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Extinction behavior and recurrence of n-type Markov branching–immigration processes

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Abstract

In this paper, we consider n-type Markov branching–immigration processes. The uniqueness criterion is first established. Then, we construct a related system of differential equations based on the branching property. Furthermore, the explicit expression of extinction probability and the mean extinction time are successfully obtained in the absorbing case by using the unique solution of the related system of differential equations and Kolmogorov forward equations. Finally, the recurrence and ergodicity criteria are given if the zero state $\mathbf{0}$ is not absorbing.

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1 Introduction

Markov branching processes occupy a major niche in the theory and applications of probability. Good general references are Asmussen and Hering [2], Athreya and Jagers [3], Athreya and Ney [4] and Harris [7]. Within the branching structure, both state-independent and state-dependent immigration have been studied. For the former, Sevast'yanov [13] and Vatutin [14] and [15] considered a branching process with state-independent immigration. Aksland [1] considered a modified birth–death process where the state-independent immigration is imposed. On the other hand, for the latter, Kulkarni and Pakes [8] discussed the total progeny of a branching process with state-dependent immigration. Foster [6] and Pakes [11] considered a discrete-time branching process with immigration at state 0. Yamazato [16] and Pakes and Tavaré [12] investigated the continuous-time version.

Let $(Z_t : t \geq 0)$ denote an n-type Markov branching process (nTMBP) with per capita birth rate and offspring distribution of the type k particle being $\theta_k > 0$ and $\{p_{\boldsymbol{j}}^{(k)} : \boldsymbol{j} \in \mathbf{Z}_+^n\}$ ($k = 1, \ldots, n$), respectively, where $\mathbf{Z}_+^n = \{\boldsymbol{j} = (j_1, \ldots, j_n) : j_1, \ldots, j_n \in \mathbf{Z}_+\}$ with $\mathbf{Z}_+ = \{0, 1, \ldots\}$. In this paper, we mainly consider a modification $(X_t : t \geq 0)$ of the nTMBP that allows it to be resurrected whenever it hits the zero state and allows immigration when it does not hit the zero state. $(X_t : t \geq 0)$ is called an n-type Markov branching–immigration process (nTMBPI). In order to clearly describe the evolution of (nTMBPI), we adopt the following conventions throughout this paper.

(C-1) For any $\boldsymbol{i} = (i_1, \ldots, i_n) \in \mathbf{Z}_+^n$, denote $|\boldsymbol{i}| = \sum_{k=1}^n i_k$.

(C-2) $[0,1]^n = \{(u_1, \ldots, u_n) : 0 \le u_1, \ldots, u_n \le 1\}$. For $\boldsymbol{u}, \boldsymbol{v} \in [0,1]^n$, $\boldsymbol{u} \le \boldsymbol{v}$ means $u_k \le v_k$ $(k = 1, \ldots, n)$, while $\boldsymbol{u} < \boldsymbol{v}$ means $u_k \le v_k$ $(k = 1, \ldots, n)$ and $u_k < v_k$ for at least one k.

(C-3) For $\boldsymbol{u} \in [0,1]^n$ and $\boldsymbol{i} \in \mathbf{Z}_+^n$, $\boldsymbol{u}^{\boldsymbol{i}} = \prod_{k=1}^n u_k^{i_k}$.

(C-4) $\chi_{\mathbf{Z}_+^n}(\cdot)$ is the indicator of $\mathbf{Z}_+^n$.

(C-5) $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{1} = (1, \ldots, 1)$, $e_i = (0, \ldots, 1_i, \ldots, 0)$ are vectors in $[0,1]^n$. $\mathbf{Z}_+^n \setminus \{\mathbf{0}\}$ is simply written as $\mathbf{Z}_{++}^n$.

The evolution of nTMBPI can be described as follows.

(i) There are n types of particles in the system. The life length of a type k particle is exponentially distributed with parameter θ_k. Upon its death, it produces offspring of the n-types according to the distribution $\{p_{\boldsymbol{j}}^{(k)} : \boldsymbol{j} \in \mathbf{Z}_+^n\}$, $k = 1, \ldots, n$. Particles live and produce independently of each other, and of the past. Without loss of generality, we assume $p_{e_k}^{(k)} = 0$ $(k = 1, \ldots, n)$.

(ii) Let $\alpha > 0$ and $\{a_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbf{Z}_{++}^n\}$ be a discrete law. When the system is nonempty, then Poisson immigration events with parameter α may occur with random numbers of immigrates according to the law $\{a_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbf{Z}_{++}^n\}$. Immigration is independent of particles in the system.

(iii) Let $\beta \ge 0$ and $\{h_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbf{Z}_{++}^n\}$ be a discrete law. When the system is empty, then Poisson resurrection events with parameter h may occur with random numbers of immigrates according to the law $\{h_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbf{Z}_{++}^n\}$. Resurrection, immigration, and particles in the system are independent of each other.

By the above description, $(X_t : t \ge 0)$ is a Markov process satisfying the following conditions:

(a) the state space is $\mathbf{Z}_+^n$;

(b) its generator $Q = (q_{\boldsymbol{ij}} : \boldsymbol{i}, \boldsymbol{j} \in \mathbf{Z}_+^n)$ satisfies

$$q_{\boldsymbol{ij}} = \begin{cases} \beta h_{\boldsymbol{j}}, & \text{if } |\boldsymbol{i}| = 0, \boldsymbol{j} \ne \mathbf{0} \\ \sum_{k=1}^n i_k \theta_k p_{\boldsymbol{j}-\boldsymbol{i}+e_k}^{(k)} + \alpha a_{\boldsymbol{j}-\boldsymbol{i}}, & \text{if } |\boldsymbol{i}| > 0, \boldsymbol{j} \ne \boldsymbol{i}, \\ -(\sum_{k=1}^n i_k \theta_k + \alpha(1 - \delta_{\boldsymbol{i}\mathbf{0}}) + \beta \delta_{\boldsymbol{i}\mathbf{0}}), & \text{if } \boldsymbol{j} = \boldsymbol{i}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Remark 1.1 θ_k, α, and β are viewed as "branching rate", "immigration rate", and "resurrection rate", respectively. The matrix Q given in (1.1) is called an n-type branching–immigration Q-matrix (nTBI Q-matrix).

Li and Chen [9] considered the one-type case. The aim of this paper is to consider the extinction behavior and recurrence property of n-type Markov branching–immigration processes. In contrast to the one-type cases, when a particle of one type in the system splits, the number of particles of different type may change. Therefore, the method used in the one-type case fails and some new approaches should be used in the current situation. In this paper, we find a new method to investigate the extinction behavior and recurrence property of the n-type Markov branching–immigration processes (see, Theorems 3.1 and 3.2).

The structure of this paper is as follows. Regularity and uniqueness criteria together with some preliminary results are first established in Sect. 2. In Sect. 3, we concentrate on discussing the extinction behavior of the absorbing nTBIP (i.e., $\beta = 0$) and the explicit extinction probability is obtained. In Sect. 4, the recurrence criterion is presented in the case $\beta > 0$.

2 Preliminaries and uniqueness

Since Q is determined by the sequences $\{p_{\boldsymbol{j}}^{(i)} : \boldsymbol{j} \in \mathbf{Z}_+^n\}$ $(i = 1, \ldots, n)$, $\{a_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbf{Z}_{++}^n\}$, and $\{h_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbf{Z}_{++}^n\}$, we define their generating functions as

$$B_i(\boldsymbol{u}) = \theta_i\left(\sum_{\boldsymbol{j}\in\mathbf{Z}_+^n} p_{\boldsymbol{j}}^{(i)}\boldsymbol{u}^{\boldsymbol{j}} - u_i\right), \quad i = 1, \ldots, n,$$

$$I(\boldsymbol{u}) = \alpha\left(\sum_{\boldsymbol{j}\in\mathbf{Z}_{++}^n} a_{\boldsymbol{j}}\boldsymbol{u}^{\boldsymbol{j}} - 1\right),$$

$$R(\boldsymbol{u}) = \beta\left(\sum_{\boldsymbol{j}\in\mathbf{Z}_{++}^n} h_{\boldsymbol{j}}\boldsymbol{u}^{\boldsymbol{j}} - 1\right).$$

It is obvious that all the generating functions are well defined at least on $[0,1]^n$. We now investigate the properties of the generating functions $\{B_i(\boldsymbol{u}); i = 1, \ldots, n\}$, $\alpha(\boldsymbol{u})$, and $\beta(\boldsymbol{u})$. Let

$$B_{ij}(\boldsymbol{u}) = \frac{\partial B_i(\boldsymbol{u})}{\partial u_j}, \quad i,j = 1, \ldots, n,$$

$$I_j(\boldsymbol{u}) = \frac{\partial I(\boldsymbol{u})}{\partial u_j}, \quad j = 1, \ldots, n,$$

$$R_j(\boldsymbol{u}) = \frac{\partial R(\boldsymbol{u})}{\partial u_j}, \quad j = 1, \ldots, n,$$

$$g_{ij}(\boldsymbol{u}) = \delta_{ij} + \frac{B_{ij}(\boldsymbol{u})}{\theta_i}, \quad i,j = 1, \ldots, n,$$

where $\boldsymbol{u} \in [0,1]^n$ and δ_{ij} is the Dirac function. The matrices $(B_{ij}(\boldsymbol{u}))$ and $(g_{ij}(\boldsymbol{u}))$ are denoted by $B(\boldsymbol{u})$ and $G(\boldsymbol{u})$, respectively.

Definition 2.1 The system $\{B_i(\boldsymbol{u}) : 1 \le i \le n\}$ is called singular if there exists an $n \times n$ matrix M such that

$$\big(B_1(\boldsymbol{u}), \ldots, B_n(\boldsymbol{u})\big)' = M \cdot \boldsymbol{u}',$$

where $\boldsymbol{u}'$ denotes the transpose of the vector $\boldsymbol{u}$.

Definition 2.2 A nonnegative $n \times n$ matrix $A = (a_{ij})$ is called positively regular if there exists an integer $N > 0$, such that $A^N > 0$.

If $\{B_i(\boldsymbol{u}) : 1 \le i \le n\}$ is singular, then each particle has exactly one offspring, and hence the branching process will be equivalent to an ordinary finite Markov chain. In order to avoid discussing such trivial cases, we shall assume throughout this paper that the following conditions are satisfied:

(**A**-1). $\{B_i(\boldsymbol{u}) : 1 \le i \le n\}$ is nonsingular;

(**A**-2). $B_{ij}(\mathbf{1}) < +\infty$, $i,j = 1, \ldots, n$;

(**A**-3). $G(\mathbf{1})$ is positively regular.

The above conditions guarantee that $\mathbf{Z}_{++}^n$ is irreducible. The following two lemmas are well known and the proofs are omitted.

Lemma 2.1 *$I(\boldsymbol{u}) < 0$ for all $\boldsymbol{u} \in [0,1)^n$ and $\lim_{\boldsymbol{u}\uparrow\mathbf{1}} I(\boldsymbol{u}) = I(\mathbf{1}) = 0$. A similar property holds for $R(\boldsymbol{u})$.*

Lemma 2.2 *Suppose $G(\mathbf{1})$ is positively regular and $\{B_i(\boldsymbol{u}) : 1 \le i \le n\}$ is nonsingular. Then, the equation*

$$\big(B_1(\boldsymbol{u}), B_2(\boldsymbol{u}), \dots, B_n(\boldsymbol{u})\big) = \mathbf{0} \tag{2.1}$$

has at most two solutions in $[0,1]^n$. Let $\boldsymbol{q} = (q_1, \dots, q_n)$ and $\rho(\boldsymbol{u})$ denote the smallest nonnegative solution to (2.1) and the maximal eigenvalue of $B(\boldsymbol{u})$, respectively. Then,

(i) *q_i is the extinction probability when the Feller minimal process starts at state e_i ($i = 1, \dots, n$). Moreover, if $\rho(\mathbf{1}) \le 0$, then $\boldsymbol{q} = \mathbf{1}$; while if $\rho(\mathbf{1}) > 0$, then $\boldsymbol{q} < \mathbf{1}$, i.e., $q_1, \dots, q_n < 1$.*

(ii) *$\rho(\boldsymbol{q}) \le 0$.*

For nTBI Q-matrix Q given in (1.1), let $P(t) = (p_{\boldsymbol{ij}}(t) : \boldsymbol{i}, \boldsymbol{j} \in \mathbf{Z}_+^n)$ and $\Phi(\lambda) = (\phi_{\boldsymbol{ij}}(\lambda) : \boldsymbol{i}, \boldsymbol{j} \in \mathbf{Z}_+^n)$ be the Feller minimal Q-function and Q-resolvent, respectively.

Lemma 2.3 *For any $\boldsymbol{i} \in \mathbf{Z}_+^n$ and $\boldsymbol{u} \in [0,1)^n$, we have*

$$\frac{\partial F_{\boldsymbol{i}}(t,\boldsymbol{u})}{\partial t} = R(\boldsymbol{u}) p_{\boldsymbol{i0}}(t) + I(\boldsymbol{u}) \sum_{\boldsymbol{j}\in\mathbf{Z}_{++}^n} p_{\boldsymbol{ij}}(t) \boldsymbol{u}^{\boldsymbol{j}} + \sum_{k=1}^n B_k(\boldsymbol{u}) \frac{\partial F_{\boldsymbol{i}}(t,\boldsymbol{u})}{\partial u_k}, \tag{2.2}$$

where $F_{\boldsymbol{i}}(t,\boldsymbol{u}) = \sum_{\boldsymbol{j}\in\mathbf{Z}_+^n} p_{\boldsymbol{ij}}(t)\boldsymbol{u}^{\boldsymbol{j}}$, or in the resolvent version

$$\lambda \Phi_{\boldsymbol{i}}(\lambda,\boldsymbol{u}) - \boldsymbol{u}^{\boldsymbol{i}} = R(\boldsymbol{u}) \phi_{\boldsymbol{i0}}(\lambda) + I(\boldsymbol{u}) \sum_{\boldsymbol{j}\in\mathbf{Z}_{++}^n} \phi_{\boldsymbol{ij}}(\lambda) \boldsymbol{u}^{\boldsymbol{j}} + \sum_{k=1}^n B_k(\boldsymbol{u}) \frac{\partial \Phi_{\boldsymbol{i}}(\lambda,\boldsymbol{u})}{\partial u_k}, \tag{2.3}$$

where $\Phi_{\boldsymbol{i}}(\lambda,\boldsymbol{u}) = \sum_{\boldsymbol{j}\in\mathbf{Z}_+^n} \phi_{\boldsymbol{ij}}(\lambda)\boldsymbol{u}^{\boldsymbol{j}}$.

Proof By the Kolmogorov forward equations, we have that for any $\boldsymbol{i}, \boldsymbol{j} \in \mathbf{Z}_+^n$,

$$\begin{aligned} &p'_{\boldsymbol{ij}}(t) \\ &\quad = \sum_{\boldsymbol{k}\ne\boldsymbol{j}} p_{\boldsymbol{ik}}(t) \left[\sum_{l=1}^n k_l \theta_l p^{(l)}_{\boldsymbol{j}-\boldsymbol{k}+e_l} \cdot \chi_{\mathbf{Z}_+^n}(\boldsymbol{j}-\boldsymbol{k}+e_l) + \alpha a_{\boldsymbol{j}-\boldsymbol{k}} \cdot \chi_{\mathbf{Z}_+^n}(\boldsymbol{j}-\boldsymbol{k})(1-\delta_{\mathbf{0}\boldsymbol{k}}) + \beta h_{\boldsymbol{j}} \cdot \delta_{\mathbf{0}\boldsymbol{k}} \right] \\ &\qquad - p_{\boldsymbol{ij}}(t) \left[\sum_{l=1}^n j_l \theta_l + \alpha(1-\delta_{\mathbf{0}\boldsymbol{j}}) + \beta \delta_{\mathbf{0}\boldsymbol{j}} \right]. \end{aligned}$$

Multiplying by $\boldsymbol{u}^{\boldsymbol{j}}$ on both sides of the above equality and summing over $\boldsymbol{j} \in \mathbf{Z}_+^n$ we immediately obtain (2.2). Taking a Laplace transform on (2.2) immediately yields (2.3). $\square$

Lemma 2.4 *Suppose that $G(\mathbf{1})$ is positively regular and $\{B_i(\boldsymbol{u}) : 1 \le i \le n\}$ is nonsingular. If $\rho(\mathbf{1}) \le 0$, then the Q-function is honest.*

Proof By Lemma 2.5 of Li and Wang [10], we know that if $\rho(\mathbf{1}) \leq 0$, then $\boldsymbol{q} = \mathbf{1}$.

Denote

$$r^* = \sup\{r \geq 0 : B_k(\boldsymbol{u}) = r, k = 1, \dots, n \text{ has a solution in } [0,1]^n\}.$$

By Lemma 2.9 of Li and Wang [10], we know that $r^* > 0$ and for any $r \in (0, r^*]$, there exist $\boldsymbol{u}(r) = (u_1(r), \dots, u_n(r)) \in [0,1)^n$ such that

$$B_k\big(\boldsymbol{u}(r)\big) = r, \quad k = 1, \dots, n$$

and, moreover,

$$\lim_{r\downarrow 0} \boldsymbol{u}(r) = \mathbf{1}.$$

Letting $\boldsymbol{u} = \boldsymbol{u}(r)$ in (2.2) and letting $r \downarrow 0$ yield

$$\sum_{\boldsymbol{j}\in \mathbf{Z}_+^n} p_{\boldsymbol{ij}}(t) \geq 1,$$

i.e., $\sum_{\boldsymbol{j}\in \mathbf{Z}_+^n} p_{\boldsymbol{ij}}(t) = 1$. Hence, $P(t)$ is honest. □

Having completed the preparation, we now prove the uniqueness of nTMBPI.

Theorem 2.1 *Let Q be given in (1.1). Then, there exists exactly one nTMBPI, i.e., the Feller minimal process.*

Proof By Lemma 2.4, We only need to consider the case that $\rho(\mathbf{1}) > 0$. For this purpose, we will show that the equations

$$\begin{cases} \eta(\lambda I - Q) = 0, \quad \eta_{\boldsymbol{j}} \geq 0, \boldsymbol{j} \in \mathbf{Z}_+^n, \\ \sum_{\boldsymbol{j}\in \mathbf{Z}_+^n} \eta_{\boldsymbol{j}} < +\infty \end{cases} \tag{2.4}$$

have only trivial solution. Suppose that the contrary is true and let $\eta = (\eta_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbf{Z}_+^n)$ be a nontrivial solution of (2.4) corresponding to $\lambda = 1$. Then, by (2.4) we have

$$\begin{aligned} \eta_{\boldsymbol{j}} = {} & \sum_{\boldsymbol{k}\neq \boldsymbol{j}} \eta_{\boldsymbol{k}} \left[\sum_{i=1}^n k_i \theta_i p^{(i)}_{\boldsymbol{j}-\boldsymbol{k}+e_i} \cdot \chi_{\mathbf{Z}_+^n}(\boldsymbol{j} - \boldsymbol{k} + e_i) + \alpha a_{\boldsymbol{j}-\boldsymbol{k}} \cdot \chi_{\mathbf{Z}_+^n}(\boldsymbol{j} - \boldsymbol{k})(1 - \delta_{\mathbf{0}\boldsymbol{k}}) + \beta h_{\boldsymbol{j}} \cdot \delta_{\mathbf{0}\boldsymbol{k}} \right] \\ & - \eta_{\boldsymbol{j}} \left[\sum_{i=1}^n k_i \theta_i + \alpha(1 - \delta_{\mathbf{0}\boldsymbol{j}}) + \beta \delta_{\mathbf{0}\boldsymbol{j}} \right], \quad \boldsymbol{j} \in \mathbf{Z}_+^n. \end{aligned} \tag{2.5}$$

Multiplying by $\boldsymbol{u}^{\boldsymbol{j}}$ on both sides of (2.5) and using some algebra yields that

$$\eta(\boldsymbol{u}) = \sum_{i=1}^n B_i(\boldsymbol{u}) \cdot \frac{\partial \eta(\boldsymbol{u})}{\partial u_i} + I(\boldsymbol{u})\big(\eta(\boldsymbol{u}) - \eta_{\mathbf{0}}\big) + R(\mathbf{0})\eta_{\mathbf{0}},$$

i.e.,

$$\big(1-I(\boldsymbol{u})\big)\big[\eta(\boldsymbol{u})-\eta_{\mathbf{0}}\big]+\big(1-R(\boldsymbol{u})\big)\eta_{\mathbf{0}}=\sum_{i=1}^{n}B_i(\boldsymbol{u})\cdot\frac{\partial\eta(\boldsymbol{u})}{\partial u_i}. \tag{2.6}$$

If $\rho(\mathbf{1})>0$, then by Lemma 2.2 and the irreducibility of $\mathbf{Z}_+^n\setminus\mathbf{0}$ we know that (2.1) has a solution $(q_1,\dots,q_n)\in(0,1)^n$. Let $\boldsymbol{u}=(q_1,\dots,q_n)$ in (2.6), we can see that the right-hand side of (2.6) is zero. Therefore, the left-hand side of (2.6) must be zero, which implies that $\eta_{\boldsymbol{j}}=0$ ($\forall\boldsymbol{j}\in\mathbf{Z}_+^n$). The proof is complete. □

3 Extinction

In this section, we shall discuss the extinction property of the absorbing nTMBPI (i.e., $\beta=0$). Let $\tilde{Q}$ denote the absorbing nTBI Q-matrix and $\tilde{P}(t)=(\tilde{p}_{\boldsymbol{ij}}(t):\boldsymbol{i},\boldsymbol{j}\in\mathbf{Z}_+^n)$ denote the Feller minimal $\tilde{Q}$-function. Also, let $a_{\boldsymbol{i0}}=\lim_{t\to\infty}\tilde{p}_{\boldsymbol{i0}}(t)$ be the extinction probability of $\tilde{P}(t)$ starting at state $\boldsymbol{i}$. In order to discuss the extinction property, we need the following important result, which plays a key role in our discussion.

Theorem 3.1 *Suppose that $G(\mathbf{1})$ is positively regular and $\{B_i(\boldsymbol{u});1\le i\le n\}$ is nonsingular. If $B_1(\mathbf{0})>0$, then the system of equations*

$$\begin{cases} u_k'(u)=\frac{B_k(u,u_2,\dots,u_n)}{B_1(u,u_2,\dots,u_n)}, & 2\le k\le n,\\ u_k|_{u=0}=0, & 2\le k\le n\end{cases} \tag{3.1}$$

has a unique solution $(u_k(u);2\le k\le n)$. Furthermore, this solution satisfies

(i) *$(u_k(u);2\le k\le n)$ is well defined on $[0,q_1]$;*
(ii) *$u_k'(0)\ge0$ and $u_k'(u)>0$ for all $u\in(0,q_1)$ and $2\le k\le n$;*
(iii) *$u_k(q_1)=q_k$, $2\le k\le n$.*

Proof Since $B_1(\mathbf{0})>0$, we know that $B_1(u,0,\dots,0)=0$ has a positive root $u^*\in(0,1]$. For any $\varepsilon>0$, $\{\frac{B_k(u,u_2,\dots,u_n)}{B_1(u,u_2,\dots,u_n)};2\le k\le n\}$ satisfy the Lipschitz condition on $[0,u^*-\varepsilon]\times[0,1]^{n-1}$, therefore, by the theory of differential equations, (3.1) has a unique solution $(u_k(u);2\le k\le n)$ defined on $[0,u^*-\varepsilon]$. Furthermore, (3.1) has a unique solution $(u_k(u);2\le k\le n)$ defined on $[0,u^*)$ since $\varepsilon>0$ is arbitrary.

We claim that $u_k'(u)\ge0$ $(2\le k\le n)$ for all $u\in[0,u^*)$. In fact, if there exist $u\in[0,u^*)$ and $2\le k\le n$ such that $u_k'(u)<0$, denote

$$\tilde{u}=\inf\big\{u\in[0,u^*):u_k'(u)<0\text{ for some }k\in\{2,\dots,n\}\big\}$$

and

$$H=\big\{k\in\{2,\dots,n\}:\exists\varepsilon>0\text{ s.t. }u_k'(u)<0\text{ for }u\in(\tilde{u},\tilde{u}+\varepsilon)\big\}.$$

It is obvious that $H\ne\emptyset$. Since $(u_k(u);2\le k\le n)$ is the solution of (3.1), we have

$$B_k\big(\tilde{u},u_2(\tilde{u}),\dots,u_n(\tilde{u})\big)=0,\quad k\in H$$

and there exists $\bar{u} \in (\tilde{u}, u^*)$ such that $u_k(\bar{u}) \geq u_k(\tilde{u})$ ($k \in H^c =: \{2, \dots, n\} \setminus H$), $u_k(\bar{u}) < u_k(\tilde{u})$ ($k \in H$) and

$$B_k\bigl(\bar{u}, u_2(\bar{u}), \dots, u_n(\bar{u})\bigr) < 0, \quad k \in H. \tag{3.2}$$

Consider

$$I = \bigl\{B_k\bigl(\bar{u}, \boldsymbol{u}_{H^c}(\bar{u}), \boldsymbol{u}_H\bigr) : k \in H\bigr\},$$

where $\boldsymbol{u}_H = (u_k : k \in H)$ and $\boldsymbol{u}_{H^c}(\bar{u}) = (u_k(\bar{u}) : k \in H^c)$. Obviously,

$$B_k\bigl(\bar{u}, \boldsymbol{u}_{H^c}(\bar{u}), \boldsymbol{u}_H(\tilde{u})\bigr) \geq 0, \quad k \in H,$$

where $\boldsymbol{u}_H(\tilde{u}) = (u_k(\tilde{u}) : k \in H)$. Therefore, the smallest nonnegative zero of I is in $\prod_{k=\tilde{k}}^{n} [u_k(\tilde{u}), 1]$. Combining with (3.2) we know that $u_k(\bar{u}) \geq u_k(\tilde{u})$ ($k \in H$), which contradicts $u_k(\bar{u}) < u_k(\tilde{u})$ ($k \in H$).

We now further claim that $u_k'(u) > 0$ ($2 \leq k \leq n$) for all $u \in (0, u^*]$. In fact, suppose that there exists $\hat{u} \in (0, u^*]$ such that

$$B_k\bigl(\hat{u}, u_2(\hat{u}), \dots, u_n(\hat{u})\bigr) = 0$$

for some $k \geq 2$. Denote

$$\hat{H} = \bigl\{k; B_k\bigl(\hat{u}, u_2(\hat{u}), \dots, u_n(\hat{u})\bigr) = 0\bigr\}$$

and

$$\hat{H}^c = \{1, 2, \dots, n\} \setminus \hat{H}.$$

It is easy to see that $\hat{H}^c \neq \emptyset$. By the irreducibility of the set of nonzero states we know that there exist $k \in \hat{H}$, $j \in \hat{H}^c$ such that

$$B_{kj}\bigl(\hat{u}, u_2(\hat{u}), \dots, u_n(\hat{u})\bigr) > 0.$$

On the other hand,

$$\lim_{u \uparrow \hat{u}} \frac{B_k(u, u_2(u), \dots, u_n(u))}{u - \hat{u}} = \sum_{i \in \hat{H}^c} B_{ki}\bigl(\hat{u}, u_2(\hat{u}), \dots, u_n(\hat{u})\bigr) \cdot u_i'(\hat{u}) > 0,$$

which contradicts $B_k(u, u_2(u), \dots, u_n(u)) \geq 0$ for all $u \in [0, u^*]$, where $u_1'(\hat{u}) = 1$.

Since $B_1(u^*, u_2(u^*), \dots, u_n(u^*)) > B_1(u^*, 0, \dots, 0) = 0$, we can apply mathematical induction to prove that the solution of (3.1) can be uniquely extended to $[0, q_1)$. Now, we claim that

$$u_k(q_1) = \lim_{u \uparrow q_1} u_k(u) = q_k, \quad k \geq 2.$$

Indeed, since $B_k(u,u_2(u),\dots,u_n(u))>0$ $(k\geq 1)$ for all $u\in(0,q_1)$, it can be easily seen that $u_k(u)\in(0,q_k)$ $(k\geq 2)$ for all $u\in(0,q_1)$ and therefore, $u_k(q_1)\in(0,q_k]$ for all $k\geq 2$. If $u_k(q_1)<q_k$ for some $k\geq 2$, denote

$$M=\{k\geq 2;u_k(q_1)<q_k\},\qquad M^c=\{1,2,\dots,n\}\setminus M.$$

It follows from the irreducibility of the set of nonzero states we know that there exists $j\in M^c$ such that

$$\lim_{u\uparrow q_1}B_j\big(u,u_2(u),\dots,u_n(u)\big)=B_j\big(q_1,u_2(q_1),\dots,u_n(q_1)\big)<0,$$

which contradicts $B_j(u,u_2(u),\dots,u_n(u))>0$ for all $u\in(0,q_1)$. The proof is complete. □

Corollary 3.1 *Suppose that $G(\mathbf{1})$ is positively regular, $\{B_i(\boldsymbol{u});1\leq i\leq n\}$ is nonsingular. If $B_1(\mathbf{0})>0$, $B_2(\mathbf{0})>0$, then the system of equations*

$$\begin{cases}u_k'(u)=\frac{B_k(u_1,u,\dots,u_n)}{B_2(u_1,u,\dots,u_n)}, & k\neq 2,\\ u_k|_{u=0}=0, & k\neq 2\end{cases}\tag{3.3}$$

has the same solution as (3.1).

Proof By Theorem 3.1, we know that (3.3) has a unique solution. For convenience, we denote the solutions to (3.3) by $(u_1(u_2),u_3(u_2),\dots,u_n(u_2))$. Since $u_1'(u_2)>0$ for all $u_2\in[0,q_2)$, we know that the function $u_1(u_2)$ $(u_2\in[0,q_2))$ has an inverse function $u_2=f_2(u_1)$, $(u_1\in[0,q_1))$ satisfying $\frac{df_2}{du_1}=1/u_1'$. Let $u_k=f_k(u_1)=u_k(f_2(u_1))$ $(u_1\in[0,q_1])$ for $k\geq 3$. It can be easily seen that $u_k=f_k(u_1)$ $(k\geq 2)$ is the solution to (3.1). □

By the irreducibility of $\mathbf{Z}^n_{++}$, Theorem 3.1, and Corollary 3.1, we can assume that $B_1(\mathbf{0})>0$ without loss of generality and let $(u_2(u),\dots,u_n(u))(u\in[0,q_1])$ denote the unique solution to (3.1).

Before stating our main result in this section, we first provide two useful lemmas.

Lemma 3.1 *Let $(\tilde{p}_{\boldsymbol{ij}}(t):\boldsymbol{i},\boldsymbol{j}\in\mathbf{Z}^n_+)$ be the Feller minimal $\tilde{Q}$-function, where $\tilde{Q}$ is an absorbing nTBI Q-matrix. Then, for any $\boldsymbol{i}\in\mathbf{Z}^n_+$,*

$$\int_0^\infty\tilde{p}_{\boldsymbol{ik}}(t)\,dt<\infty,\quad \boldsymbol{k}\neq\mathbf{0}\tag{3.4}$$

and thus

$$\lim_{t\to\infty}\tilde{p}_{\boldsymbol{ik}}(t)=0,\quad \boldsymbol{i}\in\mathbf{Z}^n_+,\boldsymbol{k}\neq\mathbf{0}.\tag{3.5}$$

Moreover, for any $\boldsymbol{i}\in\mathbf{Z}^n_{++}$ and $\boldsymbol{u}\in[0,1)^n$, we have

$$\sum_{\boldsymbol{k}\neq\mathbf{0}}\left(\int_0^\infty\tilde{p}_{\boldsymbol{ik}}(t)\,dt\right)\cdot\boldsymbol{u}^{\boldsymbol{k}}<\infty.\tag{3.6}$$

Proof By the construction of $\tilde{Q}$, all the states in $\mathbf{Z}^n_{++}$ are transient. Hence, (3.4) and (3.5) hold.

We now prove (3.6). For this purpose, we shall consider two different cases separately.

First, consider the case $\rho(\mathbf{1}) > 0$. By Lemma 2.1(ii), (2.1) has a root $\boldsymbol{q} \in (0,1)^n$. Let $\tilde{\boldsymbol{u}} \in \prod_{i=1}^n (q_i, 1)$. We claim that there exists $\bar{\boldsymbol{u}} \in \prod_{i=1}^n [\tilde{u}_i, 1)$ such that

$$B_i(\bar{\boldsymbol{u}}) < 0, \quad \forall i = 1, 2, \dots, n. \tag{3.7}$$

Indeed, let $H_1 = \{i : B_i(\tilde{\boldsymbol{u}}) > 0\}$. By Li and Wang [10] we know that $H_1 \neq \{1, 2, \dots, n\}$ since $\rho(\mathbf{1}) > 0$. If $H_1 = \emptyset$, then $B_i(\tilde{u}_1, \dots, \tilde{u}_n) \le 0$ ($\forall i = 1, \dots, n$). If $H_1 \neq \emptyset$, then by Lemma 2.2, we know that there exists $\boldsymbol{u}^{(1)} \in \prod_{i=1}^n [\tilde{u}_i, 1)$ such that $B_i(u_1^{(1)}, \dots, u_n^{(1)}) = 0$ for all $i \in H_1$. Let

$$H_2 = \bigl\{i : B_i\bigl(\boldsymbol{u}^{(1)}\bigr) > 0\bigr\},$$

then $H_2 \subset \{1, 2, \dots, n\} \setminus H_1$. It is obvious that $H_1 \cup H_2 \neq \{1, 2, \dots, n\}$. If $H_2 = \emptyset$, then $B_i(\boldsymbol{u}^{(1)}) \le 0$ ($\forall i = 1, \dots, n$). If $H_2 \neq \emptyset$, then by Lemma 2.2, we know that there exists $\boldsymbol{u}^{(2)} \in \prod_{i=1}^n [u_i^{(1)}, 1)$ such that $B_i(\boldsymbol{u}^{(2)}) = 0$ for all $i \in H_1 \cup H_2$. By repeatedly using the same argument and noting $\{1, 2, \dots, n\}$ is a finite set, we can obtain $H_1, H_2, \dots, H_m$ such that $H_{m+1} = \emptyset$ and hence $B_i(\boldsymbol{u}^{(m)}) \le 0$ ($\forall i = 1, \dots, n$). It is obvious that $H_1 \cup \cdots \cup H_m \neq \{1, 2, \dots, n\}$, i.e., $B_i(\boldsymbol{u}^{(m)}) < 0$ for all $i \in \{1, \dots, n\} \setminus H_1 \cup \cdots \cup H_m$. By the irreducibility of $\mathbf{Z}^n_{++}$, we can see that (3.7) holds for $\bar{\boldsymbol{u}}$ smaller than (if necessary) but closing to $\boldsymbol{u}^{(m)}$.

By (2.2) we know that

$$\frac{\partial \tilde{F}_{\boldsymbol{i}}(t, \bar{\boldsymbol{u}})}{\partial t} = I(\bar{\boldsymbol{u}}) \sum_{\boldsymbol{j} \in \mathbf{Z}^n_{++}} \tilde{p}_{\boldsymbol{ij}}(t) \bar{\boldsymbol{u}}^{\boldsymbol{j}} + \sum_{k=1}^n B_k(\bar{\boldsymbol{u}}) \frac{\partial \tilde{F}_{\boldsymbol{i}}(t, \bar{\boldsymbol{u}})}{\partial u_k},$$

which implies (3.6), where $\tilde{F}_{\boldsymbol{i}}(t, \bar{\boldsymbol{u}}) = \sum_{\boldsymbol{j} \in \mathbf{Z}^n_+} \tilde{p}_{\boldsymbol{ij}}(t) \bar{\boldsymbol{u}}^{\boldsymbol{j}}$.

Next, consider the case that $\rho(\mathbf{1}) \le 0$. Let $\tilde{\boldsymbol{u}} \in (0,1)^n$. By Theorem 3.1, there exists $v \in (\tilde{u}_1, 1)$ such that $(v, u_2(v), \dots, u_n(v)) \in \prod_{i=1}^n (\tilde{u}_i, 1)$ and hence by (2.2) and Theorem 3.1 we have

$$1 \ge I\bigl(v, u_2(v), \dots, u_n(v)\bigr) G_{\boldsymbol{i}}(T, v) + B_1\bigl(v, u_2(v), \dots, u_n(v)\bigr) \cdot \frac{\partial G_{\boldsymbol{i}}(T, v)}{\partial v},$$

where $G_{\boldsymbol{i}}(T, v) = \sum_{\boldsymbol{j} \in \mathbf{Z}^n_{++}} (\int_0^T \tilde{p}_{\boldsymbol{ij}}(t)\, dt) v^{j_1} u_2^{j_2}(v) \cdots u_n^{j_n}(v)$. Equation (3.6) can be obtained immediately from the above inequality. The proof is complete. □

For any $\boldsymbol{i} \neq \mathbf{0}$, denote $G_{\boldsymbol{i}}(v) = G_{\boldsymbol{i}}(\infty, v)$. From Lemma 3.1, $G_{\boldsymbol{i}}(v)$ is well defined at least for $v \in [0, 1)$.

Theorem 3.2 *For any* $\boldsymbol{i} \neq \mathbf{0}$, $a_{\boldsymbol{i0}} = 1$ *if and only if* $\rho(\mathbf{1}) \le 0$ *and* $J = +\infty$ *where*

$$J := \int_0^1 \frac{1}{B_1(y, u_2(y), \dots, u_n(y))} \cdot e^{\int_0^y \frac{I(x, u_2(x), \dots, u_n(x))}{B_1(x, u_2(x), \dots, u_n(x))} dx}\, dy. \tag{3.8}$$

More specifically,

(i) *If* $\rho(\mathbf{1}) \le 0$ *and* $J = +\infty$, *then* $a_{\boldsymbol{i0}} = 1$ ($\boldsymbol{i} \neq \mathbf{0}$).

(ii) *If* $\rho(\mathbf{1}) \le 0$ *and* $J < +\infty$, *then*

$$a_{\boldsymbol{i}\mathbf{0}} = \frac{\int_0^1 \frac{y^{i_1} u_2^{i_2}(y)\cdots u_n^{i_n}(y)}{B_1(y,u_2(y),\ldots,u_n(y))} \cdot e^{\int_0^y \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))}\,dx}\,dy}{\int_0^1 \frac{1}{B_1(y,u_2(y),\ldots,u_n(y))} \cdot e^{\int_0^y \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))}\,dx}\,dy} < 1. \tag{3.9}$$

(iii) *If* $0 < \rho(\mathbf{1}) \le +\infty$ *and thus equation* (2.1) *possesses a smallest nonnegative root* $\boldsymbol{q} = (q_1, u_2(q_1), \ldots, u_n(q_1)) \in (0,1)^n$, *then*

$$a_{\boldsymbol{i}\mathbf{0}} = \frac{\int_0^{q_1} \frac{y^{i_1} u_2^{i_2}(y)\cdots u_n^{i_n}(y)}{B_1(y,u_2(y),\ldots,u_n(y))} \cdot e^{\int_0^y \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))}\,dx}\,dy}{\int_0^{q_1} \frac{1}{B_1(y,u_2(y),\ldots,u_n(y))} \cdot e^{\int_0^y \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))}\,dx}\,dy} < \prod_{k=1}^n q_k^{i_k} < 1, \quad \boldsymbol{i} \ne \mathbf{0}.$$

Proof Integrating the equality (2.2) with respect to $t \in [0,\infty)$ and using Theorem 3.1, we have that for any $v \in [0,1)$ and $\boldsymbol{i} \ne \mathbf{0}$,

$$\begin{aligned} &a_{\boldsymbol{i}\mathbf{0}} - v^{i_1} u_2^{i_2}(v)\cdots u_n^{i_n}(v) \\ &\quad = B_1\bigl(v, u_2(v), \ldots, u_n(v)\bigr) \cdot G_{\boldsymbol{i}}'(v) + I\bigl(v, u_2(v), \ldots, u_n(v)\bigr) \cdot G_{\boldsymbol{i}}(v), \end{aligned} \tag{3.10}$$

where $G_{\boldsymbol{i}}(v) < +\infty$. First, consider the case $\rho(1,\ldots,1) \le 0$. Solving the ordinary differential equation (3.10) for $v \in [0,1)$ immediately yields

$$\begin{aligned} &G_{\boldsymbol{i}}(v) \cdot e^{\int_0^v \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))}\,dx} \\ &\quad = \int_0^v \frac{a_{\boldsymbol{i}\mathbf{0}} - y^{i_1} u_2^{i_2}(y)\cdots u_n^{i_n}(y)}{B_1(y,u_2(y),\ldots,u_n(y))} \cdot e^{\int_0^y \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))}\,dx}\,dy, \end{aligned} \tag{3.11}$$

which implies that if $J = +\infty$, then $a_{\boldsymbol{i}\mathbf{0}} = 1$. Indeed, if $a_{\boldsymbol{i}\mathbf{0}} < 1$, then by letting $v \uparrow 1$ in (3.11) we see that the right-hand side of (3.11) tends to $-\infty$, while the left-hand side is always nonnegative, which is a contradiction. Hence, (i) is proven.

Now, we turn to (ii). First, note that $J < +\infty$ implies $\int_0^1 \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))}\,dx = -\infty$. Since the left-hand side of (3.11) is always nonnegative so is the right-hand side. It follows that $a_{\boldsymbol{i}\mathbf{0}} \ge J^{-1} \cdot \int_0^1 \frac{y^{i_1} u_2^{i_2}(y)\cdots u_n^{i_n}(y)}{B_1(y,u_2(y),\ldots,u_n(y))} \cdot e^{\int_0^y \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))}\,dx}\,dy$. Therefore, in order to prove (ii), we only need to show that

$$a_{\boldsymbol{i}\mathbf{0}} \le J^{-1} \cdot \int_0^1 \frac{y^{i_1} u_2^{i_2}(y)\cdots u_n^{i_n}(y)}{B_1(y,u_2(y),\ldots,u_n(y))} \cdot e^{\int_0^y \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))}\,dx}\,dy.$$

Take $x_{\boldsymbol{j}}^* = J^{-1} \cdot \int_0^1 \frac{y^{j_1} u_2^{j_2}(y)\cdots u_n^{j_n}(y)}{B_1(y,u_2(y),\ldots,u_n(y))} \cdot e^{\int_0^y \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))}\,dx}\,dy$ ($\boldsymbol{j} \ne \mathbf{0}$), then for any $\boldsymbol{i} \ne \mathbf{0}$,

$$\begin{aligned} &\sum_{\boldsymbol{j}\ne\mathbf{0}} q_{\boldsymbol{i}\boldsymbol{j}} x_{\boldsymbol{j}}^* + q_{\boldsymbol{i}\mathbf{0}} \\ &\quad = J^{-1} \cdot \int_0^1 \frac{\sum_{\boldsymbol{j}\in\mathbf{Z}_+^n} q_{\boldsymbol{i}\boldsymbol{j}} \cdot y^{j_1} u_2^{j_2}(y)\cdots u_n^{j_n}(y)}{B_1(y,u_2(y),\ldots,u_n(y))} \cdot e^{\int_0^y \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))}\,dx}\,dy \\ &\quad = J^{-1} \cdot \int_0^1 \sum_{k=1}^\infty i_k y^{i_1} u_2^{i_2}(y)\cdots u_k^{i_k-1}(y) u_k'(y)\cdots u_n^{j_n}(y) \cdot e^{\int_0^y \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))}\,dx}\,dy \end{aligned}$$

$$+ J^{-1} \cdot \int_0^1 \frac{y^{j_1} u_2^{j_2}(y) \cdots u_n^{j_n}(y) I(y, u_2(y), \ldots, u_n(y))}{B_1(y, u_2(y), \ldots, u_n(y))} \cdot e^{\int_0^y \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))} dx} dy$$
$$= 0.$$

Here, the last equality follows from the integration by parts. Hence, $(x_{\boldsymbol{j}}^* : \boldsymbol{j} \neq \mathbf{0})$ is a solution of the equation

$$\sum_{\boldsymbol{j} \neq \mathbf{0}} q_{\boldsymbol{ij}} x_{\boldsymbol{j}}^* + q_{\boldsymbol{i0}} = 0, \quad 0 \le x_{\boldsymbol{j}}^* \le 1, \boldsymbol{i} \neq \mathbf{0}.$$

By Lemma 3.2 in Li and Chen [9], we then have $a_{\boldsymbol{i0}} \le x_{\boldsymbol{i}}^*$ ($\boldsymbol{i} \neq \mathbf{0}$) since $a_{\boldsymbol{i0}}$ is the minimal solution of the above equation. (ii) is proved.

Finally, we consider (iii). Suppose that $\rho(\mathbf{1}) > 0$. By Lemma 2.1, we know that equation (2.1) has a root $(q_1, u_2(q_1), \ldots, u_n(q_1)) \in (0,1)^n$ and $G_{\boldsymbol{i}}(v) < \infty$ for all $v \in [0, q_1]$. Similarly as in the above, we only need to show that

$$a_{\boldsymbol{i0}} \le \lim_{v \uparrow q_1} \left[\int_0^v \frac{1}{B_1(y, u_2(y), \ldots, u_n(y))} \cdot e^{\int_0^y \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))} dx} dy \right]^{-1}$$
$$\cdot \int_0^v \frac{y^{j_1} u_2^{j_2}(y) \cdots u_n^{j_n}(y)}{B_1(y, u_2(y), \ldots, u_n(y))} \cdot e^{\int_0^y \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))} dx} dy.$$

By Lemma 2.1 we know that $\int_0^{q_1} \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))} dx = -\infty$ and

$$\int_0^y \frac{I(x, u_2(x), \ldots, u_n(x))}{B_1(x, u_2(x), \ldots, u_n(x))} dx \le \int_0^y \frac{I(q_1, q_2, \ldots, q_n)}{B_1(x, q_2, \ldots, q_n)} dx \le C \ln \frac{q_1 - y}{q_1}$$

for $y \in [0, q_1)$, where C is a positive constant. Hence, the integral $\int_0^{q_1} \frac{1}{B_1(y,u_2(y),\ldots,u_n(y))} \cdot e^{\int_0^y \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))} dx} dy$, denoted by D, is convergent. Now, by letting

$$y_{\boldsymbol{j}}^* = D^{-1} \cdot \int_0^{q_1} \frac{1}{B_1(y, u_2(y), \ldots, u_n(y))} \cdot e^{\int_0^y \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))} dx} dy, \quad \boldsymbol{j} \neq \mathbf{0},$$

we may prove similarly as above that $(y_{\boldsymbol{j}}^* : \boldsymbol{j} \neq \mathbf{0})$ is a solution of the equation

$$\sum_{\boldsymbol{j} \neq \mathbf{0}} q_{\boldsymbol{ij}} x_{\boldsymbol{i}} + q_{\boldsymbol{i0}} = 0, \quad 0 \le x_{\boldsymbol{j}} \le 1, \boldsymbol{i} \neq \mathbf{0}.$$

Again, by Lemma 3.2 in Li and Chen [9], we have $a_{\boldsymbol{i0}} \le y_{\boldsymbol{i}}^*$ ($\boldsymbol{i} \neq \mathbf{0}$), which proves the first equality in (3.5). The last two assertions in (3.5) are obvious. The proof is complete. □

By Theorem 3.2, we see that when immigration occurs then the condition $\rho(\mathbf{1}) \le 0$ (i.e., the death rate is not less than the mean birth rate) is no longer sufficient for the process to be finally extinct. A further condition $J = \infty$, which reflects the effect of immigration, is necessary to guarantee the final extinction.

Having obtained the extinction probability, we are now in a position to consider the extinction time. We shall use $E_{\boldsymbol{i}}[\tau_0]$ to denote the mean extinction time when the process starts at state $\boldsymbol{i} \neq \mathbf{0}$.

Theorem 3.3 *Suppose that $\rho(\mathbf{1}) \le 0$ and $J = \infty$, where J is given in (3.8) and thus the extinction probability $a_{\boldsymbol{i}\mathbf{0}} = 1$ ($\boldsymbol{i} \ne \mathbf{0}$). Then, for any $\boldsymbol{i} \ne \mathbf{0}$, $E_{\boldsymbol{i}}[\tau_0] < \infty$ if and only if*

$$\int_0^1 \frac{1 - yu_2(y)\cdots u_n(y) - I(y, u_2(y), \dots, u_n(y))}{B_1(y, u_2(y), \dots, u_n(y))}\,dy < \infty \tag{3.12}$$

and in which case, $E_{\boldsymbol{i}}[\tau_0]$ is given by

$$E_{\boldsymbol{i}}[\tau_0] = \int_0^1 \frac{1 - y^{i_1}u_2^{i_2}(y)\cdots u_n^{i_n}(y)}{B_1(y, u_2(y), \dots, u_n(y))} \cdot e^{-\int_y^1 \frac{I(x,u_2(x),\dots,u_n(x))}{B_1(x,u_2(x),\dots,u_n(x))}\,dx}\,dy. \tag{3.13}$$

Proof It follows from (3.11) that

$$\begin{aligned}
&\sum_{\boldsymbol{j}\ne\mathbf{0}} \left(\int_0^\infty p_{\boldsymbol{ij}}(t)\,dt\right) \cdot u^{j_1}u_2^{j_2}(u)\cdots u_n^{j_n}(u) \\
&\quad= \int_0^u \frac{1 - y^{i_1}u_2^{i_2}(y)\cdots u_n^{i_n}(y)}{B_1(y, u_2(y), \dots, u_n(y))} \cdot e^{-\int_y^u \frac{I(x,u_2(x),\dots,u_n(x))}{B_1(x,u_2(x),\dots,u_n(x))}\,dx}\,dy.
\end{aligned}$$

Letting $u \uparrow 1$, using the honesty condition and applying the Monotone Convergence Theorem then yields

$$\begin{aligned}
E_{\boldsymbol{i}}[\tau_0] &= \int_0^\infty \bigl(1 - p_{\boldsymbol{i}\mathbf{0}}(t)\bigr)\,dt \\
&= \sum_{\boldsymbol{j}\in\mathbf{Z}_{++}^n} \int_0^\infty p_{\boldsymbol{ij}}(t)\,dt \\
&= \int_0^1 \frac{1 - y^{i_1}u_2^{i_2}(y)\cdots u_n^{i_n}(y)}{B_1(y, u_2(y), \dots, u_n(y))} \cdot e^{-\int_y^1 \frac{I(x,u_2(x),\dots,u_n(x))}{B_1(x,u_2(x),\dots,u_n(x))}\,dx}\,dy.
\end{aligned}$$

Thus, (3.13) is proved. Finally, it is fairly easy to show that the expression in (3.13) is finite if and only if (3.12) holds. □

4 Recurrence Property

In this section we consider the recurrence property of nTMBPI in the case that $\beta \ne 0$ and thus $\mathbf{0}$ is no longer an absorbing state. We shall assume that the nTBI Q-matrix Q is regular.

It is well known that the nTMBPI is recurrent if and only if the extinction probability of the related absorbing nTMBPI (i.e., $\beta = 0$) equals 1. Therefore, by Theorem 3.2 we have the following result.

Theorem 4.1 *The nTMBPI is recurrent if and only if $\rho(\mathbf{1}) \le 0$ and $J = +\infty$, where J is given in (3.8).*

Now, we consider the positive recurrence of nTMBPI.

Theorem 4.2 *The nTMBPI is positive recurrent (i.e., ergodic) if and only if $\rho(\mathbf{1}) \le 0$ and*

$$\int_0^1 \frac{-I(y, u_2(y), \dots, u_n(y)) - R(y, u_2(y), \dots, u_n(y))}{B_1(y, u_2(y), \dots, u_n(y))}\,dy < \infty. \tag{4.1}$$

Moreover, if $\rho(\mathbf{1}) < 0$ *and* $\sum_{j=1}^{n}(I_j(\mathbf{1}) + R_j(\mathbf{1})) < \infty$, *then the process is exponentially ergodic.*

Proof Denote $\tilde{R}(x) := R(x, u_2(x), \dots, u_n(x))$, $\tilde{I}(x) := I(x, u_2(x), \dots, u_n(x))$, and $\tilde{B}_k(x) := B_k(x, u_2(x), \dots, u_n(x))$ $(k = 1, \dots, n)$.

Suppose that $\rho(\mathbf{1}) \le 0$ and (4.1) holds. By Chen [5], in order to prove the positive recurrence, we only need to show that the equation

$$\begin{cases} \sum_{\boldsymbol{j}\in\mathbf{Z}_+^n} q_{\boldsymbol{ij}} y_{\boldsymbol{j}} \le -1, & \boldsymbol{i} \ne \mathbf{0}, \\ \sum_{\boldsymbol{j}\ne\mathbf{0}} q_{\mathbf{0}\boldsymbol{j}} y_{\boldsymbol{j}} < \infty \end{cases}$$

has a finite nonnegative solution. By the irreducibility property and the fact that $\rho(\mathbf{1}) \le 0$, we may obtain from (4.1) that

$$\int_0^1 \frac{1 - y^{i_1} u_2^{i_2}(y) \cdots u_n^{i_n}(y)}{\tilde{B}_1(y)} \cdot e^{\int_0^y \frac{\tilde{I}(x)}{\tilde{B}_1(x)}\,dx}\,dy < \infty, \quad \boldsymbol{i} \in \mathbf{Z}_+^n.$$

Indeed, since $\beta > 0$, it is easy to see that there exists a positive constant L such that $1 - yu_2(y) \cdots u_n(y) \le L \cdot \tilde{R}(y)$. Hence,

$$\int_0^1 \frac{1 - y^{j_1} u_2^{j_2}(y) \cdots u_n^{j_n}(y)}{\tilde{B}_1(y)}\,dy < \infty$$

for any $\boldsymbol{j} \in \mathbf{Z}_+^n$. Now, let

$$y_{\boldsymbol{j}} = e^{-\int_0^1 \frac{\tilde{I}(x)}{\tilde{B}_1(x)}\,dx} \cdot \int_0^1 \frac{1 - y^{j_1} u_2^{j_2}(y) \cdots u_n^{j_n}(y)}{\tilde{B}_1(y)} \cdot e^{\int_0^y \frac{\tilde{I}(x)}{\tilde{B}_1(x)}\,dx}\,dy, \quad \boldsymbol{j} \in \mathbf{Z}_+^n,$$

then $0 \le y_{\boldsymbol{j}} < \infty$ ($\boldsymbol{j} \in \mathbf{Z}_+^n$) and it can be checked that $\sum_{\boldsymbol{j}\in\mathbf{Z}_+^n} q_{\boldsymbol{ij}} y_{\boldsymbol{j}} = -1$ ($\boldsymbol{i} \ne \mathbf{0}$) and

$$\sum_{\boldsymbol{j}\ne\mathbf{0}} q_{\mathbf{0}\boldsymbol{j}} y_{\boldsymbol{j}} \le e^{-\int_0^1 \frac{\tilde{I}(x)}{\tilde{B}_1(x)}\,dx} \cdot \int_0^1 \frac{-\tilde{R}(y)}{\tilde{B}_1(y)}\,dy < \infty.$$

Therefore, the nTMBPI is positive recurrent.

Conversely, suppose that the process is positive recurrent and thus possesses an equilibrium distribution $(\pi_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbf{Z}_+^n)$. Letting $t \to \infty$ in (2.2) and using the dominated convergence theorem yields

$$\begin{aligned} &\tilde{R}(s)\pi_{\mathbf{0}} + \tilde{I}(s) \sum_{\boldsymbol{j}\ne\mathbf{0}} \pi_{\boldsymbol{j}} s^{j_1} u_2^{j_2}(s) \cdots u_n^{j_n}(s) \\ &\quad + \sum_{k=1}^{n} \tilde{B}_k(s) \sum_{\boldsymbol{j}\ne\mathbf{0}} \pi_{\boldsymbol{j}} j_k s^{j_1} u_2^{j_2}(s) \cdots u_k^{j_k-1}(s) \cdots u_n^{j_n}(s) = 0 \end{aligned} \tag{4.2}$$

for $s \in [0, 1)$.

Since $\tilde{R}(s) < 0$ and $\tilde{I}(s) < 0$ for all $s \in [0, 1)$, by (4.2) and the proof of Theorem 3.1, we know that $\rho(\mathbf{1}) \le 0$. Denote

$$\pi(s) = \sum_{\boldsymbol{j}\in\mathbf{Z}_+^n} \pi_{\boldsymbol{j}} s^{j_1} u_2^{j_2}(s) \cdots u_n^{j_n}(s).$$

It follows from (4.2) that

$$\pi(s)=\pi_{\mathbf{0}}\left[1+\int_0^s \frac{-\tilde{R}(y)}{\tilde{B}_1(y)}\cdot e^{-\int_y^s \frac{\tilde{I}(x)}{\tilde{B}_1(x)}\,dx}\,dy\right],\quad s\in[0,1). \tag{4.3}$$

Since $\int_0^s \frac{-\tilde{R}(y)}{\tilde{B}_1(y)}\cdot e^{\int_0^y \frac{\tilde{I}(x)}{\tilde{B}_1(x)}\,dx}\,dy \geq \int_0^{\frac{1}{2}} \frac{-\tilde{R}(y)}{\tilde{B}_1(y)}\cdot e^{\int_0^y \frac{\tilde{I}(x)}{\tilde{B}_1(x)}\,dx}\,dy > 0$ for $s \geq \frac{1}{2}$, we must have $\int_0^1 \frac{-\tilde{I}(x)}{\tilde{B}_1(x)}\,dx < \infty$. Hence,

$$\lim_{s\uparrow 1}\int_0^s \frac{-\tilde{R}(y)}{\tilde{B}_1(y)} \leq \lim_{s\uparrow 1}\frac{\int_0^s \frac{-\tilde{R}(y)}{\tilde{B}_1(y)}\cdot e^{\int_0^y \frac{\tilde{I}(x)}{\tilde{B}_1(x)}\,dx}\,dy}{e^{\int_0^s \frac{\tilde{I}(x)}{\tilde{B}_1(x)}\,dx}} < \infty.$$

Hence, (4.1) holds. The first part is proved.

Now, suppose that $\rho(\mathbf{1}) < 0$ and $\sum_{j=1}^n (I_j(\mathbf{1}) + R_j(\mathbf{1})) < \infty$. We prove that the nTBIP is exponentially ergodic. Since $\rho(\mathbf{1})$ has a positive eigenvector $(x_1,\ldots,x_n)$, let

$$C_1 := \left(\sum_{j=1}^n I_j(\mathbf{1})\right)\vee\left(\sum_{j=1}^n R_j(\mathbf{1})\right)\cdot\max\{x_1,\ldots,x_n\} > 0, \qquad C_2 := -\rho(\mathbf{1}) > 0$$

and $f_{\boldsymbol{i}} = \sum_{k=1}^n i_k x_k$ ($\boldsymbol{i}\in\mathbf{Z}_+^n$). We can see that for any $\boldsymbol{i}\in\mathbf{Z}_+^n$,

$$\begin{aligned}
&\sum_{\boldsymbol{j}\in\mathbf{Z}_+^n} q_{\boldsymbol{ij}}(f_{\boldsymbol{j}}-f_{\boldsymbol{i}})\\
&\quad=\sum_{k=1}^n i_k\sum_{l=1}^n B_{kl}(1,\ldots,1)x_l+\sum_{l=1}^n\big[\delta_{\mathbf{0}\boldsymbol{i}}R_l(\mathbf{1})+(1-\delta_{\mathbf{0}\boldsymbol{i}})I_l(\mathbf{1})\big].\\
&\quad\leq C_1-C_2 f_{\boldsymbol{i}}.
\end{aligned}$$

By Corollary 4.49 of Chen [5], the process is exponentially ergodic. The proof is complete. □

Theorem 4.3 *Suppose that the nTMBPI is positive recurrent. Then, its equilibrium distribution* $(\pi_{\boldsymbol{j}} : \boldsymbol{j}\in\mathbf{Z}_+^n)$ *is given by*

$$\pi(s)=\pi_{\mathbf{0}}\left[1+\int_0^s \frac{-R(y,u_2(y),\ldots,u_n(y))}{B_1(y,u_2(y),\ldots,u_n(y))}\cdot e^{-\int_y^s \frac{I(x,u_2(x),\ldots,u_n(x))}{B_1(x,u_2(x),\ldots,u_n(x))}\,dx}\,dy\right],\quad s\in[0,1), \tag{4.4}$$

where $\pi(s)=\sum_{\boldsymbol{j}\in\mathbf{Z}_+^n}\pi_{\boldsymbol{j}} s^{j_1}u_2^{j_2}(s)\cdots u_n^{j_n}(s)$.

Proof (4.4) follows directly from the proof of Theorem 4.2 (see (4.3)). □

The following conclusion follows immediately from Theorem 3.3.

Theorem 4.4 *The nTMBPI is never strongly ergodic.*

Finally, we give an example to illustrate our results.

Example 4.1 Consider a two-type Markov branching–immigration process with $B_1(u,v) = p - u + (1-p)v^2$, $B_2(u,v) = p - v + (1-p)u^2$, $I(u,v) = \alpha(uv-1)$, and $R(u,v) = \beta(uv-1)$, where $\alpha > 0$, $\beta \geq 0$ and $p \in (0,1)$.

It is easy to see that $\rho(1,1) = 1 - 2p$. Moreover, the solution of (3.1) is $v(u) = u$ and the smallest nonnegative solution of (2.1) is $q_1 = q_2 = \min(1, \frac{p}{1-p})$.

(i) For the case $\beta = 0$, by Theorem 3.1,

$$a_{\boldsymbol{i}0} = \frac{\int_0^{q_1} \frac{y^{i_1+i_2}}{p-y+(1-p)y^2} \cdot e^{\int_0^y \frac{\alpha(x^2-1)}{p-x+(1-p)x^2}\,dx}\,dy}{\int_0^{q_1} \frac{1}{p-y+(1-p)y^2} \cdot e^{\int_0^y \frac{\alpha(x^2-1)}{p-x+(1-p)x^2}\,dx}\,dy},$$

which is equal to 1 if and only if $p > \frac{1}{2}$ or that $p = \frac{1}{2}$ and $\alpha \leq \frac{1}{4}$. Furthermore, if $p = \frac{1}{2}$ and $\alpha \leq \frac{1}{4}$, then $E_{e_1}[\tau_0] = +\infty$. While if $p > \frac{1}{2}$, then

$$\begin{aligned} E_{e_1}[\tau_0] &= \int_0^1 \frac{1}{p-(1-p)y} \cdot e^{\int_y^1 \frac{\alpha(1+x)}{p-(1-p)x}\,dx}\,dy \\ &= (2p-1)^{-\frac{\alpha}{(1-p)^2}} \int_0^1 \bigl[p-(1-p)y\bigr]^{\frac{\alpha}{(1-p)^2}-1} e^{-\frac{\alpha(1-y)}{1-p}}\,dy. \end{aligned}$$

(ii) For the case $\beta > 0$, by Theorem 4.2, the process is positive recurrent if and only if $p > \frac{1}{2}$.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

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The authors declare no competing interests.

Author contributions
Junping Li proved Theorems 3.1-3.3 and wrote the main manuscript, Juan Wang proved Theorems 4.2-4.3. Both authors reviewed the manuscript.

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