Wang and Jiang *Boundary Value Problems* (2024) 2024:7
https://doi.org/10.1186/s13661-023-01808-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Time decay of solutions for compressible isentropic non-Newtonian fluids

Jialiang Wang[1] and Han Jiang[1*]

*Correspondence:
jianghan19940116@163.com
[1]College of Mathematics and Statistics, Fuzhou University, Fuzhou, 350108, China

Abstract
In this paper, we consider the Cauchy problem of a compressible Navier–Stokes system of Eills-type non-Newtonian fluids. We investigate the time decay properties of classical solutions for the compressible non-Newtonian fluid equations. More specifically, we construct a new linearized system in terms of a combination of the solutions, and then we investigate the long-time behavior of the Cauchy problem for the three-dimensional isentropic compressible Eills-type non-Newtonian fluids with an initial perturbation.

Keywords: Non-Newtonian fluids; Cauchy problem; Classical solution; Time decay

1 Introduction

Motivated by the well-posedness result of non-Newtonian fluids in [1] and the long-time behavior result of the Navier–Stokes system in [2], we investigate the time decay properties of solutions for compressible non-Newtonian fluid equations. More specifically, we investigate the optimal decay rate of the highest-order derivative of solutions to the equations of compressible non-Newtonian fluids, which defined in a bounded domain $\Omega \subset \mathbb{R}^3$ is governed by the following equations [3]:

$$\begin{cases} \rho_t + \operatorname{div}(\rho \boldsymbol{v}) = 0, \\ (\rho v)_t + \operatorname{div}(\rho \boldsymbol{v} \otimes \boldsymbol{v}) + \nabla p(\rho) = \operatorname{div} \mathbb{S}, \end{cases} \tag{1.1}$$

where the unknowns $\rho := \rho(x,t)$, $\boldsymbol{v} := \boldsymbol{v}(x,t)$ denote the density and the velocity of the non-Newtonian fluids, respectively, $p := p(\rho)$ is the fluid pressure, which is a smooth function depending on ρ, and $\mathbb{S}$ represents the viscous stress tensor, which depends on the rate of strain $D_{ij}(\nabla \boldsymbol{v})$, where $D_{ij}(\nabla \boldsymbol{v})$ is given as $D_{ij}(\nabla \boldsymbol{v}) = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}$. We mention that $(1.1)_1$ is the continuity equation and $(1.1)_2$ describes the balance law of momentum.

If the relation between the stress and the rate of strain is linear, i.e., $S_{ij} = \mu(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$, then the fluid is called Newtonian. The coefficient μ is called the viscosity coefficient, which depends on temperature, density, and pressure. For example, water, alcohols, and simple hydrocarbon compounds turn out to be Newtonian fluids. The governing equations of motion can be written by the Navier–Stokes equations.

If the relation between the stress and the rate of strain is nonlinear, the fluid is called non-Newtonian. Examples of non-Newtonian fluids include molten plastics, greases, paper pulp, and biological fluids like blood. The simplest stress–strain relation in non-Newtonian fluids is given by $S_{ij} = \mu(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})^q$ for $0 < q < 1$ [4].

Recently, the following stress–strain relation has been widely investigated [3]:

$$S_{ij} = \left(\mu_0 + \mu_1 \left|D(\nabla \boldsymbol{v})\right|^{p-2}\right) D_{ij}(\nabla \boldsymbol{v}),$$

where

$$\begin{cases} \text{Newtonian,} & \text{for } \mu_0 > 0, \mu_1 = 0; \\ \text{Rabinowitsch,} & \text{for } \mu_0, \mu_1 > 0, p = 4; \\ \text{Eills,} & \text{for } \mu_0, \mu_1 > 0, p > 2; \\ \text{Ostwald–de Waele,} & \text{for } \mu_0 = 0, \mu_1 > 0, p > 1; \\ \text{Bingham,} & \text{for } \mu_0, \mu_1 > 0, p = 1. \end{cases}$$

The above five types of stress–strain relation can be found in non-Newtonian fluids. For example, for $\mu_0 = 0$, it is a pseudo-plastic fluid in the case of $p < 2$ and it is a dilatant fluid in the case of $p > 2$. From a physical point of view, the stress–strain relation describes a shear thickening fluid if $p > 2$ and a shear thinning fluid if $1 < p < 2$. The values of the parameters p and μ_1 of the pseudo-plastic Ostwald–de Waele models are presented in [3].

On account of the physical importance of non-Newtonian fluids, they have attracted attention from many engineers, mathematicians, physicists, and so on. However, both the problems of well-posedness and dynamical behaviors of motion equations of non-Newtonian fluids are very difficult to investigate because of the singularity.

Even so, important progress has been made in the theoretical analysis of non-Newtonian fluid systems: Bothe–Pruss studied a class of non-Newtonian fluids based on L_p-theory [5]; Feireisl–Kwon studied the long-time behavior of dissipative solutions to models of non-Newtonian compressible fluids [6]; Moscariello–Porzio investigated the behavior in time of solutions to motion of Non-Newtonian fluids [7]. More results of non-Newtonian fluids can be found in [1, 3, 8–11] and the references cited therein.

In this paper, we investigate the corresponding Eills-type non-Newtonian fluids, i.e.,

$$\begin{aligned} &\mathbb{S} = \left(\mu_0 + \mu_1 \left|D(\nabla \boldsymbol{v})\right|^{p-2}\right) D(\nabla \boldsymbol{v}), \\ &\operatorname{div} \mathbb{S} = \mu_0(\Delta \boldsymbol{v} + \nabla \operatorname{div} \boldsymbol{v}) + \mu_1 \operatorname{div}\left(\left|D(\nabla \boldsymbol{v})\right|^{p-2} D(\nabla \boldsymbol{v})\right). \end{aligned}$$

Thus, equation (1.1) reduces to

$$\begin{cases} \rho_t + \operatorname{div}(\rho \boldsymbol{v}) = 0, \\ \rho \boldsymbol{v}_t + \operatorname{div}(\rho \boldsymbol{v} \otimes \boldsymbol{v}) + \nabla p(\rho) - \mu_0(\Delta \boldsymbol{v} + \nabla \operatorname{div} \boldsymbol{v}) = \mu_1 \operatorname{div}(|D(\nabla \boldsymbol{v})|^{p-2} D(\nabla \boldsymbol{v})). \end{cases} \tag{1.2}$$

We can reformulate the system (1.2). Without loss of generality, we take $P'(1) = 1$ and thus derive the following equations from (1.2) by applying the formulas above:

$$\begin{cases} \rho_t + \operatorname{div}(\rho \boldsymbol{v}) = 0, \\ \boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + p'(\rho)\nabla\rho/\rho - \mu_0(\Delta \boldsymbol{v} + \nabla \operatorname{div} \boldsymbol{v})/\rho \\ \quad = \mu_1 \operatorname{div}(|D(\nabla \boldsymbol{v})|^{p-2} D(\nabla \boldsymbol{v}))/\rho. \end{cases} \tag{1.3}$$

Much important progress has been made in the investigation of long-time behaviors of global smooth solutions to compressible Navier–Stokes systems. For instance, the global existence of strong solutions to compressible Navier–Stokes equations in multidimensional whole space was obtained first by Matsumura–Nishida [12, 13], who also showed that the global solution tends to its equilibrium state in large time. The optimal $L^p (p \geq 2)$ decay rates were established later by Ponce [14]. To conclude, the optimal L^2 time decay rate for isentropic compressible Navier–Stokes equations in three dimensions is

$$\left\| (\rho - \bar{\rho}, \boldsymbol{v})(t) \right\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}},$$

where $(\bar{\rho}, \mathbf{0})$ is the constant state.

Then Liu–Wang [15] investigated the properties of the Green's function for isentropic Navier–Stokes systems and showed an interesting pointwise convergence of global solutions to the diffusive waves with the optimal time decay rate in odd dimension where the important phenomenon of the weaker Huygen principle is also justified due to the dispersion effects of compressible viscous fluids in multidimensional odd space. This was generalized to the full system later in [16], where the wave motions of other types are also introduced. Therein, the optimal L^∞ time decay rate in three dimensions is

$$\left\| (\rho - \bar{\rho}, \boldsymbol{v})(t) \right\|_{L^\infty(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}}.$$

The same decay property also appears in the exterior domain problem [17] and an infinite layer [18]. Li–Matsumura–Zhang [19] obtained the optimal time decay of the Navier–Stokes–Poisson system, which is different from the pure Navier–Stokes equations.

Next, Li–Zhang investigated the long-time behavior and optimal decay rates of global strong solutions to three-dimensional isentropic compressible Navier–Stokes systems; when the regular initial data also belong to some Sobolev space $H^l(\mathbb{R}^3) \cap \dot{B}^{-s}_{1,\infty}(\mathbb{R}^3)$ with $l \geq 4$ and $s \in [0,1]$, they show that the global solution to this system converges to the equilibrium state at a faster decay rate in time.

Later, Tan–Wang derived the optimal time decay rates for the higher-order spatial derivatives of solutions to magnetohydrodynamic equations [20]. Besides, Gao–Chen–Yao [21] further deduced higher time decay rates for the higher-order spatial derivatives of solutions, which improve the result of Tan–Wang [20]. Recently, exploiting the technique of decomposition of solutions into low and high frequencies in [22], Huang–Lin–Wang proved that this result also holds for $k = N$ [23]. At present, the above isentropic results have been further extended to non-isentropic cases; see [21, 24] for examples.

There are many other important results on time decay estimation. Abdallah–Jiang–Tan studied the decay estimates for isentropic compressible magnetohydrodynamic equations in a bounded domain [25], Fan–Jiang studied the long-time behavior of liquid crystal flows with a trigonometric condition in two dimensions [26], Guo–Tan studied the long-time behavior of solutions to a class of non-Newtonian compressible fluids [27], Chen–Tan–Wu have given the time decay rates for the equations of compressible heat-conductive flow through porous media [28], and Tan–Wu studied the long-time behavior of solutions for compressible Euler equations with damping in $\mathbb{R}^3$ [29]. Zhang studied the decay of the three-dimensional inviscid liquid–gas two-phase flow model [30] and the decay of the three-dimensional viscous liquid–gas two-phase flow model with damping [31]. Zhang–Wu studied the global well-posedness and long-time behavior of the viscous liquid–gas two-phase flow model [32] and the global existence and asymptotic behavior for the three-dimensional compressible non-isentropic Euler equations with damping [33]. Zhang–Tan studied the existence [34] and asymptotic behavior of global smooth solutions for p-systems with damping and boundary effects and the asymptotic behavior of solutions to the Navier–Stokes equations of a two-dimensional compressible flow [35]. Zhang–Tan–Ming studied the global existence and asymptotic behavior of smooth solutions to a coupled hyperbolic-parabolic system [36]. Jiang–Zhang studied the existence and asymptotic behavior of global smooth solutions for p-systems with nonlinear damping and fixed boundary effects [37]. Qiu–Zhang studied the decay of the three-dimensional quasilinear hyperbolic equations with nonlinear damping [38]. Then Hu–Qiu–Wang–Yang studied the incompressible limit for compressible viscoelastic flows with large velocity [39], Zhao–Li–Yan studied the global Sobolev regular solution for Boussinesq systems [40], and Panasenko–Pileckas studied the non-stationary Poiseuille flow of a non-Newtonian fluid with the shear rate-dependent viscosity [41].

In this paper, we are further interested in non-Newtonian fluids. The well-posedness problem of non-Newtonian fluids has been widely investigated; see [42] for the existence results. Motivated by the optimal time decay rates of solutions of Newtonian fluids and the result of the long-time behavior of solutions to non-Newtonian compressible fluids, using the methods in [43], we further investigate the global well-posedness and optimal time decay rates of the solutions for the system (1.2). To this purpose, we provide the following initial condition for the system (1.2):

$$(\rho, \boldsymbol{v})|_{t=0} = \left(\rho^0, \boldsymbol{v}^0\right). \tag{1.4}$$

We will consider the existence problem and the optimal time decay rates of small perturbation solutions around the rest state $(\bar{\rho}, \mathbf{0})$ of the system (1.2), where $\bar{\rho}$ is always taken to be equal to one for the sake of simplicity.

By taking the new change of variables

$$\varrho = \rho - 1, \qquad \boldsymbol{u} = \boldsymbol{v} - \mathbf{0},$$

we can further rewrite the system (1.3)–(1.4) into the perturbation forms:

$$\begin{cases} \varrho_t + \operatorname{div} \boldsymbol{u} = -\operatorname{div}(\varrho \boldsymbol{u}), \\ \boldsymbol{u}_t - \mu_0(\Delta \boldsymbol{u} + \nabla \operatorname{div} \boldsymbol{u}) + \nabla \varrho = \mathcal{N}, \\ (\varrho, \boldsymbol{u})|_{t=0} = (\varrho^0, \boldsymbol{u}^0) = (\rho^0 - 1, \boldsymbol{u}^0), \end{cases} \tag{1.5}$$

where the nonlinear terms $\mathcal{N}$ are defined by

$$\begin{aligned}\mathcal{N} = &-\boldsymbol{u}\cdot\nabla\boldsymbol{u} - h_3(\varrho)\nabla\varrho - h_2(\varrho)(\Delta\boldsymbol{u} + \nabla\operatorname{div}\boldsymbol{u})\\ &+ h_1(\varrho)\operatorname{div}\left(\left|D(\nabla\boldsymbol{u})\right|^{p-2}D(\nabla\boldsymbol{u})\right),\end{aligned}$$

where the nonlinear functions are defined as follows:

$$h_1(\varrho) = \frac{\mu_1}{\varrho+1}, \qquad h_2(\varrho) = \frac{\mu_0\varrho}{\varrho+1}, \qquad h_3(\varrho) = \frac{p'(\varrho+1)}{\varrho+1} - 1.$$

Before presenting our main result, we introduce the following notations, which are used frequently throughout the paper.

1.1 Notations

(1) Basic notations:

The notation $\langle\cdot,\cdot\rangle$ stands for the inner product in $L^2(\mathbb{R}^3)$; $a \lesssim b$ means that $a \leq Cb$ for some constant $C > 0$. For simplicity, we also denote $a \approx b$ if $a \lesssim b$ and $a \gtrsim b$. The symbol ∇^l with an integer $l \geq 1$ represents as usual any spatial derivatives of order l; $C_i > 0$ represents a generic constant that may vary from line to line for $i \in \mathbb{Z}^+$; and the integral symbol $\int := \int_{\mathbb{R}^3}$.

(2) Notations of function spaces:

We employ $L^r(\mathbb{R}^3)$ to denote the usual L^r spaces and $H^s(\mathbb{R}^3)$ to denote the Sobolev spaces with norm $\|\cdot\|_{L^r}$ and $\|\cdot\|_{H^s}$, respectively, where $1 \leq r \leq \infty, s \in \mathbb{R}$.

In addition, Λ is a pseudo-differential operator, which is defined by

$$\Lambda^s f = \mathcal{F}^{-1}\left(|\xi|^s\hat{f}\right) \quad \text{for } s \in \mathbb{R},$$

where $\hat{f}$ and $\mathcal{F}^{-1}(f)$ denote the Fourier transform and the inverse Fourier transform, respectively.

Let $\varphi(\xi)$ be a smooth cut-off function, which satisfies $0 \leq \varphi(\xi) \leq 1(\xi \in \mathbb{R}^3)$ and $\varphi(\xi) = 1, |\xi| \leq 1, \varphi(\xi) = 0, |\xi| > 1$. Then we can define a frequency decomposition for the function $f(x) \in L^2(\mathbb{R}^3)$ as follows:

$$f^L(x) = \varphi(D_x)f(x), \qquad f^H(x) = \left(I - \varphi(D_x)\right)f(x),$$

where $D_x := \frac{1}{\sqrt{-1}}(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ and $\varphi(D_x)$ is a pseudo-differential operator with respect to $\varphi(\xi)$. Moreover, $f(x)$ can be expressed as

$$f(x) = f^L(x) + f^H(x). \tag{1.6}$$

1.2 Main result

Now, we are in a position to present our main result.

Theorem 1.1 *Suppose* $(\rho^0 - 1, \boldsymbol{u}^0) \in H^3(\mathbb{R}^3)$ *and*

$$\left\|\left(\rho^0 - 1, \boldsymbol{u}^0\right)(t)\right\|_{H^3(\mathbb{R}^3)} \leq \epsilon,$$

where ϵ is a sufficiently small constant. Then the Cauchy problem (1.5) *admits a unique global solution* $(\rho, \boldsymbol{u})$ *satisfying*

$$\begin{aligned}
&\rho - 1 \in C^0\big([0,\infty); H^3\big(\mathbb{R}^3\big)\big) \cap C^1\big([0,\infty); H^2\big(\mathbb{R}^3\big)\big),\\
&\boldsymbol{u} \in C^0\big([0,\infty); H^3\big(\mathbb{R}^3\big)\big) \cap C^1\big([0,\infty); H^1\big(\mathbb{R}^3\big)\big).
\end{aligned}$$

Moreover, if the initial data $(\rho^0 - 1, \boldsymbol{u}^0)$ *are bounded in* $L^1(\mathbb{R}^3)$ *space, then, for any* $t \geq 0$, *the classical solution* $(\rho, \boldsymbol{u})$ *enjoys*

$$\big\|\nabla^k(\rho - 1, \boldsymbol{u})(t)\big\|_{H^3(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad k = 0, 1, 2, 3.$$

The first step involves constructing a new linearized system by combining the solutions. Subsequently, our focus is on establishing the global existence and uniqueness of the solution for the Cauchy problem (1.5). The a priori estimates will be provided in Proposition 3.1, where our main focus lies on estimating the nonlinear terms of a specific class of compressible Eills-type non-Newtonian fluids with $\mathbb{S} = (\mu_0 + \mu_1 |D(\nabla \boldsymbol{u})|^{p-2}) D(\nabla \boldsymbol{u})$. We employ an energy estimation method to address the challenges posed by nonlinear structures, thereby enabling us to obtain the energy estimation under the H^3 norm. Subsequently, we present a technique for eliminating the low frequency component and provide a decay estimate for this particular part. Finally, we establish decay rates for the nonlinear system.

The rest of this paper is organized as follows. First, in Sect. 2, we list some well-known mathematical results, which will be used in Sects. 3 and 4. In Sect. 3, we establish a priori estimates of solutions and then prove the existence of the global-in-time solution based on the local existence of unique solutions. In Sect. 4, we will obtain the optimal time decay rates of the non-homogeneous system by constructing some decay estimates of the linearized system based on the technique of decomposition of solutions into low and high frequencies [22].

2 Basic analysis tools

This section is devoted to providing some important mathematical results, which will be used in the next sections.

Lemma 2.1 ([22]). *For any given integers i, j, k, we have*

$$\begin{aligned}
&\big\|\nabla^j f^L\big\|_{L^2} \leq r_0^{j-i}\big\|\nabla^i f^L\big\|_{L^2}, \qquad \big\|\nabla^j f^H\big\|_{L^2} \leq \frac{1}{R_0^{k-i}}\big\|\nabla^k f^L\big\|_{L^2},\\
&\big\|\nabla^j f^L\big\|_{L^2} \leq \big\|\nabla^k f\big\|_{L^2} \quad \text{and} \quad \big\|\nabla^j f^H\big\|_{L^2} \leq \big\|\nabla^k f\big\|_{L^2},
\end{aligned}$$

where $f \in H^n(\mathbb{R}^3)$ and $i \leq j \leq k \leq n$. Moreover, we have

$$r_0^j\|f^n\|_{L^2} \leq \big\|\nabla^j f^n\big\|_{L^2} \leq R_0^j\|f^n\|_{L^2},$$

for some constant $r_0 > 0$ and $R_0 > 0$.

Lemma 2.2 ([44]). *Let $f \in H^2(\mathbb{R}^3)$. Then we can have*

$$\|f\|_{L^p} \le \|f\|_{H^1} \quad \text{for } 2 \le p \le 6,$$
$$\|f\|_{L^\infty} \le \|\nabla f\|_{L^2}^{\frac{1}{2}} \|\nabla f\|_{H^1}^{\frac{1}{2}} \le \|\nabla f\|_{H^1},$$
$$\|f\|_{L^6} \le \|\nabla f\|_{L^2}.$$

Lemma 2.3 ([45]). *We have*

$$\left\|\nabla^l (fg)\right\|_{L^p} \le \|f\|_{L^{q_1}} \left\|\nabla^l g\right\|_{L^{q_2}} + \left\|\nabla^l f\right\|_{L^{q_3}} \|g\|_{L^{q_4}},$$

where $l \ge 1$, $1 \le q_i \le +\infty$, and

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}.$$

Lemma 2.4 (*Gagliardo–Nirenberg inequality*). *Suppose $0 \le i, j \le k$. Then we have*

$$\left\|\nabla^i g\right\|_{L^p} \le \left\|\nabla^j g\right\|_{L^{p_1}}^{1-\sigma} \left\|\nabla^k g\right\|_{L^{p_2}}^{\sigma},$$

where $0 \le \sigma \le 1$ and

$$\frac{i}{3} - \frac{1}{p} = \left(\frac{j}{3} - \frac{1}{p_1}\right)(1-\sigma) + \left(\frac{k}{3} + \frac{1}{p_2}\right)\sigma.$$

In particular, if $p = \infty$, then $0 < \sigma < 1$ is required.

Lemma 2.5 ([46]). *Let $\psi(\omega)$ be a smooth function of ω with bounded derivatives of any order. If $\|\omega\|_{L^\infty(\mathbb{R}^3)} \le 1$, then for any integer $i \ge 1$, we have*

$$\left\|\nabla^i \psi(\omega)\right\|_{L^p(\mathbb{R}^3)} \le \left\|\nabla^i \omega\right\|_{L^p(\mathbb{R}^3)},$$

for $1 \le p \le \infty$.

For the decay estimates of solutions, we further introduce the following basic inequalities.

Lemma 2.6 ([47]). *Suppose $c_1, c_2, c_3 \in \mathbb{R}^3$ and $0 \le c_1 \le c_2, c_3 > 0$. We have, for $t \in \mathbb{R}_+$,*

$$\int_0^t (1+t-\tau)^{-c_1}(1+\tau)^{-c_2}\, d\tau \le C(c_1, c_2)(1+t)^{-c_1}$$

and

$$\int_0^t (1+\tau)^{-c_1} e^{-c_3(t-\tau)}\, d\tau \le C(c_1, c_3)(1+t)^{-c_1},$$

where constants $C(c_1, c_2) > 0$, $C(c_1, c_3) > 0$ only depend on c_1, c_2, c_3.

3 Global existence and uniqueness for the nonlinear system

This section is devoted to establishing the global existence and uniqueness of the solution for the Cauchy problem (1.5). More precisely, with the help of a priori estimates, we extend the local classical solution to the global one by the standard continuity method.

3.1 Global existence of solutions

First, we provide the relevant space for the system (1.5) by

$$\begin{aligned}\Omega(0,T) = \big\{(\varrho,\boldsymbol{u}) | \varrho \in C^0\big(0,T;H^3\big(\mathbb{R}^3\big)\big) \cap C^1\big(0,T;H^2\big(\mathbb{R}^3\big)\big),\\ \boldsymbol{u} \in C^0\big(0,T;H^3\big(\mathbb{R}^3\big)\big) \cap C^1\big(0,T;H^1\big(\mathbb{R}^3\big)\big),\\ \nabla\varrho \in L^2\big(0,T;H^2\big(\mathbb{R}^3\big)\big); \nabla\boldsymbol{u} \in L^2\big(0,T;H^3\big(\mathbb{R}^3\big)\big)\big\}\end{aligned}$$

for any $0 \le T \le \infty$.

By a method similar to the one in [13, 48], we can get the local existence of unique solutions to (1.5).

Proposition 3.1 (*Local existence*). *Suppose* $(\varrho^0,\boldsymbol{u}^0) \in H^3(\mathbb{R}^3)$ *and* $\inf\{\varrho^0 + 1\} > 0$. *Then there exists a constant* $T_0 > 0$ *depending on* $\|\varrho^0,\boldsymbol{u}^0)\|_{H^3(\mathbb{R}^3)}$ *such that the system* (1.5) *has a unique solution* $(\varrho,\boldsymbol{u}) \in \Omega(0,T_0)$, *which satisfies*

$$\inf_{x\in\mathbb{R}^3, 0\le t\le T_0} \{\varrho + 1\} > 0$$

and

$$\big\|(\varrho,\boldsymbol{u})(t)\big\|_{H^3} + \left(\int_0^t \big\|\nabla(\varrho,\boldsymbol{u})\big\|_{H^3}^2 \,\mathrm{d}\tau\right)^{\frac{1}{2}} \le \sqrt{C_1}\big\|\big(\varrho^0,\boldsymbol{u}^0\big)\big\|_{H^3},$$

where $C_1 > 0$ *is a constant.*

Proof With the iteration technique and the fixed point theorem in hand, the conclusion is obvious; please refer to [13, 48] for the details. □

Proposition 3.2 (*A priori estimates*). *Assume that the Cauchy problem* (1.5) *has a solution* $(\varrho,\boldsymbol{u}) \in \Omega(0,T)$ *with a constant* $T > 0$. *Then there exists a sufficiently small constant* $\epsilon_0 > 0$ *such that if*

$$\sup_{0\le t\le T} \big\|(\varrho,\boldsymbol{u})(t)\big\|_{H^3} \le \epsilon_0, \tag{3.1}$$

then for any $t \in [0,T]$ *we have*

$$\big\|(\varrho,\boldsymbol{u})(t)\big\|_{H^3}^2 + \int_0^t \big(\big\|\nabla\varrho(\tau)\big\|_{H^2}^2 + \big\|\nabla\boldsymbol{u}(\tau)\big\|_{H^3}^2\big)\,\mathrm{d}\tau \le C_2\big\|\big(\varrho^0,\boldsymbol{u}^0\big)\big\|_{H^3}^2, \tag{3.2}$$

where the constant $C_2 > 0$ *is independent of* T.

The details of the proof of Proposition 3.2 will be given in Sect. 3.2.

Theorem 3.1 (*Global existence*). *Suppose* $(\varrho^0, \boldsymbol{u}^0) \in H^3(\mathbb{R}^3)$. *Then there exists a constant* $\epsilon > 0$ *such that when*

$$\mathcal{C}_0 < \min\left\{ \frac{\epsilon}{\sqrt{C_1}}, \frac{\epsilon}{\sqrt{C_1 C_2}} \right\} < \infty,$$

the Cauchy problem of (1.5) *admits a unique solution* $(\varrho, \boldsymbol{u})$, *which satisfies for any* $t > 0$

$$\left\|(\varrho, \boldsymbol{u})(t)\right\|_{H^3}^2 + \int_0^t \left(\left\|\nabla \varrho(\tau)\right\|_{H^2}^2 + \left\|\nabla \boldsymbol{u}(\tau)\right\|_{H^3}^2 \right) \mathrm{d}\tau \le C_2 \mathcal{C}_0^2,$$

where $\mathcal{C}_0 := \|(\varrho^0, \boldsymbol{u}^0)\|_{H^3}$ *and* $C_1, C_2 > 0$ *are constants.*

Proof With Propositions 3.1 and 3.2 in hand, we can easily derive Theorem 3.1 by a classical method. We omit it here due to space constraints; please refer to [13, 48] for the details. □

Remark 3.1 With the Sobolev imbedding inequality in hand, it is easy to get $\frac{1}{2} \le \rho + 1 \le \frac{3}{2}$. Then under the assumptions in Proposition 3.2, we can have

$$\left|h_1(\varrho)\right| \le C_3, \left|(h_2, h_3)(\varrho)\right| \le C_3|\varrho|, \qquad \left|\left(h_1^{(l)}, h_2^{(l)}, h_3^{(l)}\right)(\varrho)\right| \le C_3 \quad \text{for } l \ge 1,$$

where $C_3 > 0$ is a constant.

3.2 Proof of Proposition 3.2

In this subsection, we aim to complete the proof of Proposition 3.2. The key step in the proof is to derive the energy estimates of the solution $(\varrho, \boldsymbol{u})$ for the transformed Cauchy problem (1.5) by the energy method.

Lemma 3.1 *We have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1 + \frac{\gamma_1}{4}\|\nabla \varrho\|_{L^2}^2 + \frac{\mu_0}{2}\|\nabla \boldsymbol{u}\|_{H^1}^2 + \frac{\mu_0}{2}\|\operatorname{div} \boldsymbol{u}\|_{H^1}^2 \le 0,$$

where

$$\mathcal{F}_1 = \frac{1}{2}\left(\|\varrho\|_{H^1}^2 + \|\boldsymbol{u}\|_{H^1}^2\right) + \gamma_1 \int \nabla \varrho \cdot \boldsymbol{u} \,\mathrm{d}x,$$

where $0 < \gamma_1 < \{\frac{1}{4}, \frac{1}{8\mu_0}\}$ *is a given constant.*

Proof Multiplying $\nabla^k(1.5)_1$, $\nabla^k(1.5)_2$ by $\nabla^k \varrho$, $\nabla^k \boldsymbol{u}$, respectively, and integrating over $\mathbb{R}^3$ by parts, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^k \varrho\right\|_{L^2}^2 + \left\|\nabla^k \boldsymbol{u}\right\|_{L^2}^2\right) + \mu_0\left\|\nabla^k \nabla \boldsymbol{u}\right\|_{L^2}^2 + \mu_0\left\|\nabla^k \operatorname{div} \boldsymbol{u}\right\|_{L^2}^2 \\ &\quad = \left\langle \nabla^k \varrho, -\nabla^k \operatorname{div}(\varrho \boldsymbol{u})\right\rangle + \left\langle \nabla^k \boldsymbol{u}, \nabla^k \mathcal{N}\right\rangle. \end{aligned} \tag{3.3}$$

By $\langle \nabla(1.5)_1, \boldsymbol{u}\rangle + \langle (1.5)_2, \nabla\varrho\rangle$, we find that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \nabla\varrho\cdot\boldsymbol{u}\,\mathrm{d}x + \|\nabla\varrho\|_{L^2}^2 \\
&\quad = \|\operatorname{div}\boldsymbol{u}\|_{L^2}^2 + \mu_0\int \nabla\varrho\cdot\Delta\boldsymbol{u}\,\mathrm{d}x + \mu_0\int\nabla\varrho\cdot\nabla\operatorname{div}\boldsymbol{u}\,\mathrm{d}x \\
&\qquad - \int \nabla\operatorname{div}(\varrho\boldsymbol{u})\cdot\boldsymbol{u}\,\mathrm{d}x + \int \mathcal{N}\cdot\nabla\varrho\,\mathrm{d}x.
\end{aligned}\tag{3.4}$$

Then using Young's inequality, we can estimate that for some fixed constant γ_1,

$$\begin{aligned}
&\gamma_1\mu_0\int\nabla\varrho\cdot\Delta\boldsymbol{u}\,\mathrm{d}x \le \frac{\gamma_1}{4}\|\nabla\varrho\|_{L^2}^2 + \gamma_1\mu_0^2\|\Delta\boldsymbol{u}\|_{L^2}^2, \\
&\gamma_1\mu_0\int\nabla\varrho\cdot\nabla\operatorname{div}\boldsymbol{u}\,\mathrm{d}x \le \frac{\gamma_1}{4}\|\nabla\varrho\|_{L^2}^2 + \gamma_1\mu_0^2\|\nabla\operatorname{div}\boldsymbol{u}\|_{L^2}^2.
\end{aligned}\tag{3.5}$$

Adding up the two identities $\gamma_1\cdot(3.4)$ and $\sum_{0\le k\le 1}(3.3)$, then using (3.5), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varrho\|_{H^1}^2 + \|\boldsymbol{u}\|_{H^1}^2 + 2\gamma_1\int\nabla\varrho\cdot\boldsymbol{u}\,\mathrm{d}x\right) \\
&\qquad + \frac{\gamma_1}{2}\|\nabla\varrho\|_{L^2}^2 + \mu_0\|\nabla\boldsymbol{u}\|_{H^1}^2 + \mu_0\|\operatorname{div}\boldsymbol{u}\|_{H^1}^2 \\
&\quad \le \|\operatorname{div}\boldsymbol{u}\|_{L^2}^2 + \gamma_1\mu_0^2\|\Delta\boldsymbol{u}\|_{L^2}^2 + \gamma_1\mu_0^2\|\nabla\operatorname{div}\boldsymbol{u}\|_{L^2}^2 - \int\varrho\operatorname{div}(\varrho\boldsymbol{u})\,\mathrm{d}x \\
&\qquad - \int\nabla\varrho\cdot\nabla\operatorname{div}(\varrho\boldsymbol{u})\,\mathrm{d}x + \int\boldsymbol{u}\cdot\mathcal{N}\,\mathrm{d}x + \int\nabla\boldsymbol{u}\cdot\nabla\mathcal{N}\,\mathrm{d}x \\
&\qquad - \gamma_1\int\boldsymbol{u}\cdot\nabla\operatorname{div}(\varrho\boldsymbol{u})\,\mathrm{d}x + \gamma_1\int\nabla\varrho\cdot\mathcal{N}\,\mathrm{d}x.
\end{aligned}\tag{3.6}$$

The nonlinear terms on the right-hand side of (3.6) can be bounded as follows. With Young's inequality and Hölder's inequality in hand, integrating by parts and using Lemmas 2.2 and 2.3 and (3.1), we obtain

$$-\int\varrho\operatorname{div}(\varrho\boldsymbol{u})\,\mathrm{d}x \le C\epsilon_0\big\|\nabla(\varrho,\boldsymbol{u})\big\|_{L^2}^2, \tag{3.7}$$

$$-\int\nabla\varrho\cdot\nabla\operatorname{div}(\varrho\boldsymbol{u})\,\mathrm{d}x \le C\epsilon_0\big(\big\|\nabla^2\varrho\big\|_{L^2}^2 + \big\|\nabla(\varrho,\boldsymbol{u})\big\|_{L^2}^2\big), \tag{3.8}$$

$$-\int\boldsymbol{u}\cdot\nabla\operatorname{div}(\varrho\boldsymbol{u})\,\mathrm{d}x = \int\operatorname{div}\boldsymbol{u}\operatorname{div}(\varrho\boldsymbol{u})\,\mathrm{d}x \le C\epsilon_0\big\|\nabla(\varrho,\boldsymbol{u})\big\|_{L^2}^2 \tag{3.9}$$

and

$$\begin{aligned}
&\int\boldsymbol{u}\cdot\mathcal{N}\,\mathrm{d}x \\
&\quad = \int\boldsymbol{u}\cdot\big(-\boldsymbol{u}\cdot\nabla\boldsymbol{u} - h_3(\varrho)\nabla\varrho - h_2(\varrho)(\Delta\boldsymbol{u} + \nabla\operatorname{div}\boldsymbol{u})\big)\,\mathrm{d}x \\
&\qquad + \int\boldsymbol{u}\cdot h_1(\varrho)\operatorname{div}\big(\big|D(\nabla\boldsymbol{u})\big|^{p-2}D(\nabla\boldsymbol{u})\big)\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\boldsymbol{u}\|_{L^6}\big(\|\boldsymbol{u}\|_{L^3}\|\nabla\boldsymbol{u}\|_{L^2}+\big\|h_3(\varrho)\big\|_{L^3}\|\nabla\varrho\|_{L^2}\big)\\
&\quad+C\|\boldsymbol{u}\|_{L^6}\big\|h_2(\varrho)\big\|_{L^3}\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}\\
&\quad+C\|\boldsymbol{u}\|_{L^6}\big\|h_1(\varrho)\big\|_{L^\infty}\big\|\operatorname{div}\big|D(\nabla\boldsymbol{u})\big|^{p-2}\big\|_{L^2}\big\|D(\nabla\boldsymbol{u})\big\|_{L^3}\\
&\quad+C\|\boldsymbol{u}\|_{L^6}\big\|h_1(\varrho)\big\|_{L^\infty}\||D(\nabla\boldsymbol{u})|^{p-2}\|_{L^3}\big\|\operatorname{div}D(\nabla\boldsymbol{u})\big\|_{L^2}\\
&\le C\epsilon_0\big(\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}^2+\big\|\nabla(\varrho,\boldsymbol{u})\big\|_{L^2}^2+\|\nabla\boldsymbol{u}\|_{H^2}^2\big),
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
\int\nabla\boldsymbol{u}\cdot\nabla\mathcal{N}\,\mathrm{d}x&=\int\nabla\boldsymbol{u}\cdot\nabla\big(-\boldsymbol{u}\cdot\nabla\boldsymbol{u}-h_3(\varrho)\nabla\varrho\big)\,\mathrm{d}x\\
&\quad-\int\nabla\boldsymbol{u}\cdot\nabla\big(h_2(\varrho)(\Delta\boldsymbol{u}+\nabla\operatorname{div}\boldsymbol{u})\big)\,\mathrm{d}x\\
&\quad+\int\nabla\boldsymbol{u}\cdot\nabla\big(h_1(\varrho)\operatorname{div}\big(\big|D(\nabla\boldsymbol{u})\big|^{p-2}D(\nabla\boldsymbol{u})\big)\big)\,\mathrm{d}x\\
&\le C\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}\big(\|\nabla\boldsymbol{u}\|_{L^2}\|\boldsymbol{u}\|_{L^\infty}+\big\|h_3(\varrho)\big\|_{L^\infty}\|\nabla\varrho\|_{L^2}\big)\\
&\quad+C\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}^2\big\|h_2(\varrho)\big\|_{L^\infty}\\
&\quad+C\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}\big\|h_1(\varrho)\big\|_{L^\infty}\big\|\operatorname{div}\big|D(\nabla\boldsymbol{u})\big|^{p-2}\big\|_{L^2}\big\|D(\nabla\boldsymbol{u})\big\|_{L^\infty}\\
&\quad+C\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}\big\|h_1(\varrho)\big\|_{L^\infty}\||D(\nabla\boldsymbol{u})|^{p-2}\|_{L^\infty}\big\|\operatorname{div}D(\nabla\boldsymbol{u})\big\|_{L^2}\\
&\le C\epsilon_0\big(\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}^2+\big\|\nabla(\varrho,\boldsymbol{u})\big\|_{L^2}^2+\|\nabla\boldsymbol{u}\|_{H^2}^2\big),
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
\gamma_1\int\nabla\varrho\cdot\mathcal{N}\,\mathrm{d}x&=\gamma_1\|\nabla\varrho\|_{L^2}\|\mathcal{N}\|_{L^2}\\
&\le C\gamma_1\|\nabla\varrho\|_{L^2}\big(\|\nabla\boldsymbol{u}\|_{L^2}\|\boldsymbol{u}\|_{L^\infty}+\big\|h_3(\varrho)\big\|_{L^\infty}\|\nabla\varrho\|_{L^2}\big)\\
&\quad+C\|\nabla\varrho\|_{L^2}\big\|h_2(\varrho)\big\|_{L^\infty}\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}\\
&\quad+C\|\nabla\varrho\|_{L^2}\big\|h_1(\varrho)\big\|_{L^\infty}\big\|\operatorname{div}\big|D(\nabla\boldsymbol{u})\big|^{p-2}\big\|_{L^2}\big\|D(\nabla\boldsymbol{u})\big\|_{L^\infty}\\
&\quad+C\|\nabla\varrho\|_{L^2}\big\|h_1(\varrho)\big\|_{L^\infty}\||D(\nabla\boldsymbol{u})|^{p-2}\|_{L^\infty}\big\|\operatorname{div}D(\nabla\boldsymbol{u})\big\|_{L^2}\\
&\le C\epsilon_0\big(\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}^2+\big\|\nabla(\varrho,\boldsymbol{u})\big\|_{L^2}^2+\|\nabla\boldsymbol{u}\|_{H^2}^2\big).
\end{aligned}
\tag{3.12}
$$

Putting the estimates (3.7)–(3.12) into (3.6), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varrho\|_{H^1}^2+\|\boldsymbol{u}\|_{H^1}^2+2\gamma_1\int\nabla\varrho\cdot\boldsymbol{u}\,\mathrm{d}x\right)\\
&\quad+\frac{\gamma_1}{4}\|\nabla\varrho\|_{L^2}^2+\frac{\mu_0}{2}\|\nabla\boldsymbol{u}\|_{H^1}^2+\frac{\mu_0}{2}\|\operatorname{div}\boldsymbol{u}\|_{H^1}^2\\
&\le C(1+\gamma_1)\epsilon_0\big(\big\|\nabla^2(\varrho,\boldsymbol{u})\big\|_{L^2}^2+\big\|\nabla(\varrho,\boldsymbol{u})\big\|_{L^2}^2+\|\nabla\boldsymbol{u}\|_{H^2}^2\big),
\end{aligned}
\tag{3.13}
$$

where $0<\gamma_1<\{\frac{1}{4},\frac{1}{8\mu_0}\}$ is a fixed constant. Then the desired estimate follows from (3.13). Thus, the proof of Lemma 3.1 is complete. □

Now we establish the energy estimate on the highest-order derivatives of the solution $(\varrho,\boldsymbol{u})$ for the Cauchy problem (1.5).

Lemma 3.2 *We have*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2+\frac{\gamma_2}{4}\big\|\nabla^2\varrho\big\|_{L^2}^2+\frac{\mu_0}{2}\big\|\nabla^2\operatorname{div}\boldsymbol{u}\big\|_{L^2}^2+\frac{\mu_0}{2}\big\|\nabla^3\boldsymbol{u}\big\|_{L^2}^2\\&\quad\le\frac{1}{4}\|\nabla\operatorname{div}\boldsymbol{u}\|_{L^2}^2+C\epsilon_0\big(\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}^2+\|\nabla\boldsymbol{u}\|_{H^2}^2\big),\end{aligned}\tag{3.14}$$

where

$$\mathcal{F}_2=\frac{1}{2}\big(\big\|\nabla^2\varrho\big\|_{L^2}^2+\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}^2\big)+\gamma_2\int\nabla^2\varrho\cdot\nabla\boldsymbol{u}\,\mathrm{d}x,$$

where $0<\gamma_2<\{\frac{1}{4},\frac{1}{8\mu_0}\}$ *is a given constant.*

Proof Multiplying $\nabla^2(1.5)_1$, $\nabla^2(1.5)_2$ by $\nabla^2\varrho$, $\nabla^2\boldsymbol{u}$, respectively, and integrating over $\mathbb{R}^3$ by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\big\|\nabla^2\varrho\big\|_{L^2}^2+\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}^2\big)+\mu_0\big\|\nabla^2\nabla\boldsymbol{u}\big\|_{L^2}^2+\mu_0\big\|\nabla^2\operatorname{div}\boldsymbol{u}\big\|_{L^2}^2\\&\quad=\big\langle\nabla^2\varrho,-\nabla^2\operatorname{div}(\varrho\boldsymbol{u})\big\rangle+\big\langle\nabla^2\boldsymbol{u},\nabla^2\mathcal{N}\big\rangle.\end{aligned}\tag{3.15}$$

Multiplying $\nabla^2(1.5)_1$ by $\nabla\boldsymbol{u}$ and then exploiting $\nabla(1.5)_2\cdot\nabla^2\varrho$ and Young's inequality, we can estimate that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\nabla^2\varrho\cdot\nabla\boldsymbol{u}\,\mathrm{d}x+\int\big|\nabla^2\varrho\big|^2\,\mathrm{d}x\\&\quad=\|\nabla\operatorname{div}\boldsymbol{u}\|_{L^2}^2+\mu_0\int\nabla^2\varrho\cdot\nabla\Delta\boldsymbol{u}\,\mathrm{d}x+\mu_0\int\nabla^2\varrho\cdot\nabla^2\operatorname{div}\boldsymbol{u}\,\mathrm{d}x\\&\qquad-\int\nabla^2\operatorname{div}(\varrho\boldsymbol{u})\cdot\nabla\boldsymbol{u}\,\mathrm{d}x+\int\nabla\mathcal{N}\cdot\nabla^2\varrho\,\mathrm{d}x\\&\quad\le\frac{1}{2}\big\|\nabla^2\varrho\big\|_{L^2}^2+\mu_0^2\|\nabla\Delta\boldsymbol{u}\|_{L^2}^2+\mu_0^2\big\|\nabla^2\operatorname{div}\boldsymbol{u}\big\|_{L^2}^2+\|\nabla\operatorname{div}\boldsymbol{u}\|_{L^2}^2\\&\qquad-\int\nabla^2\operatorname{div}(\varrho\boldsymbol{u})\cdot\nabla\boldsymbol{u}\,\mathrm{d}x+\int\nabla\mathcal{N}\cdot\nabla^2\varrho\,\mathrm{d}x.\end{aligned}\tag{3.16}$$

Thus, summing up (3.15) and $\gamma_2\cdot$(3.16), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\big\|\nabla^2\varrho\big\|_{L^2}^2+\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}^2+2\gamma_2\int\nabla^2\varrho\cdot\nabla\boldsymbol{u}\,\mathrm{d}x\right)\\&\qquad+\frac{\gamma_2}{2}\big\|\nabla^2\varrho\big\|_{L^2}^2+\mu_0\big\|\nabla^2\nabla\boldsymbol{u}\big\|_{L^2}^2+\mu_0\big\|\nabla^2\operatorname{div}\boldsymbol{u}\big\|_{L^2}^2\\&\quad\le\gamma_2\mu_0^2\|\nabla\Delta\boldsymbol{u}\|_{L^2}^2+\gamma_2\mu_0^2\big\|\nabla^2\operatorname{div}\boldsymbol{u}\big\|_{L^2}^2+\gamma_2\|\nabla\operatorname{div}\boldsymbol{u}\|_{L^2}^2\\&\qquad-\int_{R^3}\nabla^2\varrho\cdot\nabla^2\operatorname{div}(\varrho\boldsymbol{u})\,\mathrm{d}x+\int\nabla^2\boldsymbol{u}\cdot\nabla^2\mathcal{N}\,\mathrm{d}x\\&\qquad-\gamma_2\int\nabla\boldsymbol{u}\cdot\nabla^2\operatorname{div}(\varrho\boldsymbol{u})\,\mathrm{d}x+\gamma_2\int\nabla^2\varrho\cdot\nabla\mathcal{N}\,\mathrm{d}x.\end{aligned}\tag{3.17}$$

Next we estimate for the nonlinear terms on the right-hand side of (3.17). Thanks to Lemmas 2.2–2.4, Hölder's inequality, and Young's inequality, after integrating by parts, we can have

$$-\int \nabla^2\varrho\cdot\nabla^2\operatorname{div}(\varrho\boldsymbol{u})\,\mathrm{d}x\le C\epsilon_0\big(\big\|\nabla^3\boldsymbol{u}\big\|_{L^2}^2+\big\|\nabla^2(\varrho,\boldsymbol{u})\big\|_{L^2}^2\big), \tag{3.18}$$

$$-\int \nabla\boldsymbol{u}\cdot\nabla^2\operatorname{div}(\varrho\boldsymbol{u})\,\mathrm{d}x\le C\epsilon_0\big\|\nabla^2(\varrho,\boldsymbol{u})\big\|_{L^2}^2 \tag{3.19}$$

and

$$\begin{aligned}
&\int \nabla^2\boldsymbol{u}\cdot\nabla^2\mathcal{N}\,\mathrm{d}x\\
&\quad=\int \nabla^2\boldsymbol{u}\cdot\nabla^2\big(-\boldsymbol{u}\cdot\nabla\boldsymbol{u}-h_3(\varrho)\nabla\varrho-h_2(\varrho)(\Delta\boldsymbol{u}+\nabla\operatorname{div}\boldsymbol{u})\big)\,\mathrm{d}x\\
&\qquad+\int \nabla^2\boldsymbol{u}\cdot\nabla^2\big(h_1(\varrho)\operatorname{div}\big(\big|D(\nabla\boldsymbol{u})\big|^{p-2}D(\nabla\boldsymbol{u})\big)\big)\,\mathrm{d}x\\
&\quad\le C\big(\big|\big\langle\nabla^3\boldsymbol{u},\nabla(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\big\rangle\big|+\big|\big\langle\nabla^3\boldsymbol{u},\nabla\big(h_3(\varrho)\nabla\varrho\big)\big\rangle\big|\big)\\
&\qquad+C\big(\big|\big\langle\nabla^3\boldsymbol{u},\nabla\big(h_2(\varrho)\Delta\boldsymbol{u}\big)\big\rangle\big|+\big|\big\langle\nabla^3\boldsymbol{u},\nabla\big(h_2(\varrho)\nabla\operatorname{div}\boldsymbol{u}\big)\big\rangle\big|\big)\\
&\qquad+C\big(\big|\big\langle\nabla^3\boldsymbol{u},\nabla\big(h_1(\varrho)\operatorname{div}\big(\big|D(\nabla\boldsymbol{u})\big|^{p-2}D(\nabla\boldsymbol{u})\big)\big)\big\rangle\big|\big)\\
&\quad\le C\big\|\nabla^3\boldsymbol{u}\big\|_{L^2}\big(\|\nabla\boldsymbol{u}\|_{L^6}\|\nabla\boldsymbol{u}\|_{L^3}+\|\boldsymbol{u}\|_{L^\infty}\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}\big)\\
&\qquad+C\big\|\nabla^3\boldsymbol{u}\big\|_{L^2}\big(\big\|\nabla h_3(\varrho)\big\|_{L^\infty}\|\nabla\varrho\|_{L^2}+\big\|h_3(\varrho)\big\|_{L^\infty}\big\|\nabla^2\varrho\big\|_{L^2}\big)\\
&\qquad+C\big\|\nabla^3\boldsymbol{u}\big\|_{L^2}\big(\big\|\nabla h_2(\varrho)\big\|_{L^\infty}\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}+\big\|h_2(\varrho)\big\|_{L^\infty}\big\|\nabla^3\boldsymbol{u}\big\|_{L^2}\big)\\
&\qquad+C\big\|\nabla^3\boldsymbol{u}\big\|_{L^2}\big(\big\|h_1(\varrho)\big\|_{L^\infty}+\big\|\nabla h_1(\varrho)\big\|_{L^\infty}\big)\|\boldsymbol{u}\|_{H^2}^{p-1}\\
&\quad\le C\epsilon_0\big(\|\nabla\boldsymbol{u}\|_{H^2}^2+\big\|\nabla^2(\varrho,\boldsymbol{u})\big\|_{L^2}^2\big),
\end{aligned}\tag{3.20}$$

$$\begin{aligned}
&\int \nabla^2\varrho\cdot\nabla\mathcal{N}\,\mathrm{d}x\\
&\quad=\int \nabla^2\varrho\cdot\nabla\big(-\boldsymbol{u}\cdot\nabla\boldsymbol{u}-h_3(\varrho)\nabla\varrho-h_2(\varrho)(\Delta\boldsymbol{u}+\nabla\operatorname{div}\boldsymbol{u})\big)\,\mathrm{d}x\\
&\qquad+\int \nabla^2\varrho\cdot\nabla\big(h_1(\varrho)\operatorname{div}\big(\big|D(\nabla\boldsymbol{u})\big|^{p-2}D(\nabla\boldsymbol{u})\big)\big)\,\mathrm{d}x\\
&\quad\le C\gamma_2\big\|\nabla^2\varrho\big\|_{L^2}\big(\|\nabla\boldsymbol{u}\|_{L^6}\|\nabla\boldsymbol{u}\|_{L^3}+\|\boldsymbol{u}\|_{L^\infty}\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}\big)\\
&\qquad+C\big\|\nabla^2\varrho\big\|_{L^2}\big(\big\|\nabla h_3(\varrho)\big\|_{L^\infty}\|\nabla\varrho\|_{L^2}+\big\|h_3(\varrho)\big\|_{L^\infty}\big\|\nabla^2\varrho\big\|_{L^3}\big)\\
&\qquad+C\big\|\nabla^2\varrho\big\|_{L^2}\big(\big\|\nabla h_2(\varrho)\big\|_{L^\infty}\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}+\big\|h_2(\varrho)\big\|_{L^\infty}\big\|\nabla^3\boldsymbol{u}\big\|_{L^2}\big)\\
&\qquad+C\big\|\nabla^2\varrho\big\|_{L^2}\big(\big\|h_1(\varrho)\big\|_{L^\infty}+\big\|\nabla h_1(\varrho)\big\|_{L^\infty}\big)\|\boldsymbol{u}\|_{H^3}^{p-1}\\
&\quad\le C\epsilon_0\big(\|\nabla\boldsymbol{u}\|_{H^2}^2+\big\|\nabla^2(\varrho,\boldsymbol{u})\big\|_{L^2}^2\big).
\end{aligned}\tag{3.21}$$

Plugging (3.18)–(3.21) into (3.17), we can deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\big\|\nabla^2\varrho\big\|_{L^2}^2+\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}^2+2\gamma_2\int \nabla^2\varrho\cdot\nabla\boldsymbol{u}\,\mathrm{d}x\bigg)\\
&\qquad+\frac{\gamma_2}{4}\big\|\nabla^2\varrho\big\|_{L^2}^2+\frac{\mu_0}{2}\big\|\nabla^2\nabla\boldsymbol{u}\big\|_{L^2}^2+\frac{\mu_0}{2}\big\|\nabla^2\operatorname{div}\boldsymbol{u}\big\|_{L^2}^2\\
&\quad\le\frac{1}{4}\|\nabla\operatorname{div}\boldsymbol{u}\|_{L^2}^2+C\epsilon_0\big(\big\|\nabla^2(\varrho,\boldsymbol{u})\big\|_{L^2}^2+\|\nabla\boldsymbol{u}\|_{H^2}^2\big),
\end{aligned}$$

where $0 < \gamma_2 < \{\frac{1}{4}, \frac{1}{8\mu_0}\}$ is a fixed constant. Consequently, we complete the proof of (3.14). □

Lemma 3.3 *We have*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_3 + \frac{\gamma_3}{4}\|\nabla^3\varrho\|_{L^2}^2 + \frac{\mu_0}{2}\|\nabla^3 \operatorname{div}\boldsymbol{u}\|_{L^2}^2 + \frac{\mu_0}{2}\|\nabla^4\boldsymbol{u}\|_{L^2}^2 \\
&\quad \le \frac{1}{4}\|\nabla^2 \operatorname{div}\boldsymbol{u}\|_{L^2}^2 + C\epsilon_0\big(\|\nabla\boldsymbol{u}\|_{H^3}^2 + \|\nabla^3\boldsymbol{u}\|_{L^2}^2\big),
\end{aligned} \tag{3.22}$$

where

$$\mathcal{F}_3 = \frac{1}{2}\big(\|\nabla^3\varrho\|_{L^2}^2 + \|\nabla^3\boldsymbol{u}\|_{L^2}^2\big) + \gamma_3\int \nabla^3\varrho\cdot\nabla^2\boldsymbol{u}\,\mathrm{d}x,$$

where $0 < \gamma_3 < \{\frac{1}{4}, \frac{1}{8\mu_0}\}$ *is a given constant.*

Proof Multiplying $\nabla^3(1.5)_1$, $\nabla^3(1.5)_2$ by $\nabla^3\varrho$, $\nabla^3\boldsymbol{u}$, respectively, and integrating over $\mathbb{R}^3$ by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla^3\varrho\|_{L^2}^2 + \|\nabla^3\boldsymbol{u}\|_{L^2}^2\big) + \mu_0\|\nabla^3\nabla\boldsymbol{u}\|_{L^2}^2 + \mu_0\|\nabla^3 \operatorname{div}\boldsymbol{u}\|_{L^2}^2 \\
&\quad = \big\langle\nabla^3\varrho, -\nabla^3 \operatorname{div}(\varrho\boldsymbol{u})\big\rangle + \big\langle\nabla^3\boldsymbol{u}, \nabla^3\mathcal{N}\big\rangle.
\end{aligned} \tag{3.23}$$

Multiplying $\nabla^3(1.5)_1$ by $\nabla^2\boldsymbol{u}$ and then exploiting $\nabla^2(1.5)_2\cdot\nabla^3\varrho$ and Young's inequality, we can estimate that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \nabla^3\varrho\cdot\nabla^2\boldsymbol{u}\,\mathrm{d}x + \int |\nabla^3\varrho|^2\,\mathrm{d}x \\
&\quad = \|\nabla^2 \operatorname{div}\boldsymbol{u}\|_{L^2}^2 + \mu_0\int \nabla^3\varrho\cdot\nabla^2\Delta\boldsymbol{u}\,\mathrm{d}x + \mu_0\int \nabla^3\varrho\cdot\nabla^3 \operatorname{div}\boldsymbol{u}\,\mathrm{d}x \\
&\qquad - \int \nabla^3 \operatorname{div}(\varrho\boldsymbol{u})\cdot\nabla^2\boldsymbol{u}\,\mathrm{d}x + \int \nabla^2\mathcal{N}\cdot\nabla^3\varrho\,\mathrm{d}x \\
&\quad \le \frac{1}{2}\|\nabla^3\varrho\|_{L^2}^2 + \mu_0^2\|\nabla^2\Delta\boldsymbol{u}\|_{L^2}^2 + \mu_0^2\|\nabla^3 \operatorname{div}\boldsymbol{u}\|_{L^2}^2 + \|\nabla^2 \operatorname{div}\boldsymbol{u}\|_{L^2}^2 \\
&\qquad - \int \nabla^3 \operatorname{div}(\varrho\boldsymbol{u})\cdot\nabla^2\boldsymbol{u}\,\mathrm{d}x + \int \nabla^2\mathcal{N}\cdot\nabla^3\varrho\,\mathrm{d}x.
\end{aligned} \tag{3.24}$$

Summing up (3.23) and $\gamma_3\cdot$(3.24), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\|\nabla^3\varrho\|_{L^2}^2 + \|\nabla^3\boldsymbol{u}\|_{L^2}^2 + 2\gamma_3\int \nabla^3\varrho\cdot\nabla^2\boldsymbol{u}\,\mathrm{d}x\bigg) \\
&\qquad + \frac{\gamma_3}{2}\|\nabla^3\varrho\|_{L^2}^2 + \|\nabla^3\nabla\boldsymbol{u}\|_{L^2}^2 + \|\nabla^3 \operatorname{div}\boldsymbol{u}\|_{L^2}^2 \\
&\quad \le \gamma_3\mu_0^2\|\nabla^2\Delta\boldsymbol{u}\|_{L^2}^2 + \gamma_3\mu_0^2\|\nabla^3 \operatorname{div}\boldsymbol{u}\|_{L^2}^2 + \gamma_3\|\nabla^2 \operatorname{div}\boldsymbol{u}\|_{L^2}^2 \\
&\qquad - \int \nabla^3\varrho\cdot\nabla^3 \operatorname{div}(\varrho\boldsymbol{u})\,\mathrm{d}x + \int \nabla^3\boldsymbol{u}\cdot\nabla^3\mathcal{N}\,\mathrm{d}x \\
&\qquad - \gamma_3\int \nabla^2\boldsymbol{u}\cdot\nabla^3 \operatorname{div}(\varrho\boldsymbol{u})\,\mathrm{d}x + \gamma_3\int \nabla^3\varrho\cdot\nabla^2\mathcal{N}\,\mathrm{d}x.
\end{aligned} \tag{3.25}$$

Next we estimate for the nonlinear terms on the right-hand side of (3.25). With Lemmas 2.2–2.4, Hölder's inequality, and Young's inequality in hand, after integrating by parts, we get

$$-\int \nabla^3\varrho \cdot \nabla^3 \operatorname{div}(\varrho \boldsymbol{u})\,\mathrm{d}x \le C\epsilon_0\big(\|\nabla^3(\varrho,\boldsymbol{u})\|_{L^2}^2 + \|\nabla^4\boldsymbol{u}\|_{L^2}^2\big), \tag{3.26}$$

$$-\int \nabla^2\boldsymbol{u} \cdot \nabla^3 \operatorname{div}(\varrho \boldsymbol{u})\,\mathrm{d}x \le C\epsilon_0\|\nabla^3(\varrho,\boldsymbol{u})\|_{L^2}^2 \tag{3.27}$$

and

$$\begin{aligned}
&\int \nabla^3\boldsymbol{u}\cdot\nabla^3\mathcal{N}\,\mathrm{d}x\\
&= \int \nabla^3\boldsymbol{u}\cdot\nabla^3\big(-\boldsymbol{u}\cdot\nabla\boldsymbol{u} - h_3(\varrho)\nabla\varrho - h_2(\varrho)(\Delta\boldsymbol{u}+\nabla\operatorname{div}\boldsymbol{u})\big)\,\mathrm{d}x\\
&\quad + \int \nabla^3\boldsymbol{u}\cdot\nabla^3\big(h_1(\varrho)\operatorname{div}\big(|D(\nabla\boldsymbol{u})|^{p-2}D(\nabla\boldsymbol{u})\big)\big)\,\mathrm{d}x\\
&\le C\big(\big|\langle\nabla^4\boldsymbol{u},\nabla^2(\boldsymbol{u}\cdot\nabla\boldsymbol{u})\rangle\big| + \big|\langle\nabla^4\boldsymbol{u},\nabla^2\big(h_3(\varrho)\nabla\varrho\big)\rangle\big|\big)\\
&\quad + C\big(\big|\langle\nabla^4\boldsymbol{u},\nabla^2\big(h_2(\varrho)\Delta\boldsymbol{u}\big)\rangle\big| + \big|\langle\nabla^4\boldsymbol{u},\nabla^2\big(h_2(\varrho)\nabla\operatorname{div}\boldsymbol{u}\big)\rangle\big|\big)\\
&\quad + C\big(\big|\langle\nabla^4\boldsymbol{u},\nabla^2\big(h_1(\varrho)\operatorname{div}\big(|D(\nabla\boldsymbol{u})|^{p-2}D(\nabla\boldsymbol{u})\big)\big)\rangle\big|\\
&\le C\|\nabla^4\boldsymbol{u}\|_{L^2}\big(\|\nabla\boldsymbol{u}\|_{L^6}\|\nabla^2\boldsymbol{u}\|_{L^3} + \|\boldsymbol{u}\|_{L^\infty}\|\nabla^3\boldsymbol{u}\|_{L^2}\big)\\
&\quad + C\|\nabla^4\boldsymbol{u}\|_{L^2}\big(\|\nabla^2h_3(\varrho)\|_{L^\infty}\|\nabla\varrho\|_{L^2} + \|\nabla h_3(\varrho)\|_{L^\infty}\|\nabla^2\varrho\|_{L^2}\big)\\
&\quad + C\|\nabla^4\boldsymbol{u}\|_{L^2}\|h_3(\varrho)\|_{L^\infty}\|\nabla^3\varrho\|_{L^2}\\
&\quad + C\|\nabla^4\boldsymbol{u}\|_{L^2}\big(\|\nabla^2h_2(\varrho)\|_{L^\infty}\|\nabla^2\boldsymbol{u}\|_{L^2} + \|\nabla h_2(\varrho)\|_{L^\infty}\|\nabla^3\boldsymbol{u}\|_{L^2}\big)\\
&\quad + C\|\nabla^4\boldsymbol{u}\|_{L^2}\|h_2(\varrho)\|_{L^\infty}\|\nabla^4\boldsymbol{u}\|_{L^2}\\
&\quad + C\|\nabla^4\boldsymbol{u}\|_{L^2}\big(\|h_1(\varrho)\|_{L^\infty} + \|\nabla h_1(\varrho)\|_{L^\infty} + \|\nabla^2h_1(\varrho)\|_{L^\infty}\big)\|\boldsymbol{u}\|_{H^3}^{p-2}\|\boldsymbol{u}\|_{H^4}\\
&\le C\epsilon_0\big(\|\nabla\boldsymbol{u}\|_{H^3}^2 + \|\nabla^3(\varrho,\boldsymbol{u})\|_{L^2}^2\big),
\end{aligned}\tag{3.28}$$

$$\begin{aligned}
&\int \nabla^3\varrho\cdot\nabla^2\mathcal{N}\,\mathrm{d}x\\
&= \int \nabla^3\varrho\cdot\nabla^2\big(-\boldsymbol{u}\cdot\nabla\boldsymbol{u} - h_3(\varrho)\nabla\varrho - h_2(\varrho)(\Delta\boldsymbol{u}+\nabla\operatorname{div}\boldsymbol{u})\big)\,\mathrm{d}x\\
&\quad + \int \nabla^3\varrho\cdot\nabla^2\big(h_1(\varrho)\operatorname{div}\big(|D(\nabla\boldsymbol{u})|^{p-2}D(\nabla\boldsymbol{u})\big)\big)\,\mathrm{d}x\\
&\le C\|\nabla^3\varrho\|_{L^2}\big(\|\nabla\boldsymbol{u}\|_{L^6}\|\nabla^2\boldsymbol{u}\|_{L^3} + \|\boldsymbol{u}\|_{L^\infty}\|\nabla^3\boldsymbol{u}\|_{L^2}\big)\\
&\quad + C\|\nabla^3\varrho\|_{L^2}\big(\|\nabla^2h_3(\varrho)\|_{L^\infty}\|\nabla\varrho\|_{L^2} + \|\nabla h_3(\varrho)\|_{L^\infty}\|\nabla^2\varrho\|_{L^2}\big)\\
&\quad + C\|\nabla^3\varrho\|_{L^2}\|h_3(\varrho)\|_{L^\infty}\|\nabla^3\varrho\|_{L^2}\\
&\quad + C\|\nabla^3\varrho\|_{L^2}\big(\|\nabla^2h_2(\varrho)\|_{L^\infty}\|\nabla^2\boldsymbol{u}\|_{L^2} + \|\nabla h_2(\varrho)\|_{L^\infty}\|\nabla^3\boldsymbol{u}\|_{L^2}\big)\\
&\quad + C\|\nabla^3\varrho\|_{L^2}\|h_2(\varrho)\|_{L^\infty}\|\nabla^4\boldsymbol{u}\|_{L^2}\\
&\quad + C\|\nabla^3\varrho\|_{L^2}\big(\|h_1(\varrho)\|_{L^\infty} + \|\nabla h_1(\varrho)\|_{L^\infty} + \|\nabla^2h_1(\varrho)\|_{L^\infty}\big)\|\boldsymbol{u}\|_{H^3}^{p-2}\|\boldsymbol{u}\|_{H^4}\\
&\le C\epsilon_0\big(\|\nabla\boldsymbol{u}\|_{H^3}^2 + \|\nabla^3(\varrho,\boldsymbol{u})\|_{L^2}^2\big).
\end{aligned}\tag{3.29}$$

Plugging (3.26)–(3.29) into (3.25), we can deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^3\varrho\right\|_{L^2}^2+\left\|\nabla^3\boldsymbol{u}\right\|_{L^2}^2+2\gamma_3\int\nabla^3\varrho\cdot\nabla^2\boldsymbol{u}\,\mathrm{d}x\right)$$
$$+\frac{\gamma_3}{4}\left\|\nabla^3\varrho\right\|_{L^2}^2+\frac{\mu_0}{2}\left\|\nabla^3\nabla\boldsymbol{u}\right\|_{L^2}^2+\frac{\mu_0}{2}\left\|\nabla^3\operatorname{div}\boldsymbol{u}\right\|_{L^2}^2$$
$$\le\frac{1}{4}\left\|\nabla^2\operatorname{div}\boldsymbol{u}\right\|_{L^2}^2+C\epsilon_0\left(\|\nabla\boldsymbol{u}\|_{H^3}^2+\left\|\nabla^3(\varrho,\boldsymbol{u})\right\|_{L^2}^2\right),$$

where $0<\gamma_3<\{\frac{1}{4},\frac{1}{8\mu_0}\}$ is a fixed constant. Consequently, we completed the proof of (3.22). □

With Lemmas 3.1–3.2 in hand, we easily further obtain Proposition 3.2. In fact, keeping in mind the Young's inequality and the definitions of $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, we have

$$\frac{1}{C_4}\left\|(\varrho,\boldsymbol{u})\right\|_{H^3}^2\le\mathcal{F}_1+\mathcal{F}_2+\mathcal{F}_3\le C_4\left\|(\varrho,\boldsymbol{u})\right\|_{H^3}^2,$$

where $C_4>0$ is a constant, which yields

$$\mathcal{F}_1+\mathcal{F}_2+\mathcal{F}_3\approx\left\|(\varrho,\boldsymbol{u})\right\|_{H^3}^2.$$

Thanks to the three lemmas above, integrating the resulting inequality over $(0,t)$, (3.2) holds for the small enough ϵ_0. This completes the proof of Proposition 3.2.

4 Time decay rates of the solution

In this section we shall show the time decay rates of the Cauchy problem (1.5). The proof will be broken up into two subsections.

4.1 Cancellation of the low frequency part

Inspired by the observation of canceling the low frequency part of the solution, we draw the following conclusion.

Lemma 4.1 *We have*

$$\left\|\nabla^3(\varrho,\boldsymbol{u})\right\|_{L^2}^2\le Ce^{C_5t}\left\|\nabla^3\left(\varrho^0,\boldsymbol{u}^0\right)\right\|_{L^2}^2+C\int_0^t e^{C_5(t-\tau)}\left\|\nabla^3\left(\varrho^L,\boldsymbol{u}^L\right)(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau,\tag{4.1}$$

where $C,C_5>0$ are constants.

Proof Multiplying $\nabla^3(1.5)_1$ by $\nabla^2\boldsymbol{u}$, exploiting $\nabla^2(1.5)_2\cdot\nabla^3\varrho^L$ in L^2, and integrating by parts, we can estimate that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\nabla^3\varrho^L\cdot\nabla^2\boldsymbol{u}\,\mathrm{d}x=\mu_0\int\nabla^3\varrho^L\cdot\nabla^2\Delta\boldsymbol{u}\,\mathrm{d}x+\mu_0\int\nabla^3\varrho^L\cdot\nabla^3\operatorname{div}\boldsymbol{u}\,\mathrm{d}x$$
$$+\int\nabla^2\operatorname{div}\boldsymbol{u}\cdot\nabla^2\operatorname{div}\boldsymbol{u}^L\,\mathrm{d}x-\int\nabla^3\varrho^L\cdot\nabla^3\varrho\,\mathrm{d}x$$
$$+\int\nabla^2\mathcal{N}\cdot\nabla^3\varrho^L\,\mathrm{d}x+\int\nabla^2\operatorname{div}(\varrho\boldsymbol{u})^L\cdot\nabla^2\operatorname{div}\boldsymbol{u}\,\mathrm{d}x.$$

Similarly to (3.5), using the Young's inequality, we have

$$
\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}\int \nabla^3\varrho^L\cdot\nabla^2\boldsymbol{u}\,\mathrm{d}x \\
&\le \frac{\mu_0}{2}\big\|\nabla^2\Delta\boldsymbol{u}\big\|_{L^2}^2+\frac{\mu_0}{2}\big\|\nabla^3\operatorname{div}\boldsymbol{u}\big\|_{L^2}^2+\big\|\nabla^2\operatorname{div}\boldsymbol{u}\big\|_{L^2}^2 \\
&\quad+\frac{1}{2}\big\|\nabla^2\operatorname{div}\boldsymbol{u}^L\big\|_{L^2}^2+\frac{5+2\mu_0}{2}\big\|\nabla^3\varrho^L\big\|_{L^2}^2+\frac{1}{8}\big\|\nabla^3\varrho\big\|_{L^2}^2 \\
&\quad+\frac{1}{2}\big\|\nabla^2\operatorname{div}(\varrho\boldsymbol{u})^L\big\|_{L^2}^2+\frac{1}{2}\big\|\nabla^2\mathcal{N}\big\|_{L^2}^2.
\end{aligned}
\tag{4.2}
$$

By virtue of the Plancherel theorem and Lemma 3.3, we estimate that

$$
\begin{aligned}
&\frac{1}{2}\big\|\nabla^2\operatorname{div}(\varrho\boldsymbol{u})^L\big\|_{L^2}^2+\frac{1}{2}\big\|\nabla^2\mathcal{N}\big\|_{L^2}^2 \\
&\le C\epsilon_0\big(\big\|\nabla^4\boldsymbol{u}\big\|_{L^2}^2+\big\|\nabla^3(\varrho,\boldsymbol{u})\big\|_{L^2}^2\big).
\end{aligned}
\tag{4.3}
$$

Adding up (4.1) and $\gamma_2\cdot$(4.2) with some positive constants, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}_3-\gamma_3\int\nabla^3\varrho^L\cdot\nabla^2\boldsymbol{u}\,\mathrm{d}x\right) \\
&\qquad+\frac{\gamma_3}{8}\big\|\nabla^3\varrho\big\|_{L^2}^2+\frac{\mu_0}{4}\big\|\nabla^3\boldsymbol{u}^H\big\|_{L^2}^2+\frac{\mu_0}{4}\big\|\nabla^4\boldsymbol{u}\big\|_{L^2}^2+\frac{\mu_0}{2}\big\|\nabla^3\operatorname{div}\boldsymbol{u}\big\|_{L^2}^2 \\
&\le\left(\frac{1}{4}+\gamma_3\right)\big\|\nabla^2\operatorname{div}\boldsymbol{u}\big\|_{L^2}^2+\frac{\gamma_3\mu_0}{2}\big\|\nabla^2\Delta\boldsymbol{u}\big\|_{L^2}^2+\frac{\gamma_3\mu_0}{2}\big\|\nabla^3\operatorname{div}\boldsymbol{u}\big\|_{L^2}^2 \\
&\quad+C\gamma_3\big(\big\|\nabla^3\varrho^L\big\|_{L^2}^2+\big\|\nabla^2\operatorname{div}\boldsymbol{u}^L\big\|_{L^2}^2\big) \\
&\quad+C\epsilon_0(1+\gamma_3)\big\|\nabla^3(\varrho,\boldsymbol{u})\big\|_{L^2}^2+C\epsilon_0\big\|\nabla^4\boldsymbol{u}\big\|_{L^2}\big\|\nabla^3\boldsymbol{u}\big\|_{L^2}^{p-2}.
\end{aligned}
$$

In addition, by the frequency decomposition (1.6), we further put $\frac{\mu_0}{4}\|\nabla^2\boldsymbol{u}^L\|_{L^2}^2$ on both sides of (4.1) to get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}_3-\gamma_3\int\nabla^3\varrho^L\cdot\nabla^2\boldsymbol{u}\,\mathrm{d}x\right) \\
&\qquad+\frac{\gamma_3}{8}\big\|\nabla^3\varrho\big\|_{L^2}^2+\frac{\mu_0}{8}\big\|\nabla^3\boldsymbol{u}\big\|_{L^2}^2+\frac{\mu_0}{4}\big\|\nabla^4\boldsymbol{u}\big\|_{L^2}^2+\frac{\mu_0}{2}\big\|\nabla^3\operatorname{div}\boldsymbol{u}\big\|_{L^2}^2 \\
&\le\left(\frac{1}{4}+\gamma_3\right)\big\|\nabla^2\operatorname{div}\boldsymbol{u}\big\|_{L^2}^2+\frac{\gamma_3\mu_0}{2}\big\|\nabla^2\Delta\boldsymbol{u}\big\|_{L^2}^2+\frac{\gamma_3\mu_0}{2}\big\|\nabla^3\operatorname{div}\boldsymbol{u}\big\|_{L^2}^2 \\
&\quad+C\gamma_3\big\|\nabla^3\varrho^L\big\|_{L^2}^2+\left(\frac{1}{4}+C\gamma_3\right)\big\|\nabla^3\boldsymbol{u}^L\big\|_{L^2}^2 \\
&\quad+C\epsilon_0(1+\gamma_3)\big\|\nabla^3(\varrho,\boldsymbol{u})\big\|_{L^2}^2+C\epsilon_0\big\|\nabla^4\boldsymbol{u}\big\|_{L^2}\big\|\nabla^3\boldsymbol{u}\big\|_{L^2}^{p-2}.
\end{aligned}
$$

Furthermore, noting the smallness of ϵ_0, we obviously have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}_3-\gamma_3\int\nabla^3\varrho^L\cdot\nabla^2\boldsymbol{u}\,\mathrm{d}x\right) \\
&\qquad+\frac{\gamma_3}{16}\big\|\nabla^3\varrho\big\|_{L^2}^2+\frac{\mu_0}{16}\big\|\nabla^3\boldsymbol{u}\big\|_{L^2}^2+\frac{\mu_0}{8}\big\|\nabla^4\boldsymbol{u}\big\|_{L^2}^2+\frac{\mu_0}{8}\big\|\nabla^3\operatorname{div}\boldsymbol{u}\big\|_{L^2}^2 \\
&\le C\big\|\nabla^3\big(\varrho^L,\boldsymbol{u}^L\big)\big\|_{L^2}^2.
\end{aligned}
\tag{4.4}
$$

In view of the frequency decomposition (1.6), we get

$$\begin{aligned}
&\mathcal{F}_3 - \gamma_3 \int \nabla^3 \varrho^L \cdot \nabla^2 \boldsymbol{u}\, \mathrm{d}x \\
&\quad = \frac{1}{2}\left(\|\nabla^3 \varrho\|_{L^2}^2 + \|\nabla^3 \boldsymbol{u}\|_{L^2}^2\right) + \gamma_3 \int \nabla^3 \varrho^H \cdot \nabla^2 \boldsymbol{u}\, \mathrm{d}x \\
&\quad \le \frac{1}{2}\left(\|\nabla^3 \varrho\|_{L^2}^2 + \|\nabla^3 \boldsymbol{u}\|_{L^2}^2\right) + \frac{\gamma_3}{2}\|\nabla^2 \varrho\|_{L^2}^2 + \frac{\gamma_3}{2}\|\nabla^3 \boldsymbol{u}\|_{L^2}^2.
\end{aligned} \tag{4.5}$$

Next, recombining (4.5), we have

$$\mathcal{F}_3 - \gamma_3 \int \nabla^3 \varrho^L \cdot \nabla^2 \boldsymbol{u}\, \mathrm{d}x \approx \|\nabla^3(\varrho, \boldsymbol{u})\|_{L^2}^2, \tag{4.6}$$

and with the help of (4.4) and (4.6), we can deduce that for a suitable constant C_5,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{F}_3 - \gamma_3 \int \nabla^3 \varrho^L \cdot \nabla^2 \boldsymbol{u}\, \mathrm{d}x\right) + C_5\left(\mathcal{F}_3 - \gamma_3 \int \nabla^3 \varrho^L \cdot \nabla^2 \boldsymbol{u}\, \mathrm{d}x\right) \\
&\quad \le C\|\nabla^3(\varrho^L, \boldsymbol{u}^L)\|_{L^2}^2.
\end{aligned}$$

Consequently, thanks to the Gronwall inequality, we conclude that

$$\begin{aligned}
&\mathcal{F}_3(t) - \gamma_3 \int \nabla^3 \varrho^L \cdot \nabla^2 \boldsymbol{u}\, \mathrm{d}x \\
&\quad \le C_5 e^{-C_5 t}\left(\mathcal{F}_3(0) - \gamma_3 \int \nabla^3 \varrho_0^L \cdot \nabla^2 \boldsymbol{u}_0\, \mathrm{d}x\right) \\
&\qquad + C\int_0^t e^{-C_5(t-\tau)}\|\nabla^3(\varrho^L, \boldsymbol{u}^L)\|_{L^2}^2\, \mathrm{d}\tau.
\end{aligned}$$

This completes the proof of Lemma 4.1. □

4.2 Decay estimate of the low frequency part

We will present the estimate of the low frequency part of the constructed solution by analyzing the structure of the semigroup of the system (1.5). To this end, by Hausdorff decomposition [49], we first decompose the velocity $\boldsymbol{u}$ into $m = \Lambda^{-1}\operatorname{div}\boldsymbol{u}$ and $M = \Lambda^{-1}\operatorname{curl}\boldsymbol{u}$, where $\operatorname{curl}_{ij} = \partial_j u_i - \partial_i u_j$ and $\Lambda = \sqrt{-\Delta}$. Then, the system (1.5) can be decoupled into the following systems:

$$\begin{cases}
\varrho_t + \Lambda m = -\operatorname{div}(\varrho \boldsymbol{u}), \\
m_t - 2\mu_0 \Delta m - \Lambda \varrho = \Lambda^{-1}\operatorname{div}\mathcal{N}, \\
(\varrho, m)|_{t=0} = (\varrho^0, m^0)(x)
\end{cases} \tag{4.7}$$

and

$$\begin{cases}
M_t - \mu_0 \Delta M = \Lambda^{-1}\operatorname{curl}\mathcal{N}, \\
M(0,x) = M^0(x),
\end{cases} \tag{4.8}$$

where $m^0 := \Lambda^{-1}\operatorname{div}\boldsymbol{u}^0$ and $M^0 =: \Lambda^{-1}\operatorname{curl}\boldsymbol{u}^0$. Then a direct calculation yields the following lemma; see [22] for the proof.

Lemma 4.2 *Let $M(t,x)$ be the solution to the linearized system of* (4.8). *Then, for all* $|\xi|^2 \geq 0$, *we have*

$$\left|\hat{M}(t,\xi)\right|^2 \leq Ce^{-|\xi|^2 t}\left|\hat{M}(0,\xi)\right|^2,$$

where $C > 0$ is a constant and $\hat{M}$ denotes the Fourier transform of M.

Now we turn to considering the linearized system of (4.7). The following system can be obtained by applying the Fourier transform:

$$\begin{cases} \hat{\varrho}_t = -|\xi|\hat{m}, \\ \hat{m}_t = -|\xi|\hat{\varrho} - 2\mu_0|\xi|^2\hat{m}, \end{cases} \tag{4.9}$$

which is also rewritten by

$$\hat{U}_t = \hat{\mathcal{A}}\big(|\xi|\big)\hat{U}, \tag{4.10}$$

where $\hat{U} = (\hat{\varrho}, \hat{m})$ and

$$\hat{\mathcal{A}}\big(|\xi|\big) = \begin{pmatrix} 0, & -|\xi| \\ |\xi|, & -2\mu_0|\xi|^2 \end{pmatrix}.$$

According to the standard theory of ordinary differential equations, the system (4.10) admits a solution which can be expressed by

$$\hat{U} = e^{t\hat{\mathcal{A}}(|\xi|)}\hat{U}(0). \tag{4.11}$$

By taking the inverse Fourier transform on both sides of (4.11), we can obtain

$$U = A(t)U(0),$$

where

$$A(t)U = \mathcal{F}^{-1}\big(e^{t\hat{\mathcal{A}}(|\xi|)}\hat{U}(\xi)\big),$$

which implies the solution to the linearized system of (4.7). Additionally, we further work out the eigenvalues $\lambda_i(\xi)$ $(i = 1,2)$ of matrix $\hat{\mathcal{A}}(|\xi|)$ and express them by

$$\det\big(\hat{\mathcal{A}}\big(|\xi|\big) - \lambda I\big) = \lambda^2 + 2\mu_0|\xi|\lambda + |\xi|^2 = 0. \tag{4.12}$$

Then the eigenvalues $\lambda_i(\xi)$ $(i = 1,2)$ of $\hat{\mathcal{A}}(|\xi|)$ can be calculated by (4.12) as follows:

$$\begin{cases} \lambda_1(\xi) = -\mu_0|\xi|^2 + |\xi|\sqrt{\mu_0^2|\xi|^2 - 1}, \\ \lambda_2(\xi) = -\mu_0|\xi|^2 - |\xi|\sqrt{\mu_0^2|\xi|^2 - 1}. \end{cases}$$

Based on the semigroup decomposition theory given in [2], we have

$$e^{t\hat{A}(|\xi|)} = e^{\lambda_1 t}P_1(\xi) + e^{\lambda_2 t}P_2(\xi),$$

where

$$P_i(\xi) = \prod_{j\neq i} \frac{A(|\xi|) - \lambda_i I}{\lambda_i - \lambda_j} \quad (i,j = 1,2)$$

is a projection operator.

Then by tedious and careful calculations, we can present the asymptotic expansions of $\lambda_i(\xi), P_i(\xi)$ $(i = 1,2)$ and $e^{t\hat{A}(|\xi|)}$ in the low and high frequency situations, which give rise to the following lemma.

Lemma 4.3 *For any* $|\xi| \leq 1$, $\lambda_i(\xi)$ $(i = 1,2)$ *has the Taylor series expansion*

$$\begin{cases} \lambda_1(\xi) = -|\xi|^2 + i(|\xi| + O(|\xi|^3)), \\ \lambda_2(\xi) = -|\xi|^2 - i(|\xi| + O(|\xi|^3)). \end{cases}$$

We omit the proof of this lemma here; please refer to [17]. With the help of Lemmas 4.2 and 4.3, one can obtain the time decay estimates of the low frequency part of the solution to the linear system (4.9).

Proposition 4.1 *Let* $1 \leq q \leq 2$. *Then for any integer* $k \geq 0$, *the solution to the linearized system of* (1.5) *satisfies*

$$\|\nabla^k(\varrho^L, m^L, \boldsymbol{u}^L)(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}} \|(\varrho^0, m^0, \boldsymbol{u}^0)(t)\|_{L^q},$$

where $C > 0$ *is a constant independent of* T.

Proof Following the arguments in [43], we can deduce that

$$\|\nabla^k(\varrho^L, m^L, M^L)(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}} \|(\varrho^0, m^0, M^0)(t)\|_{L^q}. \tag{4.13}$$

Note that

$$\boldsymbol{u} = \Delta^{-1}(\nabla \operatorname{div} \boldsymbol{u} - \operatorname{curl}\operatorname{curl} \boldsymbol{u}) = -\Lambda^{-1}\nabla m + \Lambda^{-1}\operatorname{curl} M, \tag{4.14}$$

which together with (4.13) implies that

$$\|\nabla^k \boldsymbol{u}^L(t)\|_{L^2} = \|\nabla^k(m^L, M^L)(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}} \|\boldsymbol{u}^0(t)\|_{L^q}. \tag{4.15}$$

Combining (4.13) and (4.15), we complete the proof of Proposition 4.1. □

4.3 Decay rates for the nonlinear system

This subsection is devoted to investigating the optimal time decay rates of the solutions for the nonlinear system (1.5). For convenience, we define

$$W(t) = \big(\varrho(t), \boldsymbol{u}(t)\big)^T$$

and

$$Q = \begin{pmatrix} 0, & \operatorname{div} \\ \nabla, & -\mu_0\Delta - \mu_0\nabla\operatorname{div} \end{pmatrix}.$$

Then it follows from the nonlinear system (1.5) that

$$W_t + QW = \mathcal{N}_1(W)$$

with the initial data $W|_{t=0} = W(0)$, where $\mathcal{N}_1(W)$ has been defined by

$$\mathcal{N}_1(W) = \big(-\operatorname{div}(\varrho\boldsymbol{u}), \mathcal{N}\big)^T.$$

Thanks to Duhamel's principle, the solution of the nonlinear system can be presented as

$$W(t) = Q(0)W(0) + \int_0^t Q(t-\tau)\mathcal{N}_1(W)(\tau)\,\mathrm{d}\tau,$$

where $Q(0)W(0)$ is the initial data of the solution to the linearized system of (1.5). In addition, with Proposition 4.1 in hand, we have the following lemma.

Lemma 4.4 *For any integer $k \geq 0$, we have*

$$\begin{aligned} \big\|\nabla^k W^L(t)\big\|_{L^2} \leq{}& C_6(1+t)^{-(\frac{3}{4}+\frac{k}{2})}\big\|W(0)\big\|_{L^1} \\ &+ C_6\int_0^{\frac{t}{2}}(1+t-\tau)^{-(\frac{3}{4}+\frac{k}{2})}\big\|\mathcal{N}_1(W)(\tau)\big\|_{L^1}\,\mathrm{d}\tau \\ &+ C_6\int_{\frac{t}{2}}^t(1+t-\tau)^{\frac{k}{2}}\big\|\mathcal{N}_1(W)(\tau)\big\|_{L^2}\,\mathrm{d}\tau, \end{aligned} \tag{4.16}$$

where $C_6 > 0$ is a constant.

Based on Lemmas 4.1 and 4.4, we are in a position to establish the optimal time decay rates of solutions.

Lemma 4.5 (*Optimal time decay rates*). *With the assumptions in Theorem 1.1, we have, for any $t \in [0,\infty)$,*

$$\big\|\nabla^l(\varrho,\boldsymbol{u})(t)\big\|_{L^2} \leq C(1+t)^{-(\frac{3}{4}+\frac{l}{2})}, \quad l = 0,1,2,3,$$

where $C > 0$ is a constant.

Proof We first define a non-decreasing Lyapunov function $\mathcal{G}(\tau)$ as

$$\mathcal{G}(\tau) := \sup_{0\le\tau\le t}\sum_{l=0}^{3}(1+\tau)^{\frac{3}{4}+\frac{l}{2}}\big\|\nabla^l(\varrho,\boldsymbol{u})(\tau)\big\|_{L^2},$$

which implies that for $0\le\tau\le t$ and $0\le l\le 3$,

$$\big\|\nabla^l(\varrho,\boldsymbol{u})(\tau)\big\|_{L^2}\le C_7(1+\tau)^{-(\frac{3}{4}+\frac{l}{2})}\mathcal{G}(\tau), \tag{4.17}$$

where the constant $C_7>0$ is independent of ϵ_0.

Due to (4.17) and Hölder's inequality, we have

$$\begin{aligned}\big\|\mathcal{N}_1(W)(\tau)\big\|_{L^1} &\le \big\|(\varrho,\boldsymbol{u})\big\|_{L^2}\big\|\nabla(\varrho,\boldsymbol{u})\big\|_{L^2}+\|\varrho\|_{L^2}\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}\\ &\quad+\|\nabla\boldsymbol{u}\|_{L^2}^2+\|\nabla\boldsymbol{u}\|_{L^{2(q-2)}}^{p-2}\|\nabla\boldsymbol{u}\|_{H^1}\\ &\le\epsilon_0\mathcal{G}(t)(1+\tau)^{-\frac{5}{4}}\end{aligned}$$

and

$$\begin{aligned}\big\|\mathcal{N}_1(W)(\tau)\big\|_{L^2} &\le \big\|(\varrho,\boldsymbol{u})\big\|_{L^3}\big\|\nabla(\varrho,\boldsymbol{u})\big\|_{L^6}+\|\varrho\|_{L^\infty}\big\|\nabla^2\boldsymbol{u}\big\|_{L^2}\\ &\quad+\|\nabla\boldsymbol{u}\|_{L^3}^2+\|\nabla\boldsymbol{u}\|_{L^6}^2+\|\nabla\boldsymbol{u}\|_{L^\infty}^{p-2}\|\nabla\boldsymbol{u}\|_{H^1}\\ &\le\epsilon_0^{1-\vartheta}\mathcal{G}^{1+\vartheta}(t)(1+\tau)^{-(\frac{7}{4}+\frac{3\vartheta}{4})},\end{aligned}$$

where $\vartheta\in(0,\frac{1}{2})$ is a given constant. Combining with Lemmas 2.6 and 4.4, we have

$$\begin{aligned}\big\|\nabla^l W^L(t)\big\|_{L^2} &\le C(1+t)^{-(\frac{3}{4}+\frac{l}{2})}\big\|W(0)\big\|_{L^1}\\ &\quad+C_6\int_0^{\frac{t}{2}}(1+t-\tau)^{-(\frac{3}{4}+\frac{l}{2})}\epsilon_0\mathcal{G}(\tau)(1+\tau)^{-\frac{5}{4}}\,\mathrm{d}\tau\\ &\quad+C_6\int_{\frac{t}{2}}^{t}(1+t-\tau)^{\frac{l}{2}}\epsilon_0^{1-\vartheta}\mathcal{G}^{1+\vartheta}(\tau)(1+\tau)^{-(\frac{7}{4}+\frac{3\vartheta}{4})}\,\mathrm{d}\tau\\ &\le C(1+t)^{-(\frac{3}{4}+\frac{l}{2})}\big(\big\|W(0)\big\|_{L^1}+\epsilon_0\mathcal{G}(\tau)+\epsilon_0^{1-\vartheta}\mathcal{G}^{1+\vartheta}(\tau)\big),\end{aligned} \tag{4.18}$$

where $0\le l\le 3$. Putting (4.18) into (4.1) and using Lemma 2.6, we can deduce that

$$\begin{aligned}\big\|\nabla^3 W(t)\big\|_{L^2}^2 &\le Ce^{-C_3t}\big\|\nabla^3W(0)\big\|_{L^2}^2\\ &\quad+C\big(\big\|W(0)\big\|_{L^1}^2+\epsilon_0^2\mathcal{G}^2(t)\big)\int_0^t e^{-C_3(t-\tau)}(1+\tau)^{-\frac{7}{2}}\,\mathrm{d}\tau\\ &\quad+C\epsilon_0^{2-2\vartheta}\mathcal{G}^{2+2\vartheta}(t)\int_0^t e^{-C_3(t-\tau)}(1+t)^{-\frac{7}{2}}\,\mathrm{d}\tau\\ &\le C(1+t)^{-\frac{7}{2}}\big(\big\|W(0)\big\|_{H^2\cap L^1}^2+\epsilon_0^2\mathcal{G}^2(t)+\epsilon_0^{2-2\vartheta}\mathcal{G}^{2+2\vartheta}(t)\big).\end{aligned} \tag{4.19}$$

Making use of (1.6) and Lemma 2.1, we have

$$\begin{aligned}\|\nabla^l W(t)\|_{L^2}^2 &\le C\|\nabla^l W^L(t)\|_{L^2}^2 + C\|\nabla^l W^H(t)\|_{L^2}^2\\ &\le C\|\nabla^l W^L\|_{L^2}^2 + C\|\nabla^3 W\|_{L^2}^2.\end{aligned}\tag{4.20}$$

Therefore, by putting (4.18)–(4.19) into (4.20), we deduce that for $0 \le l \le 3$,

$$\|\nabla^l W(t)\|_{L^2}^2 \le C(1+t)^{-(\frac{3}{2}+l)}\big(\|W(0)\|_{H^2\cap L^1}^2 + \epsilon_0^2\mathcal{G}^2(t) + \epsilon_0^{2-2\vartheta}\mathcal{G}^{2+2\vartheta}(t)\big).$$

Recalling the definition of $\mathcal{G}(t)$, we can derive for sufficiently small ϵ_0 that

$$\mathcal{G}^2(t) \le \frac{C_8}{2}\big(\|(\varrho,u)(0)\|_{H^2\cap L^1}^2 + \epsilon_0^2\mathcal{G}^2(t) + \epsilon_0^{2-2\vartheta}\mathcal{G}^{2+2\vartheta}(t)\big),\tag{4.21}$$

where C_8 is independent of ϵ_0.

For the last term on the right-hand side of (4.21), by Young's inequality, we obtain

$$C_8\epsilon_0^{2-2\vartheta}\mathcal{G}^{2+2\vartheta}(t) \le \frac{1-\vartheta}{2}C_8^{\frac{2}{1-\vartheta}} + \frac{1+\vartheta}{2}\epsilon_0^{\frac{1-\vartheta}{1+\vartheta}}\mathcal{G}^4(t).\tag{4.22}$$

From (4.21)–(4.22), we have

$$\mathcal{G}^2(t) \le \mathcal{I}_0 + C_{\epsilon_0}\mathcal{G}^4(t),$$

where

$$C_{\epsilon_0} := \frac{1+\vartheta}{2}\epsilon_0^{\frac{4(1-\vartheta)}{1+\vartheta}}$$

and

$$\mathcal{I}_0 := C_8\|(\varrho,u)(0)\|_{H^2\cap L^1}^2 + \frac{1-\vartheta}{2}C_8^{\frac{2}{1-\vartheta}}.$$

Now we prove that $\mathcal{G}(t)$ is a bounded function by contradiction. Suppose $\mathcal{G}^2(t) > 2\mathcal{I}_0$ for any $t \ge t_1$ with a constant $t_1 > 0$. Noting that $\mathcal{G}(t) \in C^0[0,+\infty)$ and $\mathcal{G}^2(0)$ is small, we have

$$\mathcal{G}^2(t_0) = 2\mathcal{I}_0\tag{4.23}$$

with some $t_0 \in (0, t_1)$. Moreover, from (4.23), we have

$$\mathcal{G}^2(t_0) \le \mathcal{I}_0 + C_{\epsilon_0}\mathcal{G}^4(t_0),$$

which implies

$$\mathcal{G}^2(t_0) \le \frac{\mathcal{I}_0}{1 - C_{\epsilon_0}\mathcal{G}^2(t_0)}.\tag{4.24}$$

Assume ϵ_0 is a small constant such that $C_{\epsilon_0} < \frac{1}{4\mathcal{I}_0}$, which leads to $C_{\epsilon_0}\mathcal{G}^2(t_0) < \frac{1}{2}$. This fact together with (4.24) implies

$$\mathcal{G}^2(t_0) < 2\mathcal{I}_0.\tag{4.25}$$

Clearly, (4.25) contradicts (4.23). Therefore, one always gets $\mathcal{G}^2(t) \leq 2\mathcal{I}_0$ for any $t \geq t_1$. Keeping in mind that $\mathcal{G}(t)$ is non-decreasing, we can deduce $\mathcal{G}(t) \leq C$ for any $t \in [0, +\infty)$. This completes the proof. □

5 Conclusion

In this paper, we investigated the time decay properties of solutions for compressible non-Newtonian fluid equations. More specifically, we investigated the long-time behavior of the Cauchy problem for Eills-type three-dimensional isentropic compressible fluids by the well-posedness result for the non-Newtonian fluid equations in [1], as well as the long-time behavior result for the Navier–Stokes system in [2]. Li–Zhang investigated the long-time behavior of Newtonian fluids; we further investigated the long-time behavior of Eills-type non-Newtonian fluids.

Acknowledgements
The research of Jialiang Wang and Han Jiang was supported by the Natural Science Foundation of China (Grant Nos. 12022102, 12231016, and 12371233) and the Natural Science Foundation of Fujian Province of China (Grant Nos. 2020J02013 and 2022J01105).

Funding
See Acknowledgments section.

Abbreviations
Not applicable.

Data availability
Not applicable.

Declarations

Ethics approval and consent to participate
There does not exist any ethical issue regarding this work.

Competing interests
The authors declare no competing interests.

Author contributions
This work was carried out in collaboration between the two authors. Jialiang Wang designed the study and guided the research. Han Jiang and Jialiang Wang performed the analysis and wrote the first draft of the manuscript. Han Jiang and Jialiang Wang managed the analysis of the study. The two authors read and approved the final manuscript.

Received: 8 May 2023 Accepted: 13 December 2023 Published online: 03 January 2024

References

1. Guo, B.L., Zhu, P.C.: Partial regularity of suitable weak solution to the system of the incompressible non-Newtonian fluids. J. Differ. Equ. **178**, 281–297 (2002)
2. Li, H.L., Zhang, T.: Large time behavior of isentropic compressible Navier–Stokes system in $\mathbb{R}^3$. Math. Methods Appl. Sci. **34**, 670–682 (2011)
3. Yuan, H.J., Si, X., Feng, Z.S.: Global strong solutions of a class of non-Newtonian fluids with small initial energy. J. Math. Anal. Appl. **474**, 72–93 (2019)
4. Bohme, G.: Non-Newtonian Fluid Mechanics. Appl. Math. Mech. North-Holland, Amsterdam (1987)
5. Bothe, D., Pruss, J.: L_p-Theory for a class of non-Newtonian fluids. SIAM J. Math. Anal. **39**, 379–421 (2007)
6. Feireisl, E., Kwon, Y.S., Novotny, A.: On the long-time behavior of dissipative solutions to models of non-Newtonian compressible fluids. Arch. Ration. Mech. Anal. **241**, 475–495 (2021)
7. Moscariello, G., Porzio, M.M.: On the behavior in time of solutions to motion of non-Newtonian fluids. Nonlinear Differ. Equ. Appl. **27**, 42 (2020)
8. Bellout, H., Bloom, F., Necas, J.: Young measure-valued solutions for non-Newtonian incompressible viscous fluids. Commun. Partial Differ. Equ. **19**, 1763–1803 (1994)
9. Ladyzhenskaya, O.A.: The Mathematical Theory of Viscous Incompressible Flow. Gordon & Breach, New York (1969)
10. Lian, S., Gao, W., Yuan, H.: Existence of solutions to an initial Dirichlet problem of evolutional $p(x)$-Laplace equations. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **29**, 377–399 (2012)

11. Liu, H., Yuan, H., Qiao, J.: Global existence of strong solutions of Navier–Stokes equations with non-Newtonian potential for one-dimensional isentropic compressible fluids. Z. Angew. Math. Phys. **63**, 865–878 (2012)
12. Matsumura, A., Nishida, T.: The initial value problem for the equation of motion of compressible viscous and heat-conductive fluids. Proc. Jpn. Acad., Ser. A, Math. Sci. **55**, 337–342 (1979)
13. Matsumura, A., Nishida, T.: The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ. **20**, 67–104 (1980)
14. Ponce, G.: Global existence of small solution to a class of nonlinear evolution equations. Nonlinear Anal. **9**, 339–418 (1985)
15. Liu, T.P., Wang, W.K.: The pointwise estimates of diffusion waves for the Navier–Stokes equations in odd multi-dimensions. Commun. Math. Phys. **196**, 145–173 (1998)
16. Li, D.L.: The Green's function of the Navier–Stokes equations for gas dynamics in $\mathbb{R}^3$. Commun. Math. Phys. **257**, 579–619 (2005)
17. Kobayashi, T.: Some estimates of solutions for the equations of motion of compressible viscous fluid in an exterior domain in $\mathbb{R}^3$. J. Differ. Equ. **184**, 587–619 (2002)
18. Kagei, Y.: Large time behavior of solutions to the compressible Navier–Stokes equation in an infinite layer. Hiroshima Math. J. **38**, 95–124 (2008)
19. Li, H.L., Matsumura, A., Zhang, G.J.: Optimal decay rate of the compressible Navier–Stokes–Poisson system in $\mathbb{R}^3$. Arch. Ration. Mech. Anal. **196**, 681–713 (2010)
20. Tan, Z., Wang, H.Q.: Optimal decay rates of the compressible magnetohydrodynamic equations. Nonlinear Anal., Real World Appl. **14**, 188–201 (2013)
21. Gao, J., Chen, Y., Yao, Z.A.: Long-time behavior of solution to the compressible magnetohydrodynamic equations. Nonlinear Anal. **128**, 122–135 (2015)
22. Wang, W.J., Wen, H.Y.: Global well-posedness and time-decay estimates for compressible Navier–Stokes equations with reaction diffusion. Sci. China Math. **64**, 1–30 (2020)
23. Huang, W.T., Lin, X.Y., Wang, W.W.: Decay-in-time of the highest-order derivatives of solutions for the compressible isentropic MHD equations. J. Math. Anal. Appl. **502**, 125–273 (2021)
24. Pu, B.L., Guo, X.: Global existence and convergence rates of smooth solutions for the full compressible MHD equations. Z. Angew. Math. Phys. **64**, 519–538 (2013)
25. Abdallah, M.A., Jiang, F., Tan, Z.: Decay estimates for isentropic compressible magnetohydrodynamic equations in bounded domain. Acta Math. Sci. **32B**(6), 2211–2220 (2012)
26. Fan, J.S., Jiang, F.: Large-time behavior of liquid crystal flows with a trigonometric condition in two dimensions. Commun. Pure Appl. Anal. **15**(1), 73–90 (2016)
27. Guo, S.S., Tan, Z.: Large-time behaviour of solutions to a class of non-Newtonian compressible fluids. Nonlinear Differ. Equ. Appl. **24**, 23 (2017)
28. Chen, Q., Tan, Z., Wu, G.C.: Time decay rates for the equations of the compressible heat-conductive flow through porous media. J. Differ. Equ. **259**(9), 4707–4731 (2015)
29. Tan, Z., Wu, G.C.: Large time behavior of solutions for compressible Euler equations with damping in $\mathbb{R}^3$. J. Differ. Equ. **252**, 1546–1561 (2012)
30. Zhang, Y.H.: Decay of the 3D inviscid liquid-gas two-phase flow model. Z. Angew. Math. Phys. **67**(54), 1–22 (2016)
31. Zhang, Y.H.: Decay of the 3D viscous liquid-gas two-phase flow model with damping. J. Math. Phys. **57**(8), 081517 (2016)
32. Wu, G.C., Zhang, Y.H.: Global well-posedness and large time behaviour of the viscous liquid-gas two-phase flow model. Proc. R. Soc. Edinb., Sect. A, Math. **150**(4), 1999–2024 (2020)
33. Zhang, Y.H., Wu, G.C.: Global existence and asymptotic behavior for the 3D compressible non-isentropic Euler equations with damping. Acta Math. Sci. **34B**(2), 424–434 (2014)
34. Zhang, Y.H., Tan, Z.: Existence and asymptotic behavior of global smooth solution for p-system with damping and boundary effect. Nonlinear Anal., Theory Methods Appl. **72**, 2499–2513 (2010)
35. Zhang, Y.H., Tan, Z.: Asymptotic behavior of solutions to the Navier–Stokes equations of a two-dimensional compressible flow. Acta Math. Appl. Sin. Engl. Ser. **27**(4), 697–712 (2011)
36. Zhang, Y.H., Tan, Z., Ming, B.S.: Global existence and asymptotic behavior of smooth solutions to a coupled hyperbolic-parabolic system. Nonlinear Anal., Real World Appl. **14**, 465–482 (2013)
37. Jiang, M.N., Zhang, Y.H.: Existence and asymptotic behavior of global smooth solution for p-system with nonlinear damping and fixed boundary effect. Math. Methods Appl. Sci. **37**, 2585–2596 (2014)
38. Qiu, H.J., Zhang, Y.H.: Decay of the 3D quasilinear hyperbolic equations with nonlinear damping. Adv. Math. Phys. **2017**, 1–13 (2017)
39. Hu, H.J., Qiu, Y.B., Wang, D.H., Yang, L.: Incompressible limit for compressible viscoelastic flows with large velocity. Adv. Nonlinear Anal. **12**(1), 20220324 (2023)
40. Zhao, X.F., Li, W.J., Yan, W.P.: Global Sobolev regular solution for Boussinesq system. Adv. Nonlinear Anal. **12**(1), 20220298 (2023)
41. Panasenko, F., Pileckas, J.: Nonstationary Poiseuille flow of a non-Newtonian fluid with the shear rate-dependent viscosity. Adv. Nonlinear Anal. **12**(1), 20220259 (2023)
42. Liu, H., Sun, C., Meng, F.: Global well-posedness of the 3D magneto-micropolar equations with damping. Appl. Math. Lett. **94**, 38–43 (2019)
43. Li, J.D., Fu, S.B., Wang, W.W.: On time-decay rates of strong solutions for the 3D magnetohydrodynamics equations with nonlinear damping. J. Math. Anal. Appl. **515**, 126450 (2022)
44. Majda, A.J., Bertozzi, A.L.: Vorticity and Incompressible Flow. Cambridge Texts in Applied Mathematics, vol. 27 (2002)
45. Ju, N.: Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space. Commun. Math. Phys. **251**, 365–376 (2004)
46. Wei, R.Y., Li, Y., Yao, Z.A.: Decay of the compressible magnetohydrodynamic equations. Z. Angew. Math. Phys. **66**, 2499–2524 (2015)
47. Zhang, J., Zhao, J.: Some decay estimates of solutions for the 3D compressible isentropic magnetohydrodynamics. Commun. Math. Sci. **8**, 835–850 (2010)

48. Chen, Q., Tan, Z.: Global existence and convergence rates of smooth solutions for the compressible magnetohydrodynamic equations. Nonlinear Anal. **72**, 4438–4451 (2010)
49. Adams, R.A.: Sobolev Spaces. Pure Appl. Math., vol. 65. Academic Press [Harcourt Brace Jovanovich], New York (1975)

Publisher's Note