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Continuity and pullback attractors for a semilinear heat equation on time-varying domains

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Abstract

We consider dynamics of a semilinear heat equation on time-varying domains with lower regular forcing term. Instead of requiring the forcing term $f(\cdot)$ to satisfy $\int_{-\infty}^{t} e^{\lambda s}\|f(s)\|_{L^2}^2\,ds < \infty$ for all $t \in \mathbb{R}$, we show that the solutions of a semilinear heat equation on time-varying domains are continuous with respect to initial data in H^1 topology and the usual (L^2, L^2) pullback $\mathscr{D}_\lambda$-attractor indeed can attract in the H^1-norm, provided that $\int_{-\infty}^{t} e^{\lambda s}\|f(s)\|_{H^{-1}(\mathcal{O}_s)}^2\,ds < \infty$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathcal{O}_s))$.

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1 Introduction

Let $\mathcal{O}$ be a nonempty bounded open subset of $\mathbb{R}^N$ with C^2 boundary $\partial\mathcal{O}$, and let $r = r(y,t)$ be a vector function

$$r \in C^1\left(\overline{\mathcal{O}} \times \mathbb{R}; \mathbb{R}^N\right) \tag{1.1}$$

such that

$$r(\cdot,t) : \mathcal{O} \to \mathcal{O}_t \quad \text{is a } C^2\text{-diffeomorphism for all } t \in \mathbb{R}. \tag{1.2}$$

We consider the following initial boundary value problem for a semilinear parabolic equation:

$$\begin{cases} u_t - \Delta u + g(u) = f(t) & \text{in } Q_\tau, \\ u = 0 & \text{on } \Sigma_\tau, \\ u(\tau,x) = u_\tau(x), & x \in \mathcal{O}_\tau, \end{cases} \tag{1.3}$$

where $\tau \in \mathbb{R}$, $u_\tau : \mathcal{O}_\tau \to \mathbb{R}$, $Q_{\tau,T} := \bigcup_{t\in(\tau,T)} \mathcal{O}_t \times \{t\}$ for all $T > \tau$, $Q_\tau := \bigcup_{t\in(\tau,+\infty)} \mathcal{O}_t \times \{t\}$, $\Sigma_{\tau,T} := \bigcup_{t\in(\tau,T)} \partial\mathcal{O}_t \times \{t\}$, $\Sigma_\tau := \bigcup_{t\in(\tau,+\infty)} \partial\mathcal{O}_t \times \{t\}$, and $f : Q_\tau \to \mathbb{R}$ are given. We assume that $g \in C^1(\mathbb{R},\mathbb{R})$ is a given function for which there exist nonnegative constants α_1, α_2, β, l, and $p \geq 2$ such that

$$-\beta + \alpha_1 |s|^p \leq g(s)s \leq \beta + \alpha_2 |s|^p, \qquad g'(s) \geq -l \quad \forall s \in \mathbb{R}, \tag{1.4}$$

and, moreover, g satisfies the Lipschitz condition: there exists a positive constant c_0 such that

$$\bigl|g(u) - g(v)\bigr| \leq c_0\bigl(1 + |u|^{p-2} + |v|^{p-2}\bigr) \cdot |u - v| \quad \forall u, v \in \mathbb{R}. \tag{1.5}$$

About the diffeomorphism $r(\cdot,\cdot)$, as in Límacoet al. [6] and Kloedenet al. [5], we assume that the function $\bar{r} = \bar{r}(x,t)$, where $\bar{r}(\cdot,t) = r^{-1}(\cdot,t)$ denotes the inverse of $r(\cdot,t)$, satisfies

$$\bar{r} \in C^{2,1}\bigl(\bar{Q}_{\tau,T};\mathbb{R}^N\bigr) \quad \text{for all } \tau < T. \tag{1.6}$$

The reaction–diffusion equation with nonlinear term $g(\cdot)$ satisfying assumptions (1.4) is one of the classical example models in the theory of infinite-dimensional dynamical systems, especially regarding to the theory of attractors; e.g., see the classical monographs in this field like [1, 8, 11].

About the dynamics of reaction–diffusion equation (1.3)–(1.4), the known results mainly concentrate in the L^2 phase space; e.g., see [8, 11] for the fixed domain case (i.e., $r(\cdot,t) \equiv Id$) and Kloeden et al. [5, 13] for time-varying domain case; and the corresponding mathematical analysis is standard to some extent. When we try to improve the corresponding results to a more regular phase space, say H^1, some essential difficulties arise, for example, the continuity with respect to the initial data and asymptotical compactness in H^1 topology. Indeed, even in the autonomous case, for any space dimension N and any growth power $p \geq 2$ (comes from (1.4)), the question about the continuity of solution with respect to initial data in H^1 remained open until 2008; see Robinson [8]. In 2008, for the autonomous case of (1.3) and with the same assumption (1.4) about the nonlinearity, Trujillo and Wang [12] used the method of differentiating the equation with respect to t to get the bounded estimate for $\|tu_t\|_{L^2}$ for $t \in [0,T]$ and then obtained the uniform boundedness of $tu(t)$ in $L^\infty(0,T;H^2)$ and, finally, obtained the continuity in H^1 for any space dimension N and any growth power $p \geq 2$ (to our knowledge, this is the first result). Later, Cao et al. [2] obtained such continuity for nonautonomous case by establishing some new a priori estimates for the difference of solutions near the initial time; see also [3, 13] for further discussion in this direction.

Note that to obtain the continuity with respect to the initial data and existence of attractors in the H^1 topology, to our knowledge, the known results always required the force term to belong to L^2; e.g., see [2, 3, 12] for autonomous and stochastic case; and in [13], to obtain similar results as in [2] in the nonautonomous case, they required $f(\cdot)$ to satisfy

$$\int_{-\infty}^{t} e^{\lambda s}\bigl\|f(s)\bigr\|_{L^2}^2\,ds < \infty \quad \text{for all } t \in \mathbb{R} \tag{1.7}$$

for some proper positive constant λ. On the other hand, it is well-known that when we consider system (1.3)–(1.4) in H^1, it is natural to require $f(\cdot) \in H^{-1}$ only.

The main aim of this paper is to establish the same continuity with respect to the initial data in the H^1 topology and H^1-attraction as that in [2, 3, 12, 13] and relax the assumption on the forcing term. To include the nonautonomous case, we consider systems (1.3)–(1.4) defined on a time-varying domain. Note that a semilinear heat equation on a time-varying domain is intrinsically nonautonomous even if the terms in the equation do not depend explicitly on time.

Assumption I r and $\bar{r}$ satisfy assumptions (1.1), (1.2), and (1.6); $\partial\mathcal{O}$ is C^2 and $N \le 2p/(p-2)$, or $\partial\mathcal{O}$ is C^j with $j \ge 2$ integer such that $j \ge N(p-2)/2p$; $g(\cdot)$ satisfies (1.4), and $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H^{-1}(\mathcal{O}_t))$.

Under Assumption I, the existence and uniqueness of strong solution and weak solution of (1.3) (see [5, 6] for the corresponding definition of solutions) were obtained by Kloeden et al. [5] and then defined the nonautonomous process $U(t,\tau): L^2(\mathcal{O}_\tau) \to L^2(\mathcal{O}_t)$, $-\infty < \tau \le t < \infty$ by $U(t,\tau)u_\tau := u(t;\tau,u_\tau) = u(t)$. Moreover, if we assume further that r satisfies

$$r \in C_b\big(\bar{\mathcal{O}} \times \mathbb{R}; \mathbb{R}^N\big) \tag{1.8}$$

and f satisfies

$$\int_{-\infty}^{t} e^{\lambda s} \big\|f(s)\big\|^2_{H^{-1}(\mathcal{O}_s)}\, ds < \infty \quad \text{for all } t \in \mathbb{R}, \tag{1.9}$$

where $\lambda := \min_{v \in H_0^1(\Omega), v \neq 0} \frac{\|\nabla v\|^2_{(L^2(\Omega))^N}}{\|v\|^2_{L^2(\Omega)}}$ is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ with $\Omega := \bigcup_{t\in\mathbb{R}} \mathcal{O}_t$, then the process $U(t,\tau)$ has an (L^2, L^2) pullback attractor $\hat{\mathscr{A}} = \{\mathscr{A}(t) : t \in \mathbb{R}\}$; see [5] for more detail.

Our main result is the following theorem.

Theorem 1.1 *Let Assumption* I, (1.5), *and* (1.8)–(1.9) *hold. If the forcing term* $f \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathcal{O}_t))$, *then the process* $U(t,\tau)$ *is continuous with respect to the initial data in the* H^1 *topology; more precisely, for all* $\tau \in \mathbb{R}$ *and* $t > \tau$, *if* $u_{n\tau} \in L^2(\mathcal{O}_\tau)$ *satisfy* $u_{n\tau} \to u_{0\tau}$ *in* $L^2(\mathcal{O}_\tau)$ *as* $n \to \infty$, *then*

$$U(t,\tau)u_{n\tau} \to U(t,\tau)u_{0\tau} \quad \textit{in } H_0^1(\mathcal{O}_t) \textit{ as } n \to \infty. \tag{1.10}$$

Moreover, the (L^2, L^2) *pullback attractor* $\hat{\mathscr{A}} = \{\mathscr{A}(t) : t \in \mathbb{R}\}$ *obtained in* [5] *can pullback attract in the topology of* H^1, *i.e., for all* $t \in \mathbb{R}$ *and* $\hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathscr{D}$,

$$\mathrm{dist}_{H_0^1(\mathcal{O}_t)}\big(U(t,\tau)D(\tau), \mathscr{A}(t)\big) \to 0 \quad \textit{as } \tau \to -\infty. \tag{1.11}$$

As mentioned previously, after the work [2], although (1.3) is defined on a time-varying domain, the continuity in (1.10) and attraction (1.11) is more or less expectable, in this paper, we give rigorous proofs about how to justify the approximation that is necessary due to relaxing the assumption on the forcing term. Note also that here we only additionally assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathcal{O}_t))$, but not (1.7), which was required in [3, 7, 9, 13] etc. for obtaining the boundedness in L^p and H^1. However, in the nonautonomous case, the question whether we can remove further the additional condition $f \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathcal{O}_t))$ remains open.

2 Preliminaries

2.1 Functional spaces

We first recall some functional spaces and notations.

For a fixed finite time interval $[\tau, T]$, let $(X_t, \|\cdot\|_{X_t})$ $(t \in [\tau, T])$ be a family of Banach spaces such that $X_t \subset L^1_{\mathrm{loc}}(\mathcal{O}_t)$ for all $t \in [\tau, T]$. For any $1 \le q \le \infty$, we denote by $L^q(\tau, T; X_t)$ the vector space of all functions $u \in L^1_{\mathrm{loc}}(Q_{\tau,T})$ such that $u(t) = u(\cdot, t) \in X_t$ for a.e. $t \in (\tau, T)$ and the function $\|u(\cdot)\|_{X_\cdot}$ defined by $t \mapsto \|u(t)\|_{X_t}$ belongs to $L^q(\tau, T)$.

On $L^q(\tau, T; X_t)$, we consider the norm given by

$$\|u\|_{L^q(\tau,T;X_t)} := \big\| \big\|u(\cdot)\big\|_{X_\cdot} \big\|_{L^q(\tau,T)}.$$

2.2 Definitions of solutions

For the readers' convenience, in this subsection, we recall the definition of different solutions of equation (1.3); see Límaco et al. [6] and Kloeden et al. [5] for more detail.

For each $T > \tau$, consider the auxiliary problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + g(u) = f(t) \quad \text{in } Q_{\tau,T}, \\ u = 0 \quad \text{on } \Sigma_{\tau,T}, \\ u(\tau, x) = u_\tau(x), \quad x \in \mathcal{O}_\tau, \end{cases} \tag{2.1}$$

where $\tau \in \mathbb{R}$ and $u_\tau : \mathcal{O}_\tau \to \mathbb{R}$.

Definition 2.1 (Strong solution) A function $u = u(x, t)$ defined in $Q_{\tau,T}$ is said to be a strong solution of problem (2.1) if

$$u \in L^2\big(\tau, T; H^2(\mathcal{O}_t)\big) \cap C\big([\tau, T]; H^1_0(\mathcal{O}_t)\big) \cap L^\infty\big(\tau, T; L^q(\mathcal{O}_t)\big), \quad u' \in L^2\big(\tau, T; L^2(\mathcal{O}_t)\big),$$

and the three equations in (2.1) are satisfied almost everywhere in their corresponding domains.

Denote

$$\mathcal{U}_{\tau,T} := \big\{\varphi \in L^2\big(\tau, T; H^1_0(\mathcal{O}_t)\big) \cap L^q\big(\tau, T; L^q(\mathcal{O}_t)\big) : \varphi' \in L^2\big(\tau, T; L^2(\mathcal{O}_t)\big), \\ \varphi(\tau) = \varphi(T) = 0\big\}.$$

Definition 2.2 Let $u_\tau \in L^2(\mathcal{O}_\tau), f \in L^2(\tau, T; H^{-1}(\mathcal{O}_t))$, and $-\infty < \tau \le T < \infty$. We say that a function u is a weak solution of (2.1) if

(1) $u \in C([\tau, T]; L^2(\mathcal{O}_t)) \cap L^2(\tau, T; H^1_0(\mathcal{O}_t)) \cap L^q(\tau, T; L^q(\mathcal{O}_t))$ with $u(\tau) = u_\tau$;

(2) there exists a sequence of regular data $u_{\tau m} \in H^1_0(\mathcal{O}_\tau) \cap L^q(\mathcal{O}_\tau)$ and $f_m \in L^2(\tau, T; L^2(\mathcal{O}_t))$, $m = 1, 2, \dots$, such that

$$u_{\tau m} \to u_\tau \quad \text{in } L^2(\mathcal{O}_\tau), \qquad f_m \to f \quad \text{in } L^2\big(\tau, T; H^{-1}(\mathcal{O}_t)\big),$$

and

$$u_m \to u \quad \text{in } C\big([\tau, T]; L^2(\mathcal{O}_t)\big),$$

where u_m is the unique strong solution of (2.1) corresponding to $(u_{\tau m},f_m)$;

(3) for all $\varphi \in \mathcal{U}_{\tau,T}$,

$$\int_\tau^T \int_{\mathcal{O}_t} u(x,t)\varphi'(x,t)\,dx\,dt + \int_\tau^T \int_{\mathcal{O}_t} \nabla_x u \cdot \nabla_x \varphi\,dx\,dt$$
$$= -\int_\tau^T \int_{\mathcal{O}_t} g\big(u(x,t)\big)\varphi(x,t)\,dx\,dt + \int_\tau^T \int_{\mathcal{O}_t} f(x,t)\varphi(x,t)\,dx\,dt.$$

Definition 2.3 (Weak solution) A function $u : \bigcup_{t\in[\tau,\infty)} \mathcal{O}_t \times \{t\} \to \mathbb{R}$ is called a weak solution of (1.3) if for any $T > \tau$, the restriction of u on $\bigcup_{t\in[\tau,T]} \mathcal{O}_t \times \{t\}$ is a weak solution of (2.1).

2.3 Preliminary lemmas

For later application, in the following, we collect some results for obtaining higher-order integrability, which can be proved by the standard methods; see [5, 10] for the detailed proofs.

Lemma 2.4 *If $u \in L^2(\tau,T;H_0^1(\mathcal{O}_t)) \cap L^\infty(Q_{\tau,t})$ and $u' \in L^2(\tau,T;L^2(\mathcal{O}_t))$, then for any $k \in [0,\infty)$,*

$$|u|^k \cdot u \in L^2\big(\tau,T;H_0^1(\mathcal{O}_t)\big) \cap L^\infty(Q_{\tau,t}), \tag{2.2}$$

and the following energy equality is satisfied:

$$\big\|u(t_2)\big\|_{L^{k+2}(\mathcal{O}_{t_2})}^{k+2} - \big\|u(t_1)\big\|_{L^{k+2}(\mathcal{O}_{t_1})}^{k+2}$$
$$= (k+2)\int_{t_1}^{t_2} \big(u'(t), \big|u(t)\big|^k \cdot u(t)\big)_t\,dt \quad \forall \tau \le t_1 \le t_2 \le T. \tag{2.3}$$

Lemma 2.5 *For any $k > 0$ and any $\phi \in H_0^1(\mathcal{O}_s) \cap L^\infty(\mathcal{O}_s)$ for some $s \in \mathbb{R}$, we the following equality*:

$$\int_{\mathcal{O}_s} \nabla\phi \cdot \nabla\big(|\phi|^k\phi\big)\,dx = (k+1)\left(\frac{2}{k+2}\right)^2 \int_{\mathcal{O}_s} \big|\nabla|\phi|^{\frac{k+2}{2}}\big|^2\,dx, \tag{2.4}$$

where $\cdot$ stands for the usual inner product in $\mathbb{R}^N$.

Lemma 2.6 *Let $f \in L^2_{\mathrm{loc}}(\mathbb{R};L^2(\mathcal{O}_s))$ satisfy (1.9). Then, for each $T \in \mathbb{R}$, there is a family $\{f_m\} \subset L^\infty_{\mathrm{loc}}(Q_{-\infty,T})$ such that*

$$\text{for any (fixed)} \quad \tau \in (-\infty,T), \qquad f_m \to f \quad \text{in } L^2\big(\tau,T;L^2(\mathcal{O}_s)\big) \tag{2.5}$$

and for any $t \in (-\infty,T)$,

$$\int_{-\infty}^t e^{\lambda s}\big\|f_m(s)\big\|_{L^2(\mathcal{O}_s)}^2\,ds \le 2\int_{-\infty}^t e^{\lambda s}\big\|f(s)\big\|_{L^2(\mathcal{O}_s)}^2\,ds + \frac{1}{4} \quad \text{for all } m = 1,2,\ldots. \tag{2.6}$$

Recall that $Q_{-\infty,T} = \bigcup_{t\in(-\infty,T)} \mathcal{O}_t \times \{t\}$ and the family $\{f_m\}$ may depend on T.

In order the test function $|u|^k \cdot u$ to make sense, we also recall the following L^∞-estimate on the nice initial data, which can be obtained by applying the standard *Stampacchia's truncation method*; see [10] for a detailed proof.

Lemma 2.7 (L^∞-estimate) *Let Assumption* I *be satisfied. Then for any* $-\infty < \tau \le T < \infty$ *and any initial data* $(u_\tau, f) \in (H_0^1(\mathcal{O}_\tau) \cap L^\infty(\mathcal{O}_\tau), L^\infty(Q_{\tau,T}))$, *the unique strong solution* u *of* (2.1) *belongs to* $L^\infty(Q_{\tau,T})$.

3 Higher-order integrability

Along the ideas in [2], as the preliminaries, in this section, we obtain some higher-order integrability of the difference of two weak solutions near the initial time, which was firstly established in [2] for the (autonomous and fixed domain) stochastic case of (1.3), and later, similar results were obtained in [13] for (1.3)–(1.4) in the stochastic case (in time-varying case, but the forcing term was required to satisfy (1.7)).

3.1 A priori estimates for approximation solutions

To make our proof rigorous, we will use the approximation techniques.

For any (fixed) $T \in \mathbb{R}$, throughout this section, we choose (we can do this by Lemma 2.6) and fix a family $\{f_m\} \subset L^\infty_{\mathrm{loc}}(Q_{-\infty,T})$ such that

$$\text{the family } \{f_m\} \text{ satisfying conditions (2.5)–(2.6) in Lemma 2.6.} \tag{3.1}$$

Then, for any $\tau < T$ and any $u_\tau, v_\tau \in L^2(\mathcal{O}_\tau)$, according to the definition of a weak solution, we know that there are two sequences $\{(u_{\tau m}, f_m)\}$ and $\{(v_{\tau m}, f_m)\}$ satisfying

$$u_{\tau m}, v_{\tau m} \in H_0^1(\mathcal{O}_\tau) \cap L^\infty(\mathcal{O}_\tau) \quad \text{and} \quad f_m \in L^\infty(Q_{\tau,T}) \tag{3.2}$$

such that

$$\begin{aligned} &u_{\tau m} \to u_\tau, \qquad v_{\tau m} \to v_\tau \quad \text{in } L^2(\mathcal{O}_\tau) \quad \text{and} \\ &f_m \to f \quad \text{in } L^2\big(\tau, T; L^2(\mathcal{O}_t)\big) \text{ as } m \to \infty \end{aligned} \tag{3.3}$$

and

$$u_m \to u \quad \text{and} \quad v_m \to v \quad \text{in } C^0\big([\tau, T]; L^2(\mathcal{O}_t)\big), \tag{3.4}$$

where u_m and v_m are the unique strong solution of (1.3) corresponding to the regular data $(u_{\tau m}, f_m)$ and $(v_{\tau m}, f_m)$, respectively.

Without loss of generality, by (3.3) we can require that

$$\|u_{\tau m}\|_\tau^2 \le \|u_\tau\|_\tau^2 + 1 \quad \text{and} \quad \|v_{\tau m}\|_\tau^2 \le \|v_\tau\|_\tau^2 + 1 \quad \text{for all } m = 1, 2, \ldots, \tag{3.5}$$

where and hereafter, $\|\cdot\|_s$ denotes the usual norm of $L^2(\mathcal{O}_s)$ ($s \in \mathbb{R}$).

Denote

$$w_m(t) = u_m(t) - v_m(t) \quad \text{for any } \tau \le t \le T. \tag{3.6}$$

Then $w_m(t)$ $(m = 1, 2, \dots)$ is the unique strong solution of the following equation:

$$\begin{cases} \frac{\partial w_m}{\partial t} - \Delta w_m + g(u_m) - g(v_m) = 0 \quad \text{in } Q_{\tau,T}, \\ w_m = 0 \quad \text{on } \Sigma_{\tau,T}, \\ w_m(\tau, x) = u_{\tau m} - v_{\tau m}, \quad x \in \mathcal{O}_\tau, \end{cases} \tag{3.7}$$

that is, $w_m \in L^2(\tau, T; H^2(\mathcal{O}_t)) \cap C([\tau, T]; H_0^1(\mathcal{O}_t)) \cap L^\infty(\tau, T; L^q(\mathcal{O}_t))$, $w_m' \in L^2(\tau, T; L^2(\mathcal{O}_t))$, and the three equations in (3.7) are satisfied almost everywhere in their corresponding domains.

The main purpose of this subsection is to prove the following uniform (with respect to m) a priori estimates of w_m defined in (3.6).

Theorem 3.1 *Let Assumption I hold. Then, for any $\tau \le T$ and any $k = 1, 2, \dots$, there exists a positive constant $M_k = M(T - \tau, k, N, l, \|u_\tau\|_\tau, \|v_\tau\|_\tau)$, such that for all $m = 1, 2, \dots$,*

$$(t-\tau)^{\frac{N}{N-2}} \left\| (t-\tau)^{b_k} w_m(t) \right\|_{L^{2(\frac{N}{N-2})^k}(\mathcal{O}_t)}^{2(\frac{N}{N-2})^k} \le M_k \quad \text{for all } t \in [\tau, T] \tag{A_k}$$

and

$$\int_\tau^T \left(\int_{\mathcal{O}_t} \left| (t-\tau)^{b_{k+1}} \cdot w_m(t) \right|^{2(\frac{N}{N-2})^{k+1}} dx \right)^{\frac{N-2}{N}} dt \le M_k, \tag{B_k}$$

where $w_m(t) = u_m(t) - v_m(t) = U(t,\tau)u_{\tau m} - U(t,\tau)v_{\tau m}$,

$$b_1 = 1 + \frac{1}{2}, \qquad b_2 = 1 + \frac{1}{2} + 1, \quad \text{and} \quad b_{k+1} = b_k + \frac{1 + \frac{N}{N-2}}{2(\frac{N}{N-2})^{k+1}} \quad \text{for } k = 2, 3, \dots, \tag{3.8}$$

and all constants M_k $(k = 1, 2, \dots)$ are independent of m.

Proof By Lemma 2.7 we know that $u_m, v_m \in L^\infty(Q_{\tau,T})$ for each $m = 1, 2, \dots$, and so

$$w_m = u_m - v_m \in C\big([\tau, T]; H_0^1(\mathcal{O}_t)\big) \cap L^\infty(Q_{\tau,T}),$$

and for any $0 \le \theta < \infty$,

$$|w_m|^\theta \cdot w_m \in L^2\big(\tau, T; H_0^1(\mathcal{O}_t)\big) \cap L^\infty(Q_{\tau,T}).$$

Consequently, we can multiply (3.7) by $|w_m|^\theta \cdot w_m$ for all $\theta \in [0, \infty)$.

In the following, we will separate our proof into two steps.

Step 1 $k = 1$

At first, multiplying (3.7) by w_m, from the definition of a strong solution and (1.4), applying Lemmas 2.4 and 2.5, we obtain that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w_m\|_t^2 + \int_{\mathcal{O}_t} \left|\nabla w_m(t)\right|^2 dx &= -\int_{\mathcal{O}_t} \big(g(u_m) - g(v_m)\big) w_m \, dx \\ &\le l \left\| w_m(t) \right\|_t^2 \quad \text{a.e. } t \in (\tau, T) \end{aligned} \tag{3.9}$$

(recall that $\|\cdot\|_s$ denotes the $L^2(\mathcal{O}_s)$-norm), which implies that

$$\|w_m(t)\|_t^2 \le e^{2l(t-\tau)}\|w_m(\tau)\|_\tau^2, \tag{3.10}$$

and then

$$\begin{aligned}\int_\tau^T \|\nabla w_m(t)\|_t^2\,dt &\le l\int_\tau^T \|w_m(s)\|_s^2\,ds + \frac{1}{2}\|w_m(\tau)\|_\tau^2 \\ &\le \frac{1}{2}\big(e^{2l(T-\tau)}+1\big)\|w_m(\tau)\|_\tau^2.\end{aligned} \tag{3.11}$$

Consequently, combining with the embedding

$$\left(\int_{\mathcal{O}_s} |v|^{\frac{2N}{N-2}}\,dx\right)^{\frac{N-2}{N}} \le c_{N,\tau,T}\int_{\mathcal{O}_s}|\nabla v|^2\,dx, \quad \forall v\in H^1(\mathcal{O}_s)\ \forall s\in[\tau,T], \tag{3.12}$$

we can deduce that

$$\begin{aligned}&\int_\tau^T \left(\int_{\mathcal{O}_t}\big|(t-\tau)^{b_1}w_m(t)\big|^{\frac{2N}{N-2}}\,dx\right)^{\frac{N-2}{N}}dt \\ &\quad\le (T-\tau)^{2b_1}\frac{c_{N,\tau,T}}{2}\big(e^{2l(T-\tau)}+1\big)\|w_m(\tau)\|_\tau^2.\end{aligned} \tag{3.13}$$

Note that here the embedding constant $c_{N,\tau,T}$ in (3.12) depends only on the domain $\bigcup_{s\in[\tau,T]}\mathcal{O}_s$.

Secondly, multiplying (3.7) by $|w_m|^{\frac{2N}{N-2}-2}\cdot w_m$, and similarly to (3.9), we have that

$$\begin{aligned}&\frac{1}{2}\left(\frac{N-2}{N}\right)\frac{d}{dt}\|w_m(t)\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)} + \frac{\frac{2N}{N-2}-1}{(\frac{N}{N-2})^2}\int_{\mathcal{O}_t}\big|\nabla|w_m(t)|^{(\frac{N}{N-2})}\big|^2\,dx \\ &\quad\le l\|w_m(t)\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)} \qquad \text{for a.e. } t\in(\tau,T).\end{aligned}$$

To simplify the calculations, we denote by c, c_i ($i=1,2,\dots$) the constants that depend only on N, $T-\tau$, k, and l and may vary from line to line. Then the above inequality can be written as

$$\frac{d}{dt}\|w_m(t)\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)} + c_1\int_{\mathcal{O}_t}\big|\nabla|w_m(t)|^{\frac{N}{N-2}}\big|^2\,dx \le c_2\|w_m(t)\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)}, \tag{3.14}$$

and by multiplying both sides with $(t-\tau)^{\frac{3N}{N-2}}$ we obtain that

$$\begin{aligned}&\frac{d}{dt}\|(t-\tau)^{b_1}w_t(t)\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)} + c_1\int_{\mathcal{O}_t}\big|\nabla|(t-\tau)^{b_1}w_m(t)|^{\frac{N}{N-2}}\big|^2\,dx \\ &\quad\le c_2\|(t-\tau)^{b_1}w_m(t)\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)} + c_3(t-\tau)^{\frac{3N}{N-2}-1}\|w_m(t)\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)} \\ &\quad\le c\left(1+\frac{1}{t-\tau}\right)\|(t-\tau)^{b_1}w_m(t)\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)}.\end{aligned} \tag{3.15}$$

Recall that $b_1 = 1+\frac{1}{2}$ was defined in (3.2).

One direct result of (3.15) is that

$$(t-\tau)\frac{d}{dt}\left\|(t-\tau)^{b_1}w_m(t)\right\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)} \le c\left\|(t-\tau)^{b_1}w_m(t)\right\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)},$$

and so

$$(t-\tau)\frac{d}{dt}\left\|(t-\tau)^{b_1}w_m(t)\right\|^{2}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)} \le c\frac{N-2}{N}\left\|(t-\tau)^{b_1}w_m(t)\right\|^{2}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)}. \tag{3.16}$$

Consequently, for any $t\in[\tau,T]$, integrating (3.16) over $[\tau,t]$, we obtain that

$$\begin{aligned}(t-\tau)\left\|(t-\tau)^{b_1}w_m(t)\right\|^{2}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)} &\le \left(c\frac{N-2}{N}+1\right)\int_\tau^T\left\|(s-\tau)^{b_1}w_m(s)\right\|^{2}_{L^{\frac{2N}{N-2}}(\mathcal{O}_s)}\,ds\\ &\le c\left\|w_m(\tau)\right\|^2_\tau \quad (\text{by } (3.13)),\end{aligned}$$

and hence

$$(t-\tau)^{\frac{N}{N-2}}\left\|(t-\tau)^{b_1}w_m(t)\right\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)} \le c\left\|w_m(\tau)\right\|^{\frac{2N}{N-2}}_\tau \quad \text{for all } t\in[\tau,T]. \tag{3.17}$$

Then, multiplying (3.15) by $(t-\tau)^{\frac{2N}{N-2}}$, we obtain that for a.e. $t\in(\tau,T)$,

$$\begin{aligned}&(t-\tau)^{\frac{2N}{N-2}}\frac{d}{dt}\left\|(t-\tau)^{b_1}w_m(t)\right\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)} + c_1\int_{\mathcal{O}_t}\left|\nabla\left|(t-\tau)^{b_1+1}w_m(t)\right|^{\frac{N}{N-2}}\right|^2dx\\ &\quad\le c(t-\tau)^{\frac{N+2}{N-2}}\left\|(t-\tau)^{b_1}w_m(t)\right\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(\mathcal{O}_t)}\\ &\quad\le c(t-\tau)^{\frac{2}{N-2}}\left\|w_m(\tau)\right\|^{\frac{2N}{N-2}}_\tau \quad (\text{by } (3.17)).\end{aligned}$$

Integrating this inequality over $[\tau,T]$ with respect to t, we obtain that

$$\int_\tau^T\int_{\mathcal{O}_t}\left|\nabla\left|(t-\tau)^{b_2}w_m(t)\right|^{\frac{N}{N-2}}\right|^2dx\,dt \le c\left\|w_m(\tau)\right\|^{\frac{2N}{N-2}}_\tau, \tag{3.18}$$

where we have used (3.17). Consequently, applying embedding (3.12) again, we can deduce that

$$\int_0^T\left(\int_\Omega\left|(t-\tau)^{b_2}w_m(t)\right|^{2(\frac{N}{N-2})^2}dx\right)^{\frac{N-2}{N}}dt \le c_{N,\tau,T}\,c\left\|w_m(\tau)\right\|^{\frac{2N}{N-2}}_\tau. \tag{3.19}$$

Therefore, noticing (3.3) and (3.5), from (3.17) and (3.19) we know that there is a positive constant M_1, which depends only on N, τ, T, l, $\|u_\tau\|_\tau$, $\|v_\tau\|_\tau$, such that (A_1) and (B_1) hold.

Step 2 Assuming that (A_k) and (B_k) hold for $k\ge 1$, we will show that (A_{k+1}) and (B_{k+1}) hold.

Multiplying (3.7) by $|w_m|^{2(\frac{N}{N-2})^{k+1}-2}\cdot w_m$, using (1.4), and applying Lemmas 2.4 and 2.5, we obtain that

$$\begin{aligned}&\frac{d}{dt}\left\|w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)} + c\int_{\mathcal{O}_t}\left|\nabla\left|w_m(t)\right|^{(\frac{N}{N-2})^{k+1}}\right|^2dx\\ &\quad\le c_1\left\|w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)} \quad \text{for a.e. } t\in(\tau,T).\end{aligned} \tag{3.20}$$

Multiplying both sides of (3.20) by $(t-\tau)^{2(\frac{N}{N-2})^{k+1}\cdot b_{k+1}}$, we deduce that

$$
\begin{aligned}
&\frac{d}{dt}\Big((t-\tau)^{2(\frac{N}{N-2})^{k+1}\cdot b_{k+1}}\|w_m\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)}\Big)+c\int_{\mathcal{O}_t}\Big|\nabla\big|(t-\tau)^{b_{k+1}}\cdot w_m(t)\big|^{(\frac{N}{N-2})^{k+1}}\Big|^2\,dx\\
&\quad\le c_1\big\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\big\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)}\\
&\qquad+c_2(t-\tau)^{2(\frac{N}{N-2})^{k+1}\cdot b_{k+1}-1}\big\|w_m(t)\big\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)},
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\frac{d}{dt}\big\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\big\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)}+c\int_{\mathcal{O}_t}\Big|\nabla\big|(t-\tau)^{b_{k+1}}\cdot w_m(t)\big|^{(\frac{N}{N-2})^{k+1}}\Big|^2\,dx\\
&\quad\le\Big(c_1+\frac{c_2}{t-\tau}\Big)\big\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\big\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)}.
\end{aligned}
\tag{3.21}
$$

At first, from (3.21) we have

$$
\begin{aligned}
&(t-\tau)\frac{d}{dt}\big\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\big\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)}\\
&\quad\le c\big\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\big\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)},
\end{aligned}
\tag{3.22}
$$

and so

$$
\begin{aligned}
&(t-\tau)\frac{d}{dt}\big\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\big\|^{2(\frac{N}{N-2})^{k}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)}\\
&\quad\le c\frac{N-2}{N}\big\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\big\|^{2(\frac{N}{N-2})^{k}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)}.
\end{aligned}
\tag{3.23}
$$

Integrating (3.23) over $[\tau,t]$ and applying (B_k), we deduce that

$$
\begin{aligned}
&(t-\tau)\big\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\big\|^{2(\frac{N}{N-2})^{k}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)}\\
&\quad\le\Big(c\frac{N-2}{N}+1\Big)\int_\tau^T\big\|(s-\tau)^{b_{k+1}}\cdot w_m(s)\big\|^{2(\frac{N}{N-2})^{k}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_s)}\,ds\\
&\quad\le\Big(c\frac{N-2}{N}+1\Big)M_k\quad\text{for all } t\in[\tau,T],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&(t-\tau)^{\frac{N}{N-2}}\big\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\big\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)}\\
&\quad\le\Big[\Big(c\frac{N-2}{N}+1\Big)M_k\Big]^{\frac{N}{N-2}}\quad\text{for all } t\in[\tau,T].
\end{aligned}
\tag{3.24}
$$

In the following, after obtained (3.24), we will return to (3.21) to deduce (B_{k+1}). Multiplying both sides of (3.21) by $(t-\tau)^{1+\frac{N}{N-2}}$, we obtain that

$$\begin{aligned}&(t-\tau)^{1+\frac{N}{N-2}}\frac{d}{dt}\left\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)}\\&\quad+c\int_{\mathcal{O}_t}\left|\nabla\left|(t-\tau)^{b_{k+1}+\frac{1+\frac{N}{N-2}}{2(\frac{N}{N-2})^{k+1}}}\cdot w_m(t)\right|^{(\frac{N}{N-2})^{k+1}}\right|^2dx\\&\le c_3(t-\tau)^{\frac{N}{N-2}}\left\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)}.\end{aligned}\tag{3.25}$$

Then from (3.24) and the definition of b_{k+2} we obtain that

$$\begin{aligned}&(t-\tau)^{1+\frac{N}{N-2}}\frac{d}{dt}\left\|(t-\tau)^{b_{k+1}}\cdot w_m(t)\right\|^{2(\frac{N}{N-2})^{k+1}}_{L^{2(\frac{N}{N-2})^{k+1}}(\mathcal{O}_t)}\\&\quad+c\int_{\mathcal{O}_t}\left|\nabla\left|(t-\tau)^{b_{k+2}}\cdot w_m(t)\right|^{(\frac{N}{N-2})^{k+1}}\right|^2dx\\&\le c_3\left[\left(c\frac{N-2}{N}+1\right)M_k\right]^{\frac{N}{N-2}}\quad\text{for all } t\in[\tau,T].\end{aligned}\tag{3.26}$$

Integrating this inequality over $[\tau,T]$ and using (3.24) again, we deduce that

$$\int_\tau^T\int_{\mathcal{O}_t}\left|\nabla\left|(t-\tau)^{b_{k+2}}\cdot w_m(t)\right|^{(\frac{N}{N-2})^{k+1}}\right|^2dx\,dt\le c_4\left[\left(c\frac{N-2}{N}+1\right)M_k\right]^{\frac{N}{N-2}}.\tag{3.27}$$

Consequently, using of the embedding inequality (3.12) again, we obtain that

$$\int_\tau^T\left(\int_\Omega\left|(t-\tau)^{b_{k+2}}\cdot w_m(t)\right|^{2(\frac{N}{N-2})^{k+2}}dx\right)^{\frac{N-2}{N}}dt\le c_5\left[\left(c\frac{N-2}{N}+1\right)M_k\right]^{\frac{N}{N-2}}.\tag{3.28}$$

Therefore by setting

$$M_{k+1}=(1+c_5)\left[\left(c\frac{N-2}{N}+1\right)M_k\right]^{\frac{N}{N-2}},$$

(3.24) and (3.28) imply that (A_{k+1}) and (B_{k+1}) hold, respectively. □

3.2 Higher-order integrability near the initial time

Based on the a priori estimate in Theorem 3.1 for the approximation solutions, we can obtain the following higher-order integrability near the initial time:

Theorem 3.2 *Let Assumption I hold, and let $u_\tau, v_\tau\in L^2(\mathcal{O}_\tau)$. Then for any $T\ge\tau$ and $k=1,2,\dots$, there exists a positive constant $M_k=M(T-\tau,k,N,l,\|u_\tau\|_\tau,\|v_\tau\|_\tau)$ such that*

$$(t-\tau)^{\frac{N}{N-2}}\left\|(t-\tau)^{b_k}w(t)\right\|^{2(\frac{N}{N-2})^k}_{L^{2(\frac{N}{N-2})^k}(\mathcal{O}_t)}\le M_k\quad\textit{for all } t\in[\tau,T],$$

where $w(t)=U(t,\tau)u_\tau-U(t,\tau)v_\tau$, and

$$b_1=1+\frac12,\qquad b_2=1+\frac12+1\quad\textit{and}\quad b_{k+1}=b_k+\frac{1+\frac{N}{N-2}}{2(\frac{N}{N-2})^{k+1}}\quad\textit{for } k=2,3,\dots.$$

Proof For any (fixed) $\tau \in \mathbb{R}$ and $T \geq \tau$, choose two sequences $(u_{\tau m}, f_m)$ and $(v_{\tau m}, f_m)$ satisfying all conditions (3.1)–(3.5).

Then from Theorem 3.1 we have that for any $k = 1, 2, \ldots$, there exists a positive constant $M_k = M(T - \tau, k, N, l, \|u_\tau\|_\tau, \|v_\tau\|_\tau)$ such that

$$(t-\tau)^{\frac{N}{N-2}} \left\| (t-\tau)^{b_k} \big(u_m(t) - v_m(t) \big) \right\|^{2(\frac{N}{N-2})^k}_{L^{2(\frac{N}{N-2})^k}(\mathcal{O}_t)} \leq M_k \quad \text{for all } t \in [\tau, T], \tag{3.29}$$

where u_m and v_m are the unique strong solutions of (1.3) corresponding to the regular data $(u_{\tau m}, f_m)$ and $(v_{\tau m}, f_m)$ on the interval $[\tau, T]$, respectively.

From (3.4) we know that for each $t \in [\tau, T]$, there are two subsequences $\{u_{m_j}(t)\} \subset \{u_m(t)\}$ and $\{v_{m_j}(t)\} \subset \{v_m(t)\}$ satisfying

$$u_{m_j}(t) \to u(t) = U(t,\tau)u_\tau \quad \text{and} \quad v_{m_j}(t) \to v(t) = U(t,\tau)v_\tau \quad \text{a.e. on } \mathcal{O}_t \text{ as } j \to \infty,$$

where the subindex m_j may depend on t.

Hence, since estimate (3.29) is independent of m, we can finish our proof by applying the Fatou lemma:

$$\begin{aligned}
&(t-\tau)^{\frac{N}{N-2}} \left\| (t-\tau)^{b_k} \big(u(t) - v(t) \big) \right\|^{2(\frac{N}{N-2})^k}_{L^{2(\frac{N}{N-2})^k}(\mathcal{O}_t)} \\
&\quad = (t-\tau)^{\frac{N}{N-2}} \int_{\mathcal{O}_t} \liminf_{j\to\infty} \left| (t-\tau)^{b_k} \big(u_{m_j}(t) - v_{m_j}(t) \big) \right|^{2(\frac{N}{N-2})^k} dx \\
&\quad \leq \liminf_{j\to\infty} (t-\tau)^{\frac{N}{N-2}} \int_{\mathcal{O}_t} \left| (t-\tau)^{b_k} \big(u_{m_j}(t) - v_{m_j}(t) \big) \right|^{2(\frac{N}{N-2})^k} dx \\
&\quad \leq M_k.
\end{aligned}$$

□

4 Proof of Theorem 1.1

We start with the following a priori estimates.

Lemma 4.1 *Let Assumption* I *hold, and let* $f \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathcal{O}_t))$. *Then for all* $\tau \in \mathbb{R}$ *and* $u_\tau \in L^2(\mathcal{O}_\tau)$, *the corresponding weak solution* $u(t) = U(t,\tau)u_\tau$ $(t \geq \tau)$ *of equation* (1.3) *satisfies the following estimates: for any* $T > \tau$,

$$\begin{aligned}
&\int_{\mathcal{O}_s} |u(s)|^p \, dx \leq M \quad \text{for all } s \in \left[\tau + \frac{T-\tau}{2}, T\right], \quad \text{and} \\
&\int_{\frac{\tau+T}{2}}^{T} \int_{\mathcal{O}_s} |u(s)|^{2p-2} \, dx \, ds \leq M
\end{aligned} \tag{4.1}$$

with constant M *depending only on* $T - \tau$, $|\bigcup_{s\in[\tau,T]} \mathcal{O}_s|$, $\lambda_{\tau T}$, $\int_\tau^T \|f(s)\|^2_{L^2(\mathcal{O}_s)} \, ds$, *and* $\|u_\tau\|_\tau$, *where* $\lambda_{\tau T}$ *is the first eigenvalue of* $-\Delta$ *on* $H^1_0(\bigcup_{s\in[\tau,T]} \mathcal{O}_s)$.

Note that since we only assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathcal{O}_t))$, we cannot obtain the uniform boundedness of the solutions in the L^p sense as that in [3, 9, 13], i.e., our constant M above depends on the time $t - \tau$. However, we will show further that such boundedness is sufficient for Theorem 1.1.

Proof Since the results of the lemma are more or less standard, we restrict ourselves by only formal derivation of estimate (4.1), which can be easily justified using, e.g., the methods as in Sect. 3: first, deduce the a priori estimates for approximation solutions and then obtain (4.1) by Fatou's lemma.

First, multiplying (1.3) by u and integrating with respect to $x \in \mathcal{O}_t$, we have that

$$\frac{1}{2}\frac{d}{dt}\|u\|_t^2 + \int_{\mathcal{O}_t}\left|\nabla u(t)\right|^2 dx + \int_{\mathcal{O}_t} g(u)u\,dx \le \left\|f(t)\right\|_t\left\|u(t)\right\|_t \quad \text{for a.e. } t \in (\tau, T); \tag{4.2}$$

recall that $\|\cdot\|_t$ denotes the $L^2(\mathcal{O}_t)$-norm; Then using (1.4) and Cauchy's inequality, we obtain that

$$\begin{aligned}&\frac{d}{dt}\|u\|_t^2 + 2\lambda_{\tau T}\left\|u(t)\right\|_t^2 + 2\alpha_1\int_\tau^t\int_{\mathcal{O}_s}\left|u(s)\right|^p dx\,ds - 2\beta|\mathcal{O}_t| \\ &\quad\le \frac{1}{2\lambda_{\tau T}}\int_\tau^t\left\|f(s)\right\|_s^2 ds + 2\lambda_{\tau T}\left\|u(t)\right\|_t^2 \quad \text{for all } t \in [\tau, T]\end{aligned}$$

(recall that $\lambda_{\tau T}$ is the first eigenvalue of $-\Delta$ on $H_0^1(\bigcup_{s\in[\tau,T]}\mathcal{O}_s)$), which implies that

$$\begin{aligned}&\left\|u(t)\right\|_t^2 + 2\alpha_1\int_\tau^t\int_{\mathcal{O}_s}\left|u(s)\right|^p dx\,ds \\ &\quad\le \frac{1}{2\lambda_{\tau T}}\int_\tau^t\left\|f(s)\right\|_s^2 ds + 2\beta\left|\bigcup_{s\in[\tau,T]}\mathcal{O}_s\right| + \|u_\tau\|_\tau^2 \quad \text{for all } t \in [\tau, T].\end{aligned} \tag{4.3}$$

Secondly, multiplying in (1.3) by $|u|^{p-2}\cdot u$ and integrating with respect to $x \in \mathcal{O}_t$, we have that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_{\mathcal{O}_t}\left|u(t)\right|^p dx + \alpha_1\int_{\mathcal{O}_t}\left|u(t)\right|^{2p-2} dx \\ &\quad\le \beta\int_{\mathcal{O}_t}\left|u(t)\right|^{p-2} dx + \left\|f(t)\right\|_{L^2(\mathcal{O}_t)}\left\|u(t)\right\|_{L^{2p-2}(\mathcal{O}_t)}^{p-1} \quad \text{a.e. } t \in (\tau, T),\end{aligned}$$

where we have used Lemmas 2.4 and 2.5 and (1.4). Consequently, using Cauchy's inequality, we have that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathcal{O}_t}\left|u(t)\right|^p dx + c_1\int_{\mathcal{O}_t}\left|u(t)\right|^{2p-2} dx \\ &\quad\le c_2 + c_3\left\|f(t)\right\|_{L^2(\mathcal{O}_t)}^2 \quad \text{for a.e. } t \in (\tau, T),\end{aligned} \tag{4.4}$$

where the constants c_1, c_2, c_3 depend only on β, α_1, and p.

Now from (4.3) we know that there is $t_0 \in [\tau, \frac{\tau+T}{2}]$ such that

$$u(t_0) \in L^p(\mathcal{O}_{t_0}) \tag{4.5}$$

and

$$\left\|u(t_0)\right\|_{L^p(\mathcal{O}_{t_0})}^p \le \frac{1}{\alpha_1(T-\tau)}\left(\frac{1}{2\lambda_{\tau T}}\int_\tau^T\left\|f(s)\right\|_s^2 ds + 2\beta\left|\bigcup_{s\in[\tau,T]}\mathcal{O}_s\right| + \|u_\tau\|_\tau^2\right). \tag{4.6}$$

Therefore, for any $t \in [\frac{T+\tau}{2}, T]$, integrating (4.4) with respect to time from t_0 to t, we deduce that

$$\begin{aligned} &\|u(t)\|^p_{L^p(\mathcal{O}_t)} + c_1 \int_{t_0}^t \int_{\mathcal{O}_s} |u(s)|^{2p-2}\,dx\,ds \\ &\quad \le c_2(t-t_0) + c_3 \int_{t_0}^t \|f(s)\|^2_{L^2(\mathcal{O}_s)}\,ds + \|u(t_0)\|^p_{L^p(\mathcal{O}_{t_0})}, \end{aligned} \tag{4.7}$$

which, combined with (4.6) and (4.3), immediately implies (4.1). □

Now we are ready to prove our main results.

Proof of Theorem 1.1 It suffices to prove the following claim: *For any $u_\tau, v_\tau \in L^2(\mathcal{O}_\tau)$, we have the following estimate for $t > \tau$:*

$$\|U(t,\tau)u_\tau - U(t,\tau)v_\tau\|^2_{H^1_0(\mathcal{O}_t)} \le c_1\|u_\tau - v_\tau\|^2_\tau + c_2\|u_\tau - v_\tau\|^{2\theta}_\tau, \tag{4.8}$$

where the constants $c_i > 0$ and $\theta \in (0,1)$ depend only on $t-\tau$, $\|u_\tau\|_\tau$, and $\|v_\tau\|_\tau$.

Indeed, the H^1-continuity (1.10) immediately follows from (4.8).

To see the H^1-pullback attraction (1.11), for each $t \in \mathbb{R}$, we denote by $B(t)$ the 1-neighborhood of $\mathscr{A}(t)$ with respect to the $L^2(\mathcal{O}_t)$-norm. Then $B(t)$ is bounded in $L^2(\mathcal{O}_t)$, and by (4.8) there are two positive constants $c_i' > 0$ and $\theta \in (0,1)$ that depend only on t and $\|B(t)\|_t$ such that, for all $u_\tau, v_\tau \in B(t-1)$,

$$\|U(t,t-1)u_\tau - U(t,t-1)v_\tau\|^2_{H^1_0(\mathcal{O}_t)} \le c_1'\|u_\tau - v_\tau\|^2_\tau + c_2'\|u_\tau - v_\tau\|^{2\theta}_\tau. \tag{4.9}$$

Now by the definition of the (L^2,L^2) pullback $\mathscr{D}_\lambda$-attractor $\mathscr{A}$, for any $\varepsilon > 0$ and any $\hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathscr{D}$, there is a time $\tau_1 (< t-1)$, which depends only on t, ε, and $\hat{D}$, such that

$$\operatorname{dist}_{L^2(\mathcal{O}_{t-1})}\big(U(t-1,\tau)D(\tau), \mathscr{A}(t-1)\big) \le \varepsilon \quad \text{for all } \tau \le \tau_1 \tag{4.10}$$

and

$$U(t-1,\tau)D(\tau) \subset B(t-1) \quad \text{for all } \tau \le \tau_1. \tag{4.11}$$

Then from (4.9)–(4.11) we have that for $\tau \le \tau_1$,

$$\begin{aligned} &\operatorname{dist}^2_{H^1_0(\mathcal{O}_t)}\big(U(t,\tau)D(\tau), \mathscr{A}(t)\big) \\ &\quad = \operatorname{dist}^2_{H^1_0(\mathcal{O}_t)}\big(U(t,t-1)U(t-1,\tau)D(\tau), U(t,t-1)\mathscr{A}(t-1)\big) \\ &\quad \le c_1' \operatorname{dist}^2_{L^2(\mathcal{O}_{t-1})}\big(U(t-1,\tau)D(\tau), \mathscr{A}(t-1)\big) \\ &\qquad + c_2' \operatorname{dist}^{2\theta}_{L^2(\mathcal{O}_{t-1})}\big(U(t-1,\tau)D(\tau), \mathscr{A}(t-1)\big) \quad (\text{by } (4.9)) \\ &\quad \le c_1'\varepsilon^2 + c_2'\varepsilon^{2\theta} \quad (\text{by } (4.10)). \end{aligned}$$

Consequently, we obtain the H^1-pullback attraction (1.11) by the arbitrariness of ε and $\hat{D}$.

In the following, we give the proof of the above claim. To make our proof rigorous, as in Sect. 3, we will prove the claim firstly for approximation solutions and then take the limit.

Fix T such that $T \geq t > \tau$. Then, for the initial data u_τ and v_τ, take $\{u_{\tau m}\}_{m=1}^{\infty}$, $\{v_{\tau m}\}_{m=1}^{\infty}$, and $\{f_m\}_{m=1}^{\infty}$ satisfying (3.1)–(3.5).

Denote

$$w_m(s) = u_m(s) - v_m(s) \quad \text{for } \tau \leq s \leq T. \tag{4.12}$$

Then $w_m(s)$ ($m = 1, 2, \dots$) is the unique strong solution of (3.7).

First, multiplying (3.7) by w_m and integrating with respect to $x \in \mathcal{O}_s$ and time, we obtain that

$$\|w_m(s)\|_s^2 \leq e^{2l(s-\tau)} \|w_m(\tau)\|_\tau^2 \quad \forall s \in [\tau, T] \tag{4.13}$$

and

$$\int_\tau^t \|\nabla w_m(s)\|_s^2 \, ds \leq \frac{1}{2} \|w_m(\tau)\|_\tau^2 + \int_\tau^t \|w_m(s)\|_s^2 \, ds \quad \forall t \in [\tau, T], \tag{4.14}$$

where we have used (1.4); recall that $\|\cdot\|_s$ denotes the usual $L^2(\mathcal{O}_s)$-norm and the constant l comes from (1.4).

Secondly, applying Lemma 4.1 to the initial data $u_{\tau m}$ and $v_{\tau m}$, we obtain that there is a constant M_0, which depends only on $t - \tau$, $|\bigcup_{s\in[\tau,t]} \mathcal{O}_s|$, $\lambda_{\tau t}$, $\int_\tau^t \|f(s)\|_{L^2(\mathcal{O}_s)}^2 \, ds$, β, α_1, p, $\|u_{\tau m}\|_\tau$, and $\|v_{\tau m}\|_\tau$, such that

$$\int_{\frac{\tau+t}{2}}^t \int_\Omega |u_m(s)|^{2p-2} \, dx \, ds + \int_{\frac{\tau+t}{2}}^t \int_\Omega |v_m(s)|^{2p-2} \, dx \, ds \leq M_0, \tag{4.15}$$

and from (3.5) we know that M_0 depends indeed only on $\|u_\tau\|_{L^2(\mathcal{O}_\tau)}$ and $\|v_\tau\|_{L^2(\mathcal{O}_\tau)}$ regarding to the initial data.

We now multiply (3.7) by $-\Delta w_m$ (since $w_m \in L^2(\tau, T; H^2(\mathcal{O}_t))$). We then have

$$-\int_{\mathcal{O}_s} w_m' \Delta w_m \, dx + \int_{\mathcal{O}_s} |\Delta w_m(s)|^2 = \int_{\mathcal{O}_s} (g(u_m(s) - g(v_m(s))\Delta w_m(s) \, dx. \tag{4.16}$$

Moreover, as in Límaco, Medeiros, and Zuazua [6], we have

$$-\int_{\mathcal{O}_s} w_m' \Delta w_m \, dx = \frac{1}{2} \frac{d}{ds} \int_{\mathcal{O}_s} |\nabla w_m(s)|^2 \, dx - \int_{\Gamma_s} |\nabla w_m(s)|^2 \psi \cdot n_s \, d\sigma, \tag{4.17}$$

where n_s denotes the unit outward normal vector to $\mathcal{O}_s$, and ψ is the velocity field $\psi = [\partial_s r](\bar{r}(x,s))$. Then, according to (1.1), (1.2), and (1.6), by classical trace results and interpolation we have (e.g., see Duvaut [4]) that

$$\left| \int_{\Gamma_s} |\nabla w_m(s)|^2 \psi \cdot n_s \, d\sigma \right| \leq c_\nu \left(\int_{\mathcal{O}_s} |\Delta w_m(s)|^2 \, dx \right)^\nu \left(\int_{\mathcal{O}_s} |\nabla w_m(s)|^2 \, dx \right)^{1-\nu} \tag{4.18}$$

for all $\nu \geq \frac{1}{2}$. In particular, taking $\nu = \frac{1}{2}$ in (4.18) and using Cauchy's inequality, we have that

$$\left| \int_{\Gamma_s} |\nabla w_m(s)|^2 \psi \cdot n_s \, d\sigma \right|$$

$$\leq \frac{1}{4}\int_{\mathcal{O}_s}\left|\Delta w_m(s)\right|^2 dx + 2c_{\frac{1}{2}}\int_{\mathcal{O}_s}\left|\nabla w_m(s)\right|^2 dx \quad \text{for all } s\in[\tau,T]. \tag{4.19}$$

At the same time, from (1.4) we have that

$$\begin{aligned}
&\left|\int_{\mathcal{O}_s}\big(g(u_m(s))-g(v_m(s))\big)\Delta w_m(s)\,dx\right| \\
&\quad\leq c\int_{\mathcal{O}_s}\left(1+\left|u_m(s)\right|^{p-2}+\left|v_m(s)\right|^{p-2}\right)\left|w_m(s)\right|\left|\Delta w_m(s)\right|dx \\
&\quad\leq c\int_{\mathcal{O}_s}\left|w_m(s)\right|\left|\Delta w_m(s)\right|dx \\
&\qquad + c\int_{\mathcal{O}_s}\left(\left|u_m(s)\right|^{p-2}+\left|v_m(s)\right|^{p-2}\right)\left|w_m(s)\right|\left|\Delta w_m(s)\right|dx \\
&\quad\leq \frac{1}{4}\int_{\mathcal{O}_s}\left|\Delta w_m(s)\right|^2 dx + c\left\|w_m(s)\right\|_s^2 \\
&\qquad + c\left(\left\|u_m(s)\right\|^{2p-4}_{L^{2p-2}(\mathcal{O}_s)}+\left\|v_m(s)\right\|^{2p-4}_{L^{2p-2}(\mathcal{O}_s)}\right)\left\|w_m(s)\right\|^2_{L^{2p-2}(\mathcal{O}_s)},
\end{aligned} \tag{4.20}$$

where, for the last inequality, we used the Hölder inequality with power $\frac{p-2}{2p-2}+\frac{1}{2p-2}+\frac{1}{2}=1$.

Therefore, inserting (4.17)–(4.20) into (4.16), we finally obtain that

$$\begin{aligned}
&\frac{d}{ds}\int_{\mathcal{O}_s}\left|\nabla w_m(s)\right|^2 dx \\
&\quad\leq 4c_{\frac{1}{2}}\int_{\mathcal{O}_s}\left|\nabla w_m(s)\right|^2 dx + 2c\left\|w_m(s)\right\|_s^2 + 2c\left(\left\|u_m(s)\right\|^{2p-4}_{L^{2p-2}(\mathcal{O}_s)}\right. \\
&\qquad \left. + \left\|v_m(s)\right\|^{2p-4}_{L^{2p-2}(\mathcal{O}_s)}\right)\left\|w_m(s)\right\|^2_{L^{2p-2}(\mathcal{O}_s)}.
\end{aligned} \tag{4.21}$$

Since $2(\frac{N}{N-2})^k \to \infty$ as $k\to\infty$, there is $k_0\in\mathbb{N}$ such that

$$2\left(\frac{N}{N-2}\right)^{k_0} > 2p-2.$$

For this k_0, by interpolation we have

$$\|w\|_{L^{2p-2}(\mathbb{R}^N)} \leq \|w\|^{1-\theta}_{L^{2(\frac{N}{N-2})^{k_0}}(\mathbb{R}^N)}\cdot\|w\|^{\theta}_{L^2(\mathbb{R}^N)},$$

where the power $\theta\in(0,1)$ depends only on p, k_0.

Hence from (4.21) we have that for a.e. $s\in[\tau,T]$,

$$\begin{aligned}
\frac{d}{ds}\int_{\mathcal{O}_s}\left|\nabla w_m(s)\right|^2 dx &\leq c\int_{\mathcal{O}_s}\left|\nabla w_m(s)\right|^2 dx + c\left\|w_m(s)\right\|_s^2 + c\left(\left\|u_m(s)\right\|^{2p-4}_{L^{2p-2}(\mathcal{O}_s)}\right. \\
&\quad \left. + \left\|v_m(s)\right\|^{2p-4}_{L^{2p-2}(\mathcal{O}_s)}\right)\left\|w_m(s)\right\|^{1-\theta}_{L^{2(\frac{N}{N-2})^{k_0}}(\mathcal{O}_s)}\cdot\left\|w_m(s)\right\|^{\theta}_{L^2(\mathcal{O}_s)}.
\end{aligned} \tag{4.22}$$

In the following, we will apply Theorem 3.2 to control the terms in (4.22).

Denoting $r_0=(\frac{N}{N-2})\frac{2-2\theta}{2(\frac{N}{N-2})^{k_0}}+(2-2\theta)b_{k_0}$ and multiplying (4.22) by $(s-\frac{t+\tau}{2})^{r_0}$, we obtain that

$$\left(s-\frac{t+\tau}{2}\right)^{r_0}\frac{d}{ds}\left\|\nabla w_m(s)\right\|_s^2$$

$$
\begin{aligned}
&\leq c\left(s-\frac{t+\tau}{2}\right)^{r_0}\left(\|\nabla w_m(s)\|_s^2+\|w_m(s)\|_s^2\right)\\
&\quad+c\left(\|u_m(s)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4}+\|v_m(s)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4}\right)\\
&\quad\cdot\left((s-\tau)^{\frac{N}{N-2}}\|(s-\tau)^{b_{k_0}}w_m(s)\|_{L^{2(\frac{N}{N-2})^{k_0}}(\mathcal{O}_s)}^{2(\frac{N}{N-2})^{k_0}}\right)^{\frac{2-2\theta}{2(\frac{N}{N-2})^{k_0}}}\cdot\|w_m(s)\|_s^{2\theta},
\end{aligned}
\tag{4.23}
$$

where b_{k_0} is given by (3.2) corresponding to k_0.

Then applying Theorem 3.2 to the initial data $u_{\tau m}$, $v_{\tau m}$, times τ, t, and k_0, we get that there is a constant M_{k_0}, which depends only on $t-\tau$, N, l, k_0, and $\|u_{m\tau}\|_\tau$, $\|v_{m\tau}\|_\tau$, such that

$$
\left((s-\tau)^{\frac{N}{N-2}}\|(s-\tau)^{b_{k_0}}w_m(s)\|_{L^{2(\frac{N}{N-2})^{k_0}}(\mathcal{O}_s)}^{2(\frac{N}{N-2})^{k_0}}\right)^{\frac{2-2\theta}{2(\frac{N}{N-2})^{k_0}}}\leq M_{k_0}^{2-2\theta}\quad\text{for all } s\in[\tau,t];
\tag{4.24}
$$

Noting (3.5) again, we see that M_{k_0} also depends only on $\|u_\tau\|_\tau$ and $\|v_\tau\|_\tau$ regarding to the initial data.

Therefore we have the following estimate: for a.e. $s\in[\frac{t+\tau}{2},t]$,

$$
\begin{aligned}
&\left(s-\frac{t+\tau}{2}\right)^{r_0}\frac{d}{ds}\|\nabla w_m(s)\|_s^2\\
&\quad\leq c\left(s-\frac{t-\tau}{2}\right)^{r_0}\left(\|\nabla w_m(s)\|_s^2+\|w_m(s)\|_s^2\right)\\
&\qquad+cM_{k_0}^{2-2\theta}\left(\|u_m(s)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4}+\|v_m(s)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4}\right)\cdot\|w_m(s)\|_s^{2\theta}.
\end{aligned}
\tag{4.25}
$$

To ensure the power of $(s-\frac{t+\tau}{2})$ to be strictly greater than 1, we may multiply both sides by $(s-\frac{t+\tau}{2})$ and then obtain that

$$
\begin{aligned}
&\left(s-\frac{t+\tau}{2}\right)^{r_0+1}\frac{d}{ds}\|\nabla w_m(s)\|_s^2\\
&\quad\leq c\left(s-\frac{t+\tau}{2}\right)^{r_0+1}\left(\|\nabla w_m(s)\|_s^2+\|w_m(s)\|_s^2\right)\\
&\qquad+c\left(s-\frac{t+\tau}{2}\right)M_{k_0}^{2-2\theta}\left(\|u_m(s)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4}+\|v_m(s)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4}\right)\cdot\|w_m(s)\|_s^{2\theta}.
\end{aligned}
\tag{4.26}
$$

Integrating (4.26) from $\frac{\tau+t}{2}$ to t, we obtain that

$$
\begin{aligned}
&\left(\frac{t-\tau}{2}\right)^{1+r_0}\|\nabla w_m(t)\|^2\\
&\quad\leq(1+r_0)\left(\frac{t-\tau}{2}\right)^{r_0}\int_{\frac{\tau+t}{2}}^{t}\|\nabla w_m(s)\|^2\,ds\\
&\qquad+c\left(\frac{t-\tau}{2}\right)^{r_0+1}\int_{\frac{\tau+t}{2}}^{t}\left(\|\nabla w_m(s)\|_s^2+\|w_m(s)\|_s^2\right)ds\\
&\qquad+c\frac{t-\tau}{2}M_{k_0}^{2-2\theta}\int_{\frac{\tau+t}{2}}^{t}\left(\|u_m(s)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4}+\|v_m(s)\|_{L^{2p-2}(\mathcal{O}_s)}^{2p-4}\right)\cdot\|w_m(s)\|_s^{2\theta}\,ds\\
&\quad:=I_1+I_2+I_3.
\end{aligned}
\tag{4.27}
$$

From (4.13) and (4.14) we have that

$$\begin{aligned} & I_1 + I_2 \\ & \quad \le \left(\frac{t-\tau}{2}\right)^{r_0}\left(\left(1 + r_0 + c\frac{t-\tau}{2}\right)\left(\frac{1}{2} + \frac{1}{2l}e^{2l(t-\tau)}\right) + c\frac{t-\tau}{2}\frac{1}{2l}e^{2l(t-\tau)}\right) \\ & \qquad \times \left\|w_m(\tau)\right\|_\tau^2 \\ & \quad \le c_{r_0,t-\tau,l}\left\|w_m(\tau)\right\|_\tau^2. \end{aligned} \tag{4.28}$$

For I_3, using the Hölder inequality and (4.15), we have that

$$\begin{aligned} I_3 & \le c\frac{t-\tau}{2}M_{k_0}^{2-2\theta}2M_0^{\frac{2p-4}{2p-2}}\left(\int_{\frac{\tau+t}{2}}^t \left\|w_m(s)\right\|_s^{2\theta(p-1)}\,ds\right)^{\frac{2}{2p-2}} \\ & \le c_{M_{k_0},p,M_0,t-\tau,\theta}\left(\int_{\frac{\tau+t}{2}}^t e^{2l(s-\tau)\theta(p-1)}\,ds\right)^{\frac{2}{2p-2}}\left\|w_m(\tau)\right\|_\tau^{2\theta} \quad \text{(by (4.13))} \\ & \le c_{M_{k_0},p,M_0,t-\tau,\theta,l}\left\|w_m(\tau)\right\|_\tau^{2\theta}. \end{aligned} \tag{4.29}$$

Putting (4.28) and (4.29) into (4.27), we finally obtain that

$$\left\|\nabla w_m(t)\right\|^2 \le c_{r_0,t-\tau,l}\left\|w_m(\tau)\right\|_\tau^2 + c_{r_0,M_{k_0},p,M_0,t-\tau,\theta,l}\left\|w_m(\tau)\right\|_\tau^{2\theta}, \tag{4.30}$$

and all the constants contained in the above inequality depend only on $\|u_\tau\|_\tau$, $\|v_\tau\|_\tau$ about initial data, and, consequently, they are independent of m.

From (4.30) we know that $\{w_m(t)\}_{m=1}^\infty$ is bounded in $H_0^1(\mathcal{O}_t)$, and therefore there is a subsequence $\{w_{m_j}(t)\}_{j=1}^\infty$ such that

$$w_{m_j}(t) \to \chi \quad \text{weakly in } H_0^1(\mathcal{O}_t) \text{ as } j \to \infty. \tag{4.31}$$

On the other hand, from (3.4) we know that

$$w_{m_j}(t) \to u(t) - v(t) \quad \text{in } L^2(\mathcal{O}_t) \text{ as } j \to \infty.$$

Hence

$$u(t) - v(t) = \chi \in H_0^1(\mathcal{O}_t),$$

and using (4.30), (4.31), and (3.3), we deduce that

$$\begin{aligned} \left\|\nabla\big(u(t) - v(t)\big)\right\|_t^2 & \le \liminf_{j\to\infty}\left\|\nabla w_{m_j}(t)\right\|_t^2 \\ & \le c_{r_0,t-\tau,l}\|u_\tau - v_\tau\|_\tau^2 + c_{r_0,M_{k_0},p,M_0,t-\tau,\theta,l}\|u_\tau - v_\tau\|_\tau^{2\theta}. \end{aligned}$$

This finishes the proof of the claim and thus the proof of the theorem. □

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