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# Ground state solutions for a kind of superlinear elliptic equations with variable exponent

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**Abstract**

In this paper, we focus on the existence of ground state solutions for the $p(x)$-Laplacian equation

$$\begin{cases} -\Delta_{p(x)}u + \lambda|u|^{p(x)-2}u = f(x,u) + h(x) & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{cases}$$

Using the constraint variational method, quantitative deformation lemma, and strong maximum principle, we proved that the above problem admits three ground state solutions, especially speaking, one solution is sign-changing, one is positive, and one is negative. Our results improve on those existing in the literature.

**Mathematics Subject Classification:** 35J60; 35J70; 35D30

**Keywords:** $p(x)$-Laplacian; Ground state solution; Sign-changing solution; Variable exponent

## 1 Introduction and main results

In this paper, we mainly study the $p(x)$-Laplacian equation with variable exponent

$$\begin{cases} -\Delta_{p(x)}u + \lambda|u|^{p(x)-2}u = f(x,u) + h(x) & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a smooth bounded domain, $\lambda > 0$ is a real parameter, and $\Delta_{p(x)}$ is the $p(x)$-Laplacian operator, that is,

$$\Delta_{p(x)} = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \sum_{i=1}^{N}\left(|\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_i}\right),$$

$p \in C(\bar{\Omega})$ is a Lipschitz function, and it satisfies $1 < p^- := \inf_{x\in\Omega} p(x) \leq p^+ := \sup_{x\in\Omega} p(x) < N$, $h(x)$ is a continuous function satisfying conditions that will be proposed later, and $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function.

A new and interesting research direction is the study of variational problems with $p(x)$-growth condition. It has many practical physical meanings, such as the nonlinear elasticity theory [1], stationary thermorheological viscous flows [2], electrorheological fluids [3], image processing [4] and nonlinear Darcy's law in porous medium [5]. Recently, many scholars have become increasingly concerned about the existence and multiplicity of solutions to the $p(x)$-Laplacian problems and have obtained many results under the following two useful conditions:

(f1) $f(x,t) = o(|t|^{p^+-2}t)$ as $t \to 0$ uniformly in $x \in \Omega$;

(f2) there exist $p^+ < r(x) < p^*(x)$ and some positive constant $C$ such that

$$\left|f(x,t)\right| \le C\left(1 + |t|^{r(x)-1}\right),$$

where $p^*(x) = \frac{Np(x)}{N-p(x)}$.

As is well known, (f1) and (f2) are standard and are important in many studies. Fan and Zhang [6] considered the cases when the nonlinear term $f(x,u)$ is $p(x)$-superlinear and $p(x)$-sublinear with $u$, respectively, and obtained the existence of infinitely many solutions for problem (1.1) with $\lambda = 0$ and $h(x) \equiv 0$. Amrouss and Kissi [7] proved that (1.1) has at least two nontrivial solutions with $\lambda = 0$ and $h(x) \equiv 0$, under adequate variational methods and a variant of the Mountain Pass lemma. The common feature of [6, 7] is that the authors used the well-known Ambrosetti-Rabinowitz's type conditions, that is

(AR) there exist $\mu > p^+$ and $M_0 > 0$ such that

$$0 < \mu F(x,t) \le tf(x,t), \quad x \in \Omega, |t| \ge M_0.$$

However, many functions are superlinear but do not satisfy the (AR) condition. As is well known, the main purpose of using (AR) is to ensure the boundedness of Palais-Smail-type sequences of the corresponding functional. Many scholars attempt to study such problems using weaker conditions. Avci [8] used a variant Fountain theorem and variational method to obtain the existence of infinitely many solutions for the Dirichlet boundary problems. Applying the Morse theory and modified functional methods, Tan and Fang [9] obtained some existence and multiplicity results. Zang [10] proved the existence and multiplicity of the solutions by Cerami condition. Yucedag [11] obtained infinitely many solutions for this problem with two superlinear terms. Liu and Pucci [12] dealt with the existence of a pair of nontrivial nonnegative and nonpositive solutions for a nonlinear weighted quasilinear equation in $\mathbb{R}^N$, which involves a double-phase operator under the Cerami condition instead of the classical Palais-Smale condition. Chu, Xie and Zhou [13] introduced new methods to show the boundedness of Cerami sequences and obtained the existence and multiplicity of solutions for a new Kirchhoff equation. Qin, Tang, and Zhang [14] developed a direct method and used approximation arguments to search for the Cerami sequences of energy functionals, estimated the minimax energy levels of these sequences, and obtained the existence of ground states and nontrivial solutions for a planar Hamiltonian elliptic system with critical exponential growth. Zhang and Zhang [15] obtained the existence of semiclassical ground state solutions via the generalized Nehari manifold method, in which nonlinearity $f$ is continuous but not necessarily of class $C^1$. Li, Nie, and Zhang [16] obtained the existence of normalized ground states by the Sobolev subcritical approximation method for the first time considering mass constraints, Kirchhof-type problems, and Schwartz symmetric rearrangement.

Next, we will continue to make the following assumptions on $f(x,t)$.

(f3) $\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^{p^+}} = \infty$ uniformly in $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s)\,ds$;

(f4) for each $x \in \Omega$, $\frac{f(x,t)}{|t|^{p^+-1}}$ is an increasing function of $t$ on $\mathbb{R} \setminus \{0\}$.

There are many nonlinear terms $f(x,t)$ that satisfy (f3) and (f4) but not (AR) (e.g., $f(x,t) = p^+|t|^{p^+-2}t\ln(1+t^2)$). There are some works that use (f3) and (f4); for example, when $\lambda = 0$ and $h(x) \equiv 0$, Ge, Zhuge, and Yuan [17] proved that (1.1) possesses one positive ground state solution, one negative ground state solution, and one sign-changing ground state solution; Ge, Zhang, and Hou [18] discussed the existence of the Nehari-type ground state solutions for a superlinear $p(x)$- Laplacian equation with potential $V(x)$ using perturbation methods. However, to the best of our knowledge, there are few results in the literature regarding ground state solutions for problem (1.1) since problem (1.1) is more complicated.

The solution of problem (1.1) is understood in the weak sense, that is, $u \in W_0^{1,p(x)}(\Omega)$ is the solution of problem (1.1) if

$$\begin{aligned} &\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u \cdot \nabla v + \lambda |u|^{p(x)-2}u \cdot v\right) dx - \int_\Omega h(x)v\,dx \\ &\quad = \int_\Omega f(x,u)v\,dx, \quad \forall v \in W_0^{1,p(x)}(\Omega), \end{aligned} \tag{1.2}$$

where $W_0^{1,p(x)}(\Omega)$ is the variable exponent Sobolev space and will be defined in Sect. 2.

The energy functional related to problem (1.1) is represented by

$$J(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \lambda |u|^{p(x)}\right) dx - \int_\Omega h(x)u\,dx - \int_\Omega F(x,u)\,dx. \tag{1.3}$$

If $u \in W_0^{1,p(x)}(\Omega)$ is a solution of problem (1.1) with $u^\pm \neq 0$, then $u$ is called a sign-changing solution of problem (1.1), where $u^\pm$ are defined as follows,

$$u^+(x) := \max\{u(x), 0\} \quad \text{and} \quad u^-(x) := \min\{u(x), 0\}. \tag{1.4}$$

For the convenience of further discussions, we set

$$\begin{aligned} \Xi &:= \{u \in W_0^{1,p(x)}(\Omega) : \langle J'(u), u^+\rangle = \langle J'(u), u^-\rangle = 0, u^\pm \neq 0\}, \\ \Psi &:= \{u \in W_0^{1,p(x)}(\Omega) : \langle J'(u), u\rangle = 0, u \neq 0\}, \end{aligned}$$

and let

$$\xi := \inf_{u\in\Xi} J(u), \qquad \psi := \inf_{u\in\Psi} J(u).$$

To obtain the desired results, the following assumption is made for $h(x)$.

(h1) for any $u \in \Psi$ and $h \in L^2(\mathbb{R}^N)$, we have $\langle h(x), u\rangle \le 0$.

Now, we present our main results:

**Theorem 1.1** *Assume that* (f1)–(f4) *and* (h1) *hold, then for any* $\lambda > 0$, *problem* (1.1) *admits a sign-changing solution* $u_0 \in \Xi$ *such that*

$$J(u_0) = \inf_{u\in\Xi} J(u).$$

**Theorem 1.2** *Assume that* $p \in C^1(\bar{\Omega})$, (f1)–(f4) *and* (h1) *hold, then for any* $\lambda > 0$, *problem* (1.1) *admits at least a positive ground state solution and a negative ground state solution.*

Combining Theorem 1.1 and Theorem 1.2, we can obtain the following result.

**Corollary 1.3** *Assume that* $p \in C^1(\bar{\Omega})$, (f1)–(f4) *and* (h1) *hold, then for any* $\lambda > 0$, *problem* (1.1) *admits at least a ground state sign-changing solution, a positive ground state solution, and a negative ground state solution.*

This paper is organized as follows. Section 2 introduces some preliminary knowledge of variable exponent spaces and gives some preliminary lemmas needed to prove our results. Section 3 presents the proof of Theorem 1.1 and Theorem 1.2.

## 2 Preliminaries

In this section, we will give out some results on the variable exponent Sobolev space, which come from [6, 19–23] and references therein.

For $p \in C(\bar{\Omega})$, let

$$C_+(\bar{\Omega}) = \left\{p \in C(\bar{\Omega}) : p(x) > 1 \text{ for all } x \in \bar{\Omega}\right\}.$$

For any $p \in C_+(\bar{\Omega})$, we introduce the variable exponent Lebesgue space defined by

$$L^{p(x)}(\Omega) = \left\{u : u \text{ is a measurable real-valued function such that} \right. \\ \left. \int_\Omega |u(x)|^{p(x)}\, dx < +\infty\right\}$$

endowed with the Luxemburg norm

$$|u|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\},$$

which is a separable and reflexive Banach space. The fundamental properties of variable exponent Lebesgue spaces can be found in [21, 24].

**Proposition 2.1** [19] *The space* $L^{p(x)}(\Omega)$ *is separable, uniformly convex, and reflexive, and its conjugate space is* $L^{q(x)}(\Omega)$, *where* $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. *For all* $u \in L^{p(x)}(\Omega)$, $v \in L^{q(x)}(\Omega)$, *the Hölder inequality*

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)}$$

*holds.*

When dealing with generalized Lebesgue and Sobolev spaces, the module $\rho(u)$ of space $L^{p(x)}(\Omega)$ plays an important role, and we set

$$\rho(u) = \int_\Omega |u|^{p(x)}\, dx.$$

**Proposition 2.2** [20] *For all $u \in L^{p(x)}(\Omega)$, the following properties are valid:*

(i) *For $u \neq 0$, $|u|_{p(x)} = \mu \Leftrightarrow \rho(\frac{u}{\mu}) = 1$;*
(ii) $|u|_{p(x)} < 1\ (= 1; > 1) \Leftrightarrow \rho(u) < 1\ (= 1; > 1)$;
(iii) *If $|u|_{p(x)} \geq 1$, then $|u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$;*
(iv) *If $|u|_{p(x)} \leq 1$, then $|u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$.*

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)} = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

and is equipped with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}. \tag{2.1}$$

Then $W_0^{1,p(x)}(\Omega)$ is defined as the completion of $C_0^{\infty}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$.

**Proposition 2.3** [21] *If $q \in C_+(\bar{\Omega})$ and $1 \leq q(x) \leq p^*(x)$, then for all $x \in \bar{\Omega}$, there is a continuous embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

*If replace $\leq$ with $<$, the embedding is compact.*

**Proposition 2.4** [21] *In $W_0^{1,p(x)}(\Omega)$, the Poincare inequality holds, that is, there is a constant $C_0 > 0$, such that*

$$\|u\|_{1,p(x)} \leq C_0 \|\nabla u\|_{L^{p(x)}(\Omega)}, \tag{2.2}$$

*for all $u \in W_0^{1,p(x)}(\Omega)$.*

*Remark* 2.5 By Proposition 2.4, there exists $c_{q(x)} > 0$ such that

$$|u|_{q(x)} \leq c_{q(x)} \|u\|_{1,p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega). \tag{2.3}$$

From Proposition 2.4, it is easy to see that $|\nabla u|_{p(x)}$ is an equivalent norm on $W_0^{1,p(x)}(\Omega)$.

For the convenience of future discussion, we will set $\|u\| = \|u\|_{1,p(x)}$.

**Proposition 2.6** [18] *Let*

$$I(u) = \int_{\Omega} \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

*Then*

(i) *For $u \neq 0$, $\|u\| = \mu \Leftrightarrow \rho(\frac{u}{\mu}) = 1$;*
(ii) $\|u\| < 1\ (= 1; > 1) \Leftrightarrow \rho(u) < 1\ (= 1; > 1)$;

(iii) *If* $\|u\| \geq 1$, *then* $\|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;
(iv) *If* $\|u\| \leq 1$, *then* $\|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$.

**Proposition 2.7** [23] *For a.e.* $x \in \Omega$, *let* $p$ *and* $q$ *be measurable functions such that* $p \in L^\infty(\Omega)$ *and* $1 < p(x)q(x) \leq \infty$. *Let* $0 \neq u \in L^{q(x)}(\Omega)$, *then*

$$|u|_{p(x)q(x)} \leq 1 \quad \Rightarrow \quad |u|^{p^+}_{p(x)q(x)} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)},$$
$$|u|_{p(x)q(x)} \geq 1 \quad \Rightarrow \quad |u|^{p^-}_{p(x)q(x)} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}.$$

To study problem (1.1), a functional in $W_0^{1,p(x)}(\Omega)$ is defined as follows:

$$T(u) := \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx.$$

From [25], we know that $T \in C^1(W_0^{1,p(x)}, \mathbb{R})$ and the double phase operator $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the derivative operator of $T$ in the weak sense. We let $\Gamma = T' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$, and we have

$$\langle \Gamma(u), v \rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v\, dx,$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$. The dual space of $W_0^{1,p(x)}(\Omega)$ is denoted as $(W_0^{1,p(x)}(\Omega))^*$, and $\langle \cdot, \cdot \rangle$ denotes the paring between $W_0^{1,p(x)}(\Omega)$ and $(W_0^{1,p(x)}(\Omega))^*$. Then, one has the following proposition.

**Proposition 2.8** [6] $\Gamma : W_0^{1,p(x)}(\Omega) \to W_0^{1,p(x)}(\Omega)^*$ *is a mapping of type* $(S)_+$, *i.e., if* $u_n \rightharpoonup u$ *in* $W_0^{1,p(x)}(\Omega)$ *and* $\limsup_{m\to+\infty} \langle \Gamma(u_n) - \Gamma(u), u_n - u \rangle \leq 0$, *then* $u_n \to u$ *in* $W_0^{1,p(x)}(\Omega)$.

To prove the Theorem 1.2, we need the following strong comparison theorem:

**Lemma 2.9** [22] *Let* $u \geq 0$ *be a weak up-solution of* $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = 0$ *and* $u \not\equiv 0$. *Then, for any compact subset* $G \subset \Omega$ *with* $G \neq \emptyset$, *there is a constant* $c > 0$ *such that* $u(x) \geq c$ *for any* $x \in G$.

In the following, some lemmas will be proved, which are very important for obtaining our main results.

**Lemma 2.10** *If assumptions* (f1)–(f4) *and* (h1) *hold, we have*

$$\begin{aligned} J(u) &\geq J(su^+ + tu^-) + \frac{1-s^{p^+}}{p^+}\langle J'(u), u^+ \rangle + \frac{1-t^{p^+}}{p^+}\langle J'(u), u^- \rangle \\ &\quad + \int_\Omega g(s)\left(\left|\nabla u^+\right|^{p(x)} + \lambda\left|u^+\right|^{p(x)}\right) dx + \int_\Omega g(t)\left(\left|\nabla u^-\right|^{p(x)} + \lambda\left|u^-\right|^{p(x)}\right) dx \\ &\quad \forall u = u^+ + u^- \in W_0^{1,p(x)}(\Omega), s, t \geq 0, \end{aligned} \tag{2.4}$$

*where* $g(i) = \frac{1-i^{p(x)}}{p(x)} - \frac{1-i^{p^+}}{p^+}$, $i \geq 0$, $x \in \Omega$.

*Proof*

$$
\begin{aligned}
&J(u) - J(su^+ + tu^-) \\
&= \int_\Omega \frac{1-s^{p(x)}}{p(x)} |\nabla u^+|^{p(x)}\,dx + \int_\Omega \frac{1-t^{p(x)}}{p(x)} |\nabla u^-|^{p(x)}\,dx \\
&\quad + \int_\Omega \frac{\lambda}{p(x)} (1-s^{p(x)}) |u^+|^{p(x)}\,dx + \int_\Omega \frac{\lambda}{p(x)} (1-t^{p(x)}) |u^-|^{p(x)}\,dx \\
&\quad + \int_\Omega \big[F(x,su^+) - F(x,u^+)\big]\,dx + \int_\Omega \big[F(x,tu^-) - F(x,u^-)\big]\,dx \\
&\quad + \int_\Omega (s-1)h(x)u^+\,dx + \int_\Omega (t-1)h(x)u^-\,dx \\
&= \int_\Omega \frac{1-s^{p(x)}}{p(x)} \big(|\nabla u^+|^{p(x)} + \lambda|u^+|^{p(x)}\big)\,dx + \int_\Omega \frac{1-t^{p(x)}}{p(x)} \big(|\nabla u^-|^{p(x)} + \lambda|u^-|^{p(x)}\big)\,dx \\
&\quad + \int_\Omega \big[F(x,su^+) - F(x,u^+)\big]\,dx + \int_\Omega \big[F(x,tu^-) - F(x,u^-)\big]\,dx \\
&\quad + \int_\Omega (s-1)h(x)u^+\,dx + \int_\Omega (t-1)h(x)u^-\,dx \\
&= \int_\Omega g(s)\big(|\nabla u^+|^{p(x)} + \lambda|u^+|^{p(x)}\big)\,dx + \int_\Omega g(t)\big(|\nabla u^-|^{p(x)} + \lambda|u^-|^{p(x)}\big)\,dx \\
&\quad + \frac{1-s^{p^+}}{p^+}\langle J'(u), u^+\rangle + \int_\Omega \left[\frac{1-s^{p^+}}{p^+} f(x,u^+)u^+ + F(x,su^+) - F(x,u^+)\right]dx \\
&\quad + \frac{1-t^{p^+}}{p^+}\langle J'(u), u^-\rangle + \int_\Omega \left[\frac{1-t^{p^+}}{p^+} f(x,u^-)u^- + F(x,tu^-) - F(x,u^-)\right]dx \\
&\quad + \int_\Omega \left(\frac{1-t^{p^+}}{p^+} + t - 1\right)h(x)u^-\,dx + \int_\Omega \left(\frac{1-s^{p^+}}{p^+} + s - 1\right)h(x)u^+\,dx.
\end{aligned}
\tag{2.5}
$$

We set $z(t) = \frac{1-t^{p^+}}{p^+} if(x,i) + F(x,ti) - F(x,i)$, and take the derivative of $z(t)$ yields

$$
\frac{\partial z(t)}{\partial t} = if(x,ti) - t^{p^+-1} if(x,i) = i|t|^{p^+-1}|i|^{p^+-1}\left[\frac{f(x,ti)}{|ti|^{p^+-1}} - \frac{f(x,i)}{|i|^{p^+-1}}\right].
\tag{2.6}
$$

From (2.6) and (f4), for any $i \in (-\infty, 0) \cup (0, +\infty)$, we have

$$
\begin{cases}
\frac{\partial z(t)}{\partial t} < 0, & \text{if } 0 < t < 1, \\
\frac{\partial z(t)}{\partial t} > 0, & \text{if } t > 1.
\end{cases}
\tag{2.7}
$$

Therefore, from (2.7), we get

$$
z(t) \geq z(1) \geq 0.
\tag{2.8}
$$

Next, through simple calculations, $\frac{1-i^{p^+}}{p^+} + i - 1 \leq 0$ can be obtained. Combined with hypothesis (h1), it can be concluded that

$$
\int_\Omega \left(\frac{1-s^{p^+}}{p^+} + s - 1\right)h(x)u^+\,dx + \int_\Omega \left(\frac{1-t^{p^+}}{p^+} + t - 1\right)h(x)u^-\,dx \geq 0.
\tag{2.9}
$$

Combining (2.5), (2.8), and (2.9) completes the proof. □

The following two corollaries come from Lemma 2.10.

**Corollary 2.11** *Assume that* (f1)–(f4) *and* (h1) *hold. From Lemma* 2.10, *if* $u = u^+ + u^- \in \Xi$, *then we have*

$$J(u) = J\left(u^+ + u^-\right) = \max_{s,t\geq 0} J\left(su^+ + tu^-\right).$$

**Corollary 2.12** *Assume that* (f1)–(f4) *and* (h1) *hold. From Lemma* 2.10, *if* $u \in \Psi$, *then we have*

$$J(u) = \max_{t\geq 0} J(tu).$$

**Lemma 2.13** *Assume that* (f1)–(f4) *and* (h1) *hold. If* $u \in W_0^{1,p(x)}(\Omega)$ *with* $u^\pm \neq 0$, *then there is a unique positive number pair* $(s_u, t_u)$ *such that*

$$s_u u^+ + t_u u^- \in \Xi.$$

*Proof* For any $u \in W_0^{1,p(x)}(\Omega)$ with $u^\pm \neq 0$, define the functions $g(s,t)$ and $h(s,t) : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ as

$$g(s,t) = \left\langle J'\left(su^+ + tu^-\right), su^+\right\rangle \quad \text{and} \quad h(s,t) = \left\langle J'\left(su^+ + tu^-\right), tu^-\right\rangle, \quad \text{respectively.}$$

By simple calculation, it can be concluded that

$$\begin{aligned} g(s,t) &= \int_\Omega s^{p(x)}\left|\nabla u^+\right|^{p(x)}dx + \int_\Omega \lambda s^{p(x)}\left|u^+\right|^{p(x)}dx \\ &\quad - \int_\Omega f\left(x, su^+\right)su^+\,dx - \int_\Omega h(x)su^+\,dx, \\ h(s,t) &= \int_\Omega t^{p(x)}\left|\nabla u^-\right|^{p(x)}dx + \int_\Omega \lambda t^{p(x)}\left|u^-\right|^{p(x)}dx \\ &\quad - \int_\Omega f\left(x, tu^-\right)tu^-\,dx - \int_\Omega h(x)tu^-\,dx. \end{aligned} \tag{2.10}$$

By assumptions (f1) and (f2), one has that for every $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ such that

$$\begin{aligned} &\left|f(x,t)\right| \leq \varepsilon|t|^{p^+-1} + C_\varepsilon|t|^{r(x)-1}, \quad \forall(x,t) \in \Omega \times \mathbb{R}, \\ &\left|F(x,t)\right| \leq \varepsilon|t|^{p^+} + C_\varepsilon|t|^{r(x)}, \quad \forall(x,t) \in \Omega \times \mathbb{R}, \end{aligned} \tag{2.11}$$

where $p^+ < r(x) < p^*$.

Therefore, for $0 < s < 1$, by Proposition 2.2, Proposition 2.4, Proposition 2.6 and (2.11), one has

$$\begin{aligned} g(s,t) \geq{}& s^{p^+}\int_\Omega \left|\nabla u^+\right|^{p(x)}dx + \lambda s^{p^+}\int_\Omega \left|u^+\right|^{p(x)}dx - \int_\Omega \left(\varepsilon s^{p^+}\left|u^+\right|^{p^+} + C_\varepsilon s^{r(x)}\left|u^+\right|^{r(x)}\right)dx \\ &- s\int_\Omega h(x)u^+\,dx \end{aligned}$$

$$\ge \min\{1,\lambda\}s^{p^+}\int_\Omega\left(\left|\nabla u^+\right|^{p(x)}+\left|u^+\right|^{p(x)}\right)dx-\int_\Omega\left(\varepsilon s^{p^+}\left|u^+\right|^{p(x)}+C_\varepsilon s^{r(x)}\left|u^+\right|^{r(x)}\right)dx$$

$$-s\left|\int_\Omega h(x)u^+\,dx\right|$$

$$\ge\begin{cases}\min\{1,\lambda\}s^{p^+}\|u^+\|^{p^+}-\varepsilon s^{p^+}c_{p^+}^{p^+}\|u^+\|^{p^+}-C_\varepsilon s^{r^-}\max\{c_{r(x)}^{r^-},c_{r(x)}^{r^+}\}\|u^+\|^{r^-}\\ \quad -sc_2|h|_2\|u^+\|,\quad \text{if } \|u^+\|<1,\\ \min\{1,\lambda\}s^{p^+}\|u^+\|^{p^-}-\varepsilon s^{p^+}c_{p^+}^{p^+}\|u^+\|^{p^+}-C_\varepsilon s^{r^-}\max\{c_{r(x)}^{r^-},c_{r(x)}^{r^+}\}\|u^+\|^{r^+}\\ \quad -sc_2|h|_2\|u^+\|,\quad \text{if } \|u^+\|>1.\end{cases}\tag{2.12}$$

Similarly, for $0<t<1$, we have

$$h(s,t)\ge t^{p^+}\int_\Omega\left|\nabla u^+\right|^{p(x)}dx+\lambda t^{p^+}\int_\Omega\left|u^+\right|^{p(x)}dx-\int_\Omega\left(\varepsilon t^{p^+}\left|u^+\right|^{p^+}+C_\varepsilon t^{r(x)}\left|u^+\right|^{r(x)}\right)dx$$

$$\ge\begin{cases}\min\{1,\lambda\}t^{p^+}\|u^-\|^{p^+}-\varepsilon t^{p^+}c_{p^+}^{p^+}\|u^-\|^{p^+}-C_\varepsilon t^{r^-}\max\{c_{r(x)}^{r^-},c_{r(x)}^{r^+}\}\|u^-\|^{r^-}\\ \quad -tc_2|h|_2\|u^-\|,\quad \text{if } \|u^-\|<1,\\ \min\{1,\lambda\}t^{p^+}\|u^-\|^{p^-}-\varepsilon t^{p^+}c_{p^+}^{p^+}\|u^-\|^{p^+}-C_\varepsilon t^{r^-}\max\{c_{r(x)}^{r^-},c_{r(x)}^{r^+}\}\|u^-\|^{r^+}\\ \quad -tc_2|h|_2\|u^-\|,\quad \text{if } \|u^-\|>1.\end{cases}\tag{2.13}$$

Because $p^+<r^-$ and $u^\pm\neq 0$, from (2.12), (2.13) and arbitrariness of $\varepsilon$, it is easy to obtain that $g(s,s)>0$ and $h(s,s)>0$ when $s$ is sufficiently small.

Next, by (2.8), let $t=0$, we have

$$\frac{1}{p^+}if(x,i)-F(x,i)\ge 0,\quad i\in\mathbb{R}\setminus\{0\}.\tag{2.14}$$

Therefore, by (2.14) and (f3), if $s>1$, we have

$$\begin{aligned}g(s,t)&\le s^{p^+}\int_\Omega\left|\nabla u^+\right|^{p(x)}dx+\lambda s^{p^+}\int_\Omega\left|u^+\right|^{p(x)}dx\\&\quad -p^+\int_\Omega F\left(x,su^+\right)dx+s\int_\Omega\left|h(x)u^+\right|dx\\&\le s^{p^+}\int_\Omega\left|\nabla u^+\right|^{p(x)}dx+\lambda s^{p^+}\int_\Omega\left|u^+\right|^{p(x)}dx\\&\quad -p^+\int_\Omega\frac{F(x,su^+)}{|su^+|^{p^+}}\left|su^+\right|^{p^+}dx+s|h|_2\left|u^+\right|_2\\&=s^{p^+}\left(\int_\Omega\left|\nabla u^+\right|^{p(x)}dx+\lambda\int_\Omega\left|u^+\right|^{p(x)}dx\right.\\&\quad\left.-p^+\int_{u^+\neq 0}\frac{F(x,su^+)}{|su^+|^{p^+}}\left|u^+\right|^{p^+}dx\right)+s|h|_2\left|u^+\right|_2.\end{aligned}\tag{2.15}$$

Similarly, for $t>1$, one obtains

$$\begin{aligned}h(s,t)&\le t^{p^+}\left(\int_\Omega\left|\nabla u^-\right|^{p(x)}dx+\lambda\int_\Omega\left|u^-\right|^{p(x)}dx-p^+\int_{u^-\neq 0}\frac{F(x,tu^-)}{|tu^-|^{p^+}}\left|u^-\right|^{p^+}dx\right)\\&\quad +s|h|_2\left|u^-\right|_2.\end{aligned}\tag{2.16}$$

By (2.15) and (2.16), when $t > 0$ is sufficiently large, we have $g(t,t) < 0$ and $h(t,t) < 0$. To sum up, there exists $0 < S < T$ such that

$$g(T,T) > 0, \qquad h(T,T) > 0 \quad \text{and} \quad g(S,S) < 0, \qquad h(S,S) < 0. \tag{2.17}$$

By (2.10) and (2.17), for any $t \in [S,T]$, we have

$$g(T,t) > 0, \qquad g(S,t) < 0, \quad \text{and} \quad h(T,t) > 0, \quad h(S,t) < 0.$$

Therefore, according to Miranda's theorem [26], one can find $(s_u,t_u) \in (S,T)\times(S,T)$ such that $g(s_u,t_u) = 0$, $h(s_u,t_u) = 0$, that is $s_u u^+ + t_u u^- \in \Xi$.

Finally, we prove the uniqueness of $(s_u,t_u)$. Let $(s_1,t_1),(s_2,t_2) \in \Xi$ be such that

$$g(s_1,t_1) = h(s_1,t_1) = g(s_2,t_2) = h(s_2,t_2) = 0. \tag{2.18}$$

By Lemma 2.10, (2.10) and (2.18), we have

$$\begin{aligned}
J(s_1u^+ + t_1u^-) \geq{}& \frac{s_1^{p^+} - s_2^{p^+}}{p^+ s_1^{p^+}}\langle J'(s_1u^+ + t_1u^-), s_1u^+\rangle + \frac{t_1^{p^+} - t_2^{p^+}}{p^+ t_1^{p^+}}\langle J'(s_1u^+ + t_1u^-), t_1u^-\rangle \\
&+ \int_\Omega \left(\frac{s_1^{p(x)} - s_2^{p(x)}}{p(x)} - \frac{s_1^{p^+} - s_2^{p^+}}{p^+ s_1^{p^+}} s_1^{p(x)}\right)\left(|\nabla u^+|^{p(x)} + \lambda|u^+|^{p(x)}\right)dx \\
&+ \int_\Omega \left(\frac{t_1^{p(x)} - t_2^{p(x)}}{p(x)} - \frac{t_1^{p^+} - t_2^{p^+}}{p^+ t_1^{p^+}} t_1^{p(x)}\right)\left(|\nabla u^-|^{p(x)} + \lambda|u^-|^{p(x)}\right)dx \\
&+ J(s_2u^+ + t_2u^-) \\
={}& \int_\Omega \left(\frac{s_1^{p(x)} - s_2^{p(x)}}{p(x)} - \frac{s_1^{p^+} - s_2^{p^+}}{p^+ s_1^{p^+}} s_1^{p(x)}\right)\left(|\nabla u^+|^{p(x)} + \lambda|u^+|^{p(x)}\right)dx \\
&+ \int_\Omega \left(\frac{t_1^{p(x)} - t_2^{p(x)}}{p(x)} - \frac{t_1^{p^+} - t_2^{p^+}}{p^+ t_1^{p^+}} t_1^{p(x)}\right)\left(|\nabla u^-|^{p(x)} + \lambda|u^-|^{p(x)}\right)dx \\
&+ J(s_2u^+ + t_2u^-)
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
J(s_2u^+ + t_2u^-) \geq{}& \frac{s_2^{p^+} - s_1^{p^+}}{p^+ s_2^{p^+}}\langle J'(s_2u^+ + t_2u^-), s_2u^+\rangle + \frac{t_2^{p^+} - t_1^{p^+}}{p^+ t_2^{p^+}}\langle J'(s_2u^+ + t_2u^-), t_2u^-\rangle \\
&+ \int_\Omega \left(\frac{s_2^{p(x)} - s_1^{p(x)}}{p(x)} - \frac{s_2^{p^+} - s_1^{p^+}}{p^+ s_2^{p^+}} s_2^{p(x)}\right)\left(|\nabla u^+|^{p(x)} + \lambda|u^+|^{p(x)}\right)dx \\
&+ \int_\Omega \left(\frac{t_2^{p(x)} - t_1^{p(x)}}{p(x)} - \frac{t_2^{p^+} - t_1^{p^+}}{p^+ t_2^{p^+}} t_2^{p(x)}\right)\left(|\nabla u^-|^{p(x)} + \lambda|u^-|^{p(x)}\right)dx \\
&+ J(s_1u^+ + t_1u^-) \\
={}& \int_\Omega \left(\frac{s_2^{p(x)} - s_1^{p(x)}}{p(x)} - \frac{s_2^{p^+} - s_1^{p^+}}{p^+ s_2^{p^+}} s_2^{p(x)}\right)\left(|\nabla u^+|^{p(x)} + \lambda|u^+|^{p(x)}\right)dx
\end{aligned}$$

$$+\int_\Omega \left(\frac{t_2^{p(x)} - t_1^{p(x)}}{p(x)} - \frac{t_2^{p^+} - t_1^{p^+}}{p^+ t_2^{p^+}} t_2^{p(x)}\right)\left(\left|\nabla u^-\right|^{p(x)} + \lambda\left|u^-\right|^{p(x)}\right) dx$$
$$+ J\left(s_1 u^+ + t_1 u^-\right). \tag{2.20}$$

Combining (2.19) and (2.20), we have $s_1 = s_2$ and $t_1 = t_2$. Therefore, one has that $(s_u, t_u)$ is the unique positive pair such that $s_u u^+ + t_u u^- \in \Xi$. The proof is completed. □

**Lemma 2.14** *Assume that* (f1)–(f4) *and* (h1) *hold. Then, we have*

$$\xi = \inf_{u\in\Xi} J(u) = \inf_{u\in W_0^{1,p(x)}(\Omega), u^\pm\neq 0} \max_{s,t\geq 0} J\left(su^+ + tu^-\right).$$

*Proof* By Corollary 2.11, we can deduce that

$$\inf_{u\in W_0^{1,p(x)}(\Omega), u^\pm\neq 0} \max_{s,t\geq 0} J\left(su^+ + tu^-\right) \leq \inf_{u\in\Xi} \max_{s,t\geq 0} J\left(su^+ + tu^-\right) = \inf_{u\in\Xi} J(u) = \xi. \tag{2.21}$$

On the other hand, by Lemma 2.13, for any $u \in W_0^{1,p(x)}(\Omega)$ with $u^\pm \neq 0$, we can deduce that

$$\max_{s,t\geq 0} J\left(su^+ + tu^-\right) \geq J\left(s_u u^+ + t_u u^-\right) \geq \inf_{u\in\Xi} J(u) = \xi, \tag{2.22}$$

which implies

$$\inf_{u\in W_0^{1,p(x)}(\Omega), u^\pm\neq 0} \max_{s,t\geq 0} J\left(su^+ + tu^-\right) \geq \xi. \tag{2.23}$$

Combining (2.21) and (2.22), we can deduce that

$$\xi = \inf_{u\in W_0^{1,p(x)}(\Omega), u^\pm\neq 0} \max_{s,t\geq 0} J\left(su^+ + tu^-\right). \tag{2.24}$$

The proof is completed. □

**Lemma 2.15** *Assume that* (f1)–(f4) *and* (h1) *hold. Then* $\xi > 0$ *can be achieved.*

*Proof* First, prove that $\inf_{u\in\Psi} J(u) > 0$. For $\forall u \in \Psi$, we have $\langle J'(u), u\rangle = 0$, that is

$$\int_\Omega \left(|\nabla u|^{p(x)} + \lambda|u|^{p(x)}\right) dx - \int_\Omega h(x)u\, dx = \int_\Omega f(x,u)u\, dx. \tag{2.25}$$

By (2.11) and Remark 2.5, we have

$$\begin{aligned}\int_\Omega f(x,u)u\, dx &\leq \int_\Omega \left(\varepsilon|u|^{p^+} + C_\varepsilon|u|^{r(x)}\right) dx \\ &\leq \varepsilon|u|_{p^+}^{p^+} + C_\varepsilon \max\left\{|u|_{r(x)}^{r^-}, |u|_{r(x)}^{r^+}\right\} \\ &\leq \varepsilon c_{p^+}^{p^+}\|u\|^{p^+} + C_\varepsilon \max\left\{c_{r(x)}^{r^-}\|u\|^{r^-}, c_{r(x)}^{r^+}\|u\|^{r^+}\right\}.\end{aligned} \tag{2.26}$$

By Proposition 2.1, Remark 2.5, Proposition 2.6 and (h1), one obtains

$$\begin{aligned} &\int_\Omega \left(|\nabla u|^{p(x)} + \lambda |u|^{p(x)}\right) dx - \int_\Omega h(x) u \, dx \\ &\quad \geq \begin{cases} \min\{1,\lambda\}\|u\|^{p^+}, & \text{if } \|u\| < 1, \\ \min\{1,\lambda\}\|u\|^{p^-}, & \text{if } \|u\| > 1. \end{cases} \end{aligned} \tag{2.27}$$

Combining (2.23), (2.26), and (2.27), for any $u \in \Psi$ with $\|u\| < 1$, we have

$$\varepsilon c_{p^+}^{p^+} \|u\|^{p^+} + C_\varepsilon \max\left\{c_{r(x)}^{r^-}\|u\|^{r^-}, c_{r(x)}^{r^+}\|u\|^{r^+}\right\} \geq \min\{1,\lambda\}\|u\|^{p^+}. \tag{2.28}$$

Due to the arbitrariness of $\varepsilon$, from (2.28), we can deduce that

$$\|u\| \geq \left(\frac{1}{2C_\varepsilon \max\{c_{r(x)}^{r^-}, c_{r(x)}^{r^+}\}}\right)^{\frac{1}{r^- - p^+}} > 0. \tag{2.29}$$

Therefore, there exists a positive constant $\kappa_0 < 1$ such that

$$\|u\| \geq \kappa_0, \quad \forall u \in \Psi. \tag{2.30}$$

By hypothesis (h1), (2.11) and (2.29), we have

$$\begin{aligned} J(tu) &= \int_\Omega \frac{t^{p(x)}}{p(x)} \left(|\nabla u|^{p(x)} + \lambda |u|^{p(x)}\right) dx - \int_\Omega F(x,tu)\, dx - t\int_\Omega h(x) u \, dx \\ &\geq \frac{\min\{1,\lambda\}}{p^+} \int_\Omega t^{p(x)} \left(|\nabla u|^{p(x)} + \lambda |u|^{p(x)}\right) dx - \varepsilon t^{p^+} c_{p^+}^{p^+} \|u\|^{p^+} - C_\varepsilon \int_\Omega t^{r(x)} |u|^{r(x)}\, dx \\ &\quad - t\int_\Omega \left|h(x)u\right| dx \\ &\geq \begin{cases} \frac{\min\{1,\lambda\}}{p^+} t^{p^+}\|u\|^{p^+} - \varepsilon t^{p^+} c_{p^+}^{p^+}\|u\|^{p^+} - C_\varepsilon \max\{c_{r(x)}^{r^-}, c_{r(x)}^{r^+}\} t^{r^-}\|u\|^{r^-} \\ \quad - tc_2|h|_2\|u\|, \quad \text{if } 0 \leq t \leq 1, \kappa_0 \leq \|u\| \leq 1, \\ \frac{\min\{1,\lambda\}}{p^+} t^{p^-}\|u\|^{p^+} - \varepsilon t^{p^+} c_{p^+}^{p^+}\|u\|^{p^+} - C_\varepsilon \max\{c_{r(x)}^{r^-}, c_{r(x)}^{r^+}\} t^{r^+}\|u\|^{r^+} \\ \quad - tc_2|h|_2\|u\|, \quad \text{if } t > 1, \kappa_0 \leq \|u\| \leq 1, \\ \frac{\min\{1,\lambda\}}{p^+} t^{p^+}\|u\|^{p^-} - \varepsilon t^{p^+} c_{p^+}^{p^+}\|u\|^{p^+} - C_\varepsilon \max\{c_{r(x)}^{r^-}, c_{r(x)}^{r^+}\} t^{r^-}\|u\|^{r^+} \\ \quad - tc_2|h|_2\|u\|, \quad \text{if } 0 \leq t \leq 1, \|u\| > 1, \\ \frac{\min\{1,\lambda\}}{p^+} t^{p^-}\|u\|^{p^-} - \varepsilon t^{p^+} c_{p^+}^{p^+}\|u\|^{p^+} - C_\varepsilon \max\{c_{r(x)}^{r^-}, c_{r(x)}^{r^+}\} t^{r^+}\|u\|^{r^+} \\ \quad - tc_2|h|_2\|u\|, \quad \text{if } t > 1, \|u\| > 1. \end{cases} \end{aligned} \tag{2.31}$$

From Corollary 2.12 and (2.31), we have

$$J(u) = \max_{t \geq 0} J(tu)$$

$$\geq \begin{cases} \max_{t\geq 0}(\frac{\min\{1,\lambda\}}{p^+}t^{p^-}\|u\|^{p^+} - \varepsilon t^{p^+}c_{p^+}^{p^+}\|u\|^{p^+} - C_\varepsilon \max\{c_{r(x)}^{r^-}, c_{r(x)}^{r^+}\}t^{r^+}\|u\|^{r^-} \\ -tc_2|h|_2\|u\|), \quad \text{if } 0\leq t\leq 1, \kappa_0\leq \|u\|\leq 1, \\ \max_{t\geq 0}(\frac{\min\{1,\lambda\}}{p^+}t^{p^-}\|u\|^{p^+} - \varepsilon t^{p^+}c_{p^+}^{p^+}\|u\|^{p^+} - C_\varepsilon \max\{c_{r(x)}^{r^-}, c_{r(x)}^{r^+}\}t^{r^+}\|u\|^{r^+} \\ -tc_2|h|_2\|u\|), \quad \text{if } t> 1, \kappa_0\leq \|u\|\leq 1, \\ \max_{t\geq 0}(\frac{\min\{1,\lambda\}}{p^+}t^{p^-}\|u\|^{p^-} - \varepsilon t^{p^+}c_{p^+}^{p^+}\|u\|^{p^+} - C_\varepsilon \max\{c_{r(x)}^{r^-}, c_{r(x)}^{r^+}\}t^{r^+}\|u\|^{r^+} \\ -tc_2|h|_2\|u\|), \quad \text{if } 0\leq t\leq 1, \|u\|> 1, \\ \max_{t\geq 0}(\frac{\min\{1,\lambda\}}{p^+}t^{p^-}\|u\|^{p^-} - \varepsilon t^{p^+}c_{p^+}^{p^+}\|u\|^{p^+} - C_\varepsilon \max\{c_{r(x)}^{r^-}, c_{r(x)}^{r^+}\}t^{r^+}\|u\|^{r^+} \\ -tc_2|h|_2\|u\|), \quad \text{if } t> 1, \|u\|> 1. \end{cases}$$

$$\geq \begin{cases} \max_{t\geq 0}(\frac{\min\{1,\lambda\}}{p^+}t^{p^-}\|u\|^{p^+} - \varepsilon t^{p^+}c_{p^+}^{p^+}\|u\|^{p^+} - C_\varepsilon \max\{c_{r(x)}^{r^-}, c_{r(x)}^{r^+}\}t^{r^+}\|u\|^{r^-} \\ -tc_2|h|_2\|u\|), \quad \text{if } 0\leq t\leq 1, \kappa_0\leq \|u\|\leq 1, \\ \max_{t\geq 0}(\frac{\min\{1,\lambda\}}{p^+}t^{p^-}\|u\|^{p^-} - \varepsilon t^{p^+}c_{p^+}^{p^+}\|u\|^{p^+} - C_\varepsilon \max\{c_{r(x)}^{r^-}, c_{r(x)}^{r^+}\}t^{r^+}\|u\|^{r^+} \\ -tc_2|h|_2\|u\|), \quad \text{if } 0\leq t\leq 1, \|u\|> 1. \end{cases} \tag{2.32}$$

Hence, through basic calculations, it can be concluded that there exists a positive constant $\kappa_1(p^-, p^+, r^-, r^+, \kappa_0)$ such that

$$J(u) \geq \kappa_1, \quad \forall u \in \Psi,$$

which implies that

$$\psi = \inf_{u\in\Psi} J(u) \geq \kappa_1 > 0.$$

And since $\Xi \subseteq \Psi$, we have

$$\xi = \inf_{u\in\Xi} J(u) \geq \inf_{u\in\Psi} J(u) = \psi > 0.$$

Next, let $\{u_n\} \subset \Xi$ be a sequence of function such that $J(u_n) \to \xi$ as $n \to +\infty$. First, we prove that $\{u_n\}$ is bounded. Arguing by contradiction, suppose that $\|u_n\| \to +\infty$ as $n \to +\infty$ and let $v_n = \frac{u_n}{\|u_n\|}$. Passing, if necessary, to a subsequence, we may assume that

$$\begin{aligned} &v_n \rightharpoonup v \quad \text{in } W_0^{1,p(x)}(\Omega), \\ &v_n \to v \quad \text{in } L^{q(x)}(\Omega), p(x)\leq q(x) < p^*(x), \\ &v_n \to v \quad \text{a.e. on } \Omega. \end{aligned} \tag{2.33}$$

If $v = 0$, then $v_n \to 0$ in $L^{q(x)}$ with $1 \leq q(x) < p^*(x)$. Fix $M > (\frac{p^+(\xi+1)}{\min\{1,\lambda\}})^{\frac{1}{p^-}} > 1$. By (f1) and (f2), there exists $C_1 > 0$ such that

$$F(x,t) \leq |t|^{p^+} + C_1|t|^{r(x)}, \quad \forall (x,t) \in \Omega\times\mathbb{R}. \tag{2.34}$$

Then, using the Lebesgue dominated convergence theorem yields

$$\begin{aligned} &\limsup_{m\to\infty} \int_\Omega F(x, Rv_n)\,dx \\ &\quad \leq M^{p^+}\lim_{n\to\infty}|v_n|_{p^+}^{p^+} + C_1M^{r^+}\lim_{n\to\infty}\max\left\{|v_n|_{r(x)}^{r^-}, |v_n|_{r(x)}^{r^+}\right\} = 0. \end{aligned} \tag{2.35}$$

Let $t_n = \frac{M}{\|u_n\|}$. Hence, by Proposition 2.1, Corollary 2.12, and (2.35), we have

$$\begin{aligned}
\xi + o(1) = J(u_n) &\geq J(t_n u_n) = J(Mv_n) \\
&= \int_\Omega \frac{\min\{1,\lambda\}}{p(x)} M^{p(x)} \left( |\nabla v_n|^{p(x)} + \lambda |v_n|^{p(x)} \right) dx \\
&\quad - M \int_\Omega h(x) v_n \, dx - \int_\Omega F(x, Mv_n) \, dx \\
&\geq \frac{\min\{1,\lambda\}}{p^+} M^{p^-} - M \int_\Omega \left| h(x) v_n \right| dx - \int_\Omega F(x, Mv_n) \, dx \\
&\geq \frac{\min\{1,\lambda\}}{p^+} M^{p^-} - M |h|_2 |v_n|_2 - \int_\Omega F(x, Mv_n) \, dx \\
&\geq \frac{\min\{1,\lambda\}}{p^+} M^{p^-} - \int_\Omega F(x, Mv_n) \, dx \\
&\geq \frac{\min\{1,\lambda\}}{p^+} M^{p^-} - o(1) \\
&\geq \xi + 1 + o(1),
\end{aligned} \tag{2.36}$$

which leads to a contradiction. Thus, $v \neq 0$. By (f3), we have

$$\lim_{n\to\infty} \frac{F(x, u_n(x))}{\|u_n\|^{p^+}} = \lim_{n\to\infty} \frac{F(x, u_n(x))}{|u_n(x)|^{p^+}} \left| v_n(x) \right|^{p^+} = +\infty, \tag{2.37}$$

for all $x \in \{x \in \mathbb{R}^N : v(x) \neq 0\}$. By (f1) and (f2), there exists $C_2 \in \mathbb{R}$ such that

$$F(x,t) \geq C_2, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{2.38}$$

Therefore, from Proposition 2.6, (2.37), (2.38) and Fatou's Lemma, it yields

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{\xi + o(1)}{\|u_n\|^{p^+}} = \lim_{n\to\infty} \frac{J(u_n)}{\|u_n\|^{p^+}} \\
&= \lim_{n\to\infty} \left[ \frac{\int_\Omega \frac{1}{p(x)} [|\nabla u_n|^{p(x)} + \lambda |u_n|^{p(x)}] \, dx}{\|u_n\|^{p^+}} - \frac{\int_\Omega h(x) u_n \, dx}{\|u_n\|^{p^+}} - \frac{\int_\Omega F(x, u_n) \, dx}{\|u_n\|^{p^+}} \right] \\
&\leq \frac{\max\{1,\lambda\}}{p^-} + \lim_{n\to\infty} \frac{\int_\Omega |h(x) u| \, dx}{\|u_n\|^{p^+}} - \lim_{n\to\infty} \int_\Omega \frac{F(x, u_n)}{\|u_n\|^{p^+}} \, dx \\
&\leq \frac{\max\{1,\lambda\}}{p^-} + \lim_{n\to\infty} \frac{c_2 |h|_2 \|u_n\|}{\|u_n\|^{p^+}} - \lim_{n\to\infty} \int_\Omega \frac{F(x, u_n) - C_2}{\|u_n\|^{p^+}} \, dx \\
&\leq \frac{\max\{1,\lambda\}}{p^-} - \liminf_{n\to+\infty} \int_\Omega \frac{F(x, u_n) - C_2}{\|u_n\|^{p^+}} \, dx \\
&\leq \frac{\max\{1,\lambda\}}{p^-} - \liminf_{n\to+\infty} \int_\Omega \frac{F(x, u_n)}{\|u_n\|^{p^+}} \, dx \\
&\leq \frac{\max\{1,\lambda\}}{p^-} - \liminf_{n\to+\infty} \frac{F(x, u_n)}{|u_n|^{p^+}} |v_n|^{p^+} \, dx \\
&= -\infty.
\end{aligned} \tag{2.39}$$

This is a contradiction; therefore, $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Without loss of generality, we can assume that

$$\begin{aligned} &u_n^{\pm} \rightharpoonup u_0^{\pm} \quad \text{in } W_0^{1,p(x)}(\Omega),\\ &u_n^{\pm} \to u_0^{\pm} \quad \text{in } L^{q(x)}(\Omega) \text{ for } 1 \le q(x) < p^*(x),\\ &u_n^{\pm} \to u_0^{\pm} \quad \text{a.e. on } \Omega. \end{aligned} \tag{2.40}$$

Next, we prove that $u_0 \in \Xi$ and $J(u_0) = \xi$. Since $\{u_n\}_{n\in N} \subset \Xi$, we have $\{u_n^{\pm}\}_{n\in N} \subset \Psi$, that is

$$\int_\Omega \left(\left|\nabla u_n^{\pm}\right|^{p(x)} dx + \lambda\left|u_n^{\pm}\right|^{p(x)}\right) dx - \int_\Omega h(x)u_n^{\pm}\,dx = \int_\Omega f\left(x,u_n^{\pm}\right)u_n^{\pm}\,dx, \quad \text{and}$$
$$\left\|u_n^{\pm}\right\| \ge \kappa_0.$$

By hypothesis (h1), (2.11) and (2.30), we have

$$\begin{aligned} \varepsilon\int_\Omega \left|u_n^{\pm}\right|^{p^+} dx + C_\varepsilon \int_\Omega \left|u_n^{\pm}\right|^{r(x)} dx &\ge \int_\Omega f\left(x,u_n^{\pm}\right)u_n^{\pm}\,dx\\ &= \int_\Omega \left(\left|\nabla u_n^{\pm}\right|^{p(x)} + \lambda\left|u_n^{\pm}\right|^{p(x)}\right) dx - \int_\Omega h(x)u_n^{\pm}\,dx\\ &\ge \min\{1,\lambda\}\min\left\{\left\|u_n^{\pm}\right\|^{p^-}, \left\|u_n^{\pm}\right\|^{p^+}\right\}\\ &\ge \min\{1,\lambda\}\min\left\{\kappa_0^{p^-}, \kappa_0^{p^+}\right\}. \end{aligned} \tag{2.41}$$

Since $\{u_n\}$ is bounded, there is a constant $C_3 > 0$ such that

$$\min\{1,\lambda\}\min\left\{\kappa_0^{p^-}, \kappa_0^{p^+}\right\} \le \varepsilon C_3 + C_\varepsilon \int_\Omega \left|u_n^{\pm}\right|^{r(x)} dx.$$

Let $\varepsilon = \frac{\min\{1,\lambda\}\min\{\kappa_0^{p^-},\kappa_0^{p^+}\}}{2C_3}$, we have

$$\int_\Omega \left|u_n^{\pm}\right|^{r(x)} dx \ge \frac{\min\{1,\lambda\}\min\{\kappa_0^{p^-},\kappa_0^{p^+}\}}{2C_\varepsilon}.$$

By the compactness of the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ with $p^+ \le r(x) \le p^*(x)$, we have

$$\int_\Omega \left|u_0^{\pm}\right|^{r(x)} dx \ge \min \frac{\min\{1,\lambda\}\min\{\kappa_0^{p^-},\kappa_0^{p^+}\}}{2C_\varepsilon},$$

which means $u_0^{\pm} \ne 0$. Afterwards, notice that $u_n^{\pm} \to u_0^{\pm}$ in $L^{q(x)}(\Omega)$ with $1 \le q(x) \le p^*(x)$, by (f1), (f2), the Hölder inequality, and Lebesgue theorem, it yields

$$\begin{aligned} &\lim_{n\to+\infty} \int_\Omega f\left(x,u_n^{\pm}\right)u_n^{\pm}\,dx = \int_\Omega f\left(x,u_0^{\pm}\right)u_0^{\pm}\,dx,\\ &\lim_{n\to+\infty} \int_\Omega F\left(x,u_n^{\pm}\right) dx = \int_\Omega F\left(x,u_0^{\pm}\right) dx. \end{aligned} \tag{2.42}$$

Therefore, by the weak lower semicontinuity of the norm and $u_n^{\pm} \in \Psi$, we can deduce that

$$\begin{aligned}\langle J'(u_0), u_0^{\pm}\rangle &= \int_\Omega \left(\left|\nabla u_0^{\pm}\right|^{p(x)} + \lambda\left|u_0^{\pm}\right|^{p(x)}\right) dx - \int_\Omega h(x)u_0^{\pm}\,dx - \int_\Omega f\left(x, u_0^{\pm}\right)u_0^{\pm}\,dx \\ &\le \liminf_{n\to+\infty} \int_\Omega \left(\left|\nabla u_n^{\pm}\right|^{p(x)} + \lambda\left|u_n^{\pm}\right|^{p(x)}\right) dx \\ &\quad - \lim_{n\to+\infty} \int_\Omega h(x)u_n^{\pm}\,dx - \int_\Omega f\left(x, u_n^{\pm}\right)u_n^{\pm}\,dx \\ &= \liminf_{n\to+\infty} \langle J'(u_n), u_n^{\pm}\rangle = 0. \end{aligned} \tag{2.43}$$

Hence, from Lemma 2.13, there exists $s_0, t_0 > 0$ such that $s_0u_0^+ + t_0u_0^- \in \Xi$. By Lemma 2.10, and (2.43), we get

$$\begin{aligned}\xi &= \lim_{n\to+\infty}\left[J(u_n) - \frac{1}{p^+}\langle J'(u_n), u_n\rangle\right] \\ &= \lim_{n\to+\infty}\int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right)\left(|\nabla u_n|^{p(x)} + \lambda|u_n|^{p(x)}\right) dx + \lim_{n\to+\infty}\int_\Omega \left(\frac{1}{p^+} - 1\right)h(x)u_n\,dx \\ &\quad + \lim_{n\to+\infty}\int_\Omega \left[\frac{1}{p^+}f(x,u_n)u_n - F(x,u_n)\right] dx \\ &\ge \liminf_{n\to+\infty}\int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right)\left(|\nabla u_n|^{p(x)} + \lambda|u_n|^{p(x)}\right) dx + \lim_{n\to+\infty}\int_\Omega \left(\frac{1}{p^+} - 1\right)h(x)u_n\,dx \\ &\quad + \lim_{n\to+\infty}\int_\Omega \left[\frac{1}{p^+}f(x,u_n)u_n - F(x,u_n)\right] dx \\ &\ge \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right)\left(|\nabla u_0|^{p(x)} + \lambda|u_0|^{p(x)}\right) dx + \int_\Omega \left(\frac{1}{p^+} - 1\right)h(x)u_0\,dx \\ &\quad + \int_\Omega \left[\frac{1}{p^+}f(x,u_0)u_0 - F(x,u_0)\right] dx \\ &= J(u_0) - \frac{1}{p^+}\langle J'(u_0), u_0\rangle \\ &\ge J\left(s_0u_0^+ + t_0u_0^-\right) + \frac{1 - s_0^{p^+}}{p^+}\langle J'(u_0), u_0^+\rangle + \frac{1 - t_0^{p^+}}{p^+}\langle J'(u_0), u_0^-\rangle - \frac{1}{p^+}\langle J'(u_0), u_0\rangle \\ &= J\left(s_0u_0^+ + t_0u_0^-\right) - \frac{s_0^{p^+}}{p^+}\langle J'(u_0), u_0^+\rangle - \frac{t_0^{p^+}}{p^+}\langle J'(u_0), u_0^-\rangle \\ &\ge \xi - \frac{s_0^{p^+}}{p^+}\langle J'(u_0), u_0^+\rangle - \frac{t_0^{p^+}}{p^+}\langle J'(u_0), u_0^-\rangle, \end{aligned}$$

that is

$$\frac{s_0^{p^+}}{p^+}\langle J'(u_0), u_0^+\rangle + \frac{t_0^{p^+}}{p^+}\langle J'(u_0), u_0^-\rangle \ge 0. \tag{2.44}$$

Combining (2.43) and (2.44), we can deduce that

$$\langle J'(u_0), u_0^{\pm}\rangle = 0 \quad \text{and} \quad J(u_0) = \xi. \tag{2.45}$$

The proof is completed. □

**Lemma 2.16** *Assume that* (f1)–(f4) *and* (h1) *hold, if* $u_0 \in \Xi$ *and* $J(u_0) = \xi$, *then* $u_0$ *is a critical point of* $J(u)$.

*Proof* Since $u_0 \in \Xi$, one has $\langle J'(u_0^{\pm}), u_0^{\pm}\rangle = 0 = \langle J'(u_0), u_0\rangle$. By assumption (f4), for $0 < s \neq 1$ and $0 < t \neq 1$, we have

$$J\big(su_0^+ + tu_0^-\big) = J\big(su_0^+\big) + J\big(tu_0^-\big) < J\big(u_0^+\big) + J\big(u_0^-\big) = J(u_0) = \xi. \tag{2.46}$$

If $J'(u_0) \neq 0$, then there exist $\delta > 0$ and $\nu > 0$, such that

$$\|v - u_0\| \leq 3\delta: \quad \big\|J'(v)\big\| \geq \nu.$$

Let $Q = (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$ and $\psi(s,t) = su_0^+ + tu_0^-$, by (2.46), we have

$$\beta = \max_{(s,t)\in\partial Q} J\big(\psi(s,t)\big) < \xi. \tag{2.47}$$

Let $\varepsilon := \min\{\frac{\xi-\beta}{4}, \frac{\nu\delta}{8}\}$ and $B(u,\delta) := \{v \in W_0^{1,p(x)}(\Omega) : \|v - u\| \leq \delta\}$, by the Quantitative deformation lemma [27], there is a deformation $\theta$ such that

(i) $\theta(1,v) = v$ if $J(v) < \xi - 2\varepsilon$ or $J(v) > \xi + 2\varepsilon$,

(ii) $\theta(1, J^{\xi+\varepsilon} \cap B(u,\delta)) \subset J^{\xi-\varepsilon}$,

(iii) $J(\theta(1,v))$ is nonincreasing, $\forall v \in W_0^{1,p(x)}(\Omega)$,

where $J^{\xi\pm\varepsilon} := \{v \in W_0^{1,p(x)}(\Omega) : J(v) \leq \xi \pm \varepsilon\}$.

It is easy to see that

$$\max_{(s,t)\in D} J\big(\theta\big(1, \psi(s,t)\big)\big) < \xi.$$

Next, we show that $\theta(1,\psi(Q)) \cap \Xi \neq \emptyset$. Let $\phi(s,t) = \theta(1,\psi(s,t))$, $J_0(s,t) = \langle J'(su_0^+)u_0^+, J'(tu_0^-)u_0^-\rangle$ and $J_1(s,t) = \langle \frac{1}{s}J'(\phi^+(s,t)), \frac{1}{t}J'(\phi^-(s,t))\rangle$. Note that

$$\begin{aligned} &\big\langle J'\big(tu_0^{\pm}\big), u_0^{\pm} > 0\big\rangle \quad \text{if } 0 < t < 1, \\ &\big\langle J'\big(tu_0^{\pm}\big), u_0^{\pm} < 0\big\rangle \quad \text{if } t > 1. \end{aligned} \tag{2.48}$$

Therefore, we have that $\deg(J_0, Q, 0) = 1$. On the other hand, by (2.47) and the property (i) of $\theta$, we have that $\psi = \phi$ on $\partial Q$. Hence, $J_0 = J_1$ on $\partial Q$ and $\deg(J_0, Q, 0) = \deg(J_1, Q, 0) = 1$. This indicates that $J_1(s,t) = 0$ with some $(s,t) \in Q$, and thus $\theta(1,\psi(s,t)) = \phi(s,t) \in \Xi$. Therefore, $u_0$ is a critical point of $J(u)$. The proof is completed. □

**Lemma 2.17**

(i) *For* $x \in \Omega$, $t \leq 0$, *if* $f(x,t) \geq 0$ *and* $u \in W_0^{1,p(x)}(\Omega)$ *is a solution of problem* (1.1), *then* $u \geq 0$ *hold.*

(ii) *For* $x \in \Omega$, $t \geq 0$, *if* $f(x,t) \leq 0$ *and* $u \in W_0^{1,p(x)}(\Omega)$ *is a solution of problem* (1.1), *then* $u \leq 0$ *hold.*

*Proof* (i) Define $\Omega_1 = \{x \in \Omega : u(x) < 0\}$ and $\Omega_2 = \Omega \setminus \Omega_1$. Since $u^- = \min\{u, 0\}$ and $u^- \in W_0^{1,p(x)}(\Omega)$, we have

$$\nabla u^- = \begin{cases} \nabla u, & \text{in } \Omega_1, \\ 0, & \text{in } \Omega_2. \end{cases}$$

Replacing $v$ in (1.2) with $u^-$, we have

$$\int_\Omega \left( |\nabla u|^{p(x)-2} \nabla u \cdot \nabla u^- + \lambda |u|^{p(x)-2} u \cdot u^- \right) dx - \int_\Omega h(x) u^- \, dx = \int_\Omega f(x, u) u^- \, dx. \tag{2.49}$$

By (h1) and (2.49), we can deduce that

$$\begin{aligned} \int_{\Omega_1} \left( |\nabla u|^{p(x)} + \lambda |u|^{p(x)} \right) dx &= \int_{\Omega_1} h(x) u \, dx + \int_{\Omega_1} f(x, u) u \, dx \\ &\leq \int_{\Omega_1} f(x, u) u \, dx \leq 0. \end{aligned}$$

Therefore, $|\Omega_1| = 0$. Similarly, replacing $v$ in (1.2) with $u^+$, we can proof (ii). The proof is completed. □

## 3 Proof of main results

*Proof of Theorem* 1.1 Combining Lemma 2.15 and Lemma 2.16, there exists $u_0 \in \Xi$ such that

$$J(u_0) = \xi \quad \text{and} \quad J'(u_0) = 0. \tag{3.1}$$

From (3.1), we know that $u_0$ is a critical point of $J$; therefore, $u_0$ is a sign-changing solution of problem (1.1). □

*Proof of Theorem* 1.2 First, we define $f^+ = f(x, t)$ for $t > 0$ and $f^+ = 0$ for $t \leq 0$, and $F^+(x, t) = \int_0^t f^+(x, s) \, ds$. Let

$$\begin{aligned} J^+(u) = {} & \int_\Omega \frac{1}{p(x)} \left( |\nabla u|^{p(x)} + \lambda |u|^{p(x)} \right) dx - \int_\Omega h(x) u \, dx \\ & - \int_\Omega F^+(x, u) \, dx, \quad \forall u \in W_0^{1,p(x)}(\Omega). \end{aligned}$$

It is easy to verify that for $f^+$ and $F^+$, conditions (f1)–(f4) still hold. There are two claims to consider.

**Claim 1** *$J^+$ satisfies the $(PS)$-condition on $\Psi$. Let $\{u_n\} \subseteq \Omega$ be a $(PS)$-sequence such that*

$$\left(J^+\right)'(u_n) \to 0, \qquad J^+(u_n) \to c, \quad \forall c > 0. \tag{3.2}$$

First, we prove that $\{u_n\}$ is bounded. Arguing by contradiction, suppose that $\|u_n\| \to +\infty$ as $n \to +\infty$ and let $v_n = \frac{u_n}{\|u_n\|}$. Passing, if necessary, to a subsequence, we suppose that

$$v_n \rightharpoonup v \quad \text{in } W_0^{1,p(x)}(\Omega),$$

$$v_n \to v \quad \text{in } L^{q(x)}(\Omega), p(x) \le q(x) < p^*(x), \tag{3.3}$$
$$v_n \to v \quad \text{a.e. on } \Omega.$$

If $v = 0$, then $v_n \to 0$ in $L^{q(x)}$ with $1 \le q(x) < p^*(x)$. Fix $M > (\frac{p^+(c+1)}{\min\{1,\lambda\}})^{\frac{1}{p^-}} > 1$. By (f1) and (f2), there exists $C_4 > 0$ such that

$$F^+(x,u) \le |u|^{p^+} + C_4|u|^{r(x)}, \quad \forall (x,u) \in \Omega \times \mathbb{R}. \tag{3.4}$$

Thanks to (3.4) and the Lebesgue dominated convergence theorem, one has

$$\limsup_{m\to\infty} \int_\Omega F(x, Mv_n)\,dx \le M^{p^+} \lim_{n\to\infty} |v_n|_{p^+}^{p^+} + C_4 M^{r^+} \lim_{n\to\infty} \max\bigl\{|v_n|_{r(x)}^{r^-}, |v_n|_{r(x)}^{r^+}\bigr\} = 0. \tag{3.5}$$

Let $t_n = \frac{M}{\|u_n\|}$. Hence, by Proposition 2.1, Corollary 2.12 and (3.5), we have

$$\begin{aligned} c + o(1) = J^+(u_n) &\ge J(t_n u_n) = J^+(Mv_n) \\ &= \int_\Omega \frac{1}{p(x)} M^{p(x)} \bigl(|\nabla v_n|^{p(x)} + \lambda |v_n|^{p(x)}\bigr)\,dx - M\int_\Omega h(x)v_n\,dx - \int_\Omega F^+(x, Mv_n)\,dx \\ &\ge \frac{\min\{1,\lambda\}}{p^+} M^{p^-} - M\int_\Omega \bigl|h(x)v_n\bigr|\,dx - \int_\Omega F^+(x, Mv_n)\,dx \\ &\ge \frac{\min\{1,\lambda\}}{p^+} M^{p^-} - M|h|_2|v_n|_2 - \int_\Omega F^+(x, Mv_n)\,dx \\ &\ge \frac{\min\{1,\lambda\}}{p^+} M^{p^-} - \int_\Omega F^+(x, Mv_n)\,dx \\ &\ge \frac{\min\{1,\lambda\}}{p^+} M^{p^-} - o(1) \\ &\ge c + 1 + o(1). \end{aligned} \tag{3.6}$$

(3.6) is a contradiction. Hence, $v \ne 0$. By (f3), we have

$$\lim_{n\to\infty} \frac{F^+(x,u_n(x))}{\|u_n\|^{p^+}} = \lim_{n\to\infty} \frac{F^+(x,u_n(x))}{|u_n(x)|^{p^+}} \bigl|v_n(x)\bigr|^{p^+} = +\infty, \tag{3.7}$$

for all $x \in \{x \in \Omega : v(x) \ne 0\}$. Hence, it follows from Proposition 2.6, (2.36), (2.37), (2.38), (3.7) and Fatou's Lemma that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{c + o(1)}{\|u_n\|^{p^+}} = \lim_{n\to\infty} \frac{J^+(u_n)}{\|u_n\|^{p^+}} \\ &\le \lim_{n\to\infty} \left[ \frac{\int_\Omega \frac{1}{p(x)} [|\nabla u_n|^{p(x)} + \lambda |u_n|^{p(x)}]\,dx}{\|u_n\|^{p^+}} - \frac{\int_\Omega h(x)u_n\,dx}{\|u_n\|^{p^+}} - \frac{\int_\Omega F^+(x,u_n)\,dx}{\|u_n\|^{p^+}} \right] \\ &\le \frac{\max\{1,\lambda\}}{p^-} + \lim_{n\to\infty} \frac{\int_\Omega |h(x)u|\,dx}{\|u_n\|^{p^+}} - \lim_{n\to\infty} \int_\Omega \frac{F^+(x,u_n)}{\|u_n\|^{p^+}}\,dx \\ &\le \frac{\max\{1,\lambda\}}{p^-} + \lim_{n\to\infty} \frac{c_2|h|_2\|u_n\|}{\|u_n\|^{p^+}} - \lim_{n\to\infty} \int_\Omega \frac{F^+(x,u_n) - C_2}{\|u_n\|^{p^+}}\,dx \\ &\le \frac{\max\{1,\lambda\}}{p^-} - \liminf_{n\to+\infty} \int_\Omega \frac{F^+(x,u_n) - C_2}{\|u_n\|^{p^+}}\,dx \end{aligned}$$

$$
\begin{aligned}
&\le \frac{\max\{1,\lambda\}}{p^-} - \liminf_{n\to+\infty} \int_\Omega \frac{F^+(x,u_n)}{\|u_n\|^{p^+}}\,dx \\
&\le \frac{\max\{1,\lambda\}}{p^-} - \liminf_{n\to+\infty} \frac{F^+(x,u_n)}{|u_n|^{p^+}}|v_n|^{p^+}\,dx \\
&= -\infty. \qquad (3.8)
\end{aligned}
$$

(3.8) implies that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Without loss of generality, we can assume that

$$
\begin{aligned}
&u_n \rightharpoonup u_0 \quad \text{in } W_0^{1,p(x)}(\Omega), \\
&u_n \to u_0 \quad \text{in } L^{q(x)}(\Omega) \text{ for } 1 \le q(x) < p^*(x), \\
&u_n \to u_0 \quad \text{a.e. on } \Omega. \qquad (3.9)
\end{aligned}
$$

By (f2), Proposition 2.1, Proposition 2.7 and the boundedness of $\{u_n\}$, we have

$$
\begin{aligned}
&\lim_{n\to+\infty} \int_\Omega \left|f^+(x,u_n)\right||u_n-u_0|\,dx \\
&\quad\le \lim_{n\to+\infty} \int_\Omega C\left(1+|u_n|^{r(x)-1}\right)|u_n-u_0|\,dx \\
&\quad\le C\lim_{n\to+\infty} \int_\Omega |u_n|^{r(x)-1}|u_n-u_0|\,dx + C\lim_{n\to+\infty}\int_\Omega |u_n-u_0|\,dx \\
&\quad\le 2C\lim_{n\to+\infty} \left||u_n|^{r(x)-1}\right|_{r'(x)}|u_n-u_0|_{r(x)} + \lim_{n\to+\infty}|u_n-u_0|_1 \\
&\quad\le 2C\lim_{n\to+\infty} \max\left\{|u_n|_{r(x)}^{r^--1}, |u_n|_{r(x)}^{r^+-1}\right\}|u_n-u_0|_{r(x)} + \lim_{n\to+\infty}|u_n-u_0|_1 \\
&\quad= 0, \qquad (3.10)
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{n\to+\infty}\int_\Omega \lambda|u_n|^{p(x)-2}u_n(u_n-u_0)\,dx &\le \lim_{n\to+\infty}\int_\Omega |u_n|^{p(x)-1}|u_n-u_0|\,dx \\
&\le 2\left||u_n|^{p(x)-1}\right|_{p'(x)}|u_n-u_0|_{p(x)} \\
&\le 2\lim_{n\to+\infty}\max\left\{|u_n|_{p(x)}^{p^--1}, |u_n|_{p(x)}^{p^+-1}\right\}|u_n-u_0|_{p(x)} \\
&= 0, \qquad (3.11)
\end{aligned}
$$

where $\frac{1}{r(x)} + \frac{1}{r'(x)} = 1$. Therefore, by (f2), (3.10) and (3.11), we can deduce that

$$
\begin{aligned}
&\left\langle \Gamma(u_n) - \Gamma(u_0), u_n - u_0\right\rangle \\
&\quad= \left\langle \left(J^+\right)'(u_n) - \left(J^+\right)'(u_0), u_n-u_0\right\rangle + \int_\Omega \lambda|u_n|^{p(x)-2}u_n(u_n-u_0)\,dx \qquad (3.12) \\
&\qquad - \int_\Omega \lambda|u_0|^{p(x)-2}u_n(u_n-u_0)\,dx + \int_\Omega f^+(x,u_0)(u_n-u_0)\,dx \\
&\qquad - \int_\Omega f^+(x,u_n)(u_n-u_0)\,dx \\
&\quad\to 0, \quad \text{as } n\to+\infty. \qquad (3.13)
\end{aligned}
$$

So, $\Gamma$ is of type $(S)_+$, and we can deduce that

$$u_n \to u_0 \quad \text{in } W_0^{1,p(x)}(\Omega). \tag{3.14}$$

The proof of Claim 1 is completed.

From Lemma 2.13, it can be seen that for any $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$, there exists a unique positive number $t_u$ such that $t_u u \in \Psi$. Therefore, one can obtain that if $\mathbb{B}$ is a unit ball in $W_0^{1,p(x)}(\Omega)$, and by setting $\gamma(u) := t_u u$ to define the homomorphism $\gamma : \mathbb{B} \to \Psi$, then $\|\gamma(u)\| = t_u$. Therefore, if $\gamma^{-1}$ is the inverse of $\gamma$, and $\gamma^{-1}$ is defined as $\gamma^{-1}(v) = \frac{v}{\|v\|}$, then $\gamma^{-1} : \Psi \to \mathbb{B}$ is Lipschitz continuous. By (2.30), for any $v_1, v_2 \in \Psi$, we can deduce that

$$\begin{aligned}
\left\|\gamma^{-1}(v_1) - \gamma^{-1}(v_2)\right\| &= \left\| \frac{v_1}{\|v_1\|} - \frac{v_2}{\|v_2\|} \right\| \\
&= \left\| \frac{v_1 - v_2}{\|v_1\|} + \frac{(\|v_2\| - \|v_1\|)v_2}{\|v_1\|\|v_2\|} \right\| \\
&\le \frac{2}{\|v_1\|}\|v_1 - v_2\| \\
&\le \frac{2}{\kappa_0}\|v_1 - v_2\|.
\end{aligned}$$

Next, we define $\Phi : \mathbb{B} \to \mathbb{R}$ by

$$\Phi(u) := J\bigl(\gamma(u)\bigr).$$

**Claim 2** *$\Phi^+$ satisfies the $(PS)$-condition on $\mathbb{B}$. Set $\{u_n\} \subset \mathbb{B}$ as a $(PS)$-sequence of $\Phi^+$. Let $v_n = \gamma(u_n)$. Similar to the proof of Lemma* 3.7 *in* [28], *we need to prove that $\{v_n\} \subset \Psi$ is a $(PS)$-sequence of $\Phi^+$. From Claim* 1, *we can take the appropriate subsequence, for convenience, still denoted by $\{v_n\}$, and suppose that $v_n \to v_0$ and $u_n = \gamma^{-1}(v_n) \to \gamma^{-1}(v_n)$ with $n \to +\infty$. We can deduce that $\Phi^+$ satisfies the $(PS)$-condition.*

Finally, we prove that problem (1.1) admits at least one positive ground state solution and one negative ground state solution. Let $\{u_n^+\}$ be a minimizing sequence for $\Phi^+$. Then, using Ekeland's variational principle [29], one can suppose that $(\Phi^+)'(u_n^+) \to 0$. By Claim 2, passing, if necessary, to a subsequence, one can suppose that $u_n^+ \to u_0^+$ in $W_0^{1,p(x)}(\Omega)$. Therefore, $u_0^+$ is a minimizer of $\Phi^+$, and from [17], we can deduce that $v_0^+ := \gamma(u_0^+)$ is a ground state solution for the equation $(\phi^+)'(v) = 0$, that is

$$\begin{aligned}
&\int_\Omega \left|\nabla v_0^+\right|^{p(x)-2} \nabla v_0^+ \nabla \eta \, dx + \int_\Omega \lambda \left|v_0^+\right|^{p(x)-2} v_0^+ \eta_0 \, dx \\
&\quad = \int_\Omega h(x) v_0^+ \, dx + \int_\Omega f^+\bigl(x, v_0^+\bigr) \eta \, dx, \quad \forall \eta \in W_0^{1,p(x)}(\Omega).
\end{aligned} \tag{3.15}$$

Since $f^+(x,t) = 0$ for $x \in \Omega$, $t \le 0$, from Lemma 2.17 (i), we can conclude that $u^+ \ge 0$. Therefore, by (3.15), we have

$$\begin{aligned}
&\int_\Omega \left|\nabla v_0^+\right|^{p(x)-2} \nabla v_0^+ \nabla \eta \, dx + \int_\Omega \lambda \left|v_0^+\right|^{p(x)-2} v_0^+ \eta_0 \, dx \\
&\quad = \int_\Omega h(x) v_0^+ \, dx + \int_\Omega f\bigl(x, v_0^+\bigr) \eta \, dx, \quad \forall \eta \in W_0^{1,p(x)}(\Omega),
\end{aligned}$$

which indicates that problem (1.1) has a nontrivial ground solution $u^+ \geq 0$. Therefore, by Lemma 2.9, we can deduce that $u^+ > 0$.

Similarly, replace $f^+$ with $f^-$, where $f^-$ is defined as $f^-(x,t) = f(x,u)$ for $t < 0$ and $f^-(x,t) = 0$ for $t \geq 0$, we can deduce that problem (1.1) has a negative ground state solution $u^- < 0$. In summary, problem (1.1) has at least one positive ground state solution and one negative ground state solution. The proof is completed. □

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