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Hybrid interpolative mappings for solving fractional Navier–Stokes and functional differential equations

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Abstract
The purpose of this study is to establish fixed-point results for new interpolative contraction mappings in the setting of Busemann space involving a convex hull. To illustrate our findings, we also offer helpful and compelling examples. Finally, the theoretical results are applied to study the existence of solutions to fractional Navier–Stokes and fractional-functional differential equations as applications.

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1 Introduction

If a metric space (MS) fulfills the four essential requirements resulting from the works of Foertsch [1] and Schroeder [2], it is named as a Busemann space (BS) and a hyperbolic space (HS) [3], indicating that the space is a complete geodesic space (GS). Herbert Busemann [4, 5] first used this style of space in 1942. The cone metric of a BS was demonstrated by Pavel [6]. The conclusion of locally nonexpansive mappings in length and geodesic spaces was demonstrated by Alghamdi et al. [7]. Observations on convex combinations in GSs were made by Alghamdi and Kirk [8]. Taking a leading position in the Busemann functions, boundary weights for the stationary process, the shape formula, infinite geodesics, solvable variational formulae, and tools for demonstrating results on fluctuation exponents are some examples of invariant measures for Markov chains.

The Laplace transform method (LTM) was used by Miller and Ross [9] to solve the Cauchy problem (CP) in this specific differential equation (DE) scenario. The CP for the fractional DEs with the Caputo fractional derivative (CFD) was proved by Luchko et al. [10] using the operational technique. By applying LTMs, Podlubny [11] demonstrated fractional DEs. Numerous mathematical fields, including digital data processing, image processing, acoustics, electrical signal processing, probability theory, and physics use the CFDs; see [12]. We direct the readers to [13–18] for extensive literature on fractional DEs.

By bridging the gap left by Banach [19] after more than three decades, Kannan [20] developed a discontinuity of contraction mappings that can have a fixed point (FP) on a complete MS in 1968. By merging the ideas of Banach and Kanann on the complete MS, Reich [21] established the FP theorem involving three metric points. Ćirić [22] extended the ideas of Banach [19] to demonstrate the FP theorem using six metrics. Kannan's, Ćirić's, and Reich's FP theorems have recently been researched and expanded in many ways.

Later, in order to accelerate the convergence of an operator to a unique FP, Karapinar [21] transformed the basic Kannan [20, 23] contraction phenomenon into an interpolative one. However, Karapinar et al. [24] identified a flaw in the study by [22] regarding the assumption that the FP is unique and developed a revised version by fusing the ideas of Reich [21, 25], Ćirić [22], and Rus [26]. They did this by providing a counter-example. To disprove the notion that the FP must be unique, they offered a counter-example. Since then, different interpolative mapping variations for single and multivalued cases in different metric spaces have been demonstrated. Through the Branciari distance, Aydi et al. [27] demonstrated an interpolative Ćirić–Reich–Rus (CRR) type contractions. FP findings on extended interpolative CCR-type F-contractions were presented by Mohammadi et al. [28] along with an application. In quasi-partial b-metric space, Mishra et al. [29] demonstrated the FP theorem for interpolative CRR and Hardy–Rogers contractions along with associated FP results.

Further, to demonstrate the reality of the coincidence point, Errai et al. [30] presented some fresh interpolative Hardy–Rogers and CCR-type contractions in a metric space setting. The investigation here is unusual because it combines ideas from Busemann [4, 5], Ćirić [31], Karapinar [22], and Alghamdi et al. [7] to demonstrate the outcomes of interpolative hybrid contractions (IHC for short) mappings in convex hull combinations of a BS.

2 Preliminaries

We introduce a few definitions and theorems in the stage that follows, which will aid in the development of our key findings.

Definition 2.1 [32] Assume that Ω is an MS, a path

$$\Upsilon : [c_1, c_2] \to \Omega$$

is called a geodesic path in Ω.

Definition 2.2 [32] Assume that Θ is a vector space if for all $\theta, \vartheta \in \Omega$ such that the affine segment

$$[\theta, \vartheta] = \left\{(1-\sigma)\theta + \sigma\vartheta : 0 \leq \sigma \leq 1\right\}$$

is contained in Ξ, then a subset $\Omega \subset \Theta$ is called affinely convex.

Definition 2.3 [33] Let Ω be a geodesic MS. If for any two affinely re-parametrized geodesics $\Xi : [c_1, c_2] \to \Omega$ and $\Xi^* : [c_1', c_2'] \to \Omega$ such that the mapping

$$\mho_{\Xi,\Xi^*} : [c_1, c_2] \times \left[c_1', c_2'\right] \to \mathbb{R}$$

is convex, then the MS Ω is called a BS.

It should be noted that, for all θ, ϑ in a uniquely GS (Ω, d), a point $\theta^* \in [\theta, \vartheta]$ iff there is $\sigma \in [0,1]$ such that $d(\theta^*, \theta) = \sigma d(\theta, \vartheta)$ and $d(\theta^*, \vartheta) = (1-\sigma)d(\theta, \vartheta)$. For ease of use, we shall write

$$\theta^* = (1-\sigma)\theta \oplus \sigma\vartheta.$$

Further, the metric d on Ω is convex, provided that Ω is a BS. This implies that, for every $\rho \in \Omega$,

$$d\big(\rho, (1-\sigma)\theta \oplus \sigma\vartheta\big) \leq (1-\sigma)d(\rho, \theta) \oplus \sigma d(\rho, \vartheta), \tag{2.1}$$

for all $\sigma \in [0,1]$. The following requirements, which are both required and sufficient for a geodesic MS Ω to be a BS, are taken from [32]:

- Assume $\mho$ and $\mho'$ are respective midpoints of two geodesic segments $[\theta, \vartheta]$ and $[\theta, \vartheta']$ in Ω that share the beginning point θ. Then

$$d\big(\mho, \mho'\big) \leq \frac{1}{2}\big(d(\theta, \vartheta) + d\big(\theta, \vartheta'\big)\big).$$

- Assume $\mho$ and $\mho'$ are the respective midpoints of two geodesic segments $[\theta, \vartheta]$ and $[\theta', \vartheta']$ in Ω that share the beginning point θ. Then

$$d\big(\mho, \mho'\big) \leq \frac{1}{2}\big(d(\theta, \vartheta) + d\big(\theta', \vartheta'\big)\big).$$

Definition 2.4 [2] Assume that (Ω, d, U) is a HS if (Ω, d) is a MS and $U : \Omega \times \Omega \to \mathbb{R}$ is a function fulfilling

(B_1) for all $\theta, \vartheta, \rho \in \Omega$ and all $\sigma \in [0,1]$,

$$d(\rho, U(\theta, \vartheta, \sigma) \leq (1-\sigma)d(\rho, \theta) + \sigma d(\rho, \vartheta),$$

(B_2) for all $\theta, \vartheta, \rho \in \Omega$ and all $\sigma_1, \sigma_2 \in [0,1]$,

$$d(U(\theta, \vartheta, \sigma_1), U(\theta, \vartheta, \sigma_2) \leq |\sigma_1 - \sigma_2| d(\rho, \theta),$$

(B_3) for all $\theta, \vartheta \in \Omega$ and all $\sigma \in [0,1]$,

$$U(\theta, \vartheta, \sigma) = U(\theta, \vartheta, 1-\sigma),$$

(B_4) for all $\theta, \vartheta, \rho_1, \rho_2 \in \Omega$ and all $\sigma \in [0,1]$,

$$d(U(\theta, \rho_1, \sigma), U(\vartheta, \rho_2, \sigma) \leq (1-\sigma)d(\theta, \vartheta) + \sigma d(\rho_1, \rho_2).$$

Then (Ω, d, U) is called a BS.

It is remarkable that if only the axiom (B_1) holds, then (Ω, d, U) is a convex MS [34]. A BS is uniquely geodesic only when it is strictly convex [1]. The space is known as the HS if requirements (B_1)–(B_3) are met [35].

On a Hadamard space, where any two points are connected by a singular geodesic segment, we provide an example of Busemann functions. The function $\Im = \Im_\tau$ is convex, that is, it is convex on the segment $[\theta, \vartheta]$. If $\rho(\sigma)$ is a point, which divides $[\theta, \vartheta]$ according to $\sigma : 1 - \sigma$ ratio, then

$$\Im\big(\rho(\sigma)\big) \leq \sigma \Im\theta + (1-\sigma)\Im\vartheta,$$

for $\theta, \vartheta \in \Omega$.

Proposition 2.5 [7] *A GS has a path metric when it is a connected and open subset.*

Proposition 2.6 [36] *Let Ω be a linear space and Q be a subset of Ω. Q is called a convex if for any finite set $\{\theta_1, \theta_2, \dots, \theta_m\} \subseteq Q$ and any scalar $\sigma_j \geq 0$, $i = 1, 2, \dots, m$ with $\sigma_1 + \sigma_2 + \cdots + \sigma_m = 1$, we have*

$$\sigma_1\theta_1 + \sigma_2\theta_2 + \cdots + \sigma_m\theta_m \in Q.$$

Definition 2.7 [36] Let Ω be a linear space and Q (not necessarily convex) be a subset of Ω. Then the intersection of all convex subsets of Ω containing Q is called a convex hull (CH) of Q in Ω and is denoted by $Q_0(Q)$ such that

$$Q_0(Q) = \cap\{V \subseteq \Omega;\ Q \subseteq V,\ V \text{ is convex}\}.$$

As a result, $Q_0(Q)$ is the unique smallest convex set that contains Q.

$$Q_0(Q) = \left\{\sigma_1\theta_1 + \sigma_2\theta_2 + \cdots + \sigma_m\theta_m : \theta_m \in Q,\ \sigma_m \geq 0,\ \sum_{j=1}^{m} \sigma_j = 1\right\},$$

corresponds to the set of all Q's convex combinations.

Definition 2.8 [36] $\overline{Q_0(Q)}$ is a closure CH of Q, where

$$\overline{Q_0(Q)} = \left\{\sigma_1\theta_1 + \sigma_2\theta_2 + \cdots + \sigma_m\theta_m : \theta_m \in Q,\ \sigma_m \geq 0,\ \sum_{j=1}^{m} \sigma_j = 1\right\}.$$

The intersection of all closed convex subsets of Ω that contain Q is known as the closed CH of Q in Ω. It is denoted by $\overline{Q}_0(Q)$ such that

$$\overline{Q}_0(Q) = \cap\{V \subseteq \Omega;\ Q \subseteq V,\ V \text{ is convex and closed}\}.$$

Clearly, $\overline{Q}_0(Q) = \overline{Q_0(Q)}$ and $[\theta, \vartheta] = Q_0(\{\theta, \vartheta\})$ is a CH of θ and ϑ for all θ, ϑ in a linear space Ω.

Now, assume that $\overline{\text{conv}}(Q)$ refers to the closure of the set Q such that

$$\overline{\text{conv}}(Q) = \bigcup_{m=0}^{\infty} Q_m,$$

where $Q_0 = Q$ and the set Q_m is made up of all points in the space that are located on geodesics that have endpoints in Q_{m-1} for $m \geq 1$.

We will now present some initial findings.

Let $\overline{\Omega}$ refer to the closure of a complete BS Ω and $\overline{\Lambda}$ be a closure to the set $\Lambda \in \Omega$. Talenti [37] provided a remark on the Busemann equation by taking into account the elliptic equation below:

$$\left(\nu_\vartheta^2 - 1\right)\nu_{\theta\theta} - 2\nu_\vartheta \nu_\theta \nu_{\theta\vartheta} + \left(\nu_\theta^2 - 1\right)\nu_{\vartheta\vartheta} = 0, \tag{2.2}$$

where $\nu_\vartheta = \frac{\partial \nu}{\partial \vartheta}$, $\nu_{\vartheta\vartheta} = \frac{\partial^2 \nu}{\partial \vartheta^2}$, ν is a real-valued function of θ and ϑ and θ, ϑ are independent variables.

The mixed elliptic-hyperbolic nature of equation (2.2) is evident. As a result of the coefficient matrix

$$\begin{vmatrix} \nu_\vartheta^2 - 1 & -\nu_\vartheta \nu_\theta \\ -\nu_\vartheta \nu_\theta & \nu_\theta^2 - 1 \end{vmatrix}$$

makes the eigenvalues equal -1 and $1 + \nu_\vartheta^2 + \nu_\theta^2$. A solution ν to (2.2) is elliptic in any area with

$$\nu_\vartheta^2 + \nu_\theta^2 < 1,$$

and is hyperbolic with

$$\nu_\vartheta^2 + \nu_\theta^2 > 1.$$

As an illustration, the formulas

$$\nu(\theta, \vartheta) = \log\left(\sqrt{\theta^2 + \vartheta^2} + \sqrt{1 + \theta^2 + \vartheta^2}\right),$$

and

$$\nu(\theta, \vartheta) = \arcsin \sqrt{\theta^2 + \vartheta^2},$$

represent, in turn, the always hyperbolic and everywhere elliptic solutions to (2.2) (Fig. 1). The form

$$\nu(\theta, \vartheta) = \log\left(\frac{\cosh \theta}{\cosh \vartheta}\right),$$

provides a solution to (2.2) with obeying streamlines

$$\sinh|\theta| \sinh|\vartheta| = \text{constant},$$

and whose kind fluctuates-elliptic near the region

$$\sinh|\theta| \sinh|\vartheta| > 1.$$

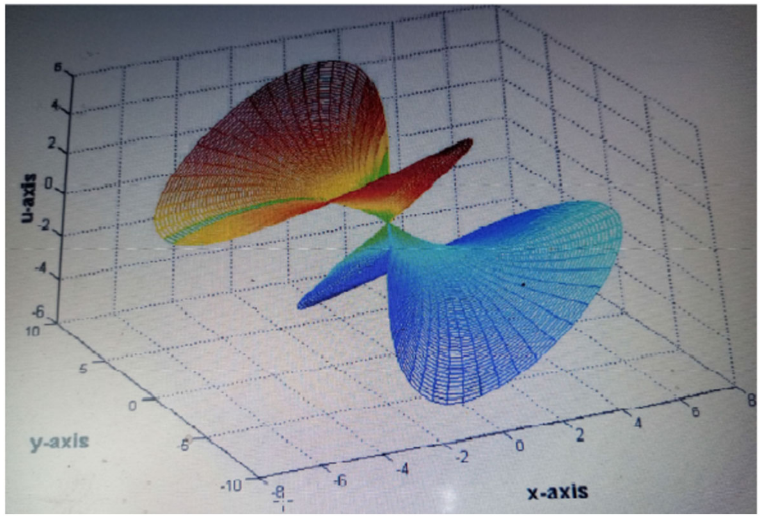

Figure 1 An elliptic solution to the equation (2.2)

The zero mean curvature equation is also known as Equation (2.2). Since the vector $(\Im\theta, \Im\vartheta, 1)$ determines normal direction of the graph $\tau = \Im(\theta, \vartheta)$, the immersion at point $\mho$ is Minkowski space if and only if $1 - v_\theta^2 + v_\vartheta^2$ is positive, negative, or zero at $\mho$. The partial Eq. (2.2) must be elliptic, hyperbolic, or parabolic in order for these criteria to apply. The elliptic solution to (2.2) obeys both

$$v_\theta^2 + v_\vartheta^2 < 1,$$

and

$$\frac{\partial}{\partial\theta}\left[\frac{v_\theta}{1 - v_\theta^2 - v_\vartheta^2}\right] + \frac{\partial}{\partial\vartheta}\left[\frac{v_\theta}{1 - v_\theta^2 - v_\vartheta^2}\right] = 0.$$

Remember the metric described as

$$d = (\partial\theta)^2 + (\partial\vartheta)^2 - (\partial v)^2,$$

area of the space-like graph

$$v = \int\int \left(1 - v_\theta^2 - v_\vartheta^2\right)\partial\theta\,\partial\vartheta,$$

holds in the Minkowski space in three dimensions. The following BS example was provided by Megginson [38]:

Example 2.9 In $\mathbb{R}^m$, for $q \in (1, \infty)$, describe $\|.\|_q$ as

$$\|.\|_q = \left(\sum_{j=1}^{m} |\theta_j|^q\right)^{\frac{1}{q}},$$

where $\theta = (\theta_1, \theta_2, \dots, \theta_m) \in \mathbb{R}^m$. Then $(\mathbb{R}^m, \|.\|)$ is strictly convex and hence a BS.

Lemma 2.10 *Assume that* (Ω, d) *is a BS and* $\{\theta_m^1\}, \{\theta_m^2\}, \ldots, \{\theta_m^M\} \subseteq \Omega$ *be M-sequence such that*

$$\lim_{m\to\infty} d(\theta_m^1, \theta_{m+1}^1) = 0, \quad \textit{for all } m = 1, 2, \ldots, M.$$

The following theorem was established in 1971 by Ćirić [31] as a generalization of the Banach contraction principle:

Theorem 2.11 *Let* (Ω, d) *be a complete MS and* $\Im : \Omega \to \Omega$ *be a given mapping. We say that* $\Im$ *is a generalized contraction if there are* $\beta_1, \beta_2, \beta_3, \beta_4 \geq 0$ *with* $\beta_1 + \beta_2 + \beta_3 + 2\beta_4 < 1$ *such that the inequality*

$$d(\Im\theta, \Im\vartheta) \leq \beta_1 d(\theta, \vartheta) + \beta_2 d(\theta, \Im\theta) + \beta_3 d(\vartheta, \Im\vartheta) + \beta_4\big[d(\theta, \Im\vartheta) + d(\vartheta, \Im\theta)\big]$$

holds for every $\theta, \vartheta \in \Omega$. *Then* $\Im$ *owns a unique FP in* Ω.

The theorem below was established by Kannan [20]:

Theorem 2.12 *Let* $\Im$ *be a self-mapping defined on a complete MS* (Ω, d). *If* $\Im$ *fulfills*

$$d(\Im\theta, \Im\vartheta) \leq \beta\big[d(\theta, \Im\theta) + d(\vartheta, \Im\vartheta)\big], \quad \textit{for every } \theta, \vartheta \in \Omega \textit{ and } \beta \in \left[0, \frac{1}{2}\right].$$

Then $\Im$ *possesses a unique FP in* Ω.

In [22], the results for interpolative Kannan contraction (IKC) were demonstrated as follows:

Definition 2.13 On a MS (Ω, d), a mapping $\Im : \Omega \to \Omega$ is called IKC, if inequality

$$d(\Im\theta, \Im\vartheta) \leq \beta\big[d(\theta, \Im\theta)\big]^{\eta}.\big[d(\vartheta, \Im\vartheta)\big]^{1-\eta},$$

is true for all $\theta, \vartheta \in \Omega$ with $\theta \neq \Im\theta$, where $\beta \in [0, 1)$ and $\eta \in (0, 1)$.

Theorem 2.14 *Assume that* (Ω, d) *is a MS. Then the mapping* $\Im : \Omega \to \Omega$ *owns a unique FP, provided that it is an IKC mapping.*

By using interpolation contraction mapping in partial MSs, Karapinar [39] proved the following interpolative Reich–Rus–Ćirić type non-unique FP theorem:

Theorem 2.15 *Suppose that* (Ω, p) *is a complete partially MS and* $\Im : \Omega \to \Omega$ *is a mapping such that*

$$p(\Im\theta, \Im\vartheta) \leq \beta\big[p(\theta, \theta)\big]^{\eta_1}.\big[p(\theta, \Im\theta)\big]^{\eta_2}.\big[p(\vartheta, \Im\vartheta)\big]^{1-\eta_1-\eta_2}, \quad \textit{for all } \theta, \vartheta \in \Omega$$

for all $\beta \in [0, 1)$, $\eta_1, \eta_2 \in (0, 1)$ and $\theta, \vartheta \in \Omega \setminus \mathrm{fix}(\Im)$, where $\mathrm{fix}(\Im) = \{\theta \in \Omega : \theta = \Im\theta\}$. Then $\Im$ has a FP in Ω.

Alghamdi et al. [7] also took into account the following theorem:

Theorem 2.16 *Assume that (Ω,d) is a complete GS, $\Lambda \in \Omega$ is a connected open set and $\Im : \Lambda \to \Lambda$ is a local radial contraction. Consider extending $\Im$ to a continuous mapping $\overline{\Im} : \overline{\Lambda} \to \overline{\Lambda}$. Then $\overline{\Im}$ possesses a FP in $\overline{\Lambda}$ and moreover $\{\Im^m\theta\}$ converges to θ^* for each $\theta \in \Lambda$.*

3 Main consequences

We start this part with the theorem below.

Theorem 3.1 *Assume that (Ω,d) is a complete GS, $\Lambda \in \Omega$ is a connected open set and $\Im : \Lambda \to \Lambda$ is a local radial IHC mapping. Consider extending $\Im$ to a continuous mapping $\overline{\Im} : \overline{\Lambda} \to \overline{\Lambda}$. If there are $\eta_1, \eta_2, \eta_3, \eta_4 \in [0,1)$ with $\eta_1 + \eta_2 + \eta_3 + \eta_4 < 1$ and $\beta \le 1$ such that*

$$d(\Im\theta, \overline{\Im}\vartheta) \le \beta G_B(\theta,\vartheta), \tag{3.1}$$

where

$$\begin{aligned} G_B(\theta,\vartheta) &= \big[d(\theta,\vartheta)\big]^{\eta_1} . \big[d(\theta,\Im\theta)\big]^{\eta_2} . \big[d(\vartheta,\overline{\Im}\vartheta)\big]^{\eta_3} . \left[\frac{d(\theta,\overline{\Im}\vartheta)}{2}\right]^{\eta_4} \\ &\quad \times \left[\frac{d(\theta,\Im\theta) + d(\vartheta,\overline{\Im}\vartheta) + d(\theta,\overline{\Im}\vartheta) + d(\vartheta,\Im\theta)}{4}\right]^{1-\eta_1-\eta_2-\eta_3-\eta_4}, \end{aligned}$$

for each $\theta,\vartheta \in \Lambda$ and $\Lambda \in \Omega$. Then $\overline{\Im}$ owns a FP in $\overline{\Lambda}$ and moreover $\{\Im^m\theta\}$ converges to θ^* for each $\theta \in \Lambda$.

Proof For $\theta_1,\theta_2 \in \Omega$ and $a_1, a_2 \in [0,1]$ fulfilling $a_1 + a_2 = 1$, let $a_1\theta_1 \oplus a_2\theta_2$ stand for the unique point of Ω. From (2.1), one has

$$\begin{aligned} d(\theta_1, a_1\theta_1 \oplus a_2\theta_2) &\le a_1 d(\theta_1,\theta_1) + a_2 d(\theta_1,\theta_2) \\ &= a_2 d(\theta_1,\theta_2), \end{aligned}$$

and

$$\begin{aligned} d(\theta_2, a_1\theta_1 \oplus a_2\theta_2) &\le a_1 d(\theta_1,\theta_2) + a_2 d(\theta_2,\theta_2) \\ &= a_1 d(\theta_1,\theta_2). \end{aligned}$$

It is specified that the ordered sum is

$$a_1\theta_1 \oplus a_2\theta_2 \oplus a_3\theta_3 \oplus \cdots \oplus a_{m-1}\theta_{m-1}.$$

For $(\theta_1,\theta_2,\ldots,\theta_{m-1}) \in \prod_{j=1}^{m-1}\Omega$, $\{a_j\}_{j=1}^{m-1} \subset [0,1]$ and $\sum_{j=1}^{m-1} a_j = 1$, for $m > 1$. Again, for $a_1, a_2, a_3 = 1$, and an ordered triple $(\theta_1,\theta_2,\theta_3) \in \prod_{j=1}^{3}\Omega$, define

$$a_1\theta_1 \oplus a_2\theta_2 \oplus a_3\theta_3 = \theta_3,$$

provided that $a_3 = 1$. Hence, the set

$$a_1\theta_1 \oplus a_2\theta_2 \oplus a_3\theta_3 \oplus \cdots \oplus a_m\theta_m = \theta_m,$$

provided that $a_m = 1$. Otherwise, we have

$$a_1\theta_1 \oplus a_2\theta_2 \oplus a_3\theta_3 = a_3\theta_3 \oplus (1-a_3)\left(\frac{a_2}{1-a_3}\theta_2 \oplus (1-a_3)\frac{a_2}{1-a_3}\theta_1\right).$$

Since d is convex, for each $\theta \in \Omega$,

$$a_1\theta_1 \oplus a_2\theta_2 \oplus a_3\theta_3 = d\left(\theta, a_3\theta_3 \oplus (1-a_3)\left(\frac{a_2}{1-a_3}\theta_2 \oplus (1-a_3)\frac{a_2}{1-a_3}\theta_1\right)\right).$$

Similarly, we have

$$\begin{aligned} a_1\theta_1 \oplus a_2\theta_2 \oplus a_3\theta_3 \oplus \cdots \oplus a_m\theta_m = a_m\theta_m \oplus (1-a_m)\bigg[&\frac{a_{m-1}}{1-a_m}\theta_1 \\ &\oplus \frac{a_{m-1}}{1-a_m}\theta_2 \oplus \cdots \oplus \frac{a_{m-1}}{1-a_m}\theta_{m-1}\bigg]. \end{aligned}$$

Now, we use the notation

$$a_1\theta_1 \oplus a_2\theta_2 \oplus a_3\theta_3 \oplus \cdots \oplus a_m\theta_m = \sum_{j=1}^{m-1} a_j\theta_j.$$

The convexity of d implies that

$$d\left(\theta, \sum_{j=1}^{m-1} a_j\theta_j\right) = \sum_{j=1}^{m-1} a_j d(\theta, \theta_j), \quad \text{for all } \theta \in \Omega. \tag{3.2}$$

Together, the completeness of Ω, let $u > 1$ and $a_1\theta_1 \oplus a_2\theta_2 \oplus a_3\theta_3 \oplus \cdots \oplus a_{u-1}\theta_{u-1}$ have been defined regardless of order, for all sets of $u-1$ points of Ω and all $\{a_1, a_2, \dots, a_{u-1}\} \subset [0,1]$, fulfilling $\sum_{j=1}^{m-1} a_j = 1$. We might further assert via inductive reasoning that $\{a_1, a_2, \dots, a_u\} \subset [0,1]$. Put

$$\begin{aligned} \theta_1^1 &= a_1\theta_1 \oplus (1-a_1)\left(\frac{a_2}{1-a_1}\theta_2 \oplus \frac{a_3}{1-a_1}\theta_3 \oplus \cdots \oplus \frac{a_u}{1-a_1}\theta_u\right), \\ \theta_2^1 &= a_2\theta_2 \oplus (1-a_2)\left(\frac{a_1}{1-a_2}\theta_1 \oplus \frac{a_3}{1-a_2}\theta_3 \oplus \cdots \oplus \frac{a_u}{1-a_2}\theta_u\right), \\ \theta_3^1 &= a_3\theta_3 \oplus (1-a_3)\left(\frac{a_1}{1-a_3}\theta_1 \oplus \frac{a_2}{1-a_3}\theta_2 \oplus \cdots \oplus \frac{a_u}{1-a_3}\theta_u\right), \\ &\vdots \\ \theta_u^1 &= a_u\theta_u \oplus (1-a_u)\left(\frac{a_1}{1-a_u}\theta_1 \oplus \frac{a_2}{1-a_u}\theta_2 \oplus \cdots \oplus \frac{a_u}{1-a_u}\theta_u\right). \end{aligned}$$

In general, assume that

$$\begin{aligned} \theta_1^m &= a_1\theta_1^{m-1} \oplus (1-a_1)\left(\frac{a_2}{1-a_1}\theta_2^{m-1} \oplus \frac{a_3}{1-a_1}\theta_3^{m-1} \oplus \cdots \oplus \frac{a_u}{1-a_1}\theta_u^{m-1}\right), \\ \theta_2^m &= a_2\theta_2^{m-1} \oplus (1-a_2)\left(\frac{a_1}{1-a_2}\theta_1^{m-1} \oplus \frac{a_3}{1-a_2}\theta_3^{m-1} \oplus \cdots \oplus \frac{a_u}{1-a_2}\theta_u^{m-1}\right), \end{aligned}$$

$$\theta_3^m = a_3\theta_3^{m-1} \oplus (1-a_3)\left(\frac{a_1}{1-a_3}\theta_1^{m-1} \oplus \frac{a_2}{1-a_3}\theta_2^{m-1} \oplus \cdots \oplus \frac{a_u}{1-a_3}\theta_u^{m-1}\right),$$
$$\vdots$$
$$\theta_u^m = a_u\theta_u^{m-1} \oplus (1-a_u)\left(\frac{a_1}{1-a_u}\theta_1^{m-1} \oplus \frac{a_2}{1-a_u}\theta_2^{m-1} \oplus \cdots \oplus \frac{a_u}{1-a_u}\theta_u^{m-1}\right).$$

Now, we estimate $d(\theta_j^m, \theta_h^m)$, $j < h$. Utilizing j and the ideas from (3.2), we obtain

$$\begin{aligned} d(\theta_j^m, \theta_h^m) &\le \sum_{j=1}^{u} a_j d(\theta_j^m, \theta_h^m) \\ &\le \sum_{j=1}^{u} a_j \sum_{h=1}^{u} a_h d(\theta_j^{m-1}, \theta_h^{m-1}) \\ &\le \sum_{j,h=1}^{u} a_j a_h d(\theta_j^{m-1}, \theta_h^{m-1}) \\ &\le 2\left[\sum_{j,h=1(j<h)}^{u} a_j a_h\right] \operatorname{diam}(\{\theta_1^{m-1}, \theta_2^{m-1}, \dots, \theta_u^{m-1}\}). \end{aligned} \tag{3.3}$$

Setting $\beta = 2(\sum_{j,h=1(j<h)}^{u} a_j a_h)$ in (3.3), we get

$$d(\theta_j^m, \theta_h^m) \le \operatorname{diam}(\{\theta_1^m, \theta_2^m, \dots, \theta_u^m\}) \le \beta \operatorname{diam}(\{\theta_1^{m-1}, \theta_2^{m-1}, \dots, \theta_u^{m-1}\}).$$

From Definition 2.8 and Definition 2.13, the set $\{\theta_1^m, \theta_2^m, \dots, \theta_u^m\}$ lies in the CH $\{\theta_1^{m-1}, \theta_2^{m-1}, \dots, \theta_u^{m-1}\}$. It yields,

$$\operatorname{conv}\{\theta_1^m, \theta_2^m, \dots, \theta_u^m\} \subset \operatorname{conv}\{\theta_1^{m-1}, \theta_2^{m-1}, \dots, \theta_u^{m-1}\}.$$

Consequently, we can write

$$\operatorname{diam}(\overline{\operatorname{conv}}\{\theta_1^m, \theta_2^m, \dots, \theta_u^m\}) \le \beta^m \operatorname{diam}(\overline{\operatorname{conv}}\{\theta_1^{m-1}, \theta_2^{m-1}, \dots, \theta_u^{m-1}\}).$$

We claim that for each $j \le h \le u$, the sequence $\{\theta_h^m\}_{h=1}^{\infty}$ is a Cauchy sequence and all these sequences reach the same limit, which is denoted by $a_1\theta_1 \oplus a_2\theta_2 \oplus \cdots \oplus a_u\theta_u$. Using Lemma 2.10, we have

$$\lim_{m\to\infty} d(\theta_j^m, \theta_h^m) = 0.$$

Next, assume that $\Im$ is a continuous IGCTC mapping of a compact BS on (Λ, d). Suppose that d is a path metric on $\Lambda \in \Omega$. Hence, the sequence $\{\Im^m\theta\}$ is Cauchy in (Λ, d). So, $\{\Im^m\theta\}$ converges to some point $\theta^* \in \overline{\Lambda}$. We shall show that $\overline{\Im}$ has a FP. For this, let the point θ_0 be an arbitrary and fixed. Describe the sequence $\{\theta_m\}_{m\in\mathbb{N}} \in \Lambda$ as $\theta_m = \Im\theta_{m-1}$ and $\theta_{m+1} = \Im\theta_m$. If $\theta_m = \theta_{m+1}$ then, there is no proof. So, assume that $\theta_m \ne \theta_{m+1}$, which yields $d(\theta_m, \theta_{m+1}) > 0$.

Now, in (3.1), put $\theta = \theta_{m-1}$, $\vartheta = \theta_m$ and $\Im = \overline{\Im}$, one has

$$d(\Im\theta_{m-1}, \Im\theta_m) \le \beta \max\{G_B(\theta_{m-1}, \theta_m), H_B(\theta_{m-1}, \theta_m)\}, \tag{3.4}$$

where

$$\begin{aligned}
&G_B(\theta_{m-1},\theta_m)\\
&\quad= \big[d(\theta_{m-1},\theta_m)\big]^{\eta_1}.\big[d(\theta_{m-1},\Im\theta_{m-1})\big]^{\eta_2}.\big[d(\theta_m,\Im\theta_m)\big]^{\eta_3}.\left[\frac{d(\theta_{m-1},\Im\theta_m)}{2}\right]^{\eta_4}\\
&\qquad\times\left[\frac{d(\theta_{m-1},\Im\theta_{m-1})+d(\theta_m,\Im\theta_{m-1})+d(\theta_{m-1},\Im\theta_m)+d(\theta_m,\Im\theta_{m-1})}{4}\right]^{1-\eta_1-\eta_2-\eta_3-\eta_4}\\
&\quad= \big[d(\theta_{m-1},\theta_m)\big]^{\eta_1}.\big[d(\theta_{m-1},\theta_m)\big]^{\eta_2}.\big[d(\theta_m,\theta_{m+1})\big]^{\eta_3}.\left[\frac{d(\theta_{m-1},\theta_{m+1})}{2}\right]^{\eta_4}\\
&\qquad\times\left[\frac{d(\theta_{m-1},\theta_m)+d(\theta_m,\theta_{m+1})+d(\theta_{m-1},\theta_{m+1})+d(\theta_m,\theta_m)}{4}\right]^{1-\eta_1-\eta_2-\eta_3-\eta_4}\\
&\quad= \big[d(\theta_{m-1},\theta_m)\big]^{\eta_1+\eta_2}.\big[d(\theta_m,\theta_{m+1})\big]^{\eta_3}.\left[\frac{d(\theta_{m-1},\theta_{m+1})}{2}\right]^{\eta_4}\\
&\qquad\times\left[\frac{d(\theta_{m-1},\theta_m)+d(\theta_m,\theta_{m+1})+d(\theta_{m-1},\theta_{m+1})}{4}\right]^{1-\eta_1-\eta_2-\eta_3-\eta_4}.
\end{aligned}\tag{3.5}$$

Suppose that $d(\theta_{m-1},\theta_m)\le d(\theta_m,\theta_{m+1})$ and $d(\theta_{m-1},\theta_{m+1})\le d(\theta_{m-1},\theta_m)+d(\theta_m,\theta_{m+1})$ for some $m\ge 1$, then we have

$$\frac{d(\theta_{m-1},\theta_{m+1})}{2}\le\frac{d(\theta_{m-1},\theta_m)+d(\theta_m,\theta_{m+1})}{2}\le d(\theta_m,\theta_{m+1}).\tag{3.6}$$

Applying (3.6) in (3.5), we have

$$G_B(\theta_{m-1},\theta_m)\le\big[d(\theta_{m-1},\theta_m)\big]^{\eta_1+\eta_2}.\big[d(\theta_m,\theta_{m+1})\big]^{1-\eta_1-\eta_2}.$$

It follows from (3.4) that

$$d(\theta_m,\theta_{m+1})\le\beta\big[d(\theta_{m-1},\theta_m)\big]^{\eta_1+\eta_2}.\big[d(\theta_m,\theta_{m+1})\big]^{1-\eta_1-\eta_2}.$$

Or,

$$d(\theta_m,\theta_{m+1})^{\eta_1+\eta_2}\le\beta\big[d(\theta_{m-1},\theta_m)\big]^{\eta_1+\eta_2},$$

which implies that

$$d(\theta_m,\theta_{m+1})\le\beta d(\theta_{m-1},\theta_m).\tag{3.7}$$

Due to mathematical induction and (3.7), one can write

$$d(\theta_m,\theta_{m+1})\le\beta^m d(\theta_0,\theta_1).$$

Since $\beta\in(0,1)$, then

$$d(\theta_m,\theta_{m+1})\to 0,\quad \text{as } m\to\infty.$$

Since Λ is complete, there is $\theta^* \in \Lambda$ such that the segment $[\theta, \Im\theta]$ lies in Λ, then $\lim_{m\to\infty} \theta_m = \theta^*$. The continuity of $\Im$ implies that

$$d\big(\theta^*, \Im\theta^*\big) = \lim_{m\to\infty} d(\theta_m, \Im\theta_m) = \lim_{m\to\infty} d(\theta_m, \theta_{m+1}) = 0.$$

Hence, θ^* is a FP of $\Im$. Ultimately, Put $\theta = \theta_m$, $\vartheta = \theta^*$ in (3.1) and using $\theta^* = \Im\theta^*$, we get

$$d\big(\theta^*, \overline{\Im}\theta^*\big) < d\big(\Im\theta_m, \overline{\Im}\theta^*\big) \le \beta G_B\big(\theta_m, \theta^*\big), \tag{3.8}$$

where

$$\begin{aligned} &G_B\big(\theta_m, \theta^*\big) \\ &\quad = \big[d\big(\theta_m, \theta^*\big)\big]^{\eta_1}.\big[d(\theta_m, \Im\theta_m)\big]^{\eta_2}.\big[d\big(\theta^*, \overline{\Im}\theta^*\big)\big]^{\eta_3}.\left[\frac{d(\theta_m, \Im\theta^*)}{2}\right]^{\eta_4} \\ &\qquad \times \left[\frac{d(\theta_m, \Im\theta_m) + d(\theta^*, \overline{\Im}\theta^*) + d(\theta_m, \overline{\Im}\theta^*) + d(\theta^*, \Im\theta_m)}{4}\right]^{1-\eta_1-\eta_2-\eta_3-\eta_4} \\ &\quad \le \big[d\big(\theta^*, \theta^*\big)\big]^{\eta_1}.\big[d\big(\theta^*, \Im\theta^*\big)\big]^{\eta_2}.\big[d\big(\theta^*, \overline{\Im}\theta^*\big)\big]^{\eta_3}.\left[\frac{d(\theta^*, \Im\theta^*)}{2}\right]^{\eta_4} \\ &\qquad \times \left[\frac{d(\theta^*, \Im\theta^*) + d(\theta^*, \overline{\Im}\theta^*) + d(\theta^*, \overline{\Im}\theta^*) + d(\theta^*, \Im\theta^*)}{4}\right]^{1-\eta_1-\eta_2-\eta_3-\eta_4} \\ &\quad \le d\big(\theta^*, \overline{\Im}\theta^*\big). \end{aligned}$$

It follows from (3.8) that

$$d\big(\theta^*, \overline{\Im}\theta^*\big) < d\big(\Im\theta_m, \overline{\Im}\theta^*\big) \le \beta d\big(\theta^*, \overline{\Im}\theta^*\big),$$

yields

$$d\big(\theta^*, \overline{\Im}\theta^*\big) \le 0,$$

which is a contradiction. Hence, θ^* is a FP of $\overline{\Im}$. This ends the proof. □

Corollary 3.2 *Assume that* (Ω, d) *is a complete GS,* $\Lambda \in \Omega$ *is a connected open set and* $\Im : \Lambda \to \Lambda$ *is a local radial IHC mapping. Consider extending* $\Im$ *to a continuous mapping* $\overline{\Im} : \overline{\Lambda} \to \overline{\Lambda}$. *If there are* $\eta_1, \eta_2, \eta_3, \eta_4 \in [0, 1)$ *with* $\eta_1 + \eta_2 + \eta_3 + \eta_4 < 1$ *and* $\beta \le 1$ *such that*

$$\begin{aligned} &\big[d(\Im\theta, \overline{\Im}\vartheta)\big]^{\eta_1}.\big[d(\theta, \Im\theta)\big]^{\eta_2}.\big[d(\vartheta, \overline{\Im}\vartheta)\big]^{\eta_3}.\left[\frac{d(\theta, \Im\vartheta)}{2}\right]^{\eta_4} \\ &\qquad \times \left[\frac{d(\theta, \Im\theta) + d(\vartheta, \overline{\Im}\vartheta) + d(\theta, \overline{\Im}\vartheta) + d(\vartheta, \Im\theta)}{4}\right]^{1-\eta_1-\eta_2-\eta_3-\eta_4} \\ &\quad \le \beta d(\theta, \vartheta), \end{aligned}$$

for each $\theta, \vartheta \in \Lambda$ *and* $\Lambda \in \Omega$. *Then* $\overline{\Im}$ *owns a FP in* $\overline{\Lambda}$ *and moreover* $\{\Im^m\theta\}$ *converges to* θ^* *for each* $\theta \in \Lambda$.

Proof The proof follows immediately from Theorem 3.1. □

The example below supports Theorem 3.1.

Example 3.3 Consider $\Omega = [0,1]$ with the standard path metric $d(\theta,\vartheta) = \|\theta - \vartheta\|$. Describe the mapping $\Im : \Lambda \to \Lambda$ as

$$S\theta = \overline{S}\theta = \begin{cases} \theta^q, & \theta \geq 0, \\ \infty, & \theta < 0. \end{cases}$$

Our current job is to confirm the inequality (3.1). For this, let $\beta = 1$, $\eta_1 = \frac{1}{3}$, $\eta_2 = \frac{1}{4}$, $\eta_3 = \frac{1}{6}$ and $\eta_4 = \frac{1}{15}$. If $\Im\theta = \overline{\Im\vartheta}$, then $d(\Im\theta, \overline{\Im}\vartheta) = 0$. Hence, (3.1) is true directly. Otherwise, we compute the subsequent path metrics. For $\theta, \vartheta \in \Omega$ with $\theta = \frac{1}{2}$, $\vartheta = \frac{1}{3}$ and $q = 2$, one has

$$\begin{aligned}
&d(\Im\theta, \overline{\Im}\vartheta) = d\big(\theta^q, \vartheta^q\big) = \big|\theta^q - \vartheta^q\big| = \left|\left(\frac{1}{2}\right)^2 - \left(\frac{1}{3}\right)^2\right| = \frac{5}{36}, \\
&d(\theta, \vartheta) = |\theta - \vartheta| = \frac{1}{6}, \\
&d(\theta, \Im\theta) = \big|\theta - \theta^q\big| = \frac{1}{4}, \\
&d(\vartheta, \overline{\Im}\vartheta) = \big|\vartheta - \vartheta^q\big| = \frac{2}{9}, \\
&d(\theta, \overline{\Im}\vartheta) = \big|\theta - \vartheta^q\big| = \frac{7}{18}, \\
&d(\vartheta, \Im\theta) = \big|\vartheta - \theta^q\big| = \frac{1}{12}.
\end{aligned}$$

Using the equality in (3.1), one may infer that

$$\big|\theta^q - \vartheta^q\big| \leq \beta G_B(\theta, \vartheta), \tag{3.9}$$

where

$$\begin{aligned}
G_B(\theta, \vartheta) &= \big[d(\theta, \vartheta)\big]^{\eta_1} . \big[d(\theta, \Im\theta)\big]^{\eta_2} . \big[d(\vartheta, \overline{\Im}\vartheta)\big]^{\eta_3} . \left[\frac{d(\theta, \Im\vartheta)}{2}\right]^{\eta_4} \\
&\quad \times \left[\frac{d(\theta, \Im\theta) + d(\vartheta, \overline{\Im}\vartheta) + d(\theta, \overline{\Im}\vartheta) + d(\vartheta, \Im\theta)}{4}\right]^{1-\eta_1-\eta_2-\eta_3-\eta_4} \\
&= \big[|\theta - \vartheta|\big]^{\eta_1} . \big[\big|\theta - \theta^q\big|\big]^{\eta_2} . \big[\big|\vartheta - \vartheta^q\big|\big]^{\eta_3} . \left[\frac{|\theta - \vartheta^q|}{2}\right]^{\eta_4} \\
&\quad \times \left[\frac{|\theta - \vartheta| + |\theta - \theta^q| + |\theta - \vartheta^q| + |\vartheta - \theta^q|}{4}\right]^{1-\eta_1-\eta_2-\eta_3-\eta_4} \\
&= \left[\frac{1}{6}\right]^{\frac{1}{3}} . \left[\frac{1}{4}\right]^{\frac{1}{4}} . \left[\frac{2}{9}\right]^{\frac{1}{6}} . \left[\frac{7}{36}\right]^{\frac{1}{15}} . \left[\frac{\frac{1}{4} + \frac{7}{18} + \frac{7}{36} + \frac{1}{12}}{4}\right]^{\frac{11}{60}} \\
&= \left[\frac{1}{6}\right]^{\frac{1}{3}} . \left[\frac{1}{4}\right]^{\frac{1}{4}} . \left[\frac{2}{9}\right]^{\frac{1}{6}} . \left[\frac{7}{36}\right]^{\frac{1}{15}} . \left[\frac{11}{48}\right]^{\frac{11}{60}} \approx 0.20726.
\end{aligned} \tag{3.10}$$

Applying (3.10) in (3.9) with $\beta = 1$ and $|\theta^q - \vartheta^q| = \frac{5}{36}$, we find that

$$\left|\theta^q - \vartheta^q\right| = \frac{5}{36} \approx 0.13889 < 0.20726 = \beta G_B(\theta, \vartheta).$$

Therefore, all requirements of Theorem (3.1) are fulfilled. Hence, $\Im$ has FPs 0 and 1 in Ω.

4 Solving the time-fractional Navier–Stokes equations

This part is considered as one of the main pillars of our paper, where theoretical results are involved to study the existence of a solution to a nonlinear partial fractional DE in the context of a BS.

The Navier–Stokes equations (NSEs) explain the conservation of mass and momentum while describing the motion of incompressible Newtonian fluid flows, which can include everything from the lubrication of ball bearings to large-scale atmospheric dynamics.

In this part, we study the time-fractional NSEs in an open set $\Omega \subset \mathbb{R}^m$, $m \geq 3$.

The time-fractional NSEs that are presented below were motivated by Zhou et al. [40]:

$$\begin{cases} \partial_\tau^\eta \theta - s\Delta\theta + (\theta.\nabla)\theta = -\nabla q + \xi, & \tau \geq 0, \\ \nabla.\theta = 0, \\ \theta|_\Omega = 0, \\ \theta(0, \nu) = \ell, \end{cases} \tag{4.1}$$

where ∂_τ^η refers to the CFD of order $\eta \in (0, 1)$,

$$\theta = \big(\theta_1(\tau, \nu), \theta_2(\tau, \nu), \theta_3(\tau, \nu), \dots, \theta_m(\tau, \nu)\big)$$

refers to the velocity fields at a point $\nu \in \Omega$ and time $\tau > 0$, $q = q(\tau, \theta)$ stands for the pressure, s is the velocity, $\xi(\tau, \nu)$ is the external force and $\ell = \ell(\nu)$ is the initial velocity. Further, Ω is considered here to have a smooth boundary.

Helmholtz projector P in [41] can be used to write equation (4.1) as follows:

$$\begin{cases} \partial_\tau^\eta \theta - sP\Delta\theta + P(\theta.\nabla)\theta = P\xi, & \tau \geq 0, \\ \nabla.\theta = 0, \\ \theta|_\Omega = 0, \\ \theta(0, \nu) = \ell. \end{cases} \tag{4.2}$$

In the divergence-free function space, Stokes operator Z is the operator $-sP\Delta$ with Dirichlet type boundary conditions. Consequently, (4.2) can be expressed as follows:

$$\begin{cases} {}_0^C D_\tau^\eta \theta = -Z\theta + R(\theta, \theta) + P\xi, & \tau > 0, \\ \theta(0) = \ell, \end{cases} \tag{4.3}$$

where $R(\theta, \vartheta) = -P(\theta.\nabla)\vartheta$. The solution of Eq. (4.3) is also a solution of Eq. (4.1), provided that one makes sense of the Stokes operator Z and the Helmholtz projector P.

Assume that $\Omega = C([0,1])$ is the space of all continuous functions on $[0,1]$. Clearly, $(\Omega, \|.\|)$ is a Banach space under the usual norm. Consider $J = [0,S]$, $S > 0$, and let $C(J,\Omega)$ be a Banach space of all continuous functions from I into Ω with the path metric

$$d(\theta,\vartheta) = \sup_{\tau\in J}\big|\theta(\tau) - \vartheta(\tau)\big|, \quad \text{for all } \theta,\vartheta \in \Omega.$$

It is clear that (Ω, d) is a complete BS.

It is possible to write the time-fractional NSE (4.3) as

$$\theta(\tau) = \ell(\tau) + \frac{1}{\Gamma(\eta)}\int_0^\tau (\tau - r)^{\eta-1}\big(Z\theta(r) + R\big(\theta(r),\theta(r)\big) + P\xi(r)\big)\,dr, \quad \tau \geq 0.$$

To present the main theorem of this part, we need the following hypotheses:

(H_1) For all $\theta,\vartheta \in \Omega$ and all $\tau \in [0,1]$, there is a continuous function $\mho : [0,1] \times \mathbb{R}^m \to \mathbb{R}$ such that

$$\big|\mho\big(\tau,\theta(r)\big) - \mho\big(\tau,\vartheta(r)\big)\big| \leq \varrho\big|\theta(r) - \vartheta(r)\big|,$$

where $\varrho > 0$ and $\mho(\tau,\theta(r)) = Z\theta(r) + R(\theta(r),\theta(r)) + P\xi(r)$,

(H_2) For $\theta,\vartheta \in \Lambda$ and $\Lambda \in \mathbb{R}^n$ there are $\eta_1,\eta_2,\eta_3,\eta_4 \in [0,1)$ with $\eta_1+\eta_2+\eta_3+\eta_4 < 1$ and $\beta \leq 1$, such that

$$d(\Im\theta,\overline{\Im}\vartheta) \leq \beta G_B(\theta,\vartheta),$$

where

$$G_B(\theta,\vartheta) = \big[d(\theta,\vartheta)\big]^{\eta_1}.\big[d(\theta,\Im\theta)\big]^{\eta_2}.\big[d(\vartheta,\overline{\Im}\vartheta)\big]^{\eta_3}.\left[\frac{d(\theta,\overline{\Im}\vartheta)}{2}\right]^{\eta_4} \\ \times \left[\frac{d(\theta,\Im\theta) + d(\vartheta,\overline{\Im}\vartheta) + d(\theta,\overline{\Im}\vartheta) + d(\vartheta,\Im\theta)}{4}\right]^{1-\eta_1-\eta_2-\eta_3-\eta_4},$$

(H_3) The parameter β takes the form

$$\beta = \frac{\varrho\tau^\eta}{\Gamma(\eta+1)} \leq 1.$$

Theorem 4.1 *Under Hypotheses* (H_1)–(H_3), *Problem* (4.3) *has a solution in* $C(J,\Omega)$.

Proof Define the mapping $\Im : C(J) \to C(J)$ by

$$\Im\theta(\tau) = \ell(\tau) + \frac{1}{\Gamma(\eta)}\int_0^\tau (\tau - r)^{\eta-1}\big(Z\theta(r) + R\big(\theta(r),\theta(r)\big) + P\xi(r)\big)\,dr, \tag{4.4}$$

for $\tau \in [0,1]$. Clearly, finding a solution to the time-fractional NSE (4.3) is equivalent to find a FP for the mapping $\Im$. We demonstrate that $\Im$ is an IHC mapping in order to demonstrate the existence of a FP of $\Im$.

From the assumptions (H_1) and (H_3), one has

$$|\Im\theta - \Im\vartheta| = \bigg|\ell(r) + \frac{1}{\Gamma(\eta)}\int_0^\tau (\tau - r)^{\eta-1}\big(Z\theta(r) + R\big(\theta(r),\theta(r)\big) + P\xi(r)\big)\,dr$$

$$
\begin{aligned}
&-\left(\ell(r)+\frac{1}{\Gamma(\eta)}\int_0^\tau(\tau-r)^{\eta-1}\big(Z\vartheta(r)+R\big(\vartheta(r),\vartheta(r)\big)+P\xi(r)\big)\,dr\right)\Bigg| \\
&\leq \frac{1}{\Gamma(\eta)}\int_0^\tau(\tau-r)^{\eta-1} \\
&\quad\times\Big|\big(Z\theta(r)+R\big(\theta(r),\theta(r)\big)+P\xi(r)\big)-\big(Z\vartheta(r)+R\big(\vartheta(r),\vartheta(r)\big)+P\xi(r)\big)\Big|\,dr \\
&= \frac{1}{\Gamma(\eta)}\int_0^\tau(\tau-r)^{\eta-1}\big|\mho\big(\tau,\theta(r)\big)-\mho\big(\tau,\vartheta(r)\big)\big|\,dr \\
&\leq \frac{\varrho\tau^\eta}{\Gamma(\eta+1)}\big|\theta(r)-\vartheta(r)\big| \\
&= \beta\big|\theta(r)-\vartheta(r)\big|.
\end{aligned}
$$

Applying Hypothesis (H_2), we can write

$$
\begin{aligned}
d(\Im\theta,\Im\vartheta) &= |\Im\theta-\Im\vartheta| \\
&\leq \beta\big|\theta(r)-\vartheta(r)\big| = \beta d(\theta,\vartheta) \\
&\leq \beta G_B(\theta,\vartheta) \\
&= \beta\Bigg(\big[d(\theta,\vartheta)\big]^{\eta_1}.\big[d(\theta,\Im\theta)\big]^{\eta_2}.\big[d(\vartheta,\overline{\Im}\vartheta)\big]^{\eta_3}.\left[\frac{d(\theta,\overline{\Im}\vartheta)}{2}\right]^{\eta_4} \\
&\quad\times\left[\frac{d(\theta,\Im\theta)+d(\vartheta,\overline{\Im}\vartheta)+d(\theta,\overline{\Im}\vartheta)+d(\vartheta,\Im\theta)}{4}\right]^{1-\eta_1-\eta_2-\eta_3-\eta_4}\Bigg),
\end{aligned}
$$

which proves that $\Im$ is an IHC. Thanks to Theorem 3.1, the mapping $\Im$ has a FP, which is a solution to the time-fractional NSE (4.3). □

5 Solving a functional-fractional differential equation

This part is considered the second pillar of our paper, where Theorem 3.1 is applied to solve a functional-fractional DE in the setting of a BS. Functional DEs are known to accurately explain a wide range of complicated phenomena in nature and industry. It is utilized, for instance, in the fields of neural networks, epidemiology, automatic control, bionomic and electronics.

Here, we take into account the initial value problems (IVPs) of fractional neuronal functional DEs with a constrained delay of the type, which is inspired by Zhou et al. [40]:

$$
\begin{cases}
{}^{C}_{\tau_0}D^{\eta}_{\tau}(\theta(\tau)-S(\tau,\theta_\tau))=G(\tau,\theta_\tau), & \tau\in(\tau_0,\tau_0+c), c>0, \\
\theta_{\tau_0}=\phi,
\end{cases}
\tag{5.1}
$$

where ${}^{C}_{\tau_0}D^{\eta}_{\tau}$ is the CFD of order $\eta\in(0,1)$, $S,G:I\times C\to\mathbb{R}^m$ are continuous functions and $\phi\in\Xi$. Let $\Xi=C(I_0,\mathbb{R}^m)$ be the space of all continuous functions on I_0 and for any $\phi\in\Xi$, define the norm

$$
\|\phi\|_\Xi=\sup_{\tau\in I_0}\big|\phi(\tau)\big|.
$$

Consider $I_0=[-c,0]$, $c>0$ and $I=[\tau_0,\tau_0-[0]+\rho]$, $\rho>0$ are two bounded and closed intervals in $\mathbb{R}^m$. Set $\aleph=[\tau_0-c,\tau_0+\rho]$.

Assume that the space $\Omega = \aleph$ of the continuous functions on I_0 and $C(\aleph, \Omega)$ is a Banach space of all continuous functions from I into Ω with the path metric

$$d(\theta, \vartheta) = \sup_{\tau \in I_0} \bigl|\theta(\tau) - \vartheta(\tau)\bigr|, \quad \text{for all } \theta, \vartheta \in \Omega.$$

It is clear that (Ω, d) is a complete BS.

The following equation represents the solution to the fractional IVP (5.1):

$$\theta(\tau) = \phi(0) - S(\tau_0, \theta_{\tau_0}) + S(\tau, \theta_\tau) + \frac{1}{\Gamma(\eta)} \int_{\tau_0}^{\tau} (\tau - r)^{\eta-1} G(r, \theta_r)\, dr, \quad \tau \in I_0,$$

where $\theta_{\tau_0} = \phi$. Also, we need Holder's inequality [42] below:

If $p \geq 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\theta_j, \vartheta_j > 0$, we have

$$\sum_{j=1}^{m} \theta_j \vartheta_j \leq \left(\sum_{j=1}^{m} \theta_j^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{m} \vartheta_j^q\right)^{\frac{1}{q}}.$$

To present our main theorem here, we strongly need the following assumptions:

(A_1) There are continuous functions $S, G : [0,1] \times C \to \mathbb{R}^m$ such that

$$\bigl|S\bigl(\tau, \theta(r)\bigr) - S\bigl(\tau, \vartheta(r)\bigr)\bigr| \leq \varrho_1 \bigl|\theta(r) - \vartheta(r)\bigr|,$$

and

$$\bigl|G\bigl(\tau, \theta(r)\bigr) - G\bigl(\tau, \vartheta(r)\bigr)\bigr| \leq \varrho(r) \bigl|\theta(r) - \vartheta(r)\bigr|,$$

for all $\tau, r \in [0,1]$ and all $\theta, \vartheta \in \Omega$, where $\varrho_1 > 0$ and $\varrho : [0,1] \to \mathbb{R}^m$ is a given function.

(A_2) The condition (H_2) holds.

(A_3) The parameter β takes the form

$$\beta = \varrho_1 + \frac{\varrho \rho^{\eta(1-\varkappa)}}{(1-\varkappa)\Gamma(\eta+1)}, \quad \text{for } \varkappa > 1.$$

Theorem 5.1 *In the light of assumptions* (A_1)–(A_3), *Problem* (5.1) *has a solution in* $C(\aleph, \Omega)$.

Proof Describe the mapping $\Im : C(J) \to C(J)$ by

$$\Im\theta(\tau) = \phi(0) - S(\tau_0, \theta_{\tau_0}) + S(\tau, \theta_\tau) + \frac{1}{\Gamma(\eta)} \int_{\tau_0}^{\tau} (\tau - r)^{\eta-1} G(r, \theta_r)\, dr, \quad \tau \in I_0, \tag{5.2}$$

where $\theta_{\tau_0} = \phi$. Clearly, finding a solution to Problem (5.2) is equivalent to find a FP for the mapping $\Im$. We demonstrate that $\Im$ is an IHC mapping to prove the existence of a FP of $\Im$. □

From the assertions (A_1), (A_3) and Holder's inequality, we get

$$\begin{aligned}
|\Im\theta - \Im\vartheta| &= \left|\phi(0) - S(\tau_0,\theta_{\tau_0}) + S(\tau,\theta_\tau) + \frac{1}{\Gamma(\eta)}\int_{\tau_0}^{\tau}(\tau-r)^{\eta-1}G(r,\theta_r)\,dr\right.\\
&\quad\left. - \left(\phi(0) - S(\tau_0,\vartheta_{\tau_0}) + S(\tau,\vartheta_\tau) + \frac{1}{\Gamma(\eta)}\int_{\tau_0}^{\tau}(\tau-r)^{\eta-1}G(r,\vartheta_r)\,dr\right)\right|\\
&\leq \left|S(\tau,\theta_\tau) - S(\tau,\vartheta_\tau)\right| + \frac{1}{\Gamma(\eta)}\int_0^{\tau}(\tau-r)^{\eta-1}\left|G(r,\theta_r) - G(r,\vartheta_r)\right|dr\\
&\leq \varrho_1|\theta_\tau - \vartheta_\tau| + \frac{1}{\Gamma(\eta)}\int_0^{\tau}(\tau-r)^{\eta-1}\varrho(r)|\theta_r - \vartheta_r|\,dr\\
&\leq \varrho_1\|\theta-\vartheta\| + \frac{1}{\Gamma(\eta)}\left(\int_0^{\tau}(\tau-r)^{\frac{\eta-1}{1-\varkappa}}\right)^{1-\varkappa}\left(\|\varrho\|^{\frac{1}{\varkappa}}\right)\|\theta-\vartheta\|\,dr\\
&\leq \left(\varrho_1 + \frac{\varrho\rho^{\eta(1-\varkappa)}}{(1-\varkappa)\Gamma(\eta+1)}\right)\|\theta-\vartheta\|\\
&= \beta\|\theta-\vartheta\|.
\end{aligned}$$

Using Assumption (H_2), we conclude that

$$\begin{aligned}
d(\Im\theta,\Im\vartheta) &= |\Im\theta - \Im\vartheta|\\
&\leq \beta\left|\theta(r) - \vartheta(r)\right| = \beta d(\theta,\vartheta)\\
&\leq \beta G_B(\theta,\vartheta)\\
&= \beta\left(\left[d(\theta,\vartheta)\right]^{\eta_1}.\left[d(\theta,\Im\theta)\right]^{\eta_2}.\left[d(\vartheta,\overline{\Im}\vartheta)\right]^{\eta_3}.\left[\frac{d(\theta,\overline{\Im}\vartheta)}{2}\right]^{\eta_4}\right.\\
&\quad\left.\times\left[\frac{d(\theta,\Im\theta) + d(\vartheta,\overline{\Im}\vartheta) + d(\theta,\overline{\Im}\vartheta) + d(\vartheta,\Im\theta)}{4}\right]^{1-\eta_1-\eta_2-\eta_3-\eta_4}\right).
\end{aligned}$$

Hence, $\Im$ is an IHC. From Theorem 3.1, the mapping $\Im$ has a FP as $\theta^* C(\aleph,\Omega)$, which is a solution to the fractional IVP (5.1).

6 Conclusion

The FP result presented in Theorem 3.1 is our main contribution to FP theory. The FP requirements for a large class of interpolative self-maps based on the output of IHC mappings in the CH combination of a BS, which is a complete GS, are provided by this theorem. This study generalizes and unifies a number of previous findings in the same direction. To support the findings, an example is derived. Finally, applications to the existence of a solution for nonlinear fractional-functional DEs and nonlinear partial fractional DEs are discussed.

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Abbreviations
MS,Metric space; BS,Busemann space; HS,Hyperbolic space; GS,Geodesic space; LTM,Laplace transform method; CP,Cauchy problem; DE,Differential equation; CFD,Caputo fractional derivative; FP,Fixed point; CRR,'Cirić-Reich–Rus;

CH,Convex hull; IKC,Interpolative Kannan contraction; IHC,Interpolative hybrid contraction; NSE,Navier–Stokes equation; IVP,Initial value problems.

Data availability
Data sharing is not applicable to this article, as no data sets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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