1(q_1)$, $U_1(q_2)$, $U_2(q_1)$, $U_2(q_2)$, let

$$Q(q) = \begin{pmatrix} U_1(q_1) & U_1(q_2) \\ U_2(q_1) & U_2(q_2) \end{pmatrix},$$

then

$$\begin{aligned}\operatorname{ddet} Q(q) &= \det_p\bigl(Q^+(q)Q(q)\bigr) \\ &= \left\|U_1(q_2)\right\|^2 \left\|U_2(q_1)\right\|^2 + \left\|U_2(q_2)\right\|^2 \left\|U_1(q_1)\right\|^2 \\ &\quad - \overline{U_1(q_2)}U_1(q_1)\overline{U_2(q_1)}U_2(q_2) - \overline{U_2(q_2)}U_2(q_1)\overline{U_1(q_1)}U_1(q_2). \end{aligned}$$

If $\operatorname{ddet} Q(q) \neq 0$, we say that BVP (1.2) is a *nonresonant* boundary value problem. BVP (1.2) called the *resonant* boundary value problem when $\operatorname{ddet} Q(q) = 0$.

3.2.1 Nonresonant problem

Theorem 3.1 *If* $\operatorname{ddet} Q(q) \neq 0$, *then the quaternion semi-homogeneous linear boundary value problem*

$$\begin{cases} q'' + a_1(t)q' + a_2(t)q = f(t), \\ U_1(q) = U_2(q) = 0, \end{cases} \tag{3.17}$$

and

$$\begin{cases} q'' + a_1(t)q' + a_2(t)q = 0, \\ U_1(q) = B_1, \qquad U_2(q) = B_2, \end{cases} \tag{3.18}$$

all have a unique solution, respectively

$$\varphi(t) = \frac{1}{\operatorname{ddet} Q(q)}\bigl[q_1(t)\bar{D}_1 + q_2(t)\bar{D}_2 + \operatorname{ddet} Q(q)\cdot z(t)\bigr], \quad t \in [a,b], \tag{3.19}$$

$$\psi(t) = \frac{1}{\operatorname{ddet} Q(q)}\bigl[q_1(t)\bar{D}_3 + q_2(t)\bar{D}_4\bigr], \quad t \in [a,b]. \tag{3.20}$$

Further,

$$q(t)=\frac{1}{\operatorname{ddet}Q(q)}\bigl[q_1(t)\bar{D}_5+q_2(t)\bar{D}_6+\operatorname{ddet}Q(q)\cdot z(t)\bigr],\quad t\in[a,b], \tag{3.21}$$

is the unique solution of inhomogeneous BVP (1.2), *where $z(t)$ is given by* (3.12),

$$\begin{aligned}\bar{D}_1={}&\|U_1(q_1)\|^2\overline{U_2(q_2)}U_2(z)+\|U_2(q_1)\|^2\overline{U_1(q_2)}U_1(z)\\&-\overline{U_2(q_2)}U_2(q_1)\overline{U_1(q_1)}U_1(z)-\overline{U_1(q_2)}U_1(q_1)\overline{U_2(q_1)}U_2(z),\end{aligned} \tag{3.22}$$

$$\begin{aligned}\bar{D}_2={}&\|U_1(q_2)\|^2\overline{U_2(q_1)}U_2(z)+\|U_2(q_2)\|^2\overline{U_1(q_1)}U_1(z)\\&-\overline{U_2(q_1)}U_2(q_2)\overline{U_1(q_2)}U_1(z)-\overline{U_1(q_1)}U_1(q_2)\overline{U_2(q_2)}U_2(z),\end{aligned} \tag{3.23}$$

$$\begin{aligned}\bar{D}_3={}&-\|U_1(q_1)\|^2\overline{U_2(q_2)}B_2-\|U_2(q_1)\|^2\overline{U_1(q_2)}B_1\\&+\overline{U_2(q_2)}U_2(q_1)\overline{U_1(q_1)}B_1+\overline{U_1(q_2)}U_1(q_1)\overline{U_2(q_1)}B_2,\end{aligned} \tag{3.24}$$

$$\begin{aligned}\bar{D}_4={}&-\|U_1(q_2)\|^2\overline{U_2(q_1)}B_2-\|U_2(q_2)\|^2\overline{U_1(q_1)}B_1\\&+\overline{U_2(q_1)}U_2(q_2)\overline{U_1(q_2)}B_1+\overline{U_1(q_1)}U_1(q_2)\overline{U_2(q_2)}B_2,\end{aligned} \tag{3.25}$$

$$\begin{aligned}\bar{D}_5={}&-\|U_1(q_1)\|^2\overline{U_2(q_2)}\bigl(B_2-U_2(z)\bigr)-\|U_2(q_1)\|^2\overline{U_1(q_2)}\bigl(B_1-U_1(z)\bigr)\\&+\overline{U_1(q_2)}U_1(q_1)\overline{U_2(q_1)}\bigl(B_2-U_2(z)\bigr)+\overline{U_2(q_2)}U_2(q_1)\overline{U_1(q_1)}\bigl(B_1-U_1(z)\bigr),\end{aligned} \tag{3.26}$$

$$\begin{aligned}\bar{D}_6={}&-\|U_1(q_2)\|^2\overline{U_2(q_1)}\bigl(B_2-U_2(z)\bigr)-\|U_2(q_2)\|^2\overline{U_1(q_1)}\bigl(B_1-U_1(z)\bigr)\\&+\overline{U_2(q_1)}U_2(q_2)\overline{U_1(q_2)}\bigl(B_1-U_1(z)\bigr)+\overline{U_1(q_1)}U_1(q_2)\overline{U_2(q_2)}\bigl(B_2-U_2(z)\bigr).\end{aligned} \tag{3.27}$$

Proof Suppose that the general solution of a second-order nonhomogeneous quaternion-value linear differential equation

$$q''+a_1(t)q'+a_2(t)q=f(t)$$

is

$$q(t)=q_1(t)c_1+q_2(t)c_2+z(t). \tag{3.28}$$

From $U_1(q)=U_2(q)=0$, we know that

$$\begin{cases}U_1(q_1)c_1+U_1(q_2)c_2+U_1(z)=0,\\U_2(q_1)c_1+U_2(q_2)c_2+U_2(z)=0.\end{cases}$$

In other words,

$$Q(q)\mathbf{C}=\boldsymbol{\beta}$$

with

$$Q(q)=\begin{pmatrix}U_1(q_1)&U_1(q_2)\\U_2(q_1)&U_2(q_2)\end{pmatrix}=\begin{pmatrix}\gamma_1&\gamma_2\end{pmatrix},$$

$$\mathbf{C}=\begin{pmatrix}c_1\\c_2\end{pmatrix},\qquad \boldsymbol{\beta}=\begin{pmatrix}-U_1(z)\\-U_2(z)\end{pmatrix}.$$

Therefore,

$$\begin{pmatrix}c_1\\c_2\end{pmatrix}=\frac{1}{\operatorname{ddet}Q(q)}\begin{pmatrix}\bar{D}_1\\\bar{D}_2\end{pmatrix},\tag{3.29}$$

where

$$\begin{aligned}
D_1 &= \det_p\begin{pmatrix}\gamma_1^+\\\beta^+\end{pmatrix}\begin{pmatrix}\gamma_2 & \gamma_1\end{pmatrix}\\
&= \det_p\begin{pmatrix}\overline{U_1(q_1)} & \overline{U_2(q_1)}\\ -\overline{U_1(z)} & -\overline{U_2(z)}\end{pmatrix}\begin{pmatrix}U_1(q_2) & U_1(q_1)\\ U_2(q_2) & U_2(q_1)\end{pmatrix}\\
&= \|U_1(q_1)\|^2\overline{U_2(z)}U_2(q_2)+\|U_2(q_1)\|^2\overline{U_1(z)}U_1(q_2)\\
&\quad-\overline{U_1(z)}U_1(q_1)\overline{U_2(q_1)}U_2(q_2)-\overline{U_2(z)}U_2(q_1)\overline{U_1(q_1)}U_1(q_2),\\
\bar{D}_1 &= \|U_1(q_1)\|^2\overline{U_2(q_2)}U_2(z)+\|U_2(q_1)\|^2\overline{U_1(q_2)}U_1(z)\\
&\quad-\overline{U_2(q_2)}U_2(q_1)\overline{U_1(q_1)}U_1(z)-\overline{U_1(q_2)}U_1(q_1)\overline{U_2(q_1)}U_2(z),\\
D_2 &= \det_p\begin{pmatrix}\gamma_2^+\\\beta^+\end{pmatrix}\begin{pmatrix}\gamma_1 & \gamma_2\end{pmatrix}\\
&= \det_p\begin{pmatrix}\overline{U_1(q_2)} & \overline{U_2(q_2)}\\ -\overline{U_1(z)} & -\overline{U_2(z)}\end{pmatrix}\begin{pmatrix}U_1(q_1) & U_1(q_2)\\ U_2(q_1) & U_2(q_2)\end{pmatrix}\\
&= \|U_1(q_2)\|^2\overline{U_2(z)}U_2(q_1)+\|U_2(q_2)\|^2\overline{U_1(z)}U_1(q_1)\\
&\quad-\overline{U_1(z)}U_1(q_2)\overline{U_2(q_2)}U_2(q_1)-\overline{U_2(z)}U_2(q_2)\overline{U_1(q_2)}U_1(q_1),\\
\bar{D}_2 &= \|U_1(q_2)\|^2\overline{U_2(q_1)}U_2(z)+\|U_2(q_2)\|^2\overline{U_1(q_1)}U_1(z)\\
&\quad-\overline{U_2(q_1)}U_2(q_2)\overline{U_1(q_2)}U_1(z)-\overline{U_1(q_1)}U_1(q_2)\overline{U_2(q_2)}U_2(z).
\end{aligned}$$

Hence,

$$\begin{aligned}
\varphi(t) &= q_1(t)c_1+q_2(t)c_2+z(t)\\
&= \frac{1}{\operatorname{ddet}Q(q)}\bigl[q_1(t)\bar{D}_1+q_2(t)\bar{D}_2+\operatorname{ddet}Q(q)z(t)\bigr].
\end{aligned}$$

According to the above discussion, the unique solution of the boundary value problem of semi-homogeneous differential equation (3.17) can be obtained.

The proofs of the remaining two conclusions are analogous to the proof above. Suppose that the general solution of a second-order homogeneous quaternion-value linear differential equation

$$q''+a_1(t)q'+a_2(t)q=0$$

is

$$q(t) = q_1(t)c_1 + q_2(t)c_2. \tag{3.30}$$

In this case, let

$$\begin{aligned}
D_3 &= \det_p \begin{pmatrix} \overline{U_1(q_1)} & \overline{U_2(q_1)} \\ \overline{B_1} & \overline{B_2} \end{pmatrix} \begin{pmatrix} U_1(q_2) & U_1(q_1) \\ U_2(q_2) & U_2(q_1) \end{pmatrix} \\
&= -\left\|U_1(q_1)\right\|^2 \overline{B_2} U_2(q_2) - \left\|U_2(q_1)\right\|^2 \overline{B_1} U_1(q_2) \\
&\quad + \overline{B_1} U_1(q_1) \overline{U_2(q_1)} U_2(q_2) + \overline{B_2} U_2(q_1) \overline{U_1}(q_1) U_1(q_2), \\
\bar{D}_3 &= -\left\|U_1(q_1)\right\|^2 \overline{U_2(q_2)} B_2 - \left\|U_2(q_1)\right\|^2 \overline{U_1(q_2)} B_1 \\
&\quad + \overline{U_2(q_2)} U_2(q_1) \overline{U_1(q_1)} B_1 + \overline{U_1(q_2)} U_1(q_1) \overline{U_2(q_1)} B_2, \\
D_4 &= \det_p \begin{pmatrix} \overline{U_1(q_2)} & \overline{U_2(q_2)} \\ \overline{B_1} & \overline{B_2} \end{pmatrix} \begin{pmatrix} U_1(q_1) & U_1(q_2) \\ U_2(q_1) & U_2(q_2) \end{pmatrix} \\
&= -\left\|U_1(q_2)\right\|^2 \overline{B_2} U_2(q_1) - \left\|U_2(q_2)\right\|^2 \overline{B_1} U_1(q_1) \\
&\quad + \overline{B_1} U_1(q_2) \overline{U_2(q_2)} U_2(q_1) + \overline{B_2} U_2(q_2) \overline{U_1(q_2)} U_1(q_1), \\
\bar{D}_4 &= -\left\|U_1(q_2)\right\|^2 \overline{U_2(q_1)} B_2 - \left\|U_2(q_2)\right\|^2 \overline{U_1(q_1)} B_1 \\
&\quad + \overline{U_2(q_1)} U_2(q_2) \overline{U_1(q_2)} B_1 + \overline{U_1(q_1)} U_1(q_2) \overline{U_2(q_2)} B_2.
\end{aligned}$$

Furthermore, suppose that

$$q(t) = q_1(t)c_1 + q_2(t)c_2 + z(t)$$

is the general solution of BVP (1.2). At the moment, let

$$\begin{aligned}
D_5 &= \det_p \begin{pmatrix} \overline{U_1(q_1)} & \overline{U_2(q_1)} \\ \overline{B_1 - U_1(z)} & \overline{B_2 - U_2(z)} \end{pmatrix} \begin{pmatrix} U_1(q_2) & U_1(q_1) \\ U_2(q_2) & U_2(q_1) \end{pmatrix} \\
&= -\left\|U_1(q_1)\right\|^2 \overline{B_2 - U_2(z)} U_2(q_2) - \left\|U_2(q_1)\right\|^2 \overline{B_1 - U_1(z)} U_1(q_2) \\
&\quad + \overline{B_1 - U_1(z)} U_1(q_1) \overline{U_2(q_1)} U_2(q_2) + \overline{B_2 - U_2(z)} U_2(q_1) \overline{U_1(q_1)} U_1(q_2), \\
\bar{D}_5 &= -\left\|U_1(q_1)\right\|^2 \overline{U_2(q_2)} \big(B_2 - U_2(z)\big) - \left\|U_2(q_1)\right\|^2 \overline{U_1(q_2)} \big(B_1 - U_1(z)\big) \\
&\quad + \overline{U_1(q_2)} U_1(q_1) \overline{U_2(q_1)} \big(B_2 - U_2(z)\big) + \overline{U_2(q_2)} U_2(q_1) \overline{U_1(q_1)} \big(B_1 - U_1(z)\big), \\
D_6 &= \det_p \begin{pmatrix} \overline{U_1(q_2)} & \overline{U_2(q_2)} \\ \overline{B_1 - U_1(z)} & \overline{B_2 - U_2(z)} \end{pmatrix} \begin{pmatrix} U_1(q_1) & U_1(q_2) \\ U_2(q_1) & U_2(q_2) \end{pmatrix} \\
&= -\left\|U_1(q_2)\right\|^2 \overline{B_2 - U_2(z)} U_2(q_1) - \left\|U_2(q_2)\right\|^2 \overline{B_1 - U_1(z)} U_1(q_1) \\
&\quad + \overline{B_1 - U_1(z)} U_1(q_2) \overline{U_2(q_2)} U_2(q_1) + \overline{B_2 - U_2(z)} U_2(q_2) \overline{U_1(q_2)} U_1(q_1), \\
\bar{D}_6 &= -\left\|U_1(q_2)\right\|^2 \overline{U_2(q_1)} \big(B_2 - U_2(z)\big) - \left\|U_2(q_2)\right\|^2 \overline{U_1(q_1)} \big(B_1 - U_1(z)\big) \\
&\quad + \overline{U_2(q_1)} U_2(q_2) \overline{U_1(q_2)} \big(B_1 - U_1(z)\big) + \overline{U_1(q_1)} U_1(q_2) \overline{U_2(q_2)} \big(B_2 - U_2(z)\big).
\end{aligned}$$

From this point of view, we completed the main proof process. □

Remark 3.2 If Theorem 3.1 studies the problem of boundary value over the domain of real numbers,equations (3.19), (3.20), (3.21) will be written in the form of a determinant expression. Owing the noncommutativity of the quaternion algebra, many properties of ordinary differential equations are not valid for quaternion-value differential equations, resulting in the complex form of the solutions of Theorem 3.1.

Remark 3.3 Theorem 3.1 only gives the method of solving when ddet $Q(q) \neq 0$. Obviously, (3.19)∼(3.21) do not hold when ddet $Q(q) = 0$.

On the basis of the above conclusions, we give some examples to solve the semi-homogeneous boundary value problem.

Example 3.1 Consider the following QDEs:

$$q'' + \mathrm{j}q' + (1 - \mathrm{k})q = 0, \quad t \in [0, 1]. \tag{3.31}$$

Setting $q_{11} = q$ and $q_{12} = q'$, the equation can be converted to

$$\begin{pmatrix} q_{11} \\ q_{12} \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \mathrm{k} - 1 & -\mathrm{j} \end{pmatrix} \begin{pmatrix} q_{11} \\ q_{12} \end{pmatrix}, \tag{3.32}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ \mathrm{k} - 1 & -\mathrm{j} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \mathrm{i} & -1 \end{pmatrix} \mathrm{j}.$$

Then

$$\phi(A) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \mathrm{i} & -1 \\ 0 & 0 & 0 & 1 \\ \mathrm{i} & 1 & -1 & 0 \end{pmatrix}$$

and

$$\left| \lambda E - \phi(A) \right| = \lambda^4 + 3\lambda^2 + 2 = 0, \tag{3.33}$$

hence, $\lambda_1 = \mathrm{i}$, $\lambda_2 = \sqrt{2}\mathrm{i}$, $\lambda_3 = -\mathrm{i}$, $\lambda_4 = -\sqrt{2}\mathrm{i}$.

According to Lemma 2.4, we can see that the eigenvector of $\lambda_1 = \mathrm{i}$ is $\varphi(\nu_1) = (0, 0, 1, \mathrm{i})^T$. Then we get

$$\nu_1 = \begin{pmatrix} -\mathrm{j} \\ \mathrm{k} \end{pmatrix}.$$

The eigenvector of $\lambda_2 = \sqrt{2}\mathrm{i}$ is $\varphi(\nu_2) = (1, \sqrt{2}\mathrm{i}, -(1 + \sqrt{2})\mathrm{i}, 2 + \sqrt{2})^T$ and

$$\nu_2 = \begin{pmatrix} 1 - (1 + \sqrt{2})\mathrm{k} \\ \sqrt{2}\mathrm{i} - (2 + \sqrt{2})\mathrm{j} \end{pmatrix}.$$

Let

$$\Phi(t) = \left(\nu_1 e^{\lambda_1 t}, \nu_2 e^{\lambda_2 t}\right) = \begin{pmatrix} -\mathrm{j}e^{\mathrm{i}t} & (1-(1+\sqrt{2})\mathrm{k})e^{\sqrt{2}\mathrm{i}t} \\ \mathrm{k}e^{\mathrm{i}t} & (\sqrt{2}\mathrm{i}-(2+\sqrt{2})\mathrm{j})e^{\sqrt{2}\mathrm{i}t} \end{pmatrix}, \tag{3.34}$$

then

$$\Phi^{+}(t) = \begin{pmatrix} \mathrm{j}e^{-\mathrm{i}t} & -\mathrm{k}e^{-\mathrm{i}t} \\ (1+(1+\sqrt{2})\mathrm{k})e^{-\sqrt{2}\mathrm{i}t} & (-\sqrt{2}\mathrm{i}+(2+\sqrt{2})\mathrm{j})e^{-\sqrt{2}\mathrm{i}t} \end{pmatrix},$$

and from (3.11) we have

$$W_{QDE}(t) = \operatorname{ddet}\Phi(t) = r\det\left(\Phi(t)\Phi^{+}(t)\right) = 4+2\sqrt{2} \neq 0. \tag{3.35}$$

Consequently, $\Phi(t)$ is a fundamental matrix of (3.32), and $-\mathrm{j}e^{\mathrm{i}t}$, $(1-(1+\sqrt{2})\mathrm{k})e^{\sqrt{2}\mathrm{i}t}$ are two linearly independent solutions to (3.31).

Example 3.2 Consider the following QDEs:

$$\begin{cases} q'' + \mathrm{j}q' + (1-\mathrm{k})q = \mathrm{i}t, \quad t \in [0,1], \\ U_1(q) = U_2(q) = 0. \end{cases} \tag{3.36}$$

From Proposition 3.6 and its proof, we can discover that

$$\bar{\omega}_{21}(t) = (4+2\sqrt{2})(1+\mathrm{k})e^{-\mathrm{i}t},$$
$$\bar{\omega}_{22}(t) = \left[-(\sqrt{2}+1)\mathrm{i}+\mathrm{j}\right]e^{-\sqrt{2}\mathrm{i}t}.$$

In addition,

$$\begin{aligned} z(t) &= \frac{1}{4+2\sqrt{2}}\int_0^t \left[-(4+2\sqrt{2})\mathrm{j}e^{\mathrm{i}(t-s)} + (4+2\sqrt{2})\mathrm{j}e^{\sqrt{2}\mathrm{i}(t-s)}\right]\mathrm{i}s\,ds \\ &= \int_0^t \mathrm{k}se^{\mathrm{i}(t-s)} - \mathrm{k}se^{\sqrt{2}\mathrm{i}(t-s)}\,ds \end{aligned} \tag{3.37}$$

is a solution to (3.36). At this time,

$$U_1(q_1) = q_1(0) = -\mathrm{j}, \qquad U_1(q_2) = q_2(0) = 1-(1+\sqrt{2})\mathrm{k},$$
$$U_2(q_1) = q_1(1) = -\mathrm{j}e^{\mathrm{i}}, \qquad U_2(q_2) = q_2(1) = \left(1-(1+\sqrt{2})\mathrm{k}\right)e^{\sqrt{2}\mathrm{i}},$$
$$U_1(z) = z(0) = 0, \qquad U_2(z) = z(1) = \frac{2-\sqrt{2}}{2}\mathrm{j} + \frac{1}{2}\mathrm{k} - \mathrm{k}e^{\mathrm{i}} + \frac{1}{2}\mathrm{k}e^{\sqrt{2}\mathrm{i}},$$

then

$$Q(q) = \begin{pmatrix} -\mathrm{j} & 1-(1+\sqrt{2})\mathrm{k} \\ -\mathrm{j}e^{\mathrm{i}} & (1-(1+\sqrt{2})\mathrm{k})e^{\sqrt{2}\mathrm{i}} \end{pmatrix}$$

and

$$\operatorname{ddet} Q(q) = \det_p\big(Q^+(q)Q(q)\big) = (4+2\sqrt{2})\big(2-\mathrm{e}^{(\sqrt{2}-1)\mathrm{i}}-\mathrm{e}^{(1-\sqrt{2})\mathrm{i}}\big) \neq 0. \tag{3.38}$$

As pointed out in Theorem 3.1, we soon show

$$\begin{aligned}
\bar{D}_1 = {} & \left(-\frac{3(1+\sqrt{2})}{2}+\frac{3\mathrm{k}}{2}\right)+\left(\frac{1+\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}-\frac{2-\sqrt{2}}{2}\mathrm{j}-\frac{1}{2}\mathrm{k}\right)\mathrm{e}^{-\mathrm{i}} \\
& + \left(\frac{1+\sqrt{2}}{2}-\frac{1}{2}\mathrm{k}\right)\mathrm{e}^{(\sqrt{2}-1)\mathrm{i}} \\
& + \left(\frac{1+\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\mathrm{i}+\frac{2-\sqrt{2}}{2}\mathrm{j}+\frac{1}{2}\mathrm{k}\right)\mathrm{e}^{-\sqrt{2}\mathrm{i}}+(1+\sqrt{2}-\mathrm{k})\mathrm{e}^{(1-\sqrt{2})\mathrm{i}}, \\
\bar{D}_2 = {} & -(6+3\sqrt{2})\mathrm{i}+\big(-2+(2+\sqrt{2})\mathrm{i}\big)\mathrm{e}^{-\mathrm{i}}+(2+\sqrt{2})\mathrm{i}\mathrm{e}^{(\sqrt{2}-1)\mathrm{i}} \\
& + \big(2-(2+\sqrt{2})\mathrm{i}\big)\mathrm{e}^{-\sqrt{2}\mathrm{i}}+(4+2\sqrt{2})\mathrm{i}\mathrm{e}^{(1-\sqrt{2})\mathrm{i}}.
\end{aligned}$$

Finally, the solution of BVP (3.36) can be obtained by substituting the above two equations and (3.37) into (3.19).

3.2.2 Resonant problem

In the case of resonance, the difficulty with BVP (1.2) is that given the general solution (3.28) to this equation, substituting into boundary conditions and having

$$\begin{cases} U_1(q_1)c_1 + U_1(q_2)c_2 + U_1(z) = B_1, \\ U_2(q_1)c_1 + U_2(q_2)c_2 + U_2(z) = B_2, \end{cases} \tag{3.39}$$

which is a question to find the solutions of c_1, c_2.

Based on what we already know about linear algebra and Lemmas 2.5 and 2.6, the condition for the above equation to have solution for c_1, c_2 is

$$\operatorname{rank}\begin{pmatrix} U_1(q_1) & U_1(q_2) \\ U_2(q_1) & U_2(q_2) \end{pmatrix} = \operatorname{rank}\begin{pmatrix} U_1(q_1) & U_1(q_2) & B_1-U_1(z) \\ U_2(q_1) & U_2(q_2) & B_2-U_2(z) \end{pmatrix} := \operatorname{rank} Q^*(q). \tag{3.40}$$

Case 1: If $\operatorname{rank} Q(q) = 0$, then the two conditions for c_1, c_2 to have solutions are

$$U_1(z) = B_1, \qquad U_2(z) = B_2. \tag{3.41}$$

Case 2: If $\operatorname{rank} Q(q) = 1$, assuming that $U_1(q_1) \neq 0$, then c_1 and c_2 have solutions, which requires that

$$\begin{vmatrix} U_1(q_1) \; B_1-U_1(z) \\ U_2(q_1) \; B_2-U_2(z) \end{vmatrix} = 0 \tag{3.42}$$

under the circumstance that above all elements commute with each other. So now we introduce the following theorem.

Theorem 3.2 *Supposing that* $\operatorname{ddet} Q(q) = 0$, *if* $\operatorname{rank} Q(q) = \operatorname{rank} Q^*(q)$, *then BVP* (1.2) *has a solution*.

If Case 1 *is true*, *for* $\forall c_1, c_2 \in \mathbb{H}$,

$$q(t) = q_1(t)c_1 + q_2(t)c_2 + z(t) \tag{3.43}$$

is a solution to BVP (1.2); *when one of the conditions in* (3.41) *is not true*, *BVP* (1.2) *has no solution*.

If Case 2 *holds*, *for* $\forall c \in \mathbb{H}$,

$$q(t) = \left[q_2(t) - q_1(t)\frac{\overline{U_1(q_1)}U_1(q_2)}{\|U_1(q_1)\|^2}\right]c + q_1(t)\frac{\overline{U_1(q_1)}(B_1 - U_1(z))}{\|U_1(q_1)\|^2} + z(t) \tag{3.44}$$

is a solution to BVP (1.2); *when one of the conditions in* Case 2 *is not satisfied*, *BVP* (1.2) *has no solution*.

Proof Based on the above analysis, it is obvious that BVP (1.2) has a solution when *Case 1* is true. Now we turn our attention to formulating the main result of *Case 2*.

From the first formula in (3.39), it easily follows that

$$c_1 = \left[U_1(q_1)\right]^{-1}\left[B_1 - U_1(z) - U_1(q_2)c_2\right]. \tag{3.45}$$

Let $c_2 = c$, then (3.43) can be denoted by

$$\begin{aligned} q(t) &= q_1(t)\big(U_1(q_1)\big)^{-1}\left[B_1 - U_1(z) - U_1(q_2)c\right] + q_2(t)c + z(t) \\ &= \left[q_2(t) - q_1(t)\big(U_1(q_1)\big)^{-1}U_1(q_2)\right]c + q_1(t)\left[U_1(q_1)\right]^{-1}\big(B_1 - U_1(z)\big) + z(t) \\ &= \left[q_2(t) - q_1(t)\frac{\overline{U_1(q_1)}U_1(q_2)}{\|U_1(q_1)\|^2}\right]c + q_1(t)\frac{\overline{U_1(q_1)}(B_1 - U_1(z))}{\|U_1(q_1)\|^2} + z(t). \end{aligned} \tag{3.46}$$

This is the end of the proof. □

4 Green's function of Sturm–Liouville type boundary value problem

Green's function plays an important role in the study of boundary value problems. With the help of Green's function, the unique solutions for the boundary value problem

$$\begin{cases} q'' + a_1(t)q' + a_2(t)q = f(t), & t \in J, \\ U_1(q) = U_2(q) = 0, \end{cases}$$

and

$$\begin{cases} q'' + a_1(t)q' + a_2(t)q = f(t), & t \in J, \\ U_1(q) = B_1, \qquad U_2(q) = B_2 \end{cases}$$

can be expressed in the form of an integral with respect to $f(t)$ at $\operatorname{ddet} Q(q) \neq 0$, thus providing a convenient way to study the connection between the solution $q(t)$ and $f(t)$ and laying the foundation for the study of nonlinear boundary problems. We will discuss each of them in two parts.

4.1 Semi-homogeneous boundary problem

In this section, we are going to discuss the semi-homogeneous BVP (3.17)

$$\begin{cases} q'' + a_1(t)q' + a_2(t)q = f(t), \\ U_1(q) = U_2(q) = 0, \end{cases}$$

and Green's function.

When $\operatorname{ddet} Q(q) \neq 0$, according to Theorem 3.1, the solution $\varphi(t)$ can be written as

$$\begin{aligned} \varphi(t) &= \frac{1}{\operatorname{ddet} Q(q)} q_1(t)\big[\|U_2(q_1)\|^2\overline{U_1(q_2)} - \overline{U_2(q_2)}U_2(q_1)\overline{U_1(q_1)}\big]U_1(z) \\ &\quad + \frac{1}{\operatorname{ddet} Q(q)} q_1(t)\big[\|U_1(q_1)\|^2\overline{U_2(q_2)} - \overline{U_1(q_2)}U_1(q_1)\overline{U_2(q_1)}\big]U_2(z) \\ &\quad + \frac{1}{\operatorname{ddet} Q(q)} q_2(t)\big[\|U_2(q_2)\|^2\overline{U_1(q_1)} - \overline{U_2(q_1)}U_2(q_2)\overline{U_1(q_2)}\big]U_1(z) \\ &\quad + \frac{1}{\operatorname{ddet} Q(q)} q_2(t)\big[\|U_1(q_2)\|^2\overline{U_2(q_1)} - \overline{U_1(q_1)}U_1(q_2)\overline{U_2(q_2)}\big]U_2(z) \\ &\quad + \frac{1}{\operatorname{ddet} Q(q)} \operatorname{ddet} Q(q)\cdot z(t). \end{aligned} \tag{4.1}$$

Due to

$$U_i(z) = \sum_{j=0}^{1} z^{(j)}(t_i),$$

namely,

$$U_1(z) = \sum_{j=0}^{1} z^{(j)}(t_1) = \frac{1}{\operatorname{ddet}\Phi(t)} \int_a^t \sum_{j=0}^{1} \big[q_1^{(j)}(t_1)\bar{\omega}_{21}(s) + q_2^{(j)}(t_1)\bar{\omega}_{22}(s)\big] f(s)\,\mathrm{d}s,$$

$$U_2(z) = \sum_{j=0}^{1} z^{(j)}(t_2) = \frac{1}{\operatorname{ddet}\Phi(t)} \int_a^t \sum_{j=0}^{1} \big[q_1^{(j)}(t_2)\bar{\omega}_{21}(s) + q_2^{(j)}(t_2)\bar{\omega}_{22}(s)\big] f(s)\,\mathrm{d}s.$$

Let

$$\begin{aligned} h_1(s) &= \sum_{j=0}^{1}\big[\|U_2(q_1)\|^2\overline{U_1(q_2)} - \overline{U_2(q_2)}U_2(q_1)\overline{U_1(q_1)}\big]\big[q_1^{(j)}(t_1)\bar{\omega}_{21}(s) + q_2^{(j)}(t_1)\bar{\omega}_{22}(s)\big] \\ &\quad + \sum_{j=0}^{1}\big[\|U_1(q_1)\|^2\overline{U_2(q_2)} - \overline{U_1(q_2)}U_1(q_1)\overline{U_2(q_1)}\big]\big[q_1^{(j)}(t_2)\bar{\omega}_{21}(s) + q_2^{(j)}(t_2)\bar{\omega}_{22}(s)\big], \\ h_2(s) &= \sum_{j=0}^{1}\big[\|U_2(q_2)\|^2\overline{U_1(q_1)} - \overline{U_2(q_1)}U_2(q_2)\overline{U_1(q_2)}\big]\big[q_1^{(j)}(t_1)\bar{\omega}_{21}(s) + q_2^{(j)}(t_1)\bar{\omega}_{22}(s)\big] \\ &\quad + \sum_{j=0}^{1}\big[\|U_1(q_2)\|^2\overline{U_2(q_1)} - \overline{U_1(q_1)}U_1(q_2)\overline{U_2(q_2)}\big]\big[q_1^{(j)}(t_2)\bar{\omega}_{21}(s) + q_2^{(j)}(t_2)\bar{\omega}_{22}(s)\big], \\ g(t,s) &= q_1(t)\bar{\omega}_{21}(s) + q_2(t)\bar{\omega}_{22}(s). \end{aligned}$$

In consequence, we get

$$\varphi(t) = \frac{1}{\operatorname{ddet} Q(q)} \frac{1}{\operatorname{ddet} \Phi(t)} \left\{ \int_a^{t_i} \bigl[q_1(t)h_1(s) + q_2(t)h_2(s)\bigr] f(s)\,\mathrm{d}s \right.$$
$$\left. + \int_a^t \operatorname{ddet} Q(q) \cdot g(t,s) f(s)\,\mathrm{d}s \right\}.$$

When $s \in (a,b)$, Green's function $G(t,s)$ is defined by

$$\begin{cases} \frac{1}{\operatorname{ddet} Q(q)} \frac{1}{\operatorname{ddet} \Phi(t)} (q_1(t)h_1(s) + q_2(t)h_2(s)), & a \le t < s < b, \\ \frac{1}{\operatorname{ddet} Q(q)} \frac{1}{\operatorname{ddet} \Phi(t)} (q_1(t)h_1(s) + q_2(t)h_2(s) + \operatorname{ddet} Q(q) \cdot g(t,s)), & a < s < t \le b. \end{cases} \tag{4.2}$$

Then the unique solution of BVP (3.17) is expressed as an integral form

$$\varphi(t) = \int_a^b G(t,s) f(s)\,\mathrm{d}s. \tag{4.3}$$

Obviously, $G(t,s)$ is only concerned with homogeneous linear boundary value problem

$$\begin{cases} q'' + a_1(t)q' + a_2(t)q = 0, \quad t \in J, \\ U_1(q) = U_2(q) = 0 \end{cases} \tag{4.4}$$

not with the nonhomogeneous term $f(t)$, $G(t,s)$ is the Green's function for homogeneous boundary value problem (4.4). The role of Green's function G(t,s) is to express the unique solution to the semi-homogeneous BVP (3.17) as integral form (4.3) in the case of $\operatorname{ddet} Q(q) \neq 0$.

Proposition 4.1 *Evidently, Green's function has the following properties* (P):

($P1$) $G(t,s)$ *is continuous on* $[a,b] \times (t_{i-1}, t_i)$, $\frac{\partial G}{\partial t}$, $\frac{\partial^2 G}{\partial t^2}$ *exist and continue on the* $[a,b] \times (t_{i-1}, t) \cup (t, t_i)$;

($P2$)

$$\left.\frac{\partial G(t,s)}{\partial t}\right|_{t=s^+} - \left.\frac{\partial G(t,s)}{\partial t}\right|_{t=s^-} = \frac{1}{\operatorname{ddet} \Phi(t)} g'(t,s);$$

($P3$)

$$\frac{\partial^2 G(t,s)}{\partial t^2} + a_1(t)\frac{\partial G(t,s)}{\partial t} + a_2(t)G(t,s) = 0 \quad \textit{if } s \neq t;$$

($P4$) *Any given* $t_{i-1} < s < t_i$,

$$U_1\bigl(G(.,s)\bigr) = U_2\bigl(G(.,s)\bigr) = 0.$$

The function $G(t,s)$ *on* $[a,b] \times [a,b]$ *satisfying property* (P) *can also be defined as Green's function of homogeneous BVP* (4.4).

Proof Taking the derivative of (4.2), when $a \leq t < s < b$, we can observe that

$$\begin{aligned} \frac{\partial G}{\partial t} &= \frac{1}{\operatorname{ddet} Q(q)} \frac{1}{\operatorname{ddet} \Phi(t)} \bigl(q_1'(t)h_1(s) + q_2'(t)h_2(s)\bigr), \\ \frac{\partial^2 G}{\partial t^2} &= \frac{1}{\operatorname{ddet} Q(q)} \frac{1}{\operatorname{ddet} \Phi(t)} \bigl(q_1''(t)h_1(s) + q_2''(t)h_2(s)\bigr), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &\frac{\partial^2 G}{\partial t^2} + a_1(t)\frac{\partial G}{\partial t} + a_2(t)G(t,s) \\ &\quad= \frac{1}{\operatorname{ddet} Q(q)} \frac{1}{\operatorname{ddet} \Phi(t)} \bigl(q_1''(t) + a_1(t)q_1'(t) + a_2(t)q_1(t)\bigr)h_1(s) \\ &\qquad+ \frac{1}{\operatorname{ddet} Q(q)} \frac{1}{\operatorname{ddet} \Phi(t)} \bigl(q_2''(t) + a_1(t)q_2'(t) + a_2(t)q_2(t)\bigr)h_2(s) \\ &\quad= 0. \end{aligned} \tag{4.6}$$

When $a < s < t \leq b$,

$$\begin{aligned} \frac{\partial G}{\partial t} &= \frac{1}{\operatorname{ddet} Q(q)} \frac{1}{\operatorname{ddet} \Phi(t)} \bigl(q_1'(t)h_1(s) + q_2'(t)h_2(s) + \operatorname{ddet} Q(q)\cdot g'(t,s)\bigr), \\ \frac{\partial^2 G}{\partial t^2} &= \frac{1}{\operatorname{ddet} Q(q)} \frac{1}{\operatorname{ddet} \Phi(t)} \bigl(q_1''(t)h_1(s) + q_2''(t)h_2(s) + \operatorname{ddet} Q(q)\cdot g''(t,s)\bigr), \end{aligned} \tag{4.7}$$

then

$$\begin{aligned} &\frac{\partial^2 G}{\partial t^2} + a_1(t)\frac{\partial G}{\partial t} + a_2(t)G(t,s) \\ &\quad= \frac{1}{\operatorname{ddet} Q(q)} \frac{1}{\operatorname{ddet} \Phi(t)} \bigl(q_1''(t) + a_1(t)q_1'(t) + a_2(t)q_1(t)\bigr)h_1(s) \\ &\qquad+ \frac{1}{\operatorname{ddet} Q(q)} \frac{1}{\operatorname{ddet} \Phi(t)} \bigl(q_2''(t) + a_1(t)q_2'(t) + a_2(t)q_2(t)\bigr)h_2(s) \\ &\qquad+ \frac{1}{\operatorname{ddet} \Phi(t)} \bigl(g''(t,s) + a_1(t)g'(t,s) + a_2(t)g(t,s)\bigr) \\ &\quad= \frac{1}{\operatorname{ddet} \Phi(t)} \bigl\{\bigl[q_1''(t) + a_1(t)q_1'(t) + a_2(t)q_1(t)\bigr]\bar{\omega}_{21}(s) \\ &\qquad+ \bigl[q_2''(t) + a_1(t)q_2'(t) + a_2(t)q_2(t)\bigr]\bar{\omega}_{21}(s)\bigr\} \\ &\quad= 0. \end{aligned} \tag{4.8}$$

In brief, (P1) and (P3) are true.

In the next moment, we give the proofs of (P2) and (P4). From (4.5) and (4.7), as we soon show,

$$\left.\frac{\partial G(t,s)}{\partial t}\right|_{t=s^+} - \left.\frac{\partial G(t,s)}{\partial t}\right|_{t=s^-} = \frac{1}{\operatorname{ddet} \Phi(t)} g'(t,s). \tag{4.9}$$

Owing to U_1, U_2 being linear with respect to q and $U_i(q) = 0$, when $a \le t < s < b$,

$$U_i\big(G(t,s)\big) = \frac{1}{\operatorname{ddet} Q(q)} \frac{1}{\operatorname{ddet} \Phi(t)} \big(U_i(q_1)h_1(s) + U_i(q_2)h_2(s)\big) = 0. \tag{4.10}$$

When $a < s < t \le b$,

$$\begin{aligned} U_i\big(G(t,s)\big) &= \frac{1}{\operatorname{ddet} Q(q)} \frac{1}{\operatorname{ddet} \Phi(t)} \big[U_i(q_1)h_1(s) + U_i(q_2)h_2(s)) + \operatorname{ddet} Q(q) \cdot U_i\big(g(t,s)\big)\big] \\ &= \frac{1}{\operatorname{ddet} \Phi(t)} \big(U_i(q_1)\bar{\omega}_{21}(s) + U_i(q_2)\bar{\omega}_{22}(s)\big) \\ &= 0. \end{aligned}$$

Above all, it is clear from the analysis of $\varphi(t)$ that there is a function G(t,s) satisfying (P) for $\operatorname{ddet} Q(q) \neq 0$. □

Theorem 4.1 *Suppose that BVP* (4.4) *satisfies* $\operatorname{ddet} Q(q) \neq 0$, $G(t,s)$ *satisfies properties* (*P*), *then the unique solution of BVP* (3.17) *is written as an integral form*

$$\varphi(t) = \int_a^b G(t,s)f(s)\,\mathrm{d}s. \tag{4.11}$$

Proof The conclusion that BVP (3.17) has a unique solution is guaranteed by Theorem 3.1. Just prove that (4.11) satisfies both the differential equation in BVP (3.17) and the boundary conditions.

Since $G(t,s)$ is discontinuous at t, integral (4.11) is divided into two integrals from a to t and from t to b, and then differentiated separately to obtain

$$\begin{aligned} \varphi'(t) &= G(t,s)f(s)|_{t=s^+} + \int_a^t \frac{\partial G(t,s)}{\partial t} f(s)\,\mathrm{d}s - G(t,s)f(s)|_{t=s^-} + \int_t^b \frac{\partial G(t,s)}{\partial t} f(s)\,\mathrm{d}s \\ &= \int_a^b \frac{\partial G(t,s)}{\partial t} f(s)\,\mathrm{d}s, \\ \varphi''(t) &= \frac{\partial G(t,s)}{\partial t} f(s)\bigg|_{t=s^+} + \int_a^t \frac{\partial^2 G(t,s)}{\partial t^2} f(s)\,\mathrm{d}s - \frac{\partial G(t,s)}{\partial t} f(s)|_{t=s^-} + \int_t^b \frac{\partial^2 G(t,s)}{\partial t^2} f(s)\,\mathrm{d}s \\ &= \int_a^b \frac{\partial^2 G(t,s)}{\partial t^2} f(s)\,\mathrm{d}s + \frac{1}{\operatorname{ddet} \Phi(t)} g'(t,s)f(t). \end{aligned}$$

Let $L(D) = D^2 + a_1(t)D + a_2(t)$, since $L(D)$ is linear, then

$$\begin{aligned} L(D)\varphi(t) &= \varphi''(t) + a_1(t)\varphi'(t) + a_2(t)\varphi(t) \\ &= \int_a^b \left[\frac{\partial^2 G(t,s)}{\partial t^2} + a_1(t)\frac{\partial G(t,s)}{\partial t} + a_2(t)G(t,s)\right] f(t)\,\mathrm{d}s + \frac{1}{\operatorname{ddet} \Phi(t)} g'(t,s)f(t) \\ &= \frac{1}{\operatorname{ddet} \Phi(t)} \big(q_1'(t)\bar{\omega}_{21}(t) + q_2'(t)\bar{\omega}_{22}(t)\big)f(t). \end{aligned}$$

From (3.15) and (3.16), we know that

$$q_1'(t)\bar{\omega}_{21}(t) + q_2'(t)\bar{\omega}_{22}(t)$$

$$
\begin{aligned}
&= q_1'(t)\bar{q}_1'(t)\|q_2(t)\|^2 - q_1'(t)\bar{q}_1(t)q_2(t)\bar{q}_2'(t) + q_2'(t)\bar{q}_2'(t)\|q_1(t)\|^2 \\
&\quad - q_2'(t)\bar{q}_2(t)q_1(t)\bar{q}_1'(t) \\
&= \|q_1'(t)\|^2\|q_2(t)\|^2 + \|q_2'(t)\|^2\|q_1(t)\|^2 - 2\mathcal{R}\{q_1'(t)\bar{q}_1(t)q_2(t)\bar{q}_2'(t)\}.
\end{aligned} \tag{4.12}
$$

Comparing the above formula with (3.11), we can find

$$
\mathcal{R}\{q_1'(t)\bar{q}_1(t)q_2(t)\bar{q}_2'(t)\} = \mathcal{R}\{q_1(t)\bar{q}_1'(t)q_2'(t)\bar{q}_2(t)\},
$$

therefore, $\frac{1}{\operatorname{ddet}\Phi(t)} = q_1'(t)\bar{\omega}_{21}(t) + q_2'(t)\bar{\omega}_{22}(t)$. To be more precise,

$$
L(D)\varphi(t) = \frac{1}{\operatorname{ddet}\Phi(t)}\big(q_1'(t)\bar{\omega}_{21}(t) + q_2'(t)\bar{\omega}_{22}(t)\big)f(t) = f(t). \tag{4.13}
$$

Meanwhile,

$$
U_i(\varphi) = \int_a^b U_i\big(G(t,s)\big)f(s)\,\mathrm{d}s = 0. \tag{4.14}
$$

Based on the above discussion, the conclusion is valid. □

Then some examples are presented to verify the validity of the above theory.

Example 4.1 Consider Green's function for the following BVP:

$$
\begin{cases}
q'' + \mathrm{j}q' + (1-\mathrm{k})q = \mathrm{i}t, & t \in [0,1], \\
U_1(q) = U_2(q) = 0.
\end{cases}
$$

Based on Example 3.1 and Example 3.2, after a series of complex calculations, we have

$$
\begin{aligned}
&\|U_2(q_1)\|^2\overline{U_1(q_2)} - \overline{U_2(q_2)}U_2(q_1)\overline{U_1(q_1)} = \big(1 + (1+\sqrt{2})\mathrm{k}\big)\big(1 - \mathrm{e}^{(1-\sqrt{2})\mathrm{i}}\big), \\
&\|U_1(q_1)\|^2\overline{U_2(q_2)} - \overline{U_1(q_2)}U_1(q_1)\overline{U_2(q_1)} = \big(1 + (1+\sqrt{2})\mathrm{k}\big)\big(\mathrm{e}^{-\sqrt{2}\mathrm{i}} - \mathrm{e}^{-\mathrm{i}}\big), \\
&\|U_2(q_2)\|^2\overline{U_1(q_1)} - \overline{U_2(q_1)}U_2(q_2)\overline{U_1(q_2)} = (4+2\sqrt{2})\mathrm{j}\big(1 - \mathrm{e}^{(\sqrt{2}-1)\mathrm{i}}\big), \\
&\|U_1(q_2)\|^2\overline{U_2(q_1)} - \overline{U_1(q_1)}U_1(q_2)\overline{U_2(q_2)} = (4+2\sqrt{2})\mathrm{j}\big(\mathrm{e}^{-\mathrm{i}} - \mathrm{e}^{-\sqrt{2}\mathrm{i}}\big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{j=0}^{1}\big(q_1^{(j)}(t_1)\bar{\omega}_{21}(s) + q_2^{(j)}(t_1)\bar{\omega}_{22}(s)\big) \\
&\quad = \big(q_1(0) + q_1'(0)\big)\bar{\omega}_{21}(s) + \big(q_2(0) + q_2'(0)\big)\bar{\omega}_{22}(s) \\
&\quad = -(4+2\sqrt{2})(\mathrm{j}-\mathrm{k})\mathrm{e}^{-\mathrm{i}s} + (4+2\sqrt{2})(1+\mathrm{j}-\mathrm{k})\mathrm{e}^{-\sqrt{2}\mathrm{i}s},
\end{aligned}
$$

$$\begin{aligned}
&\sum_{j=0}^{1}\bigl(q_1^{(j)}(t_2)\bar{\omega}_{21}(s)+q_2^{(j)}(t_2)\bar{\omega}_{22}(s)\bigr)\\
&\quad=\bigl(q_1(1)+q_1'(1)\bigr)\bar{\omega}_{21}(s)+\bigl(q_2(1)+q_2'(1)\bigr)\bar{\omega}_{22}(s)\\
&\quad=-(4+2\sqrt{2})(\mathrm{j}-\mathrm{k})\mathrm{e}^{\mathrm{i}-\mathrm{i}s}+(4+2\sqrt{2})(1+\mathrm{j}-\mathrm{k})\mathrm{e}^{\sqrt{2}\mathrm{i}-\sqrt{2}\mathrm{i}s},
\end{aligned}$$

then

$$h_1(s)=(4+2\sqrt{2})(2+\sqrt{2}-\mathrm{i}+\mathrm{j}+\sqrt{2}\mathrm{k})\bigl(2-\mathrm{e}^{(1-\sqrt{2})\mathrm{i}}-\mathrm{e}^{(\sqrt{2}-1)\mathrm{i}}\bigr)\mathrm{e}^{-\sqrt{2}\mathrm{i}s}, \tag{4.15}$$

$$h_2(s)=(24+16\sqrt{2})(1+\mathrm{i})\bigl(2-\mathrm{e}^{(1-\sqrt{2})\mathrm{i}}-\mathrm{e}^{(\sqrt{2}-1)\mathrm{i}}\bigr)\mathrm{e}^{-\mathrm{i}s}, \tag{4.16}$$

$$g(t,s)=(4+2\sqrt{2})\mathrm{j}\bigl(\mathrm{e}^{\sqrt{2}\mathrm{i}(t-s)}-\mathrm{e}^{\mathrm{i}(t-s)}\bigr). \tag{4.17}$$

By taking advantage of these results and (3.38), (3.43), Green's function can be obtained in the following form.

When $0\le t<s<1$,

$$\begin{aligned}
G(t,s)=\frac{1}{4+2\sqrt{2}}&\bigl(1-\sqrt{2}\mathrm{i}-(2+\sqrt{2})\mathrm{j}-\mathrm{k}\bigr)\mathrm{e}^{\mathrm{i}(t-\sqrt{2}s)}\\
&+\bigl(1+\mathrm{i}-(1+\sqrt{2})(\mathrm{j}+\mathrm{k})\bigr)\mathrm{e}^{\mathrm{i}(\sqrt{2}t-s)}.
\end{aligned}$$

When $0<t<s\le 1$,

$$\begin{aligned}
G(t,s)=\frac{1}{4+2\sqrt{2}}&\bigl(1-\sqrt{2}\mathrm{i}-(2+\sqrt{2})\mathrm{j}-\mathrm{k}\bigr)\mathrm{e}^{\mathrm{i}(t-\sqrt{2}s)}\\
&+\bigl(1+\mathrm{i}-(1+\sqrt{2})(\mathrm{j}+\mathrm{k})\bigr)\mathrm{e}^{\mathrm{i}(\sqrt{2}t-s)}+\mathrm{j}\bigl(\mathrm{e}^{\sqrt{2}\mathrm{i}(t-s)}-\mathrm{e}^{\mathrm{i}(t-s)}\bigr).
\end{aligned}$$

Example 4.2 Consider Green's function for the following BVP:

$$\begin{cases} q''+\mathrm{j}q'=f(t,q), & t\in[0,1],\\ U_1(q)=U_2(q)=0. \end{cases} \tag{4.18}$$

By conversion, the above equation is transformed into

$$\begin{pmatrix} q_{11}\\ q_{12}\end{pmatrix}'=\begin{pmatrix} 0 & 1\\ 0 & -\mathrm{j}\end{pmatrix}\begin{pmatrix} q_{11}\\ q_{12}\end{pmatrix}+\begin{pmatrix} 0\\ f(t,q)\end{pmatrix}, \tag{4.19}$$

where

$$A=\begin{pmatrix} 0 & 1\\ 0 & -\mathrm{j}\end{pmatrix}=\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}+\begin{pmatrix} 0 & 0\\ 0 & -1\end{pmatrix}\mathrm{j}.$$

Then

$$\phi(A)=\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\end{pmatrix},$$

and

$$\left|\lambda E - \phi(A)\right| = \lambda^2\left(\lambda^2 + 1\right) = 0, \tag{4.20}$$

thus, $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \mathrm{i}$, $\lambda_4 = -\mathrm{i}$.

When $\lambda_1 = 0$, we can arrive at $\varphi(\nu_1) = (1,0,0,0)^T$ and $\nu_1 = (1,0)^T$. When $\lambda_3 = \mathrm{i}$, we can arrive at $\varphi(\nu_2) = (1,\mathrm{i},-\mathrm{i},1)^T$ and $\nu_2 = (1-\mathrm{k},\mathrm{i}-\mathrm{j})^T$. Let

$$\Phi(t) = \left(\nu_1 \mathrm{e}^{\lambda_1 t}, \nu_2 \mathrm{e}^{\lambda_2 t}\right) = \begin{pmatrix} 1 & (1-\mathrm{k})\mathrm{e}^{\mathrm{i}t} \\ 0 & (\mathrm{i}-\mathrm{j})\mathrm{e}^{\mathrm{i}t} \end{pmatrix}, \tag{4.21}$$

then

$$\Phi^+(t) = \begin{pmatrix} 1 & 0 \\ (1+\mathrm{k})\mathrm{e}^{-\mathrm{i}t} & (-\mathrm{i}+\mathrm{j})\mathrm{e}^{-\mathrm{i}t} \end{pmatrix},$$

and from (3.11), we get

$$W_{QDE}(t) = \operatorname{ddet}\Phi(t) = r\det\left(\Phi(t)\Phi^+(t)\right) = 2 \neq 0. \tag{4.22}$$

As a consequence, $\Phi(t)$ is a fundamental matrix of homogeneous equation, and 1, $(1-\mathrm{k})\mathrm{e}^{\mathrm{i}t}$ are the two linearly independent solutions to the homogeneous equation of (4.19).

After the same calculation as Example 3.2 and Example 4.1, we get the following results:

$$\bar{\omega}_{21}(t) = -2\mathrm{j}, \qquad \bar{\omega}_{22}(t) = (-\mathrm{i}+\mathrm{j})\mathrm{e}^{-\mathrm{i}t},$$

and the solution of (4.18) is

$$z(t) = \frac{1}{2}\int_0^t 2\mathrm{j}\left(\mathrm{e}^{\mathrm{i}(t-s)} - 1\right) f(s,q)\,\mathrm{d}s. \tag{4.23}$$

Moreover,

$$\begin{aligned} U_1(q_1) &= q_1(0) = 1, & U_1(q_2) &= q_2(0) = 1-\mathrm{k}, \\ U_2(q_1) &= q_1(1) = 1, & U_2(q_2) &= q_2(1) = (1-\mathrm{k})\mathrm{e}^{\mathrm{i}}, \end{aligned}$$

then

$$Q(q) = \begin{pmatrix} 1 & 1-\mathrm{k} \\ 1 & (1-\mathrm{k})\mathrm{e}^{\mathrm{i}} \end{pmatrix},$$

and

$$\operatorname{ddet} Q(q) = \det_p\left(Q^+(q)Q(q)\right) = 2\left(2 - \mathrm{e}^{\mathrm{i}} - \mathrm{e}^{-\mathrm{i}}\right) \neq 0. \tag{4.24}$$

In addition,

$$\left\|U_2(q_1)\right\|^2\overline{U_1(q_2)} - \overline{U_2(q_2)}U_2(q_1)\overline{U_1(q_1)} = (1+\mathrm{k})\left(1-\mathrm{e}^{-\mathrm{i}}\right),$$

$$\begin{aligned}
&\|U_1(q_1)\|^2\overline{U_2(q_2)} - \overline{U_1(q_2)}U_1(q_1)\overline{U_2(q_1)} = (1+\mathrm{k})\big(\mathrm{e}^{-\mathrm{i}}-1\big),\\
&\|U_2(q_2)\|^2\overline{U_1(q_1)} - \overline{U_2(q_1)}U_2(q_2)\overline{U_1(q_2)} = 2\big(1-\mathrm{e}^{\mathrm{i}}\big),\\
&\|U_1(q_2)\|^2\overline{U_2(q_1)} - \overline{U_1(q_1)}U_1(q_2)\overline{U_2(q_2)} = 2\big(1-\mathrm{e}^{-\mathrm{i}}\big),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=0}^{1}\big(q_1^{(j)}(t_1)\bar{\omega}_{21}(s) + q_2^{(j)}(t_1)\bar{\omega}_{22}(s)\big)\\
&\quad = \big(q_1(0)+q_1'(0)\big)\bar{\omega}_{21}(s) + \big(q_2(0)+q_2'(0)\big)\bar{\omega}_{22}(s)\\
&\quad = -2\mathrm{j} + (2+2\mathrm{j})\mathrm{e}^{-\mathrm{i}s},\\
&\sum_{j=0}^{1}\big(q_1^{(j)}(t_2)\bar{\omega}_{21}(s) + q_2^{(j)}(t_2)\bar{\omega}_{22}(s)\big)\\
&\quad = \big(q_1(1)+q_1'(1)\big)\bar{\omega}_{21}(s) + \big(q_2(1)+q_2'(1)\big)\bar{\omega}_{22}(s)\\
&\quad = -2\mathrm{j} + (2+2\mathrm{j})\mathrm{e}^{\mathrm{i}-\mathrm{i}s},
\end{aligned}$$

then

$$h_1(s) = 2(1-\mathrm{i}+\mathrm{j}+\mathrm{k})\big(2-\mathrm{e}^{-\mathrm{i}}-\mathrm{e}^{\mathrm{i}}\big)\mathrm{e}^{-\mathrm{i}s}, \tag{4.25}$$

$$h_2(s) = -4\mathrm{j}\big(2-\mathrm{e}^{\mathrm{i}}-\mathrm{e}^{-\mathrm{i}}\big), \tag{4.26}$$

$$g(t,s) = 2\mathrm{j}\big(\mathrm{e}^{\mathrm{i}(t-s)}-1\big). \tag{4.27}$$

Based on all of these calculations, we can write out the form of Green's function, which is

$$G_1(t,s) = \begin{cases} \frac{1}{2}(1-\mathrm{i}+\mathrm{j}+\mathrm{k})\mathrm{e}^{-\mathrm{i}s} - (\mathrm{i}+\mathrm{j})\mathrm{e}^{\mathrm{i}t}, & 0\le t<s<1,\\ \frac{1}{2}(1-\mathrm{i}+\mathrm{j}+\mathrm{k})\mathrm{e}^{-\mathrm{i}s} - (\mathrm{i}+\mathrm{j})\mathrm{e}^{\mathrm{i}t} + \mathrm{j}(\mathrm{e}^{\mathrm{i}(t-s)}-1), & 0<s<t\le 1. \end{cases} \tag{4.28}$$

So the solution to BVP (4.18) can be uniquely expressed as

$$\varphi_1(t) = \int_0^1 G_1(t,s)f(s,q)\,\mathrm{d}s. \tag{4.29}$$

Corollary 4.1 *From what has been discussed above, it is not difficult for us to show that the Green's function of*

$$q'' + a_1(t)q' = f(t,q) \tag{4.30}$$

can also be obtained, and (4.29) *is the solution of* (4.30).

4.2 Inhomogeneous boundary value problem

In the following content, we continue to consider the inhomogeneous boundary value problem

$$\begin{cases} q'' + a_1(t)q' + a_2(t)q = f(t), & t \in J, \\ U_1(q) = B_1, \qquad U_2(q) = B_2. \end{cases} \tag{4.31}$$

Similar to the method for the semi-homogeneous boundary value problem, when $\operatorname{ddet} Q(q) \neq 0$, the solution $q(t)$ can be denoted by

$$\begin{aligned} q(t) &= \frac{1}{\operatorname{ddet} Q(q)} q_1(t)\big[\overline{U_2(q_2)}U_2(q_1)\overline{U_1(q_1)} - \|U_2(q_1)\|^2\overline{U_1(q_2)}\big]\big(B_1 - U_1(z)\big) \\ &\quad + \frac{1}{\operatorname{ddet} Q(q)} q_1(t)\big[\overline{U_1(q_2)}U_1(q_1)\overline{U_2(q_1)} - \|U_1(q_1)\|^2\overline{U_2(q_2)}\big]\big(B_2 - U_2(z)\big) \\ &\quad + \frac{1}{\operatorname{ddet} Q(q)} q_2(t)\big[\overline{U_2(q_1)}U_2(q_2)\overline{U_1(q_2)} - \|U_2(q_2)\|^2\overline{U_1(q_1)}\big]\big(B_1 - U_1(z)\big) \\ &\quad + \frac{1}{\operatorname{ddet} Q(q)} q_2(t)\big[\overline{U_1(q_1)}U_1(q_2)\overline{U_2(q_2)} - \|U_1(q_2)\|^2\overline{U_2(q_1)}\big]\big(B_2 - U_2(z)\big) \\ &\quad + \frac{1}{\operatorname{ddet} Q(q)} \operatorname{ddet} Q(q) \cdot z(t), \end{aligned} \tag{4.32}$$

where

$$B_1 - U_1(z) = B_1 - \frac{1}{\operatorname{ddet} \Phi(t)} \int_a^t \sum_{j=0}^{1} \big[q_1^{(j)}(t_1)\bar{\omega}_{21}(s) + q_2^{(j)}(t_1)\bar{\omega}_{22}(s)\big] f(s)\,\mathrm{d}s,$$

$$B_2 - U_2(z) = B_2 - \frac{1}{\operatorname{ddet} \Phi(t)} \int_a^t \sum_{j=0}^{1} \big[q_1^{(j)}(t_2)\bar{\omega}_{21}(s) + q_2^{(j)}(t_2)\bar{\omega}_{22}(s)\big] f(s)\,\mathrm{d}s.$$

Let

$$\begin{aligned} h_{11}(s) &= \sum_{j=0}^{1} \big[\overline{U_2(q_2)}U_2(q_1)\overline{U_1(q_1)} - \|U_2(q_1)\|^2\overline{U_1(q_2)}\big]\big[q_1^{(j)}(t_1)\bar{\omega}_{21}(s) + q_2^{(j)}(t_1)\bar{\omega}_{22}(s)\big] \\ &\quad + \sum_{j=0}^{1} \big[\overline{U_1(q_2)}U_1(q_1)\overline{U_2(q_1)} - \|U_1(q_1)\|^2\overline{U_2(q_2)}\big]\big[q_1^{(j)}(t_2)\bar{\omega}_{21}(s) + q_2^{(j)}(t_2)\bar{\omega}_{22}(s)\big], \\ h_{22}(s) &= \sum_{j=0}^{1} \big[\overline{U_2(q_1)}U_2(q_2)\overline{U_1(q_2)} - \|U_2(q_2)\|^2\overline{U_1(q_1)}\big]\big[q_1^{(j)}(t_1)\bar{\omega}_{21}(s) + q_2^{(j)}(t_1)\bar{\omega}_{22}(s)\big] \\ &\quad + \sum_{j=0}^{1} \big[\overline{U_1(q_1)}U_1(q_2)\overline{U_2(q_2)} - \|U_1(q_2)\|^2\overline{U_2(q_1)}\big]\big[q_1^{(j)}(t_2)\bar{\omega}_{21}(s) + q_2^{(j)}(t_2)\bar{\omega}_{22}(s)\big] \end{aligned}$$

and

$$\begin{aligned} h_3 &= \big[\overline{U_2(q_2)}U_2(q_1)\overline{U_1(q_1)} - \|U_2(q_1)\|^2\overline{U_1(q_2)}\big]B_1 \\ &\quad + \big[\overline{U_1(q_2)}U_1(q_1)\overline{U_2(q_1)} - \|U_1(q_1)\|^2\overline{U_2(q_2)}\big]B_2, \end{aligned}$$

$$
\begin{aligned}
h_4 &= \big[\overline{U_2(q_1)}U_2(q_2)\overline{U_1(q_2)} - \big\|U_2(q_2)\big\|^2\overline{U_1(q_1)}\big]B_1 \\
&\quad + \big[\overline{U_1(q_1)}U_1(q_2)\overline{U_2(q_2)} - \big\|U_1(q_2)\big\|^2\overline{U_2(q_1)}\big]B_2, \\
g(t,s) &= q_1(t)\bar{\omega}_{21}(s) + q_2(t)\bar{\omega}_{22}(s).
\end{aligned}
$$

Then

$$
\begin{aligned}
q(t) &= \frac{1}{\operatorname{ddet} Q(q)}\frac{1}{\operatorname{ddet}\Phi(t)}\bigg\{-\int_a^{t_i}\big[q_1(t)h_{11}(s) + q_2(t)h_{22}(s)\big]f(s)\,\mathrm{d}s \\
&\quad + \int_a^t \operatorname{ddet} Q(q)\cdot g(t,s)f(s)\,\mathrm{d}s\bigg\} + \frac{1}{\operatorname{ddet} Q(q)}\frac{1}{\operatorname{ddet}\Phi(t)}\big[q_1(t)h_3 + q_2(t)h_4\big],
\end{aligned}
\tag{4.33}
$$

and Green's function $G_2(t,s)$ is defined by

$$
\begin{cases}
\frac{1}{\operatorname{ddet} Q(q)}\frac{1}{\operatorname{ddet}\Phi(t)}[-q_1(t)h_{11}(s) - q_2(t)h_{22}(s)], & a \le t < s < b, \\
\frac{1}{\operatorname{ddet} Q(q)}\frac{1}{\operatorname{ddet}\Phi(t)}[-q_1(t)h_{11}(s) - q_2(t)h_{22}(s) + \operatorname{ddet} Q(q)\cdot g(t,s)], & a < s < t \le b.
\end{cases}
\tag{4.34}
$$

Therefore the solution of BVP (4.31) can be expressed as a form of

$$
q(t) = \int_a^b G_2(t,s)f(s)\,\mathrm{d}s + \frac{1}{\operatorname{ddet} Q(q)}\frac{1}{\operatorname{ddet}\Phi(t)}\big(q_1(t)h_3 + q_2(t)h_4\big). \tag{4.35}
$$

Remark 4.1 It can be seen from the above that not all solutions of boundary value problems can be written in the form of integral of Green's function, but they can be expressed in the form related to Green's function.

5 Green's functions for periodic boundary problem

In this section, we introduce Green's function for periodic boundary problems and use the Green's function to transform differential equation into integral equation. Due to the noncommutativity of quaternion algebra, we cannot easily find the Green's function for second-order quaternion differential equations. If $a(t)$ is a real function, then the commutativity will be satisfied, and Green's function will be obtained. After that, we consider the Green's function for second-order quaternion differential equations with the real variable coefficients.

Let us take the following equation as an example:

$$
\begin{cases}
q'' = a^2(t)q + f(t,q), & t \in [0,T], \\
q(0) = q(T), & q'(0) = q'(T),
\end{cases}
\tag{5.1}
$$

where $a(t) : [0,T] \to \mathbb{R}$, $A(t) = \int_0^t a(s)\,\mathrm{d}s : [0,T] \to \mathbb{R}$, and $f(t,q) : [0,T] \times \mathbb{H} \to \mathbb{H}$.

By using the reduced order method, we convert the above second-order differential equation into the following two first-order differential equations:

$$
q'(t) = a(t)q(t) + q_1(t), \tag{5.2}
$$

$$
q_1'(t) = -a(t)q_1(t) + f(t,q). \tag{5.3}
$$

The solution of quaternion differential equation $q'(t) = a(t)q(t)$ is

$$q(t) = \big[\exp A(t)\big]C, \quad C \in \mathbb{H}, t \in [0,T], \tag{5.4}$$

and satisfies $q'(t) = a(t)[\exp A(t)]C = a(t)q(t)$. Using the constant change method, we can obtain the solutions of (5.2), that is,

$$q(t) = \big[\exp A(t)\big]C(t). \tag{5.5}$$

Differentiating the above equation, we get

$$q'(t) = a(t)\big[\exp A(t)\big]C(t) + \big[\exp A(t)\big]C'(t).$$

According to this, we find that

$$\begin{aligned} C'(t) &= \big[\exp A(t)\big]^{-1} q_1(t), \\ C(t) &= C(0) + \int_0^t \big[\exp A(s)\big]^{-1} q_1(s)\,\mathrm{d}s. \end{aligned}$$

Due to $q(0) = q(T)$, using (5.4), as we soon show,

$$C(0) = \big[\exp A(T)\big]C(T),$$

and

$$\begin{aligned} C(T) &= C(0) + \int_0^T \big[\exp A(s)\big]^{-1} q_1(s)\,\mathrm{d}s \\ &= \big[\exp A(T)\big]C(T) + \int_0^T \big[\exp A(s)\big]^{-1} q_1(s)\,\mathrm{d}s. \end{aligned}$$

In other words,

$$\begin{aligned} C(T) &= \big[1 - \exp A(T)\big]^{-1} \int_0^T \big[\exp A(s)\big]^{-1} q_1(s)\,\mathrm{d}s, \\ C(0) &= \big[\exp A(T)\big]\big[1 - \exp A(T)\big]^{-1} \int_0^T \big[\exp A(s)\big]^{-1} q_1(s)\,\mathrm{d}s. \end{aligned}$$

On account of the above equations, we have

$$\begin{aligned} q(t) = {} & \exp A(t)\bigg\{\exp A(T)\big[1 - \exp A(T)\big]^{-1} \int_0^T \big[\exp A(s)\big]^{-1} q_1(s)\,\mathrm{d}s \\ & + \int_0^t \big[\exp A(s)\big]^{-1} q_1(s)\,\mathrm{d}s\bigg\} \\ = {} & \exp A(t)\bigg\{\big[1 - \exp A(T)\big]^{-1} \int_0^t \big[\exp A(s)\big]^{-1} q_1(s)\,\mathrm{d}s \\ & + \big[1 - \exp A(T)\big]^{-1} \exp A(T) \int_t^T \big[\exp A(s)\big]^{-1} q_1(s)\,\mathrm{d}s\bigg\} \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \exp A(t)\bigl[1-\exp A(T)\bigr]^{-1}\bigl[\exp\bigl(-A(s)\bigr)\bigr]q_1(s)\,\mathrm{d}s \\
&\quad + \int_t^T \exp A(t)\bigl[1-\exp A(T)\bigr]^{-1}\exp A(T)\bigl[\exp\bigl(-A(s)\bigr)\bigr]q_1(s)\,\mathrm{d}s \\
&= \int_0^t \frac{\mathrm{e}^{A(t)-A(s)}}{1-\mathrm{e}^{A(T)}}q_1(s)\,\mathrm{d}s + \int_t^T \frac{\mathrm{e}^{A(t)-A(s)+A(T)}}{1-\mathrm{e}^{A(T)}}q_1(s)\,\mathrm{d}s.
\end{aligned}
$$

It is easy to say that the Green's functions of (5.2) and (5.3) are

$$
g_1(t,s) = \begin{cases} \frac{\mathrm{e}^{A(t)-A(s)}}{1-\mathrm{e}^{A(T)}}, & 0 \le s \le t \le T, \\ \frac{\mathrm{e}^{A(t)-A(s)+A(T)}}{1-\mathrm{e}^{A(T)}}, & 0 \le t \le s \le T, \end{cases} \tag{5.6}
$$

$$
g_2(t,s) = \begin{cases} \frac{\mathrm{e}^{A(s)-A(t)+A(T)}}{\mathrm{e}^{A(T)}-1}, & 0 \le s \le t \le T, \\ \frac{\mathrm{e}^{A(s)-A(t)}}{\mathrm{e}^{A(T)}-1}, & 0 \le t \le s \le T. \end{cases} \tag{5.7}
$$

Hence,

$$
\begin{aligned}
q(t) &= \int_0^T g_1(t,s)q_1(s)\,\mathrm{d}s, \\
q_1(t) &= \int_0^T g_2(t,s)f\bigl(s,q(s)\bigr)\,\mathrm{d}s.
\end{aligned}
$$

Combining the two formulas above, we have

$$
\begin{aligned}
q(t) &= \int_0^T g_1(t,\tau)\int_0^T g_2(\tau,s)f\bigl(s,q(s)\bigr)\,\mathrm{d}s\,\mathrm{d}\tau \\
&= \int_0^T \int_0^T g_1(t,\tau)g_2(\tau,s)f\bigl(s,q(s)\bigr)\,\mathrm{d}s\,\mathrm{d}\tau \\
&= \int_0^T \biggl[\int_0^T g_1(t,s)g_2(s,\tau)\,\mathrm{d}s\biggr]f\bigl(\tau,q(\tau)\bigr)\,\mathrm{d}\tau \\
&= \int_0^T \biggl[\int_0^T g_1(t,\tau)g_2(\tau,s)\,\mathrm{d}\tau\biggr]f\bigl(s,q(s)\bigr)\,\mathrm{d}s.
\end{aligned}
$$

We define the Green's function of the second-order quaternion differential equation as

$$
G_3(t,s) = \int_0^T g_1(t,\tau)g_2(\tau,s)\,\mathrm{d}\tau. \tag{5.8}
$$

Remark 5.1 If we replace $a(t)$ in the example above with a and $a \in \mathbb{R}$, we are going to get a simpler form of Green's function like this

$$
\begin{aligned}
G_4(t,s) &= \int_0^T g_3(t,\tau)g_4(\tau,s)\,\mathrm{d}\tau, \\
g_3(t,s) &= \begin{cases} \frac{\mathrm{e}^{a(t-s)}}{1-\mathrm{e}^{aT}}, & 0 \le s \le t \le T, \\ \frac{\mathrm{e}^{a(t-s+T)}}{1-\mathrm{e}^{aT}}, & 0 \le t \le s \le T, \end{cases} \\
g_4(t,s) &= \begin{cases} \frac{\mathrm{e}^{a(s-t+T)}}{\mathrm{e}^{aT}-1}, & 0 \le s \le t \le T, \\ \frac{\mathrm{e}^{a(s-t)}}{\mathrm{e}^{aT}-1}, & 0 \le t \le s \le T. \end{cases}
\end{aligned}
$$

Proof Using the same method as in the example above, this proof is analogous to that of the proof above. □

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The authors declare no competing interests.

Author contributions
Jie Liu, Siyu Sun, and Zhibo Cheng all contributed to this study. Jie Liu and Siyu Sun proved the results and drafted the article. Jie Liu, Siyu Sun, and Zhibo Cheng reviewed and edited the manuscript. All the authors have read and approved the final manuscript.

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