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Existence and multiplicity of solutions for boundary value problem of singular two-term fractional differential equation with delay and sign-changing nonlinearity

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Abstract

In this paper, we consider the existence of solutions for a boundary value problem of singular two-term fractional differential equation with delay and sign-changing nonlinearity. By means of the Guo–Krasnosel'skii fixed point theorem and the Leray–Schauder nonlinear alternative theorem, we obtain some results on the existence and multiplicity of solutions, respectively.

Mathematics Subject Classification: 34A08; 34B18

Keywords: Two-term fractional differential equation; Delay; Singular; Sign-changing nonlinearity

1 Introduction

In this paper, we study the following two-term fractional differential equation with delay:

$$\begin{cases} -D_{0^+}^{\alpha}x(t)+ax(t)=\lambda f\bigl(t,x(t-\tau)\bigr), \quad t\in(0,1)\setminus\{\tau\}, \\ x(t)=\eta(t), \quad t\in[-\tau,0], \\ x(0)=x'(0)=x''(0)=\cdots=x^{(n-2)}(0)=0, \\ x(1)=0, \end{cases} \tag{1.1}$$

where $n-1<\alpha<n$, $n=[\alpha]+1$, $n\geq 3$ $(n\in N)$, $a>0$, $D_{0^+}^{\alpha}$ is the standard Riemann–Liouville fractional derivative, λ is a positive constant, $f(t,x):(0,1)\times R^+\to R$ is continuous, may change sign, and be singular at $t=0$, $t=1$, and $x=0$, where $R^+=(0,+\infty)$, $\eta\in C[-\tau,0]$, and $\eta(t)>0$ for $t\in[-\tau,0)$, $\eta(0)=0$.

Recently, fractional differential equations have been extensively studied, among which the existence of positive solutions to fractional differential equations was considered in [1, 8, 10–12, 15, 16, 19, 20] and [6, 22]. In particular, the nonlinear terms of the problems studied in [8, 10–12, 19] can change sign and are singular at time or space variables. In practical problems, delay is a nonnegligible factor, which can reasonably express the influence of the past on the present. Therefore the delay differential equation has a wide range of applications in control theory, signal processing, biology, finance, and many other fields [4, 10, 12, 14, 21]. In addition, unlike the above research problems, the fractional differential equations of two terms are studied in [2, 3, 17, 18].

Mu et al. [12] investigated the singular boundary value problems for the following nonlinear fractional differential equations with delay by the Guo–Krasnosel'skii fixed point theorem:

$$\begin{cases} D^\alpha x(t) + \lambda f\big(t, x(t-\tau)\big) = 0, & t \in (0,1) \setminus \{\tau\}, \\ x(t) = \eta(t), & t \in [-\tau, 0], \\ x'(1) = x'(0) = 0, \end{cases} \tag{1.2}$$

where $2 < \alpha \leq 3$, D^α is the standard Riemann–Liouville derivative, λ is a positive constant, and $f(t,x)$ may change sign and be singular at $t = 1$, $t = 0$, and $x = 0$.

Liu and Zhang [10] considered the existence of a positive solution for the following problem:

$$\begin{cases} D^\alpha_{0^+} x(t) + f\big(t, x(t-\tau)\big) = 0, & t \in (0,1) \setminus \{\tau\}, \\ x(t) = \eta(t), & t \in [-\tau, 0], \\ x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \\ x^{(n-2)}(1) = 0, \end{cases} \tag{1.3}$$

where $n - 1 < \alpha \leq n$, $n = [\alpha] + 1$, $D^\alpha_{0^+}$ is the standard Riemann–Liouville fractional derivative, $\tau \in (0,1)$, $f(t,x) : (0,1) \times R^+ \to R$ is continuous, may change sign, and be singular at $t = 0$, $t = 1$, and $x = 0$, where $R^+ = (0, +\infty)$, $\eta \in C[-\tau, 0]$, $\eta(t) > 0$ for $t \in [-\tau, 0)$, and $\eta(0) = 0$.

Wang [17] considered a class of Riemann–Liouville type two-term fractional boundary value problems:

$$\begin{cases} -D^\alpha_{0^+} u(t) + au(t) = y(t), & 0 < t < 1, \\ u(0) = u'(0) = 0, \quad u(1) = 0, \end{cases} \tag{1.4}$$

where $2 < \alpha < 3$, $a > 0$, and $D^\alpha_{0^+}$ is the standard Riemann–Liouville derivative. Some positive properties of the Green's function are deduced by using techniques of analysis, and two applications are given by the Guo–Krasnosel'skii fixed point theorem and monotone iterative technique.

Compared with problems (1.2) and (1.3), we discuss the two-term fractional differential equation and show that problem (1.1) has at least two positive solutions. Problem (1.1) is a generalization of the problem studied in [17] when $\tau = 0$ and $n = 3$. By means of the Guo–Krasnosel'skii fixed point theorem and the Leray–Schauder nonlinear alternative theorem we obtaim the existence of at least two positive solutions or three nontrivial solutions of (1.1), respectively.

This paper is organized as follows. In Sect. 2, we introduce some definitions and give preliminary results to be used in the proof of our main theorems. In Sect. 3, we establish the existence and multiplicity of solutions for problem (1.1) based on some fixed point theorems.

2 Basic definitions and preliminaries

In this section, we introduce some basic definitions, theorems, and lemmas.

Definition 2.1 ([13]) The Riemann–Liouville fractional integral of order $\alpha > 0$ for a function f is defined as

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds,$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$, where Γ is the gamma function defined by

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1}\,dt, \quad \alpha > 0.$$

Definition 2.2 ([13]) The Riemann–Liouville fractional derivative of order α ($n-1 < \alpha < n$) for a function f is defined as

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\,ds,$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$, where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, and Γ is the gamma function.

For convenience, we give the following notations:

$$h(x) = \sum_{k=0}^{+\infty} \frac{(k\alpha + \alpha - (n-1))(k\alpha + \alpha - n)x^k}{\Gamma((k+1)\alpha)}, \tag{2.1}$$

$$g(t) = t^{\alpha-1} E_{\alpha,\alpha}\left(at^{\alpha}\right), \tag{2.2}$$

$$g_{n-2}(t) = t^{\alpha-(n-1)} E_{\alpha,\alpha}\left(at^{\alpha}\right), \tag{2.3}$$

where

$$E_{\alpha,\alpha}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma((k+1)\alpha)}$$

is the Mittag-Leffler function.

By (2.1) we know that h is strictly increasing on $[0,+\infty)$, $h(0) < 0$, and

$$\lim_{x\to+\infty} h(x) = +\infty.$$

Then h has a unique positive root a^*, that is, $h(a^*) = 0$.

($\mathbf{H}_1$) $a \in (0, a^*]$ is a constant.

Lemma 2.3 ([9]) *Let $n-1 < \alpha \le n$ ($n \in \mathbb{N}$), let $\lambda \in \mathbb{R}$, and let y be a real function on $\mathbb{R}$. Then the equation*

$$D^{\alpha} u(t) - \lambda u(t) = y(t), \quad t > 0,$$

is solvable, and its general solution is given by

$$u(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big[\lambda(t-s)^\alpha\big] y(s)\,ds + \sum_{j=1}^{n} c_j t^{\alpha-1} E_{\alpha,\alpha+1-j}\big(\lambda t^\alpha\big)$$

with arbitrary $c_j \in \mathbb{R}$, $j = 1,\dots,n$, *where* $E_{\alpha,\beta}$ *is the Mittag-Leffler function.*

Lemma 2.4 *Let* $n-1 < \alpha \le n$ *and* $y \in L^1[0,1] \cap C(0,1)$. *Then the unique solution of the two-term boundary value problem*

$$\begin{cases} -D_{0^+}^{\alpha} u(t) + au(t) = \lambda y(t), & t \in (0,1), \\ u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = 0 \end{cases} \tag{2.4}$$

is given by

$$u(t) = \lambda \int_0^1 G(t,s) y(s)\,ds, \quad t \in [0,1],$$

where

$$G(t,s) = \frac{1}{g(1)} \begin{cases} g(t)g(1-s), & 0 \le t \le s \le 1, \\ g(t)g(1-s) - g(t-s)g(1), & 0 \le s \le t \le 1. \end{cases} \tag{2.5}$$

Proof By Lemma 2.3 we know that the general solution of the equation

$$-D_{0^+}^{\alpha} u(t) + au(t) = \lambda y(t)$$

can be expressed by

$$u(t) = -\lambda \int_0^t g(t-s) y(s)\,ds + c_1 g(t) + c_2 g'(t) + c_3 g''(t) + \cdots + c_n g^{(n-1)}(t).$$

Since $u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0$, we deduce that $c_n = c_{n-1} = \cdots = c_2 = 0$.
It follows from $u(1) = 0$ that

$$c_1 = \frac{\lambda \int_0^1 g(1-s) y(s)\,ds}{g(1)}.$$

Therefore the solution of (2.4) is

$$\begin{aligned} u(t) &= -\lambda \int_0^t g(t-s) y(s)\,ds + \frac{\lambda \int_0^1 g(1-s) y(s)\,ds}{g(1)} g(t) \\ &= \frac{\lambda \int_0^1 g(t) g(1-s) y(s)\,ds - \lambda \int_0^t g(t-s) g(1) y(s)\,ds}{g(1)} \\ &= \frac{\lambda \int_0^t [g(t)g(1-s) - g(t-s)g(1)] y(s)\,ds + \lambda \int_t^1 g(t) g(1-s) y(s)\,ds}{g(1)} \end{aligned}$$

$$= \lambda \int_0^1 G(t,s)y(s)\,ds, \quad t \in [0,1].$$

This completes the proof. □

Lemma 2.5 *For* $0 \le s \le t \le 1$, *we have* $g_{n-2}(t)g_{n-2}(1-s) \ge g_{n-2}(t-s)g_{n-2}(1)$.

Proof For $t > 0$, by (2.3) we have

$$g'_{n-2}(t) = \sum_{k=0}^{+\infty} \frac{[k\alpha + \alpha - (n-1)]a^k t^{k\alpha+\alpha-n}}{\Gamma((k+1)\alpha)} > 0.$$

Therefore $g_{n-2}(t)$ is strictly increasing on $[0,1]$.

By calculation we have

$$\begin{aligned} g''_{n-2}(t) &= \sum_{k=0}^{+\infty} \frac{[k\alpha + \alpha - (n-1)](k\alpha + \alpha - n)a^k t^{k\alpha+\alpha-n-1}}{\Gamma((k+1)\alpha)} \\ &= t^{\alpha-n-1}h\bigl(at^{\alpha}\bigr) \\ &< t^{\alpha-n-1}h\bigl(a^*\bigr) = 0. \end{aligned}$$

Then $g'_{n-2}(t)$ is strictly decreasing on $[0,1]$, and

$$\begin{aligned} &\frac{\partial}{\partial s}\bigl[g_{n-2}(t)g_{n-2}(1-s) - g_{n-2}(t-s)g_{n-2}(1)\bigr] \\ &\quad = g'_{n-2}(t-s)g_{n-2}(1) - g_{n-2}(t)g'_{n-2}(1-s) \\ &\quad \ge g'_{n-2}(1-s)\bigl[g_{n-2}(1) - g_{n-2}(t)\bigr] \ge 0. \end{aligned}$$

Therefore we get

$$\begin{aligned} &g_{n-2}(t)g_{n-2}(1-s) - g_{n-2}(t-s)g_{n-2}(1) \\ &\quad \ge g_{n-2}(t)g_{n-2}(1-0) - g_{n-2}(t-0)g_{n-2}(1) = 0, \end{aligned}$$

that is,

$$g_{n-2}(t)g_{n-2}(1-s) \ge g_{n-2}(t-s)g_{n-2}(1).$$

This completes the proof. □

Lemma 2.6 *The Green's function* $G(t,s)$ *satisfies the following properties*:

(1) $G(t,s) > 0$, $t,s \in (0,1)$;
(2) $G(t,s) = G(1-s,1-t)$, $t,s \in [0,1]$;
(3) $G(t,s) \ge M_1 s(1-s)^{\alpha-1}(1-t)t^{\alpha-1}$, $t,s \in [0,1]$;
(4) $G(t,s) \le M_2 s(1-s)^{\alpha-1}$ $t,s \in [0,1]$.

where

$$M_1 = \frac{1}{g(1)[\Gamma(\alpha)]^2},$$

$$M_2 = \frac{[g'(1)]^2}{g(1)s^*},$$

and $s^* \in (0,1)$ *satisfies* $s^* = (1-s^*)^{\alpha-2}$.

Proof Since (2) is obviously true, (1) can be deduced from (3), and the proof for (4) is the same as that in [17], so that we just verify (3).

(3) For $t \in [0,1]$, by (2.3) we have

$$g_{n-2}(t) = \sum_{k=0}^{+\infty} \frac{a^k t^{k\alpha+\alpha-1-(n-2)}}{\Gamma((k+1)\alpha)} \geq \frac{t^{\alpha-1-(n-2)}}{\Gamma(\alpha)}.$$

For $0 \leq t \leq s \leq 1$, the proof is similar to that in [17], and we omit it here.

For $0 \leq s \leq t \leq 1$, in view of (2.2), (2.3), and Lemma 2.5, we have that

$$g(t) = t^{n-2}g_{n-2}(t), \qquad g(1) = g_{n-2}(1),$$

and then

$$\begin{aligned}
G(t,s) &= \frac{g(t)g(1-s) - g(t-s)g(1)}{g(1)} \\
&= \frac{t^{n-2}g_{n-2}(t)(1-s)^{n-2}g_{n-2}(1-s) - (t-s)^{n-2}g_{n-2}(t-s)g_{n-2}(1)}{g(1)} \\
&\geq \frac{g_{n-2}(t)g_{n-2}(1-s)[t^{n-2}(1-s)^{n-2} - (t-s)^{n-2}]}{g(1)} \\
&\geq \frac{t^{\alpha-1-(n-2)}(1-s)^{\alpha-1-(n-2)}[t^{n-2}(1-s)^{n-2} - (t-s)^{n-2}]}{g(1)[\Gamma(\alpha)]^2} \\
&= M_1 t^{\alpha-1}(1-s)^{\alpha-(n-1)}\left[(1-s)^{n-2} - \left(1-\frac{s}{t}\right)^{n-2}\right] \\
&\geq M_1 t^{\alpha-1}(1-s)^{\alpha-(n-1)} s\left(\frac{1}{t}-1\right)(1-s)^{n-3} \\
&= M_1 t^{\alpha-1}(1-s)^{\alpha-2} s\left(\frac{1}{t}-1\right) \\
&= M_1 t^{\alpha-2}(1-s)^{\alpha-2} s(1-t) \\
&> M_1 s(1-s)^{\alpha-1}(1-t)t^{\alpha-1}.
\end{aligned}$$

This completes the proof. □

Remark 2.7 $G(t,s) \leq M_2(1-t)t^{\alpha-1}$, $t,s \in [0,1]$.

Lemma 2.8 ([7]) *Let E be a Banach space, and let $K \subset E$ be a cone. Let Ω_1 and Ω_2 be open bounded subsets of E with $\theta \in \Omega_1$ such that $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ be a completely continuous operator such that*

(i) $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$;

or

(ii) $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Lemma 2.9 ([5]) *Let Ω be a relatively open subset of a convex set C in a Banach space E. Let $T:\overline{\Omega}\to C$ be a compact map, and let $p\in\Omega$. Then either*

(1) *T has a fixed point in $\overline{\Omega}$,*

or

(2) *there are $x\in\partial\Omega$ and $\lambda\in(0,1)$ such that $x=(1-\lambda)p+\lambda Tx$.*

3 Main results

In this section, we discuss the existence and multiplicity of positive solutions for the boundary value problem (1.1).

For convenience, we always suppose that the following two conditions hold:

($\mathbf{H}_2$) There exists a nonnegative function $\rho\in C(0,1)\cap L[0,1]$ such that $\int_0^1\rho(s)\,ds>0$,

$$f(t,x)>-\rho(t),$$

and

$$\varphi_2(t)h_2(x)\le f(t,x)+\rho(t)\le\varphi_1(t)\bigl(J(x)+h_1(x)\bigr),\quad (t,x)\in(0,1)\times R^+,$$

where $\varphi_1,\varphi_2\in L[0,1]$ are nonnegative, $h_1,h_2\in C(R_0^+,R^+)$ are nondecreasing, $J\in C(R^+,R^+)$ is nonincreasing ($R_0^+=[0,+\infty)$);

($\mathbf{H}_3$)

$$0<\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)J\bigl(\eta(s-\tau)\bigr)\,ds<+\infty,$$

and there exists a constant $b>0$ such that

$$\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)J\left((s-\tau)^{\alpha-1}\bigl[1-(s-\tau)\bigr]\frac{3M_1b}{4M_2}\right)ds<+\infty,$$

where M_1 and M_2 are as in Lemma 2.6.

Remark 3.1 Let $A=\max_{-\tau\le t\le 0}\eta(t)$; when $s\in[0,\tau]$, we have $-\tau\le s-\tau\le 0$, and then $0\le\eta(s-\tau)\le A$.

Let $X=\{x|x\in C[-\tau,1]\}$. Then $(X,\|\cdot\|)$ is a Banach space with the maximum norm

$$\|x\|_{[-\tau,1]}=\max_{-\tau\le t\le 1}\bigl|x(t)\bigr|,\quad x\in X.$$

Define the cone

$$K=\left\{x\in X\Big|x(t)=0,t\in[-\tau,0],x(t)\ge\frac{M_1}{M_2}(1-t)t^{\alpha-1}\|x\|,t\in[0,1]\right\},$$

and let

$$\bar{\eta}(t)=\begin{cases}\eta(t), & t\in[-\tau,0],\\ 0, & t\in(0,1],\end{cases}$$

$$\omega(t) = \begin{cases} 0, & t \in [-\tau, 0], \\ \lambda \int_0^1 G(t,s)\rho(s)\,ds, & t \in (0,1], \end{cases} \tag{3.1}$$

$$x^*(t) = \max\{x(t) - \omega(t) + \bar{\eta}(t), 0\} = \begin{cases} \eta(t), & t \in [-\tau, 0], \\ \max\{x(t) - \omega(t), 0\}, & t \in (0,1]. \end{cases}$$

The restriction $\omega|_{[0,1]}$ of ω on $[0,1]$ is the solution of the following liner equation:

$$\begin{cases} -D_{0+}^{\alpha} x(t) + ax(t) = \lambda\rho(t), \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \\ x(1) = 0. \end{cases}$$

Then $\omega(t) = \lambda \int_0^1 G(t,s)\rho(s)\,ds$, and by Remark 2.7 we get

$$\omega(t) \le \lambda M_2 t^{\alpha-1}(1-t) \int_0^1 \rho(s)\,ds = t^{\alpha-1}(1-t)c, \tag{3.2}$$

where $c = \lambda M_2 \int_0^1 \rho(s)\,ds$.

It is easy to see that x is a solution of boundary value problem (1.1) if and only if it satisfies

$$x(t) = \begin{cases} \lambda \int_0^1 G(t,s) f(s, x(s-\tau))\,ds, & t \in (0,1), \\ \eta(t), & t \in [-\tau, 0]. \end{cases}$$

Then we consider the following operator:

$$(Tx)(t) = \begin{cases} \lambda \int_0^1 G(t,s)(f(s, x^*(s-\tau)) + \rho(s))\,ds, & t \in (0,1), \\ 0, & t \in [-\tau, 0]. \end{cases} \tag{3.3}$$

Let

$$\Omega_{r_i} = \{x \in X : \|x\| < r_i\}, \quad i = 1,2,3,$$

where r_1, r_2, r_3 satisfies

$$r_1 \ge \max\left\{\frac{4cM_2}{M_1}, b\right\}, \quad r_2 > r_1 + 1, r_3 > r_2 + 1.$$

Lemma 3.2 *Suppose* (H_1)–(H_3) *hold. Then the operator* $T : K \cap (\bar{\Omega}_{r_3} \backslash \Omega_{r_1}) \to K$ *is completely continuous.*

Proof Step 1. We will show that T is well-defined on $K \cap (\bar{\Omega}_{r_3} \backslash \Omega_{r_1})$.

For any $x \in K \cap (\bar{\Omega}_{r_3} \backslash \Omega_{r_1})$, we have $r_1 \le \|x\| \le r_3$, and

$$x(t) \ge \frac{M_1}{M_2}(1-t)t^{\alpha-1}\|x\| \ge \frac{M_1}{M_2}(1-t)t^{\alpha-1} r_1,$$

and by (3.2) we get

$$\begin{aligned} x(t)-\omega(t) &\geq \frac{M_1}{M_2}(1-t)t^{\alpha-1}r_1-(1-t)t^{\alpha-1}c \\ &= \left(\frac{M_1}{M_2}r_1-c\right)(1-t)t^{\alpha-1} \\ &\geq \frac{3M_1r_1}{4M_2}(1-t)t^{\alpha-1} \\ &\geq \frac{3M_1b}{4M_2}(1-t)t^{\alpha-1}>0. \end{aligned} \tag{3.4}$$

Then by (H_2), (H_3), (3.4), Remark 3.1, and Lemma 2.6 we obtain

$$\begin{aligned} &(Tx)(t) \\ &\quad= \lambda\int_0^1 G(t,s)\big(f\big(s,x^*(s-\tau)\big)+\rho(s)\big)\,ds \\ &\quad= \lambda\int_0^\tau G(t,s)\big(f\big(s,x^*(s-\tau)\big)+\rho(s)\big)\,ds+\lambda\int_\tau^1 G(t,s)\big(f\big(s,x^*(s-\tau)\big)+\rho(s)\big)\,ds \\ &\quad= \lambda\int_0^\tau G(t,s)\big(f\big(s,\eta(s-\tau)\big)+\rho(s)\big)\,ds \\ &\qquad+\lambda\int_\tau^1 G(t,s)\big(f\big(s,x(s-\tau)-\omega(s-\tau)\big)+\rho(s)\big)\,ds \\ &\quad\leq \lambda M_2\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)\big(J\big(\eta(s-\tau)\big)+h_1\big(\eta(s-\tau)\big)\big)\,ds \\ &\qquad+\lambda M_2\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\big(J\big(x(s-\tau)-\omega(s-\tau)\big) \\ &\qquad+h_1\big(x(s-\tau)-\omega(s-\tau)\big)\big)\,ds \\ &\quad\leq \lambda M_2\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)\big(J\big(\eta(s-\tau)\big)+h_1(A)\big)\,ds \\ &\qquad+\lambda M_2\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\left(J\left((s-\tau)^{\alpha-1}\big[1-(s-\tau)\big]\frac{3M_1r_1}{4M_2}\right)+h_1\big(x(s-\tau)\big)\right)ds \\ &\quad\leq \lambda M_2\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)\big(J\big(\eta(s-\tau)\big)+h_1(A)\big)\,ds \\ &\qquad+\lambda M_2\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\left(J\left((s-\tau)^{\alpha-1}\big[1-(s-\tau)\big]\frac{3M_1b}{4M_2}\right)+h_1(r_3)\right)ds \\ &\quad<+\infty. \end{aligned} \tag{3.5}$$

Therefore T is well-defined on $K\cap(\bar{\Omega}_{r_3}\setminus\Omega_{r_1})$.

Step 2. We will show that $T:K\cap(\bar{\Omega}_{r_3}\backslash\Omega_{r_1})\to K$.

For any $x\in K\cap(\bar{\Omega}_{r_3}\backslash\Omega_{r_1})$ and $t\in[-\tau,0]$, by (3.3) we know that $Tx(t)=0$.

For any $x\in K\cap(\bar{\Omega}_{r_3}\backslash\Omega_{r_1})$ and $t\in[0,1]$, we have

$$(Tx)(t)=\lambda\int_0^1 G(t,s)\big(f\big(s,x^*(s-\tau)\big)+\rho(s)\big)\,ds$$

$$\leq \lambda M_2 \int_0^1 s(1-s)^{\alpha-1}\big(f\big(s, x^*(s-\tau)\big) + \rho(s)\big)\, ds,$$

that is,

$$\begin{aligned}
\frac{\|Tx\|}{\lambda M_2} &\leq \int_0^1 s(1-s)^{\alpha-1}\big(f\big(s, x^*(s-\tau)\big) + \rho(s)\big)\, ds, \\
(Tx)(t) &= \lambda \int_0^1 G(t,s)\big(f\big(s, x^*(s-\tau)\big) + \rho(s)\big)\, ds \\
&\geq \lambda M_1 t^{\alpha-1}(1-t) \int_0^1 s(1-s)^{\alpha-1}\big(f\big(s, x^*(s-\tau)\big) + \rho(s)\big)\, ds \\
&\geq \frac{\lambda M_1 t^{\alpha-1}(1-t)\|Tx\|}{\lambda M_2} \\
&= \frac{M_1}{M_2} t^{\alpha-1}(1-t)\|Tx\|.
\end{aligned}$$

Hence $T : K \cap (\bar{\Omega}_{r_3} \backslash \Omega_{r_1}) \to K$.

Step 3. Now let us prove that $T : K \cap (\bar{\Omega}_{r_3} \backslash \Omega_{r_1}) \to K$ is a continuous operator.

For all $x_n, x \in K \cap (\bar{\Omega}_{r_3} \backslash \Omega_{r_1})$, $n = 1, 2, \ldots$ with $\|x_n - x\|_{[-\tau,1]} \to 0$ as $n \to +\infty$, we have

$$r_1 \leq \|x_n\| \leq r_3, \qquad r_1 \leq \|x\| \leq r_3,$$

and for any $t \in [0,1]$,

$$\begin{aligned}
x_n(t) - \omega(t) &\geq \frac{3M_1 r_1}{4M_2}(1-t)t^{\alpha-1} \geq \frac{3M_1 b}{4M_2}(1-t)t^{\alpha-1} \geq 0, \\
x(t) - \omega(t) &\geq \frac{3M_1 r_1}{4M_2}(1-t)t^{\alpha-1} \geq \frac{3M_1 b}{4M_2}(1-t)t^{\alpha-1} \geq 0.
\end{aligned}$$

By (H_2) we get that

$$\begin{aligned}
& f\big(s, x_n(s-\tau) - \omega(s-\tau)\big) + \rho(s) \\
&\quad \leq \varphi_1(s)\big(J\big(x_n(s-\tau) - \omega(s-\tau)\big) + h_1\big(x_n(s-\tau) - \omega(s-\tau)\big)\big) \\
&\quad \leq \varphi_1(s)\bigg(J\bigg((s-\tau)^{\alpha-1}\big[1-(s-\tau)\big]\frac{3M_1 r_1}{4M_2}\bigg) + h_1\big(x_n(s-\tau)\big)\bigg) \\
&\quad \leq \varphi_1(s)\bigg(J\bigg((s-\tau)^{\alpha-1}\big[1-(s-\tau)\big]\frac{3M_1 b}{4M_2}\bigg) + h_1(r_3)\bigg)\, ds
\end{aligned}$$

and, similarly,

$$f\big(s, x(s-\tau) - \omega(s-\tau)\big) + \rho(s) \leq \varphi_1(s)\bigg(J\bigg((s-\tau)^{\alpha-1}\big[1-(s-\tau)\big]\frac{3M_1 b}{4M_2}\bigg) + h_1(r_3)\bigg)\, ds.$$

Then

$$\begin{aligned}
&\big|f\big(s, x_n(s-\tau) - \omega(s-\tau)\big) + \rho(s) - \big(f\big(s, x(s-\tau) - \omega(s-\tau)\big) + \rho(s)\big)\big| \\
&\quad \leq 2\varphi_1(s)\bigg(J\bigg((s-\tau)^{\alpha-1}\big[1-(s-\tau)\big]\frac{3M_1 b}{4M_2}\bigg) + h_1(r_3)\bigg)\, ds,
\end{aligned}$$

and

$$\begin{aligned}
&\left|(Tx_n)(t)-(Tx)(t)\right| \\
&\quad=\left|\lambda\int_0^1 G(t,s)\bigl(f\bigl(s,x_n^*(s-\tau)\bigr)+\rho(s)\bigr)\,ds-\lambda\int_0^1 G(t,s)\bigl(f\bigl(s,x^*(s-\tau)\bigr)+\rho(s)\bigr)\,ds\right| \\
&\quad=\left|\lambda\int_\tau^1 G(t,s)\bigl(f\bigl(s,x_n(s-\tau)-\omega(s-\tau)\bigr)+\rho(s)\bigr)\,ds\right. \\
&\qquad\left.-\lambda\int_\tau^1 G(t,s)\bigl(f\bigl(s,x(s-\tau)-\omega(s-\tau)\bigr)+\rho(s)\bigr)\,ds\right| \\
&\quad\le\lambda\int_\tau^1 G(t,s)\bigl|\bigl(f\bigl(s,x_n(s-\tau)-\omega(s-\tau)\bigr)+\rho(s)\bigr)\,ds \\
&\qquad-\bigl(f\bigl(s,x(s-\tau)-\omega(s-\tau)\bigr)+\rho(s)\bigr)\,ds\bigr| \\
&\quad\le 2\lambda M_2\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\left(J\left((s-\tau)^{\alpha-1}\bigl[1-(s-\tau)\bigr]\frac{3M_1b}{4M_2}\right)+h_1(r_3)\right)ds.
\end{aligned}$$

By the Lebesgue dominated convergence theorem it follows that

$$\begin{aligned}
&\|Tx_n-Tx\| \\
&\quad=\max_{t\in[0,1]}\left|(Tx_n)(t)-(Tx)(t)\right| \\
&\quad=\max_{t\in[0,1]}\left|\lambda\int_\tau^1 G(t,s)\bigl(f\bigl(s,x_n(s-\tau)-\omega(s-\tau)\bigr)+\rho(s)\bigr)\,ds\right. \\
&\qquad\left.-\lambda\int_\tau^1 G(t,s)\bigl(f\bigl(s,x(s-\tau)-\omega(s-\tau)\bigr)+\rho(s)\bigr)\,ds\right| \\
&\quad\le\max_{t\in[0,1]}\lambda\int_\tau^1 G(t,s)\bigl|\bigl(f\bigl(s,x_n(s-\tau)-\omega(s-\tau)\bigr)+\rho(s)\bigr) \\
&\qquad-\bigl(f\bigl(s,x(s-\tau)-\omega(s-\tau)\bigr)+\rho(s)\bigr)\bigr|\,ds \\
&\quad\le\lambda M_2\int_t^1 s(1-s)^{\alpha-1}\bigl|\bigl(f\bigl(s,x_n(s-\tau)-\omega(s-\tau)\bigr)+\rho(s)\bigr) \\
&\qquad-\bigl(f\bigl(s,x(s-\tau)-\omega(s-\tau)\bigr)+\rho(s)\bigr)\bigr|\,ds\to 0,
\end{aligned}$$

which implies that $\|Tx_n-Tx\|_{[-\tau,1]}\to 0$ as $n\to+\infty$. Hence T is a continuous operator.

Step 4. Finally, we will prove that T is a compact operator.

Let $B\subset K\cap(\bar{\Omega}_{r_3}\backslash\Omega_{r_1})$ be any nonempty bounded set.

Firstly, we prove that $T(B)$ is uniformly bounded.

For any $x\in B$, by (3.5) we can easily get that

$$\begin{aligned}
(Tx)(t)&\le\lambda M_2\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)\bigl(J\bigl(\eta(s-\tau)\bigr)+h_1(A)\bigr)\,ds \\
&\quad+\lambda M_2\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\left(J\left((s-\tau)^{\alpha-1}\bigl[1-(s-\tau)\bigr]\frac{3M_1b}{4M_2}\right)+h_1(r_3)\right)ds \\
&<+\infty.
\end{aligned}$$

Therefore $T(B)$ is uniformly bounded.

Secondly, we prove that $T(B)$ is equicontinuous.

Since $G(t,s)$ is uniformly continuous for $(t,s) \in [0,1] \times [0,1]$, it follows that for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for $t_1, t_2, s \in [0,1]$, if $|t_1 - t_2| < \delta_0$, then

$$\begin{aligned} &\bigl|G(t_1,s) - G(t_2,s)\bigr| \\ &\quad < \varepsilon \biggl(\lambda \int_0^\tau \varphi_1(s) \bigl(J\bigl(\eta(s-\tau)\bigr) + h_1(A)\bigr)\,ds \\ &\qquad + \lambda \int_\tau^1 \varphi_1(s) \biggl(J\biggl((s-\tau)^{\alpha-1}\bigl[1-(s-\tau)\bigr]\frac{3M_1 b}{4M_2}\biggr) + h_1(r_3)\biggr)\,ds \biggr)^{-1}. \end{aligned}$$

Therefore by (H_2) and Remark 3.1 we get that for any $x \in B$,

$$\begin{aligned} &\bigl|(Tx)(t_1) - (Tx)(t_2)\bigr| \\ &\quad \le \biggl| \lambda \int_0^1 \bigl(G(t_1,s) - G(t_2,s)\bigr)\bigl(f\bigl(s, x^*(s-\tau)\bigr) + \rho(s)\bigr)\,ds \biggr| \\ &\quad \le \lambda \int_0^1 \bigl|G(t_1,s) - G(t_2,s)\bigr| \bigl(f\bigl(s, x^*(s-\tau)\bigr) + \rho(s)\bigr)\,ds \\ &\quad \le \lambda \int_0^\tau \bigl|G(t_1,s) - G(t_2,s)\bigr| \varphi_1(s)\bigl(J\bigl(\eta(s-\tau)\bigr) + h_1(A)\bigr)\,ds \\ &\qquad + \lambda \int_\tau^1 \bigl|G(t_1,s) - G(t_2,s)\bigr| \varphi_1(s) \biggl(J\biggl((s-\tau)^{\alpha-1}\bigl[1-(s-\tau)\bigr]\frac{3M_1 b}{4M_2}\biggr) + h_1(r_3)\biggr)\,ds \\ &\quad < \varepsilon. \end{aligned}$$

Thus $T(B)$ is equicontinuous.

By the Ascoli–Arzelà theorem, $T(B)$ is a sequentially compact set, and thus T is a completely continuous operator. This completes the proof. □

It is clear that if $\tilde{x}$ is a fixed point of operator T in (3.3), then by Lemma 2.4 we obtain that

$$\begin{cases} -D_{0^+}^\alpha \tilde{x}(t) + a\tilde{x}(t) = \lambda(f(t, \tilde{x}^*(t-\tau)) + \rho(t)), & t \in (0,1) \setminus \{\tau\}, \\ \tilde{x}(t) = 0, & t \in [-\tau, 0], \\ \tilde{x}(0) = \tilde{x}'(0) = \tilde{x}''(0) = \cdots = \tilde{x}^{(n-2)}(0) = 0, \\ \tilde{x}(1) = 0. \end{cases} \tag{3.6}$$

If

$$\tilde{x}(t-\tau) - \omega(t-\tau) + \bar{\eta}(t-\tau) \ge 0, \quad t \in [0,1], \tag{3.7}$$

then

$$\tilde{x}^*(t-\tau) = \tilde{x}(t-\tau) - \omega(t-\tau) + \bar{\eta}(t-\tau).$$

Let

$$x(t) = \tilde{x}(t) - \omega(t) + \bar{\eta}(t). \tag{3.8}$$

Lemma 3.3 *If $\tilde{x}(t) = x(t) + \omega(t) - \bar{\eta}(t)$ is a positive solution of boundary value problem* (3.6) *and the inequality $\tilde{x}(t) - \omega(t) + \bar{\eta}(t) \geq 0$ holds for $t \in (0,1) \setminus \{\tau\}$, then $x(t)$ is a positive solution of boundary value problem* (1.1).

Proof If $\tilde{x}(t)$ is a positive solution of boundary value problem (3.6) and $\tilde{x}(t) - \omega(t) + \bar{\eta}(t) \geq 0$ for $t \in (0,1) \setminus \{\tau\}$, then for $t \in [-\tau, 0]$, we have

$$\begin{aligned} x(t) &= \tilde{x}(t) - \omega(t) + \bar{\eta}(t) \\ &= 0 + \eta(t) - 0 = \eta(t). \end{aligned}$$

For $t \in (0,1) \setminus \{\tau\}$, by (3.6) and (3.8) we have

$$\begin{aligned} -D^{\alpha}_{0^+} x(t) + a x(t) &= -D^{\alpha}_{0^+} \big(\tilde{x}(t) - \omega(t) + \bar{\eta}(t)\big) + a\big(\tilde{x}(t) - \omega(t) + \bar{\eta}(t)\big) \\ &= -D^{\alpha}_{0^+} \big(\tilde{x}(t) - \omega(t) + 0\big) + a\big(\tilde{x}(t) - \omega(t) + 0\big) \\ &= -D^{\alpha}_{0^+} \tilde{x}(t) + a\tilde{x}(t) - \big(-D^{\alpha}_{0^+}\omega(t) + a\omega(t)\big) \\ &= \lambda f\big(t, \tilde{x}^*(t-\tau) + \rho(t)\big) - \lambda\rho(t) \\ &= \lambda f\big(t, \tilde{x}^*(t-\tau)\big) \\ &= \lambda f\big(t, x(t-\tau)\big). \end{aligned}$$

It is clear that $x(t)$ is the solution of problem (1.1). This completes the proof. □

To prove the main results, we give the following two conditions.

(H$_4$) There exists $[d,e] \subset (\tau, 1)$ such that $\int_d^e s(1-s)^{\alpha-1}\varphi_2(s)\,ds > 0$.

(H$_5$)

$$\lim_{x \to +\infty} \frac{h_2(x)}{x} = +\infty.$$

In view of (H_5), we know that there exists $M > 0$ such that $h_2(x) \geq x$ for any $x > M$.

For convenience, we introduce the following notations:

$$\begin{aligned} \zeta_1 &= \min_{t \in [d,e]} (t-\tau)^{\alpha-1}(1-t+\tau), \\ \zeta_2 &= \min_{t \in [d,e]} t^{\alpha-1}(1-t), \\ \xi_r &= M_2 \int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)\big(J\big(\eta(s-\tau)\big) + h_1(A)\big)\,ds \\ &\quad + M_2 \int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\left(J\left((s-\tau)^{\alpha-1}\big[1-(s-\tau)\big]\frac{3M_1 b}{4M_2}\right) + h_1(r)\right)ds, \end{aligned}$$

where $r \in (0, +\infty)$, and M_1 and M_2 are as in Lemma 2.6.

In the following proofs, we always choose $r_2 > \max\{M+1, r_1+1\}$.

Theorem 3.4 *Let (H_1)–(H_5) hold. Then the boundary value problem (1.1) has at least two positive solutions, provided that*

$$\lambda \in \left(\frac{4M_2}{3M_1^2\zeta_1\zeta_2\int_d^e s(1-s)^{\alpha-1}\varphi_2(s)\,ds}, \lambda^*\right),$$

where $\lambda^* = \min\{\xi_{r_1}^{-1}r_1, \xi_{r_3}^{-1}r_3\}$.

Proof For any $x \in \partial\Omega_{r_1}$ and $t \in (0,1)$, we have $\|x\| = r_1$ and

$$\begin{aligned} x(t) - \omega(t) &\geq \frac{M_1}{M_2}(1-t)t^{\alpha-1}r_1 - (1-t)t^{\alpha-1}c \\ &\geq \frac{3M_1r_1}{4M_2}(1-t)t^{\alpha-1} \\ &\geq \frac{3M_1b}{4M_2}(1-t)t^{\alpha-1} > 0. \end{aligned}$$

Then it follows from (3.5) that

$$\begin{aligned} &(Tx)(t) \\ &\quad\leq \lambda M_2\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)\big(J\big(\eta(s-\tau)\big) + h_1\big(\eta(s-\tau)\big)\big)\,ds \\ &\qquad + \lambda M_2\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\big(J\big(x(s-\tau) - \omega(s-\tau)\big) \\ &\qquad + h_1\big(x(s-\tau) - \omega(s-\tau)\big)\big)\,ds \\ &\quad\leq \lambda M_2\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)\big(J\big(\eta(s-\tau)\big) + h_1(A)\big)\,ds \\ &\qquad + \lambda M_2\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\left(J\left((s-\tau)^{\alpha-1}\big[1-(s-\tau)\big]\frac{3M_1r_1}{4M_2}\right) + h_1\big(x(s-\tau)\big)\right)ds \\ &\quad\leq \lambda^* M_2\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)\big(J\big(\eta(s-\tau)\big) + h_1(A)\big)\,ds \\ &\qquad + \lambda^* M_2\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\left(J\left((s-\tau)^{\alpha-1}\big[1-(s-\tau)\big]\frac{3M_1b}{4M_2}\right) + h_1(r_1)\right)ds \\ &\quad\leq \lambda^*\xi_{r_1} \\ &\quad< r_1. \end{aligned}$$

Therefore $\|Tx\| < \|x\|$.

On the other hand, for any $x \in \partial\Omega_{r_2}$ and $t \in (0,1)$, we have $\|x\| = r_2$ and

$$\begin{aligned} x(t) - \omega(t) &\geq \frac{M_1}{M_2}(1-t)t^{\alpha-1}r_2 - (1-t)t^{\alpha-1}c \\ &\geq \frac{3M_1r_2}{4M_2}(1-t)t^{\alpha-1} > 0. \end{aligned}$$

From (H_2), (H_4), and (H_5) we obtain that

$$\begin{aligned}
&(Tx)(t)\\
&\quad\geq \lambda\int_d^e G(t,s)\bigl(f\bigl(s,x(s-\tau)-\omega(s-\tau)\bigr)+\rho(s)\bigr)\,ds\\
&\quad\geq \lambda\int_d^e M_1 s(1-s)^{\alpha-1}(1-t)t^{\alpha-1}\varphi_2(s)h_2\bigl(x(s-\tau)-\omega(s-\tau)\bigr)\,ds\\
&\quad\geq \lambda M_1(1-t)t^{\alpha-1}\int_d^e s(1-s)^{\alpha-1}\varphi_2(s)h_2\biggl(\frac{3M_1r_2}{4M_2}(s-\tau)^{\alpha-1}\bigl[1-(s-\tau)\bigr]\biggr)\,ds\\
&\quad\geq \lambda M_1\zeta_2\int_d^e s(1-s)^{\alpha-1}\varphi_2(s)h_2\biggl(\frac{3M_1r_2}{4M_2}\zeta_1\biggr)\,ds\\
&\quad\geq \lambda M_1\zeta_2\frac{3M_1r_2}{4M_2}\zeta_1\int_d^e s(1-s)^{\alpha-1}\varphi_2(s)\,ds\\
&\quad= \frac{3\lambda M_1^2\zeta_1\zeta_2r_2}{4M_2}\int_d^e s(1-s)^{\alpha-1}\varphi_2(s)\,ds\\
&\quad> r_2.
\end{aligned}$$

Therefore $\|Tx\| > \|x\|$.

Then for any $x\in\partial\Omega_{r_3}$ and $t\in(0,1)$, we have $\|x\|=r_3$ and

$$\begin{aligned}
x_n(t)-\omega(t) &\geq \frac{M_1}{M_2}t^{\alpha-1}(1-t)r_3-t^{\alpha-1}(1-t)c\\
&\geq \frac{3M_1b}{4M_2}t^{\alpha-1}(1-t)>0.
\end{aligned}$$

Therefore

$$\begin{aligned}
&(Tx)(t)\\
&\quad= \lambda\int_0^1 G(t,s)\bigl(f\bigl(s,x^*(s-\tau)\bigr)+\rho(s)\bigr)\,ds\\
&\quad= \lambda\int_0^\tau G(t,s)\bigl(f\bigl(s,\eta(s-\tau)\bigr)+\rho(s)\bigr)\,ds\\
&\qquad+\lambda\int_\tau^1 G(t,s)\bigl(f\bigl(s,x(s-\tau)-\omega(s-\tau)\bigr)+\rho(s)\bigr)\,ds\\
&\quad\leq \lambda M_2\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)\bigl(J\bigl(\eta(s-\tau)\bigr)+h_1\bigl(\eta(s-\tau)\bigr)\bigr)\,ds\\
&\qquad+\lambda M_2\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\bigl(J\bigl(x(s-\tau)-\omega(s-\tau)\bigr)\\
&\qquad+h_1\bigl(x(s-\tau)-\omega(s-\tau)\bigr)\bigr)\,ds\\
&\quad\leq \lambda^*M_2\int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)\bigl(J\bigl(\eta(s-\tau)\bigr)+h_1(A)\bigr)\,ds\\
&\qquad+\lambda^*M_2\int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\biggl(J\biggl((s-\tau)^{\alpha-1}\bigl[1-(s-\tau)\bigr]\frac{3M_1b}{4M_2}\biggr)+h_1(r_3)\biggr)\,ds\\
&\quad\leq \lambda^*\xi_{r_3}
\end{aligned}$$

$$< r_3.$$

Therefore $\|Tx\| < \|x\|$.

Then it follows from Lemma 2.8 that T has at least two fixed points $\tilde{x}_1 \in K \cap (\Omega_{r_2} \backslash \Omega_{r_1})$ and $\tilde{x}_2 \in K \cap (\Omega_{r_3} \backslash \Omega_{r_2})$, that is, $r_1 < \|\tilde{x}_1\| < r_2 < \|\tilde{x}_2\| < r_3$. Then we have

$$\begin{aligned}
\tilde{x}_1(t) - \omega(t) + \bar{\eta}(t) &\geq \frac{M_1}{M_2}(1-t)t^{\alpha-1}r_1 - (1-t)t^{\alpha-1}c + 0 \\
&\geq \frac{3M_1 b}{4M_2}(1-t)t^{\alpha-1} > 0, \\
\tilde{x}_2(t) - \omega(t) + \bar{\eta}(t) &\geq \frac{M_1}{M_2}(1-t)t^{\alpha-1}r_2 - (1-t)t^{\alpha-1}c + 0 \\
&\geq \frac{3M_1 b}{4M_2}(1-t)t^{\alpha-1} > 0.
\end{aligned}$$

By Lemma 3.3 we know that $x_1(t) = \tilde{x}_1(t) - \omega(t) + \bar{\eta}(t)$ and $x_2(t) = \tilde{x}_2(t) - \omega(t) + \bar{\eta}(t)$ are two positive solutions of equation (1.1). The proof is completed. □

Theorem 3.5 *Let* (H_1)–(H_4) *hold. Then the boundary value problem* (1.1) *has at least one nontrivial solution when* $\lambda < \xi_r^{-1} r$, *where* $r > \max\{\frac{4cM_2}{M_1}, b\}$.

Proof It is easy to show that exists r that satisfies

$$r > \lambda \xi_r. \tag{3.9}$$

Then we can choose $n_0 \in \{1, 2, \dots\}$ such that

$$r > \lambda \xi_r + \frac{1}{n_0}.$$

For $n \geq n_0$, we consider the family of integral equations

$$(T_n x)(t) = \begin{cases} \lambda \int_0^1 G(t,s)(f_n(s, x^*(s-\tau)) + \rho(s))\,ds + \frac{1}{n}, & t \in (0,1), \\ \frac{1}{n}, & t \in [-\tau, 0], \end{cases} \tag{3.10}$$

where

$$f_n\bigl(s, x^*(s-\tau)\bigr) = \begin{cases} f(s, x^*(s-\tau)), & x^*(s-\tau) \geq \frac{1}{n}, \\ f(s, \frac{1}{n}), & x^*(s-\tau) \leq \frac{1}{n}. \end{cases}$$

Let $\Omega = \Omega_r = \{x \in K, \|x\| < r\}$. Then by the extension theorem of a completely continuous operator and the proof of Lemma 3.2 we get that $T_n : K \cap \bar{\Omega} \to K$ is a completely continuous operator.

We consider the following operator equation:

$$x = \kappa T_n x + (1-\kappa)\frac{1}{n},$$

that is,

$$x(t) = \kappa\lambda \int_0^1 G(t,s)\bigl(f_n\bigl(s, x^*(s-\tau)\bigr) + \rho(s)\bigr)\,ds + \frac{1}{n}, \tag{3.11}$$

where $\kappa \in (0,1)$.

Then we will show that if $x(t)$ is a solution of (3.11) for $\kappa \in (0,1)$, then it satisfies $\|x\| \neq r$. Contrarily, if there exists $\kappa \in (0,1)$ such that $\|x\| = r$, then from (3.1) we have

$$\begin{aligned} &x^*(s-\tau) = \eta(s-\tau), \qquad s-\tau \in [-\tau, 0], \\ &x^*(s-\tau) = \max\bigl\{x(s-\tau) - \omega(s-\tau), 0\bigr\}, \qquad s-\tau \in [0,1], \end{aligned}$$

and

$$\begin{aligned} &\|x\| \le \kappa\lambda M_2 \int_0^1 s(1-s)^{\alpha-1}\bigl(f_n\bigl(s, x^*(s-\tau)\bigr) + \rho(s)\bigr)\,ds + \frac{1}{n}, \\ &x(t) \ge \kappa\lambda M_1 t^{\alpha-1}(1-t) \int_0^1 s(1-s)^{\alpha-1}\bigl(f_n\bigl(s, x^*(s-\tau)\bigr) + \rho(s)\bigr)\,ds + \frac{1}{n} \\ &\quad \ge \kappa\lambda M_1 t^{\alpha-1}(1-t)\frac{\|x\| - \frac{1}{n}}{\kappa\lambda M_2} + \frac{1}{n} \\ &\quad = \frac{M_1}{M_2} t^{\alpha-1}(1-t)\left(\|x\| - \frac{1}{n}\right) + \frac{1}{n} \\ &\quad = \frac{M_1}{M_2} t^{\alpha-1}(1-t)\left(r - \frac{1}{n}\right) + \frac{1}{n}. \end{aligned} \tag{3.12}$$

By (3.2) we get

$$\begin{aligned} x(t) - \omega(t) &\ge \frac{M_1}{M_2} t^{\alpha-1}(1-t)\left(r - \frac{1}{n}\right) + \frac{1}{n} - t^{\alpha-1}(1-t)c \\ &= t^{\alpha-1}(1-t)\left(\frac{M_1}{M_2} r - \frac{M_1}{M_2}\frac{1}{n} - c\right) + \frac{1}{n} \\ &\ge t^{\alpha-1}(1-t)\left(\frac{3M_1 r}{4M_2} - \frac{M_1}{M_2}\frac{1}{n}\right) + \frac{1}{n} \\ &= \frac{M_1}{M_2} t^{\alpha-1}(1-t)\left(\frac{3r}{4} - \frac{1}{n}\right) + \frac{1}{n}. \end{aligned}$$

Therefore $x^*(s-\tau) \ge \frac{1}{n}$ when n is sufficiently large.

Then by (3.12) we have

$$\begin{aligned} x(t) &\ge \frac{M_1}{M_2} t^{\alpha-1}(1-t) r + \left[1 - \frac{M_1}{M_2} t^{\alpha-1}(1-t)\right]\frac{1}{n} \\ &\ge \frac{M_1}{M_2} t^{\alpha-1}(1-t) r, \end{aligned}$$

and, similarly to (3.4), we have

$$x(t) - \omega(t) \ge \frac{3M_1 b}{4M_2}(1-t)t^{\alpha-1} > 0.$$

Then

$$
\begin{aligned}
x(t) &\\
&= \kappa\lambda \int_0^1 G(t,s)\bigl(f_n\bigl(s,x^*(s-\tau)\bigr)+\rho(s)\bigr)\,ds+\frac{1}{n}\\
&\le \lambda M_2 \int_0^1 s(1-s)^{\alpha-1}\bigl(f\bigl(s,x^*(s-\tau)\bigr)+\rho(s)\bigr)\,ds+\frac{1}{n}\\
&\le \lambda M_2 \int_0^\tau s(1-s)^{\alpha-1}\bigl(f\bigl(s,\eta(s-\tau)\bigr)+\rho(s)\bigr)\,ds\\
&\quad+\lambda M_2 \int_\tau^1 s(1-s)^{\alpha-1}\bigl(f\bigl(s,x(s-\tau)-\omega(s-\tau)\bigr)+\rho(s)\bigr)\,ds+\frac{1}{n}\\
&\le \lambda M_2 \int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)\bigl(J\bigl(\eta(s-\tau)\bigr)+h_1(A)\bigr)\,ds\\
&\quad+\lambda M_2 \int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\biggl(J\biggl((s-\tau)^{\alpha-1}\bigl[1-(s-\tau)\bigr]\frac{3M_1b}{4M_2}\biggr)+h_1(r)\biggr)\,ds+\frac{1}{n}\\
&\le \lambda M_2 \int_0^\tau s(1-s)^{\alpha-1}\varphi_1(s)\bigl(J\bigl(\eta(s-\tau)\bigr)+h_1(A)\bigr)\,ds\\
&\quad+\lambda M_2 \int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\biggl(J\biggl((s-\tau)^{\alpha-1}\bigl[1-(s-\tau)\bigr]\frac{3M_1b}{4M_2}\biggr)+h_1(r)\biggr)\,ds+\frac{1}{n_0}\\
&= \lambda\xi_r+\frac{1}{n_0}\\
&< r.
\end{aligned}
$$

This is a contradiction to $\|x\|=r$, and then equation $x=\kappa T_n x+(1-\kappa)\frac{1}{n}$ has no solution on $x\in\partial(K\cap\Omega)$.

Therefore by the Lemma 2.9 we get that T_n have fixed points x_n in $K\cap\bar{\Omega}$, that is,

$$x_n(t)=\lambda\int_0^1 G(t,s)\bigl(f_n\bigl(s,x_n^*(s-\tau)\bigr)+\rho(s)\bigr)\,ds+\frac{1}{n},\quad t\in(0,1),$$

and $\|x_n\|\le r$. Thus $\{x_n\}$ is a uniformly bounded set on $(0,1)$.

By (H_2), (H_4), and Lemma 2.6 we have

$$
\begin{aligned}
x_n(t) &\ge \lambda\int_d^e G(t,s)\bigl(f_n\bigl(s,x_n^*(s-\tau)\bigr)+\rho(s)\bigr)\,ds+\frac{1}{n}\\
&\ge \lambda M_1 t^{\alpha-1}(1-t)\int_d^e s(1-s)^{\alpha-1}\bigl(f\bigl(s,x_n^*(s-\tau)\bigr)+\rho(s)\bigr)\,ds+\frac{1}{n}\\
&\ge \lambda M_1 t^{\alpha-1}(1-t)\int_d^e s(1-s)^{\alpha-1}\varphi_2(s)h_2\bigl(x_n(s-\tau)-\omega(s-\tau)\bigr)\,ds+\frac{1}{n}\\
&\ge \lambda M_1 t^{\alpha-1}(1-t)\int_d^e s(1-s)^{\alpha-1}\varphi_2(s)h_2(0)\,ds\\
&\ge \lambda M_1\zeta_2 h_2(0)\int_d^e s(1-s)^{\alpha-1}\varphi_2(s)\,ds\\
&> 0,
\end{aligned}
$$

and so $x_n(t)$ has a lower bound.

Next, we will prove $\{x_n\}$ is an equicontinuous set on $(0,1)$.

Since G is uniformly continuous for $t \in [0,1]$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $t_1, t_2, s \in [0,1]$, if $|t_1 - t_2| < \delta$, then

$$\left|G(t_1,s) - G(t_2,s)\right| < \frac{\varepsilon M_2}{\lambda \xi_r},$$

and

$$\begin{aligned}
&\left|x_n(t_1) - x_n(t_2)\right| \\
&\quad\le \lambda \int_0^1 \left|G(t_1,s) - G(t_2,s)\right| \left(f_n\left(s, x^*(s-\tau)\right) + \rho(s)\right) ds \\
&\quad\le \lambda \int_0^\tau \left|G(t_1,s) - G(t_2,s)\right| \varphi_1(s) \left(J\left(\eta(s-\tau)\right) + h_1(A)\right) ds \\
&\qquad + \lambda \int_\tau^1 \left|G(t_1,s) - G(t_2,s)\right| \varphi_1(s) \left(J\left((s-\tau)^{\alpha-1}\left[1-(s-\tau)\right]\frac{3M_1 b}{4M_2}\right) + h_1(r)\right) ds \\
&\quad= \lambda \frac{\varepsilon M_2}{\lambda \xi_r} \frac{\xi_r}{M_2} \\
&\quad= \varepsilon.
\end{aligned}$$

Therefore $\{x_n\}$ is an equicontinuous set on $(0,1)$, and then by the Arzelà–Ascoli theorem we get that $\{x_n\}$ is a sequentially compact set and has a subsequence $\{x_{n_k}\}$ $(n_k \ge n)$ uniformly convergent to $\tilde{x} \in K \cap \bar{\Omega}$, where

$$x_{n_k}(t) = \lambda \int_0^1 G(t,s)\left(f_n\left(s, x_n^*(s-\tau)\right) + \rho(s)\right) ds + \frac{1}{n_k}.$$

By the Lebesgue dominated convergence theorem we get

$$\tilde{x}(t) = \lambda \int_0^1 G(t,s)\left(f\left(s, x^*(s-\tau)\right) + \rho(s)\right) ds.$$

Therefore $\tilde{x}$ is a fixed point of T, and then $x(t) = \tilde{x}(t) - \omega(t) + \bar{\eta}(t)$ is a nontrivial solution of (1.1). The proof is completed. □

Remark 3.6 Inequality (3.9) can be derived from the following condition:

(H_6)

$$\lim_{x \to +\infty} \frac{h_1(x)}{x} < \frac{1}{\lambda M_2 \int_\tau^1 s(1-s)^{\alpha-1}\varphi_1(s)\, ds}.$$

Corollary 3.7 *Let (H_1)–(H_6) hold. Then the boundary value problem (1.1) has at least three nontrivial solutions.*

Proof Choose $r = r_1$ in Theorem 3.5. Then it follows from Theorems 3.4 and 3.5 that the boundary value problem (1.1) has at least three nontrivial solutions. The proof is completed. □

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