Xiao and Chen *Boundary Value Problems* (2024) 2024:3
https://doi.org/10.1186/s13661-023-01812-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Periodic solutions for second-order even and noneven Hamiltonian systems

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Abstract

In this paper, we consider the second-order Hamiltonian system

$$\ddot{x} + V'(x) = 0, \quad x \in \mathbb{R}^N.$$

We use the monotonicity assumption introduced by Bartsch and Mederski (Arch. Ration. Mech. Anal. 215:283–306, 2015). When V is even, we can release the strict convexity hypothesis, which is used by Bartsch and Mederski combined with the monotonicity assumption. When V is noneven, we weaken the strict convexity assumption and introduce another hypothesis (see (V10)). Then in both cases, we can build the homomorphism between the Nehari manifold and the unit sphere of some suitable space. Using the Nehari manifold method introduced by Szulkin (J. Funct. Anal. 257:3802–3822 2009), we prove the existence of T-periodic solutions with minimal period T.

Keywords: Periodic solution; Second-order Hamiltonian system; Nehari manifold; Minimal period

1 Introduction

Consider the second-order Hamiltonian system

$$\ddot{x} + V'(x) = 0, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where N is a positive integer, $V : \mathbb{R}^N \to \mathbb{R}$ is a potential function, and V' denotes the gradient of V. In 1978, Rabinowitz [27] proved that, for any $T > 0$, system (1.1) possesses a nonconstant T-periodic solution under the following assumptions:

$(V1)$ $V \in C^1(\mathbb{R}^N, \mathbb{R})$, $V(0) = 0$, and $V(x) \geq 0$ for all $x \in \mathbb{R}^N$;

$(V2)$ $V(x) = o(|x|^2)$ as $|x| \to 0$ in $\mathbb{R}^N$;

$(V3)$ ((AR)-condition) there exist constants $\mu > 2$ and $r_0 > 0$ such that

$$0 < \mu V(x) \leq \big(V'(x), x\big), \quad |x| \geq r_0.$$

Since the minimal period of this solution may be T/k for some positive integer k, Rabinowitz conjectured that system (1.1) possesses a T-periodic solution with minimal period

T under assumptions $(V1)$–$(V3)$. This is the so-called Rabinowitz minimal periodic solution conjecture. Since then, this conjecture has been studied by many mathematicians [1, 10, 12–15, 20–24, 34].

If V is convex and superquadratic, then its Fenchel conjugate function is subquadratic. Then the dual variational functional attains its minimum energy at some point, which corresponds to a T-periodic solution with minimal period T. Using this fact, Ambrosetti and Mancini [1] studied the following second-order Hamiltonian system:

$$-\ddot{x} = Qx + V'(x). \tag{1.2}$$

If Q is positive definite and V is convex, using the Clark dual, they proved that for any $T > 0$, system (1.2) possesses a sequence of solutions v_σ with minimal period T, where $\sigma = 2\pi/T > \omega_n$, and ω_n denotes the eigenvalues of Q. Moreover, $\|v_\sigma\| \to 0$ as $T \to 2\pi/\omega_n$, whereas $\|v_\sigma\|_{L^\infty} \to \infty$ as $T \to 0$. For more results on the assumption, we refer to [1, 12, 13].

Releasing the convexity assumption, many mathematicians assumed that $V(x)$ is twice continuously differentiable. Then we can define a Maslov-type index and prove an iterative formula. This formula can be used to estimate the minimal period of periodic solutions. This method was firstly introduced by Long [21], who studied second-order even Hamiltonian system (1.1) under assumptions $(V2)$, $(V3)$, and

$(V1')$ $V \in C^2(\mathbb{R}^N, \mathbb{R})$, and $V(x) \geq 0$ for all $x \in \mathbb{R}^N$;

$(V4)$ $V(x)$ is even, i.e., $V(-x) = V(x)$ for all $x \in \mathbb{R}^N$.

He proved that system (1.1) possesses a T-periodic solution with minimal period T or $T/3$ when V satisfies $(V1')$ and $(V2)$ – $(V4)$. For more results for even Hamiltonian systems, we refer to [14, 15] and references therein. For a second-order noneven Hamiltonian system, Long showed that system (1.1) possesses a T-periodic solution with minimal periodic T/k for some integer k satisfying $1 \leq k \leq n + 2$ [22] or $1 \leq k \leq n + 1$ [23]. For more results for noneven Hamiltonian systems, we refer to [20] and references therein.

There is a third-type condition, the global (AR)-condition:

$(V3')$ there exists a constant $\theta > 1$ such that

$$0 < \theta\big(V'(x), x\big) \leq \big(V''(x)x, x\big), \quad x \in \mathbb{R}^N \setminus \{0\}.$$

For V satisfying $(V1')$, $(V3')$, and $(V4)$, Xiao [34] proved the existence of a T-periodic solution with minimal period T. In 2020, Xiao and Shen [33] generalized $(V3')$ and assumed that

$(V2')$ $V'(x) = o(|x|)$ as $x \to 0$ in $\mathbb{R}^N$;

$(V5)$ $V(x)/|x|^2 \to \infty$ as $|x| \to \infty$;

$(V6)$ there exist $p > 2$ and $C > 0$ such that $|V'(x)| \leq C(1 + |x|^{p-1})$;

$(V7)$ for all $x \in \mathbb{R}^N$ with $|x| = 1$, the map $s \mapsto (V'(sx), x)/s$ is nondecreasing on $(0, \infty)$.

Using the Nehai manifold method and a disturbed technique, they still proved the existence of a T-periodic solution with minimal period T under assumptions $(V1)$, $(V2')$, and $(V4)$–$(V7)$. As is well known, the Nehari manifold method [26] can be used to study the existence of ground state solutions to partial differential equations [2–4, 6, 16–19, 25, 28–31]) and periodic solutions to ordinary differential equations and difference equations [33]. To use the Nehari manifold method, we need to build a homomorphism between the

Nehari manifold and a suitable subspace. To do this, we need to introduce some monotonicity assumptions on F to prove the following inequality:

$$g(s,v) = f(x,u)\left[\frac{1}{2}\left(s^2-1\right)u + sv\right] + F(x,u) - F(x, su+v) < 0, \tag{1.3}$$

where $F(x,u)$ is nonlinear term, and $f(x,u) = \nabla_u F(x,u)$, $s \in \mathbb{R}^+$ and $u,v \in \mathbb{R}$ ($u,v \in \mathbb{R}^N$, respectively). Those monotonicity hypotheses are divided into two cases: the low-dimensional and high-dimensional cases. In the low-dimensional case, Szulkin and Weth [29] introduced the following assumption:

(S) the map $u \mapsto f(x,u)/|u|$ is strictly increasing on $(-\infty,0)$ and $(0,\infty)$.

They proved inequality (1.3) and built the homomorphism mentioned above. For more results on this direction, we refer to [8, 17, 18, 29, 30]. In the high-dimensional case, the proof of (1.3) is more complicated. In 2015, Bartsch and Mederski [3] introduced the following assumptions:

($BM1$) if $(f(x,u),v) = (f(x,v),u) > 0$, then $F(x,u) - F(x,v) \le \frac{(f(x,u),u)^2-(f(x,u),v)^2}{2(f(x,u),u)}$.
If, in addition, $F(x,u) \ne F(x,v)$, then the strict inequality holds;

($BM2$) F is convex with respect to u.

Then they also built the homeomorphism. For more results in this direction, we refer to [3, 5, 7, 9, 25, 31, 33, 37].

Inspired by [3, 29, 30, 35], our aim is twofold. Firstly, we try to restudy system (1.1) under ($V2'$), ($V4$), ($V5$), ($V6$), and the following assumptions:

($V1''$) $V \in C^1(\mathbb{R}^N,\mathbb{R})$, $V(x) > 0$ for all $x \in \mathbb{R}^N \setminus \{0\}$, and $V(0) = 0$.

($V8$) if $(V'(x),y) = (V'(y),x) > 0$, then $V(x) - V(y) \le \frac{(V'(x),x)^2-(V'(x),y)^2}{2(V'(x),x)}$.
If, in addition, $V(x) \ne V(y)$, then the strict inequality holds.

($V9$) $(V'(x),x) > 2V(x)$ for all $x \in \mathbb{R}^N \setminus \{0\}$,

where ($V8$) is another version of ($BM1$) corresponding to system (1.1). Without the help of ($BM2$), we can still build the homomorphism mentioned above when V is even. When V is not even, we fail to built such a homomorphism without assumption ($BM2$). To overcome this difficulty, we introduce the following technical condition:

($V10$) If $V(x+y) = V(x)$ and $(V'(x),y) = 0$, then $y = 0$.

Now let us state our main results.

Theorem 1.1 *Assume that V satisfies* ($V1''$), ($V2'$), ($V4$)–($V6$), ($V8$), *and* ($V9$). *Then for any given $T > 0$, system* (1.1) *possesses a nonconstant T-periodic solution with minimal period T.*

Theorem 1.2 *Assume that V satisfies* ($V1''$), ($V2'$), ($V4$)–($V6$), ($V8$), *and* ($V9$). *Then for any given $T > 0$, system* (1.1) *has infinitely many pairs of T-periodic solutions.*

Corollary 1.3 *Assume that V satisfies* ($V1$), ($V3'$), *and* ($V4$). *Then for any given $T > 0$, system* (1.1) *possesses a nonconstant T-periodic solution with minimal period T.*

Theorem 1.4 *Assume that V satisfies* ($V1''$), ($V2'$), ($V5$), ($V6$), *and* ($V8$)–($V10$), *Then for any given $T > 0$, system* (1.1) *possesses a nonconstant T-periodic solution with minimal period T.*

The rest part of this paper is divided into two parts. In Sect. 2, we study system (1.1) with an even potential functional $V(x)$. In Sect. 3, we study system (1.1) with noneven potential functional $V(x)$.

2 The even case

Given $T > 0$, let $S_T = \mathbb{R}/(T\mathbb{Z})$. The Solobev space H^1 is defined as

$$H^1 = W^{1,2}(S_T, \mathbb{R}^N) = \{x \in L^2(S_T, \mathbb{R}^N) : \dot{x} \in L^2(S_T, \mathbb{R}^N)\}, \tag{2.1}$$

where $\dot{x}$ is the weak derivative of x. The space H^1 is equipped with the usual norm

$$\|x\|_1^2 = \int_0^T \left(|\dot{x}|^2 + |x|^2\right) dt, \quad x \in H^1,$$

and the corresponding inner product

$$\langle x, y\rangle_1 = \int_0^T \left[(\dot{x}, \dot{y}) + (x, y)\right] dt, \quad x, y \in H^1,$$

where $|\cdot|$ and $(\cdot,\cdot)$ denote the standard norm and inner product in $\mathbb{R}^N$ respectively.

The variational functional corresponding to system (1.1) is

$$\varphi(x) = \int_0^T \left[\frac{1}{2}|\dot{x}|^2 - V(x)\right] dt, \quad x \in H^1. \tag{2.2}$$

According to Lemma 2.1 in [33], since V satisfies $(V1'')$, $(V6)$, and $(V9)$, φ is continuously differentiable on H^1, and

$$\langle \varphi'(x), y\rangle_1 = \int_0^T \left[(\dot{x}, \dot{y}) - (V'(x), y)\right] dt, \quad x, y \in H^1.$$

Set $\phi(x) = \int_0^T V(x)\, dt$. Then $\phi' : H^1 \to (H^1)^*$ is compact.

By the Fourier series theory, for any $x \in H^1$, we have

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{2k\pi t}{T} + b_k \sin\frac{2k\pi t}{T}\right),$$

where $a_0, a_k, b_k \in \mathbb{R}^N$, $k = 1, 2, \ldots$. Then we define a the following subspace E of H^1:

$$E = \{x \in H^1 : x(-t) = -x(t), t \in \mathbb{R}\}. \tag{2.3}$$

Obviously, E is a closed subspace of H^1.

Define the inner product $\langle \cdot,\cdot\rangle$ on E by setting

$$\langle x, y\rangle = \int_0^T (\dot{x}, \dot{y})\, dt, \quad x, y \in E,$$

which induces a new norm $\|\cdot\|$ on E as follows:

$$\|x\|^2 = \int_0^T |\dot{x}|^2\, dt, \quad x \in E.$$

It is well known that $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent norms on E. Moreover, by the Sobolev embedding theorem there exists $M_i > 0$ such that

$$\|x\|_{L^i} \leq M_i\|x\|, \quad i = 2, p, \tag{2.4}$$

$$\|x\|_\infty \leq M_\infty\|x\|, \tag{2.5}$$

where $\|\cdot\|_{L^i}$ and $\|\cdot\|_\infty$ denote the usual norm in $L^i(S_T, \mathbb{R}^N)$ and $C(S_T, \mathbb{R}^N)$, respectively.

Restricted to E, φ can be rewritten as

$$\varphi(x) = \int_0^T \left[\frac{1}{2}|\dot{x}|^2 - V(x)\right] dt = \frac{1}{2}\|x\|^2 - \phi(x), \quad x \in E.$$

Obviously, φ is invariant by translations of $\mathbb{Z}_2$.

Lemma 2.1 [36] *Critical points of φ restricted to E are critical points of φ on the whole space H^1, which correspond to periodic solutions of system* (1.1).

According to the lemma, the critical points of φ correspond to T-periodic solutions of system (1.1), but not certainly with minimal period T. Observing that the lower the energies of the solutions, the larger the minimal periods [32], we work on a manifold of E. The critical point of φ with least energy on such a manifold gives rise to a solution of system (1.1) with minimal period T.

Define the Nehari manifold

$$\mathcal{M} = \{x \in E\backslash\{0\} : \langle\varphi'(x), x\rangle = 0\}.$$

A point $x \in E$ is called critical point of φ if $\varphi'(x) = 0$. Hence $\mathcal{M}$ contains all nontrivial critical points of φ.

Lemma 2.2 *Assume that V satisfies $(V1'')$, $(V4)$, $(V5)$, $(V8)$, and $(V9)$. Then for $x \in E \setminus \{0\}$ and $s \geq 0$, $s \neq 1$, we have*

$$\varphi(x) > \varphi(sx) - \left\langle \varphi'(x), \frac{s^2-1}{2}x \right\rangle.$$

Proof Let us show that

$$\varphi(sx) - \varphi(x) - \left\langle \varphi'(x), \frac{s^2-1}{2}x \right\rangle = \int_0^T h(s)\, dt < 0,$$

where $h(s) := V(x) - V(sx) + (V'(x), \frac{s^2-1}{2}x)$ for $s \geq 0$, $s \neq 1$.

We first claim that $h(s) \leq 0$. Then by $(V9)$ we have

$$h(0) = V(x) - \frac{1}{2}\left(V'(x), x\right) < 0 \quad \text{and} \quad h(1) = 0.$$

It follows from $(V5)$ that $\lim_{s\to\infty} h(s) = -\infty$. Therefore $h(s)$ attains its maximum on $[0, +\infty)$. Let $s_0 \geq 0$ be such that $h(s_0) = \max_{s\geq 0} h(s)$. We may assume that $s_0 > 0$. Then

$$h'(s_0) = \left(V'(x), s_0x\right) - \left(V'(s_0x), x\right) = 0,$$

that is, $(V'(x), s_0x) = (V'(s_0x), x)$. Therefore, using $(V8)$, we have

$$\begin{aligned} h(s_0) &= V(x) - V(s_0x) + \left(V'(x), \frac{s_0^2-1}{2}x\right) \\ &\le \frac{(V'(x),x)^2 - (V'(x),s_0x)^2}{2(V'(x),x)} + \frac{s_0^2-1}{2}\big(V'(x),x\big) = 0. \end{aligned}$$

Then by $(V8)$ again, $h(s_0) = 0$ if and only if $V(s_0x) = V(x)$. By the definition of h, $\frac{s_0^2-1}{2}(V'(x),x) = 0$. Since $(V1'')$ and $(V9)$ imply that $(V'(x),x) > 0$, we have that $h(s_0) = 0$ if and only if $s_0 = 1$. Hence $h(s) < 0$ for all $s \ge 0$, $s \ne 1$. □

From Lemma 2.2 we have the following lemma.

Lemma 2.3 *Assume that V satisfies $(V1'')$, $(V4)$, $(V5)$, $(V8)$, and $(V9)$. Then for $x \in \mathcal{M}$ and $s \ge 0$, $s \ne 1$, we have $\varphi(x) > \varphi(sx)$.*

For any $x \in E \setminus \{0\}$, we define

$$E(x) = \{rx : r \in \mathbb{R}\}, \qquad \hat{E}(x) = \left\{rx : r \in \mathbb{R}^+\right\},$$

where $\mathbb{R}^+ := [0, +\infty)$. Obviously, by Lemma 2.3, s_xx is the unique critical point on $\varphi|_{\hat{E}(x)}$. Then we have $s_xx \in \hat{E}(x)$ and $\langle \varphi'(s_xx), s_xx\rangle = 0$. Hence $s_xx \in \mathcal{M} \cap \hat{E}(x)$. Define the map $\hat{m} : E \setminus \{0\} \to \mathcal{M}$ by

$$\hat{m}(x) = s_xx.$$

The above discussion yields the following lemma.

Lemma 2.4 *Suppose that all assumptions of Theorem 1.1 hold. Then for any $x \in E \setminus \{0\}$, the set $\mathcal{M} \cap \hat{E}(x)$ consists of precisely one point $\hat{m}(x)$, which is the unique global maximum of $\varphi|_{\hat{E}(x)}$.*

Lemma 2.5 *If V satisfies $(V1'')$, $(V2')$, $(V4)$, and $(V6)$, then there exists $\alpha_0 > 0$ such that $\|x\| \ge \alpha_0$ for all $x \in \mathcal{M}$.*

Proof First, $(V2')$ and $(V6)$ imply that for each $\varepsilon > 0$, there is $M_\varepsilon > 0$ such that

$$\left|V'(x)\right| \le \varepsilon|x| + M_\varepsilon|x|^{p-1}, \quad x \in \mathbb{R}^N,$$

where $p > 2$ is the parameter in $(V6)$. Then for all $x \in \mathcal{M}$, we have

$$\begin{aligned} 0 &= \left\langle \varphi'(x), x\right\rangle \\ &= \|x\|^2 - \int_0^T \big(V'(x), x\big)\, dt \\ &\ge \|x\|^2 - \int_0^T \left(\varepsilon|x|^2 + M_\varepsilon|x|^p\right) dt \\ &= \|x\|^2 - \varepsilon\|x\|_{L^2}^2 - M_\varepsilon\|x\|_{L^p}^p \end{aligned}$$

$$\geq \left(1 - \varepsilon M_2^2\right)\|x\|^2 - M_\varepsilon M_p^p \|x\|^p.$$

Setting $\alpha_0 = (\frac{1-\varepsilon M_2^2}{M_\varepsilon M_p^p})^{\frac{1}{p-2}}$, we obtain $\|x\| \geq \alpha_0 > 0$. □

Lemma 2.6 *Assume that* $(V2')$ *and* $(V4)$ *hold. Then* $\mathcal{M}$ *is bounded away from* 0 *and closed. Moreover, there exists* $\rho > 0$ *such that* $c = \inf_{\mathcal{M}} \varphi \geq \inf_{S_\rho} \varphi > 0$, *where* $S_\rho = \{x \in E : \|x\| = \rho\}$.

Proof By $(V2')$, for any $\epsilon = \frac{1}{4M_2^2} > 0$, there exists $\delta_\epsilon > 0$, such that $|V'(z)| \leq \epsilon|z|$, $|z| \leq \delta_\epsilon$. Thus we have

$$V(z) \leq \epsilon|z|^2, \quad |z| \leq \delta_\epsilon.$$

For any $x \in E$, without loss of generality, we can assume that $\|x\| = 1$. Then there exists a constant ρ such that $\rho = \frac{\delta_\epsilon}{M_\infty} > 0$ and $|\rho x| \leq \delta_\epsilon$. Then

$$\begin{aligned}\max_{z\in\hat{E}(x)} \varphi(z) &\geq \max_{r\in\mathbb{R}^+} \varphi(rx) \geq \varphi(\rho x) = \frac{1}{2}\rho^2\|x\|^2 - \int_0^T V(\rho x)\,dt \\ &\geq \frac{1}{2}\rho^2\|x\|^2 - \epsilon M_2^2 \rho^2 \|x\|_{L_2}^2 = \frac{\delta_\epsilon^2}{4M_\infty^2} = \frac{1}{4}\rho^2 > 0,\end{aligned}$$

that is,

$$c = \inf_{\mathcal{N}} \varphi = \inf_{x\in E\setminus\{0\}} \max_{z\in\hat{E}(x)} \varphi(z) \geq \inf_{S_\rho} \varphi \geq \frac{1}{4}\rho^2 > 0.$$ □

Lemma 2.7 *If* $(V4)$ *and* $(V5)$ *hold, then* φ *is coercive on* $\mathcal{M}$, *i.e.,* $\varphi(x) \to \infty$ *as* $\|x\| \to \infty$, $x \in \mathcal{M}$.

Proof Assume on the contrary that φ is not coercive on $\mathcal{M}$, i.e., there exists $(x_n) \subset \mathcal{M}$ such that $\varphi(x_n) \leq d$ for some $d > 0$ as $\|x_n\| \to \infty$. Let $v_n := x_n/\|x_n\|$; passing to a subsequence, we may assume that $v_n \rightharpoonup v$.

(a) Let $v \neq 0$. Then there exists $K > 0$ such that $\|v\|_\infty \geq K$. Since $v(t)$ is uniformly continuous on $[0, T]$, there exist $t_0 \in [0, T]$ and a neighborhood $U = U(t_0) \subset [0, T]$ such that $|v(t_0)| = \|v\|_\infty$, $|v(t)| \geq K/2$ for all $t \in U$, and $\text{meas}(U) \geq \delta$ for some $\delta > 0$, where $\text{meas}(\cdot)$ denotes the Lebesgue measure. By $(V5)$ and Fatou's lemma we have

$$\int_0^T \frac{V(v_n\|x_n\|)}{(|v_n|\|x_n\|)^2}|v_n|^2\,dt \geq \int_U \frac{V(v_n\|x_n\|)}{(|v_n|\|x_n\|)^2}|v_n|^2\,dt \to +\infty. \tag{2.6}$$

Thus

$$0 \leftarrow \frac{\varphi(x_n)}{\|x_n\|^2} = \frac{1}{2} - \int_0^T \frac{V(v_n\|x_n\|)}{(|v_n|\|x_n\|)^2}|v_n|^2\,dt \to -\infty,$$

which is a contradiction.

(b) Let $\nu = 0$. We can write $x_n = \tau \nu_n$, where $\tau := \|x_n\|$. It follows from the dominated convergence theorem that $\int_0^T V(s\nu_n)\,dt \to 0$ for every $s \in \mathbb{R}^+$. Hence, for any $s > 0$, we have

$$d \geq \varphi(x_n) = \varphi(\tau \nu_n) \geq \varphi(s\nu_n) = \frac{1}{2}s^2 - \int_0^T V(s\nu_n)\,dt \to \frac{1}{2}s^2, \tag{2.7}$$

which is a contradiction when we choose $s > \sqrt{2d}$.

Hence φ is coercive on $\mathcal{M}$. □

Lemma 2.8 *If* $(V4)$ *and* $(V5)$ *hold, and* $U \subset E \setminus \{0\}$ *is a compact set, then there exists* $R > 0$ *such that* $\varphi \leq 0$ *on* $E(x) \setminus B_R(0)$ *for every* $x \in U$.

Proof Without loss of generality, we may assume that $\|x\| = 1$ for every $x \in U$. Suppose, on the contrary, that there exist $(x_n) \subset U$ and $(s_n) \subset \mathbb{R}^+$ such that $y_n := s_n x_n \in \hat{E}(x_n)$, $\varphi(y_n) > 0$ for all n, and $s_n \to +\infty$ as $n \to \infty$. Passing to a subsequence, we may assume that $x_n \to x \in E$. Then

$$0 \leq \frac{\varphi(y_n)}{s_n^2} = \frac{1}{2} - \int_0^T \frac{V(s_n x_n)}{|s_n x_n|^2}|x_n|^2\,dt. \tag{2.8}$$

Arguing similarly as for (2.6), we have

$$\frac{1}{2} - \int_0^T \frac{V(s_n x_n)}{|s_n x_n|^2}|x_n|^2\,dt \leq \frac{1}{2} - \int_U \frac{V(s_n x_n)}{|s_n x_n|^2}|x_n|^2\,dt \to -\infty, \tag{2.9}$$

which contracts to (2.8). □

Lemma 2.9 *Suppose that all assumptions of Theorem* 1.1 *hold, Then the map* $E \setminus \{0\} \to \mathcal{M}$, $x \mapsto \hat{m}(x)$, *is continuous.*

Proof For a sequence $(x_n) \subset E \setminus \{0\}$ such that $x_n \to x$, we show that $\hat{m}(x_n) \to \hat{m}(x)$ for a subsequence.

Without loss of generality, we may assume that $\|x_n\| = \|x\| = 1$ for all n, so that $\hat{m}(x_n) = \|\hat{m}(x_n)\| x_n$. By Lemma 2.8 there exists $R > 0$ such that

$$\varphi\big(\hat{m}(x_n)\big) = \sup_{\hat{E}(x_n)} \varphi \leq \sup_{B_R(0)\cap \hat{E}(x_n)} \varphi \leq \sup_{x \in B_R(0)\cap \hat{E}(x_n)} \|x\|^2 = R^2 \quad \text{for every } n.$$

Hence by the coercivity of φ, $\hat{m}(x_n)$ is bounded. Passing to a subsequence, we may assume that $t_n := \|\hat{m}(x_n)\| \to t$, and by Lemma 2.5 we have $t \geq \alpha_0 > 0$. Since $\mathcal{M}$ is closed and $\hat{m}(x_n) \to tx$, we have $tx \in \mathcal{M}$. Hence $tx = \hat{m}(x)$ and $\hat{m}(x_n) \to \hat{m}(x)$. □

Next, we consider the unit sphere $S := \{x \in E : \|x\| = 1\}$. Note that the restriction of the map $\hat{m}$ to S is a homeomorphism with inverse given by

$$m : \mathcal{M} \to S, \quad m(x) = \frac{x}{\|x\|}.$$

We will also consider the functionals $\hat{\Psi}: E\setminus\{0\}\to\mathbb{R}$ and $\Psi: S\to\mathbb{R}$ defined by

$$\hat{\Psi}(x) := \varphi\bigl(\hat{m}(x)\bigr) \quad \text{and} \quad \Psi := \hat{\Psi}|_S.$$

Arguing similarly as in Proposition 9 and Corollary 10 in [30], we have the following conclusions. Since the proofs are basically the same, we omit them.

Lemma 2.10 *Suppose that all assumptions of Theorem 1.1 hold. Then* $\hat{\Psi}\in C^1(E\setminus\{0\},\mathbb{R})$, *and* $\langle\hat{\Psi}'(w),z\rangle = \frac{\|\hat{m}(w)\|}{\|w\|}\langle\varphi'(\hat{m}(w)),z\rangle$ *for all* $w,z\in E$, $w\neq 0$.

A sequence (x_n) is called a Palais–Smale sequence (PS-sequence for short) for $\hat{\Psi}$ if $\hat{\Psi}(x_n)$ is bounded and there exist $\hat{\Psi}'(x_n)\to 0$ as $n\to\infty$. We say that $\hat{\Psi}$ satisfies the PS-condition if every PS-sequence for $\hat{\Psi}$ contains a convergent subsequence.

Lemma 2.11 (*see* [30]) *Suppose that all assumptions of Theorem 1.1 hold. Then*

(a) $\Psi\in C^1(S,\mathbb{R})$, *and*

$$\bigl\langle\Psi'(w),z\bigr\rangle = \bigl\|m(w)\bigr\|\bigl\langle\varphi'\bigl(m(w)\bigr),z\bigr\rangle \quad \textit{for all } z\in T_wS := \bigl\{v\in E: \langle w,v\rangle = 0\bigr\};$$

(b) $(w_n)_n$ *is a PS-sequence for* Ψ *if and only if* $(m(w_n))_n$ *is a PS-sequence for* φ;

(c) w *is a critical point of* Ψ *if and only if* $m(w)$ *is a nontrivial critical point of* φ. *Moreover, the corresponding values of* Ψ *and* φ *coincide, and* $\inf_S\Psi = \inf_{\mathcal{M}}\varphi$.

(d) *if* φ *is even, then so is* Ψ.

Lemma 2.12 *Assume that all assumptions of Theorem 1.1 hold. Then* φ *satisfies PS-condition on* $\mathcal{M}$, *and so does* Ψ.

Proof Let $(x_n)\subset\mathcal{M}$ be a PS-sequence of φ. Then $(\varphi(x_n))$ is bounded. According to Lemma 2.7, (x_n) is bounded. Since $\phi': H^1\to(H^1)^*$ is compact and

$$\varphi'(x_n) = x_n - \phi'(x_n)\to 0,$$

(x_n) has a convergent subsequence. Thus φ satisfies *PS*-condition.

Next, assume that (y_n) is a PS-sequence for Ψ. According to Lemma 2.11, $(m(y_n))\subset\mathcal{M}$ is a PS-sequence for φ. Since φ satisfies PS-condition, passing to a subsequence, $m(y_n)\to z$. Thus $y_n\to\hat{m}(z)$. Hence Ψ satisfies PS-condition. □

Proof of Theorem 1.1 According to Lemmas 2.6 and 2.11, $\inf_{x\in S}\Psi(x) = \inf_{x\in\mathcal{M}}\varphi(x) = c$. Let (y_n) be a minimizing sequence for Ψ restricted to S. By Ekeland's variational principle [11] we may assume that $\Psi'(y_n)\to 0$ as $n\to\infty$. It is clear that $(\Psi(y_n))$ is bounded. Then (y_n) is a PS-sequence for Ψ. Since Ψ satisfies PS-condition, (y_n) contains a subsequence converging to some limit y. Thus y is a critical point of Ψ. According to Lemma 2.11 again, $x := \hat{m}(y)\in\mathcal{M}$ is a critical point of φ.

It remains to show that $\varphi(x) = c$. Obviously, $\varphi(x) \geq c$. By $(V9)$ and Fatou's lemma

$$\begin{aligned}
c + o(1) &= \varphi(x_n) - \frac{1}{2}\langle \varphi'(x_n), x_n\rangle \\
&= \int_0^T \left[\frac{1}{2}\big(V'(x_n), x_n\big) - V(x_n)\right] dt \\
&\geq \int_0^T \left[\frac{1}{2}\big(V'(x), x\big) - V(x)\right] dt + o(1) \\
&= \varphi(x) - \frac{1}{2}\langle \varphi'(x), x\rangle + o(1) \\
&= \varphi(x) + o(1),
\end{aligned} \tag{2.10}$$

where $x_n := \hat{m}(y_n)$. Hence $\varphi(x) \leq c$. So $\varphi(x) = c$, and x is a nonconstant T-periodic solution for system (1.1).

Finally, we will show that x has T as its minimal period. Suppose that x has a minimal period T/k, where $k \geq 2$ is an integer. Denote $w(t) = x(t/k)$. Obviously, $w \in E$, and there exists $\bar{r} > 0$ such that $\bar{r}w \in \mathcal{M}$. Hence

$$\begin{aligned}
\inf_{x\in\mathcal{M}} \varphi(x) &\leq \varphi(\bar{r}w) \\
&= \int_0^T \left[\frac{1}{2}\left|\bar{r}\dot{w}(t)\right|^2 - V\big(\bar{r}w(t)\big)\right] dt \\
&= \int_0^T \left[\frac{1}{2k^2}\left|\bar{r}\dot{x}\left(\frac{t}{k}\right)\right|^2 - V\left(\bar{r}x\left(\frac{t}{k}\right)\right)\right] dt \\
&= \int_0^T \left[\frac{1}{2k^2}\left|\bar{r}\dot{x}(\tau)\right|^2 - V\big(\bar{r}x(\tau)\big)\right] d\tau \\
&< \int_0^T \left[\frac{1}{2}\left|\bar{r}\dot{x}(\tau)\right|^2 - V\big(\bar{r}x(\tau)\big)\right] d\tau \\
&= \varphi(\bar{r}x) \leq \varphi(x) = \inf_{x\in\mathcal{M}} \varphi(x),
\end{aligned}$$

which is a contradiction. Hence x has T as its minimal period. □

Let X be a Banach space such that the unit sphere S in X is a submanifold of class (at least) C^1, and let $\psi \in C^1(S, \mathbb{R})$. We have the following result.

Lemma 2.13 [30] *If X is infinite-dimensional and $\psi \in C^1(S, \mathbb{R})$ is bounded below and satisfies PS-condition, then ψ has infinitely many pairs of critical points.*

Proof of Theorem 1.2 Since V is even, so do φ and Ψ. By Lemmas 2.6, 2.11, and 2.12, $\inf_{x\in S} \Psi(x) = c > 0$, and Ψ satisfies PS-condition. Then Lemma 2.13 yields that Ψ has infinitely many pairs of critical points. Applying Proposition 2.11 again, φ has infinitely many pairs of critical points. Hence system (1.1) has infinitely many pairs of T-periodic solutions. □

Proof of Corollary 1.3 Referring to the proof of Corollary 1.1 in [33], here we can prove Corollary 1.3. We need only to check that V satisfies $(V1'')$, $(V2')$, $(V5)$, $(V6)$, $(V8)$, and

$(V9)$ under assumptions $(V1)$ and $(V3')$. It is easy to check that V satisfies $(V2')$, $(V5)$, and $(V6)$. Therefore we only verify $(V1)$, $(V8)$, and $(V9)$.

For $s > 0$, set $k(s) = (V'(sx), x)/s$. Computing directly, by $(V3')$ we have

$$k'(s) = \frac{(V''(sx)sx, x) - (V'(sx), x)}{s^2} = \frac{(V''(sx)sx, sx) - (V'(sx), sx)}{s^3} > 0.$$

Hence $k(s)$ is strictly increasing on $(0, +\infty)$. By $(V2')$ we have $k(s) \to 0$ as $s \to 0$ and $k(0) = 0$.

For $x \in \mathbb{R}^N \setminus \{0\}$, set $s = |x|$ and $y = x/|x|$, i.e., $|y| = 1$. Noting that $s > 0$, we have

$$\begin{aligned} V(x) &= V(sy) \\ &= \int_0^s \frac{(V'(\tau y), y)}{\tau} \tau \, d\tau \\ &= \int_0^s k(\tau)\tau \, d\tau \\ &< k(s) \int_0^s \tau \, d\tau \\ &= \frac{1}{2}\big(V'(sy), sy\big) = \frac{1}{2}\big(V'(x), x\big). \end{aligned}$$

Hence $(V9)$ holds. Moreover, since

$$V(x) = V(sy) = \int_0^s k(\tau)\tau \, d\tau > \int_{\frac{s}{2}}^s k(\tau)\tau \, d\tau > 0,$$

$(V1'')$ holds.

Next, we consider $(V8)$. For any $x \in E \setminus \{0\}$ and $s > 0$, if $(V'(x), sx) = (V'(sx), x) > 0$, then

$$\frac{(V'(sx), x)}{s} = \big(V'(x), x\big).$$

Since $k(s)$ is strictly increasing on $[0, +\infty)$, we have $s = 1$. Thus, obviously,

$$0 = V(x) - V(sx) \leq \frac{(V'(x), x)^2 - (V'(x), sx)^2}{2(V'(x), x)} = 0,$$

which shows that $(V8)$ is satisfied. Corollary 1.3 holds by Theorem 1.1. □

3 The noneven case

Recall that $H^1 = W^{1,2}(S_T, \mathbb{R}^N)$ is a Hilbert space. Denote by $\mathcal{A}$ the operator $-\frac{d^2}{dt^2}$ on $L^2(S_T, \mathbb{R}^N)$ with domain $D(\mathcal{A}) = H^2(S^T, \mathbb{R}^N)$. The operator $\mathcal{A}$ is a self-adjoint operator with a sequence of eigenvalues (counted with multiplicity)

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty.$$

Denote by $|\mathcal{A}|$ the absolute value of $\mathcal{A}$, and let $|\mathcal{A}|^{1/2}$ be the square root of $|\mathcal{A}|$ with domain $D(|\mathcal{A}^{1/2})$. Decompose the space H^1 as follows:

$$H^1 = H^0 \oplus H^+,$$

where H^0 and H^+ are the null space and the positive eigenvalue space of $\mathcal{A}$. Obviously, $H^0 = \mathbb{R}^N$. For any $x \in H^1$, $x = x^0 + x^+$, where $x^0 \in H^0$ and $x^+ \in H^+$. Define a new inner product and the associated norm by

$$(x, y)_2 = \left(\mathcal{A}^{1/2}x, \mathcal{A}^{1/2}y\right)_{L^2} + \left(x^0, y^0\right)_{L^2},$$
$$\|x\|_2^2 = \left\|\mathcal{A}^{1/2}x\right\|_{L^2}^2 + \left\|x^0\right\|_{L^2}^2.$$

Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on H^1.

We can rewrite φ on H^1 by

$$\varphi(x) = \frac{1}{2}\left\|x^+\right\|_2^2 - \int_0^T V(x)\,dt, \quad x \in H^1. \tag{3.1}$$

Moreover, for all $y = y^0 + y^+ \in H^1 = H^0 \oplus H^+$,

$$\left\langle \varphi'(x), y\right\rangle_2 = \left\langle x^+, y^+\right\rangle_2 - \int_0^T \left(V'(x), y\right)dt.$$

Now, for $x \in H^1 \setminus H^0$, set

$$H(x) = H^0 \oplus \mathbb{R}x, \qquad \hat{H}(x) = H^0 \oplus \mathbb{R}^+x.$$

Obviously, $H(x)$ is an $(N+1)$-dimensional space. According to the Sobolev embedding theorem, there exist $C_i > 0$ such that

$$\|x\|_{L^i} \le C_i\|x\|_2, \quad i = 2, p, \tag{3.2}$$
$$\|x\|_\infty \le C_\infty\|x\|_2, \tag{3.3}$$

where $\|\cdot\|_{L^i}$ and $\|\cdot\|_\infty$ are the usual norms in $L^i(S_T, \mathbb{R}^N)$ and $C(S_T, \mathbb{R}^N)$, respectively. Define the generalized Nehari manifold

$$\mathcal{N} := \left\{x \in H^1 \backslash H^0 : \left\langle \varphi'(x), x\right\rangle_2 = 0 \text{ and } \left\langle \varphi'(x), y\right\rangle_2 = 0 \text{ for all } y \in H^0\right\}.$$

Obviously, $\mathcal{N}$ contains all nonconstant solutions of φ. Moreover, our assumptions on V imply that solutions of (1.1) are critical points of the functional (3.1).

Lemma 3.1 *Suppose that* $(V1'')$, $(V5)$, $(V8)$, $(V9)$, *and* $(V10)$ *hold. Let* $x \in H^1 \setminus H^0$, $y \in H^0$, *and* $s \ge 0$ *with* $x \ne sx + y$. *Then*

$$\varphi(x) > \varphi(sx + y) - \left\langle \varphi'(x), \frac{s^2 - 1}{2}x + sy\right\rangle_2.$$

Proof Let x, y, and s be as in the statement. Obviously, for such x, y, and s, $x \ne sx + y$ implies $s \ne 1$ or $y \ne 0$. Then we need to show that

$$\varphi(sx + y) - \left\langle \varphi'(x), \frac{s^2 - 1}{2}x + sy\right\rangle_2 - \varphi(x) = \int_0^T g(s, y)\,dt < 0,$$

where

$$g(s,y) := V(x) - V(sx+y) + \left(V'(x), \frac{s^2-1}{2}x + sy\right).$$

We next claim that $g(s,y) < 0$. Obviously, $g(1,0) = 0$, but in this case, $x = sx + y$, which contradicts to the assumption. Using $(V1'')$ and $(V9)$, we have

$$g(0,y) = V(x) - V(y) - \frac{1}{2}\big(V'(x), x\big) < -V(y) \le 0.$$

Using $(V9)$ again, we have

$$\begin{aligned}
g(s,y) &= V(x) - V(sx+y) + \left(V'(x), \frac{s^2-1}{2}x + sy\right) \\
&< \frac{1}{2}\big(V'(x), x\big) - V(sx+y) + \left(V'(x), \frac{s^2-1}{2}x + sy\right) \\
&= -\frac{1}{2}s^2\big(V'(x), x\big) + s\big(V'(x), sx+y\big) + M|sx+y|^2 \\
&\quad - M|sx+y|^2 - V(sx+y) \\
&\le -s^2\left[M|x|^2 - \frac{1}{2}\big(V'(x), x\big) - \big|V'(x)\big|\right] \\
&\quad - |y|^2\big[M - \big|V'(x)\big|\big] - \big[V(sx+y) - M|sx+y|^2\big].
\end{aligned} \tag{3.4}$$

If M is large enough, then the quadratic form (in s and y) above is positive definite, and $V(sx+y) - M|sx+y|^2$ is bounded below. Then we have $g(s,y) \to -\infty$ as $s + |y| \to \infty$. Therefore $g(s,y)$ attains its maximum on the set $\{(s,y) | s \ge 0, y \in \mathbb{R}^N\}$. Suppose that g attains its maximum at some point (s,y) with $s > 0$. Then

$$g'_s(s,y) = \big(V'(x), sx+y\big) - \big(V'(sx+y), x\big) = 0 \tag{3.5}$$

and

$$g'_y(s,y) = sV'(x) - V'(sx+y) = 0. \tag{3.6}$$

By (3.5) and (3.6) we have $(V'(x), y) = 0$ and

$$\big(V'(x), sx+y\big) = \big(V'(sx+y), x\big). \tag{3.7}$$

If $V(x) \ne V(sx+y)$, then by (3.7) and $(V8)$ we have

$$\begin{aligned}
g(s,y) &= V(x) - V(sx+y) + \left(V'(x), \frac{s^2-1}{2}x + sy\right) \\
&< \frac{(V'(x),x)^2 - (V'(x), sx+y)^2}{2(V'(x),x)} + \left(V'(x), \frac{s^2-1}{2}x + sy\right) \\
&= \frac{(V'(x),x)^2 - (V'(x), sx)^2}{2(V'(x),x)} + \left(V'(x), \frac{s^2-1}{2}x\right) = 0.
\end{aligned}$$

If $V(x) = V(sx+y)$, then by $(V8)$, $V(x) - V(sx+y) \le \frac{(V'(x),x)^2-(V'(x),sx+y)^2}{2(V'(x),x)}$, and thus $(V'(x), sx + y)^2 \le (V'(x),x)^2$. Since $(V'(x),y) = 0$, we have $s^2 \le 1$. However, if $V(x) = V(sx+y)$ and $s = 1$, then by (V10) we get $y = 0$, which contradicts the assumption $x \ne sx + y$. Thus $s < 1$.

Hence

$$\begin{aligned} g(s,y) &= V(x) - V(sx+y) + \left(V'(x), \frac{s^2-1}{2}x + sy\right) \\ &= \left(V'(x), \frac{s^2-1}{2}x + sy\right) = \left(V'(x), \frac{s^2-1}{2}x\right) < 0. \end{aligned}$$

Hence the claim holds, and the conclusion follows. □

Lemma 3.2 *Suppose that* $(V1'')$, $(V5)$, $(V8)$, $(V9)$, *and* $(V10)$ *hold. Let* $x \in \mathcal{N}$, $y \in H^0$, *and* $s \ge 0$ *with* $x \ne sx + y$. *Then*

$$\varphi(x) > \varphi(sx+y).$$

Proof Since $x \in \mathcal{N}$, by the definition of $\mathcal{N}$, $\langle \varphi'(x), \frac{s^2-1}{2}x + sy\rangle_2 = 0$. Hence by Lemma 3.1 we have $\varphi(x) > \varphi(sx+y)$. Therefore the maximum point of φ is unique, and the lemma holds. □

Obviously, Lemma 3.2 implies that if $x \in \mathcal{N}$, then x is a unique maximum of $\varphi|_{\hat{H}(x)}$.

Lemma 3.3 *Assume that* $(V2')$ *holds. Then*

(i) *there is a constant* $\rho > 0$ *such that* $\inf_{\mathcal{N}} \varphi \ge \inf_{S_\rho^+} \varphi > 0$, *where* $S_\rho^+ := \{x \in H^+ : \|x\|_2 = \rho\}$;

(ii) *for every* $x \in \mathcal{N}$, $\|x^+\|_2 \ge \sqrt{2c}$. *Moreover,* $\mathcal{N}$ *is closed and bounded away from* H^0.

Proof (i) Arguing similarly as in the proof of Lemma 2.6, we can show this conclusion.

(ii) Denote $c = \inf_{\mathcal{N}} \varphi$. For all $x \in \mathcal{N}$, by (i) we have

$$c \le \frac{1}{2}\|x^+\|_2^2 - \int_0^T V(x)\,dt \le \frac{1}{2}\|x^+\|_2^2.$$

Hence $\|x^+\|_2 \ge \sqrt{2c}$. Clearly, $\varphi|_{H^0} \le 0$. Then $\mathcal{N}$ is bounded away from H^0. We can take a sequence $(y_n) \subset \mathcal{N}$ and prove that its limit $y \in \mathcal{N}$. Then $\mathcal{N}$ is closed. □

Lemma 3.4 *Assume that* $(V5)$ *holds. Then* φ *is coercive on* $\mathcal{N}$, *i.e.,* $\varphi(x) \to \infty$ *as* $\|x\|_2 \to \infty$, $x \in \mathcal{N}$.

Proof If not, then there exists $(x_n) \subset \mathcal{N}$ such that $\varphi(x_n) \le d$ for some $d > 0$ as $\|x_n\|_2 \to \infty$. Let $s_n := \|x_n\|_2$, $v_n := x_n/\|x_n\|_2$. Passing to a subsequence, (v_n) converges weakly to some point v.

(a) Let $v \ne 0$. Arguing similarly as for (2.6), we have

$$0 \leftarrow \frac{\varphi(x_n)}{\|x_n\|_2^2} = \frac{1}{2} - \int_0^T \frac{V(v_n\|x_n\|_2)}{(|v_n|\|x_n\|_2)^2}|v_n|^2\,dt \to -\infty,$$

which is a contradiction.

(b) Let $v = 0$. If $\|v_n^+\|_2 \to 0$, then $\|v_n^0\|_2^2 = 1 - \|v_n^+\|_2^2 \to 1$. Since $\dim H^0 = N < \infty$, we have $v_n \to v$ and $\|v\|_2 = 1$, which contradicts to $v = 0$. Otherwise, if $\|v_n^+\|_2 \nrightarrow 0$, then there exist some $\alpha > 0$ and N such that $\|v_n^+\|_2 \geq \alpha$ for all $n \geq N$. It follows from the dominated convergence theorem that $\int_0^T V(sv_n)\,dt \to 0$ for every $s \in (0,\infty)$. Hence, for any $s > 0$ and $n \geq N$, we have

$$d \geq \varphi(x_n) = \varphi(s_n v_n) \geq \varphi(sv_n) \geq \frac{1}{2}s^2\alpha^2 - \int_0^T V(sv_n)\,dt \to \frac{1}{2}s^2\alpha^2,$$

which is a contradiction when we choose $s > \sqrt{2d}/\alpha$. Hence φ is coercive on $\mathcal{N}$. □

Lemma 3.5 *If $(V5)$ holds and $U \subset H^+ \setminus \{0\}$ is a compact subset, then there exists $R > 0$ such that $\varphi(z) \leq 0$ on $z \in \hat{H}(y) \setminus B_R(0)$ for every $y \in U$.*

Proof Without loss of generality, we may assume that $\|y\|_2 = 1$ for every $y \in U$. Suppose, on the contrary, that there exist $(y_n) \subset U$ and $z_n \in \hat{H}(y_n)$ such that $\varphi(z_n) > 0$ for all n and $\|z_n\|_2 \to \infty$ as $n \to \infty$. Passing to a subsequence, we may assume that $y_n \to y \in H^+$, $\|y\|_2 = 1$. Set $v_n = z_n/\|z_n\|_2 = s_n y_n + v_n^0$. Then $|s_n|^2 = 1 - \|v_n^0\|_2^2 \leq 1$. Passing to a subsequence, $v_n \to s_0 y + v^0 \neq 0$ as $n \to \infty$. Arguing as for (2.6), we have

$$0 \leq \frac{\varphi(z_n)}{\|z_n\|_2^2} \leq \frac{1}{2} - \int_0^T \frac{V(\|z_n\|_2 v_n)}{\|z_n\|_2^2 |v_n|^2} |v_n|^2\,dt \to -\infty,$$

which is a contradiction. This finishes the proof of the lemma. □

Lemma 3.6 *Assume that all assumptions of Theorem 1.4 hold. Then for each $x \in H^1 \setminus H^0$, the set $\mathcal{N} \cap \hat{H}(x)$ consists of precisely one point $\hat{n}(x)$, which is the unique global maximum of $\varphi|_{\hat{H}(x)}$.*

Proof Let x be given in this lemma. According to Lemmas 3.3 and 3.5, $\varphi|_{\hat{H}(x)}$ attains its maximum on $B_R(0)$ for some R large enough. Since $\hat{H}(x)$ is a closed subset of a finite-dimensional space, there exist r_x, v_x, and $\hat{n}(x) := r_x x^+ + v_x$ such that

$$\varphi|_{\hat{H}(x)}\bigl(\hat{n}(x)\bigr) = \max_{z \in \hat{H}(x)} \varphi(z).$$

Then $\langle \varphi'(\hat{n}(x)), z\rangle_2 = 0$ for $z \in \hat{H}(x)$. Thus $\langle \varphi'(\hat{n}(x)), \hat{n}(x)\rangle_2 = \langle \varphi'(\hat{n}(x)), z\rangle_2 = 0$ for $z \in H^0$, i.e., $\hat{n}(x) \in \mathcal{N}$. Lemma 3.2 yields that the maximum point of $\varphi_{\hat{H}(x)}$ is unique. □

Arguing similarly as in [29], we can prove the following two lemmas.

Lemma 3.7 *Assume that all assumptions of Theorem 1.4 hold. Then the map $H^+ \setminus \{0\} \to \mathcal{N}$, $x \mapsto \hat{n}(x)$, is continuous.*

Define the following maps:

$$\hat{\Phi} : H^+ \setminus \{0\} \to \mathbb{R}, \qquad \hat{\Phi}(x) := \varphi\bigl(\hat{n}(x)\bigr),$$

$$\Phi : S^+ \to \mathbb{R}, \qquad \Phi = \hat{\Phi}|_{S^+},$$

$$n : \mathcal{N} \to S^+, \qquad n(x) = \frac{x^+}{\|x^+\|_2},$$

where $S^+ := \{x \in H^+ : \|x\|_2 = 1\}$ in H^+. Then $\hat{\Phi}$ is continuous by Lemma 3.7. Moreover, n is a homeomorphism between S^+ and $\mathcal{N}$.

Lemma 3.8 *Assume that all assumptions of Theorem 1.4 hold. Then*

(a) $\Phi \in C^1(S^+, \mathbb{R})$, *and*

$$\langle \Phi'(w), z\rangle_2 = \|\hat{n}(w)^+\|_2 \langle \varphi'(\hat{n}(w)), z\rangle_2, \quad z \in T_w S^+ := \{v \in H^+ : \langle w, v\rangle_2 = 0\};$$

(b) $(w_n)_n$ *is a PS-sequence for* Φ *if and only if* $(\hat{n}(x_n))_n$ *is a PS-sequence for* φ;

(c) *We have*

$$\inf_{S^+} \Phi = \inf_{\mathcal{N}} \varphi = c.$$

Moreover, $x \in S^+$ *is a critical point of* Φ *if and only if* $\hat{n}(x) \in \mathcal{N}$ *is a critical point of* φ, *and the corresponding critical values coincide.*

Lemma 3.9 *Assume that all assumptions of Theorem 1.4 hold. Then* φ *satisfies PS-condition on* $\mathcal{N}$, *and so does* Φ.

Proof Let $(x_n) \subset \mathcal{N}$ be a PS-sequence of φ. Then $(\varphi(x_n))$ is bounded. By the coercivity of φ, (x_n) is bounded. Set $x_n = x_n^0 + x_n^+$ for all n, where $x_n^0 \in H^0$ and $x_n^+ \in H^+$. So both (x_n^0) and (x_n^+) are bounded. Since $\phi' : H^1 \to (H^1)^*$ is compact and

$$\varphi'(x_n) = x_n^+ - \phi'(x_n) \to 0,$$

(x_n^+) has a convergent subsequence. As $\dim H^0 = N$, (x_n^0) has a convergent subsequence. Hence (x_n) has a convergent subsequence. Thus φ satisfies *PS*-condition. Following the same way as in the proof of Lemma 2.12, we can attain that Φ also satisfies PS-condition. □

Proof of Theorem 1.4 We know that

$$c = \inf_{\mathcal{N}} \varphi = \inf_{S^+} \Phi > 0.$$

Let (w_n) be a minimizing sequence for Φ on S^+. Then $\Phi(w_n) \to c$ as $n \to \infty$. By Ekeland's variational principle we have $\Phi'(w_n) \to 0$ as $n \to \infty$. Put $x_n := \hat{n}(w_n) \in \mathcal{N}$. Then $\varphi(x_n) \to c$ and $\varphi'(x_n) \to 0$ as $n \to \infty$. Hence (x_n) is a PS-sequence of φ. Since φ satisfies PS-condition, (x_n) contains a converging subsequence; denote its limit by x. Since $\mathcal{N}$ is closed, $x \in \mathcal{N}$, and x is a critical point of φ. Clearly, $\varphi(x) \geq c$. Arguing similarly as for (2.10), we can show that $\varphi(x) \leq c$. Consequently, $\varphi(x) = c$, and x is a nonconstant T-period for system (1.1).

Next, we claim that x has T as its minimal period. Suppose, on the contrary, that x has a minimal period T/k, where $k \geq 2$ is an integer. We write $x = x^+ + x^0 \in H^+ \oplus H^0$. By Lemma 3.3, $\|x^+\|_2 \geq \sqrt{2c} > 0$, i.e., $x^+ \neq 0$. Denote $y(t) = x(t/k)$. Obviously, $y^+(t) = x^+(t/k)$, and $y^0 = x^0$. Subsequently, $y \in H^1 \setminus H^0$. By Lemma 3.6 there exists $z \in \hat{H}(y)$ such that $z :=$

$\hat{n}(y) \subset \mathcal{N}$. Denote $z(t) = z^+(t) + z^0$. Then $z^+(t) = sx^+(t/k)$, and $z^0 \in H^0$, where $s \in \mathbb{R}^+$. Hence

$$
\begin{aligned}
\varphi(x) &= \inf_{y\in\mathcal{N}} \varphi(y) \\
&\leq \varphi(z) \\
&= \int_0^T \left[\frac{1}{2} \left|\dot{z}(t)\right|^2 - V\bigl(z(t)\bigr) \right] dt \\
&= \int_0^T \left[s^2 \frac{1}{2k^2} \left| \dot{x}^+ \left(\frac{t}{k} \right) \right|^2 - V\left(sx^+ \left(\frac{t}{k} \right) + z^0 \right) \right] dt \\
&= \int_0^T \left[s^2 \frac{1}{2k^2} \left| \dot{x}^+(\tau) \right|^2 - V\bigl(sx^+(\tau) + z^0 \bigr) \right] d\tau \\
&< \int_0^T \left[\frac{s^2}{2} \left| \dot{x}^+(\tau) \right|^2 - V\bigl(sx^+(\tau) + z^0 \bigr) \right] d\tau \\
&= \varphi\bigl(sx^+ + z^0 \bigr) \leq \varphi(x) = \inf_{y\in\mathcal{N}} \varphi(y),
\end{aligned}
$$

which is a contradiction. Hence x is a nonconstant T-periodic solution with minimal period T. □

Funding
This paper is supported by National Natural Science Foundation of China (No. 11871171).

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
J. Xiao and X. Chen wrote the main manuscript text. Both authors reviewed the manuscript.

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Received: 16 November 2023 Accepted: 18 December 2023 Published online: 02 January 2024

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