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# RESEARCH

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# Abstract

Spectral element discretization of the

time-dependent Stokes problem with

nonstandard boundary conditions

This work deals with the spectral element discretization of the time-dependent Stokes problem in two- and three-dimensional domains. The boundary condition is defined on the normal component of the velocity and the tangential components of the vorticity. The discretization related to the time variable is processed by a Backward Euler method. We prove through a detailed numerical analysis the well-posedness of the full discrete problem.

**Keywords:** Time-dependent Stokes problem; Spectral element method; Implicit Euler scheme

# **1** Introduction

In this article we are interested in the discretization of the nonstationary Stokes problem by the spectral element method in two or three dimensions. Boundary conditions are proposed, in dimension 2, on the normal component of the velocity and on the vorticity. In dimension 3, the boundary conditions are on the normal component of the velocity and on the tangential components of the vector-field vorticity.

Among other examples of this type of boundary conditions, we can cite: a fluid on each side of a membrane, water in a crack in a rigid casting medium, and the coupling of several turbulent fluids. These nonstandard boundary conditions for the Stokes problem were first proposed in [1, 2] (see also [3] and [4]). The new formulation is proposed with three unknowns, vorticity, velocity, and pressure. This formulation is best adapted to this type of boundary conditions even though it is expensive in terms of implementation. We cite [5] for the extension to simply connected three-dimensional domains that concerns the stationary problem and [6] for the time-dependent problem. We also cite the article [7] for the processing of multiconnected domains.

We can refer to article [2] for the discretization of this formulation by the finite-element method. It was extended to the case of spectral methods in [5] for the stationary problem and in [6, 8] for the nonstationnairy problem. The discrete spectral spaces are polynomial spaces defined by analogy to the finite-element spaces introduced by Nedelec in [9]. We carry out a conforming and nonoverlapping partition of the domain by rectangles for dimension 2 and by a rectangular parallelepiped in dimension 3. We use the spectral element

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method for the discretization of this problem. We assume that the discrete solution is a polynomial of degree N on each subdomain. We enforced the continuity of the solution at the interfaces to have a conforming method. Then, the discrete problem is obtained by the Galerkin method with numerical integration.

The numerical analysis of the discrete problem is based on the standard properties of the spectral element method [10]. We begin by demonstrating that the time semidiscrete problem discretized by the implicit Euler scheme is well-posed. We then show that the full problem discretized by the spectral element method admits a unique solution.

The outline of the paper is as follows:

- In Sect. 2, we present the continuous problem and the new formulation of the time-dependent Stokes problem.
- Section 3 deals with the analysis of the time semidiscrete problem.
- The analysis of the spectral element discretization is carried out in Sect. 4.

# 2 The continuous problem

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We consider  $\Omega$  a bounded and simply connected domain of  $\mathbb{R}^d$ , d = 2, d = 3, and  $\partial\Omega$  is its connected Lipschitz continuous boundary. The generic point in  $\Omega$  is  $\mathbf{x} = (x, y)$  or  $\mathbf{x} = (x, y, z)$  according to the dimension. We introduce  $\mathbf{n}$  as the unit outward vector to  $\Omega$  on  $\partial\Omega$  and [0, T] an interval of  $\mathbb{R}$  such that T is a positive constant. The nonstationary Stokes problem is:

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t}(\mathbf{x},t) &- \nu \Delta \mathbf{v}(\mathbf{x},t) + \nabla p(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t) & \text{in } \Omega \times [0,T], \\ \text{div } \mathbf{v}(\mathbf{x},t) &= 0 & \text{in } \Omega \times [0,T], \\ \mathbf{v}(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x}) &= 0 & \text{on } \partial \Omega \times [0,T], \\ \gamma(\mathbf{curl } \mathbf{v})(\mathbf{x},t) &= \mathbf{0} & \text{on } \partial \Omega \times [0,T], \\ \mathbf{v}(\mathbf{x},0) &= \mathbf{v}_0 & \text{in } \Omega, \end{aligned}$$
(1)

where **f** is a data function,  $\nu > 0$  is the viscosity, the unknown **v** is the velocity of the fluid and *p* is its pressure. The operator  $\gamma$  represents the tangential boundary defined as:

- For d = 2: when  $\mathbf{v} = (v_x, v_y)$ , curl  $\mathbf{v} = \partial_x v_y \partial_y v_x$ , then  $\gamma$  (curl  $\mathbf{v}$ ) is the trace on  $\partial \Omega$  of the scalar function curl  $\mathbf{v}$ .
- For d = 3: when  $\mathbf{v} = (v_x, v_y, v_z)$ , **curl**  $\mathbf{v} = (\partial_y v_z \partial_z v_y, \partial_z v_x \partial_x v_y, \partial_x v_y \partial_y v_x)$ , then  $\gamma$  (**curl**  $\mathbf{v}$ ) = **curl**  $\mathbf{v} \times \mathbf{n}$  on the boundary  $\partial \Omega$  where  $\times$  represents the vectorial cross-product.

We define the vorticity  $\varpi = \operatorname{curl} v$  as a new unknown and by using the property

 $-\Delta \mathbf{v} = \mathbf{curl}(\mathbf{curl}\,\mathbf{v}) - \nabla(\operatorname{div}\,\mathbf{v}),$ 

we show that the system (1) is equivalent to:

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t}(\mathbf{x},t) + \nu \operatorname{curl} \boldsymbol{\varpi}(\mathbf{x},t) + \nabla p(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t) & \text{in } \Omega \times [0,T], \\ \operatorname{div} \mathbf{v}(\mathbf{x},t) = 0 & \operatorname{in } \Omega \times [0,T], \\ \boldsymbol{\varpi}(\mathbf{x},t) = \operatorname{curl} \mathbf{v}(\mathbf{x},t) & \operatorname{in } \Omega \times [0,T], \\ \mathbf{v}(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x}) = 0 & \operatorname{on } \partial \Omega \times [0,T], \\ \boldsymbol{\gamma}(\boldsymbol{\varpi})(\mathbf{x},t) = \mathbf{0} & \operatorname{on } \partial \Omega \times [0,T], \\ \mathbf{v}(\mathbf{x},0) = \mathbf{v}_0 & \operatorname{in } \Omega. \end{cases}$$

$$(2)$$

Let  $(\cdot, \cdot)$  be the  $L^2(\Omega)$  scalar product. We consider  $L^2_0(\Omega)$  the space of functions in  $L^2(\Omega)$  that have a null integral on  $\Omega$  and  $\mathcal{D}(\Omega)$  the space of indefinitely differentiable functions with a compact support in  $\Omega$ .

In the case when the domain  $\Omega$  is multiply connected the conditions

curl  $\mathbf{v} = \mathbf{0}$ , div  $\mathbf{v} = 0$  in  $\Omega$ , and  $\mathbf{v} \cdot \mathbf{n} = 0$  on  $\partial \Omega$ ,

are not enough to prove the uniqueness of the velocity, see [11]. An explicit example was given in [12, Chap. 3]. Let  $\Upsilon_i$ ,  $1 \le i \le I$  be open connected curves or surfaces called "cuts" that satisfy:

- $\Upsilon_i$ ,  $1 \le i \le I$  is included in  $\Omega$  and  $\partial \Upsilon_i$  is included in  $\partial \Omega$ ;
- $\Upsilon_i \cap \Upsilon_j = \emptyset, 1 \le i \ne j \le I;$
- $\Omega^{\diamond} = \Omega/(\bigcup_{i=1}^{I} \Upsilon_{i})$  is a simply connected domain.

The further conditions that we need for the uniqueness of the velocity are, for  $1 \le i \le I$ 

 $\mathbf{v} \cdot \mathbf{n} = 0$  on  $\Upsilon_i$ .

To prove the well-posedness of problem (2), we need to define the following spaces:

$$H(\operatorname{div}, \Omega) = \left\{ \mathbf{u} \in \left( L^2(\Omega) \right)^d; \operatorname{div} \mathbf{u} \in L^2(\Omega) \right\},\$$

defined with the norm:

$$\|\mathbf{u}\|_{H(\operatorname{div},\Omega)} = \left(\|\mathbf{u}\|_{L^2(\Omega)^d}^2 + \|\operatorname{div}\mathbf{u}\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}.$$

The normal trace operator  $\mathbf{u} \longrightarrow \mathbf{u}.\mathbf{n}$  is defined from  $H(\operatorname{div}, \Omega)$  into  $H^{-\frac{1}{2}}(\partial \Omega)$ , such that for any function  $\kappa \in H^1(\Omega)$ :

$$\langle \mathbf{u}.\mathbf{n},\kappa\rangle = \int_{\Omega} \operatorname{div} \mathbf{u}(\mathbf{x})\kappa(\mathbf{x})\,d\mathbf{x} + \int_{\Omega} \mathbf{u}(\mathbf{x}).\nabla\kappa(\mathbf{x})\,d\mathbf{x},$$

where  $\langle \cdot, \cdot \rangle$  is the duality product between  $H^{-\frac{1}{2}}(\partial \Omega)$  and  $H^{\frac{1}{2}}(\partial \Omega)$ , see [11, Chap I, Sect. 2]. Then, the kernel of the normal trace operator in  $H(\operatorname{div}, \Omega)$  is the space:

$$H_0(\operatorname{div}, \Omega) = \left\{ \mathbf{u} \in H(\operatorname{div}, \Omega); \mathbf{u}.\mathbf{n} = 0 \text{ on } \partial \Omega \right\}.$$

Let

$$H(\operatorname{\mathbf{curl}},\Omega) = \left\{ \mathbf{u} \in L^2(\Omega)^d; \operatorname{\mathbf{curl}} \mathbf{u} \in L^2(\Omega)^{\frac{d(d-1)}{2}} \right\},\$$

defined with the norm:

$$\|\mathbf{u}\|_{H(\mathbf{curl},\Omega)} = \left(\|\mathbf{u}\|_{L^{2}(\Omega)^{d}}^{2} + \|\mathbf{curl}\,\mathbf{u}\|_{L^{2}(\Omega)}^{2} \frac{d(d-1)}{2}\right)^{\frac{1}{2}}.$$

For d = 3, the tangential trace operator  $\mathbf{u} \times \mathbf{n}$  belongs to  $H^{-\frac{1}{2}}(\partial \Omega)^3$  such that  $\forall \boldsymbol{\mu} \in H(\mathbf{curl}, \Omega)$ ,

$$\langle \mathbf{u} \times \mathbf{n}, \boldsymbol{\mu} \rangle = \int_{\Omega} \mathbf{u}(\mathbf{x}) \cdot \mathbf{curl} \, \boldsymbol{\mu}(\mathbf{x}) \, d\mathbf{x} - \int_{\Omega} \mathbf{curl} \, \mathbf{u}(\mathbf{x}) \cdot \boldsymbol{\mu}(\mathbf{x}) \, d\mathbf{x}$$

Then, the kernel of the tangential operator in  $H(\mathbf{curl}, \Omega)$  is:

$$H_0(\operatorname{\mathbf{curl}},\Omega) = \left\{ \mathbf{u} \in H(\operatorname{\mathbf{curl}},\Omega); \mathbf{u} \times \mathbf{n} = 0 \text{ on } \partial \Omega \right\}.$$

We remark that the spaces  $H(\operatorname{curl}, \Omega)$  and  $H_0(\operatorname{curl}, \Omega)$  coincide, respectively, with the spaces  $H^1(\Omega)$  and  $H_0^1(\Omega)$  if d = 2.

We introduce the space:

$$\mathbb{K}(\Omega) = \left\{ \mathbf{v} \in H_0(\operatorname{div}, \Omega), \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_i, 1 \le i \le I \right\}$$

as the space of the velocity. Then, we consider the following variational formulation: If **f** belongs to the space  $L^2(0, T; L^2(\Omega)^d)$ , find  $(\varpi, \mathbf{v}, p) \in L^2(0, T; H_0(\operatorname{curl}, \Omega)) \times L^2(0, T; \mathbb{K}(\Omega)) \times L^2(0, T; L^2_0(\Omega))$  such that

$$\begin{cases} \forall \mathbf{w} \in \mathbb{K}(\Omega), \quad (\frac{\partial \mathbf{v}}{\partial t}, \mathbf{w}) + a(\varpi, \mathbf{v}; \mathbf{w}) + b(\mathbf{w}, p) = (\mathbf{f}, \mathbf{w}), \\ \forall q \in L_0^2(\Omega), \quad b(\mathbf{v}, q) = 0, \\ \forall \vartheta \in H_0(\mathbf{curl}, \Omega), \quad c(\varpi, \mathbf{v}; \vartheta) = 0, \end{cases}$$
(3)

where the bilinear forms  $a(\cdot, \cdot; \cdot)$ ,  $b(, \cdot, \cdot)$  and  $c(\cdot, \cdot; \cdot)$  are defined as follows:

$$a(\varpi, \mathbf{v}; \mathbf{w}) = \nu \int_{\Omega} \mathbf{curl}(\varpi)(\mathbf{x}, t) \cdot \mathbf{w}(\mathbf{x}) \, d\mathbf{x}, \qquad b(\mathbf{v}, q) = -\int_{\Omega} \operatorname{div} \mathbf{v}(\mathbf{x}, t) q(\mathbf{x}) \, d\mathbf{x} \quad \text{and}$$
$$c(\varpi, \mathbf{v}; \vartheta) = \int_{\Omega} \varpi(\mathbf{x}, t) \cdot \vartheta(\mathbf{x}) \, d\mathbf{x} - \int_{\Omega} \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{curl} \, \vartheta(\mathbf{x}) \, d\mathbf{x}.$$

See [11, 13, 14], for the proof of the equivalence of system (3) with system (2). To show the well-posedness of problem (3), we need to define the following kernels: the kernel of the bilinear form  $b(\cdot, \cdot)$ 

$$\mathbb{V} = \left\{ \boldsymbol{\varphi} \in \mathbb{K}(\Omega); \forall q \in L_0^2(\Omega), b(\boldsymbol{\varphi}, q) = 0 \right\} = \left\{ \boldsymbol{\varphi} \in \mathbb{K}(\Omega); \operatorname{div} \boldsymbol{\varphi} = 0 \right\},\$$

and the kernel of the bilinear form c(.,.;.)

$$\mathbb{U} = \left\{ (\boldsymbol{\vartheta}, \boldsymbol{\varphi}) \in H_0(\operatorname{\mathbf{curl}}, \Omega) \times \mathbb{V}; \forall \boldsymbol{\psi} \in H_0(\operatorname{\mathbf{curl}}, \Omega), c(\boldsymbol{\vartheta}, \boldsymbol{\varphi}; \boldsymbol{\psi}) = 0 \right\}$$
$$= \left\{ (\boldsymbol{\vartheta}, \boldsymbol{\varphi}) \in H_0(\operatorname{\mathbf{curl}}, \Omega) \times \mathbb{V}; \boldsymbol{\vartheta} = \operatorname{\mathbf{curl}} \boldsymbol{\varphi} \right\}.$$

Then,  $(\varpi, \mathbf{v})$  is the solution of the following reduced problem: Find  $(\varpi, \mathbf{v}) \in L^2(0, T; \mathbb{U})$  such that

$$\forall \mathbf{w} \in \mathbb{V}, \quad \left(\frac{\partial \mathbf{v}}{\partial t}, \mathbf{w}\right) + a(\varpi, \mathbf{v}; \mathbf{w}) = (\mathbf{f}, \mathbf{w}). \tag{4}$$

The arguments to prove the well-posedness of problem (4) are exactly the same as in [5, Lem 2.3], and in [15, Chap III, Thm 1.1].

**Proposition 1** For  $\mathbf{f} \in L^2(0, T; L^2(\Omega)^d)$  and  $\mathbf{v}_0 \in \mathbb{K}(\Omega)$ , problem (4) has a unique solution  $(\varpi, \mathbf{v}) \in L^2(0, T; \mathbb{U})$ , which satisfies:

$$\begin{split} \|\varpi\|_{L^{\infty}(0,T;L^{2}(\Omega)^{\frac{d(d-1)}{2}})}^{2} + \|\mathbf{v}\|_{L^{\infty}(0,T;L^{2}(\Omega)^{d})}^{2} \\ & \leq c \big(\|\mathbf{v}_{0}\|_{L^{2}(\Omega)^{d}}^{2} + \|\mathbf{f}\|_{L^{2}(0,T;L^{2}(\Omega)^{d})}^{2} \big), \end{split}$$

where c > 0 and depend only on  $\Omega$  and T.

We recall the following inf-sup condition on bilinear form  $b(\cdot, \cdot)$  see [11, Chap I, Lem 4.1] for its proof. There exists a constant  $\delta > 0$  such that:

$$\forall q \in L_0^2(\Omega), \quad \sup_{\varphi \in \mathbb{K}(\Omega)} \frac{b(\varphi, q)}{\|\varphi\|_{H(\operatorname{div}, \Omega)}} \ge \delta \|q\|_{L^2(\Omega)}.$$
(5)

Based on the inf-sup (5) and the arguments in [16, 17] we conclude the following theorem proved in [18, 19].

**Proposition 2** If data **f** belongs to  $L^2(0, T; L^2(\Omega)^d)$  and  $\mathbf{v}_0$  belongs to  $\mathbb{K}(\Omega)$ , problem (3) has a unique solution  $(\varpi, \mathbf{v}, p)$  in  $L^2(0, T; H_0(\operatorname{curl}, \Omega)) \times L^2(0, T; \mathbb{K}(\Omega)) \times L^2(0, T; L^2_0(\Omega))$ , such that:

$$\begin{split} \|\varpi\|_{L^{\infty}(0,T;L^{2}(\Omega)}^{2} \frac{d(d-1)}{2}) + \|\mathbf{v}\|_{L^{\infty}(0,T;L^{2}(\Omega)^{d})}^{2} + \|p\|_{L^{2}(0,T;L_{0}^{2}(\Omega))}^{2} \\ & \leq c \big(\|\mathbf{v}_{0}\|_{L^{2}(\Omega)^{d}}^{2} + \|\mathbf{f}\|_{L^{2}(0,T;L^{2}(\Omega)^{d})}^{2} \big). \end{split}$$

## 3 The time semidiscrete problem

In this section, we use the implicit Euler method for the dicretization of the derivative in time of the problem (3). We partition the interval [0, T] in subintervals  $[t_{j-1}, t_j]$ , for  $1 \le j \le J$ , J > 0 integer such that  $0 = t_0 < t_1 < \cdots < t_J = T$ . Let  $\tau_j = t_j - t_{j-1}$ ,  $\tau = (\tau_1, \tau_2, \dots, \tau_j)$  and  $|\tau| = \max_{1 \le j \le J} \tau_j$ .

If the data functions  $(\mathbf{f}, \mathbf{v}_0) \in L^2(0, t; L^2(\Omega)^d) \times \mathbb{K}(\Omega)$ , the discrete time problem using the Euler implicit scheme is:

Find  $(\varpi^j)_{1 \le j \le J} \in (H_0(\operatorname{curl}, \Omega))^{J+1}, (\mathbf{v}^j)_{1 \le j \le J} \in (\mathbb{K}(\Omega))^{J+1}$  and  $(p^j)_{1 \le j \le J} \in (L^2_0(\Omega))^J$  such that:

$$\mathbf{v}^0 = \mathbf{v}_0 \quad \text{in } \Omega \tag{6}$$

and for all  $1 \leq j \leq J$ ,  $\mathbf{f}^j = \mathbf{f}(., t_j)$ ,

$$\begin{cases} \forall \mathbf{w} \in \mathbb{K}(\Omega), \quad (\mathbf{v}^{j}, \mathbf{w}) + \tau_{j} a(\varpi^{j}, \mathbf{v}^{j}; \mathbf{w}) + \tau_{j} b(\mathbf{w}, p^{j}) = (\mathbf{v}^{j-1}, \mathbf{w}) + \tau_{j}(\mathbf{f}^{j}, \mathbf{w}), \\ \forall q \in L_{0}^{2}(\Omega), \quad b(\mathbf{v}^{j}, q) = 0, \\ \forall \vartheta \in H_{0}(\mathbf{curl}, \Omega), \quad c(\varpi^{j}, \mathbf{v}^{j}; \vartheta) = 0. \end{cases}$$

$$\tag{7}$$

Then, we conclude that the couple  $(\varpi^j, \mathbf{v}^j) \in \mathbb{U}$  is a solution of the following problem:

$$\forall \mathbf{w} \in \mathbb{V}, \quad \left(\mathbf{v}^{j}, \mathbf{w}\right) + \tau_{j} a\left(\varpi^{j}, \mathbf{v}^{j}; \mathbf{w}\right) = \left(\mathbf{v}^{j-1}, \mathbf{w}\right) + \tau_{j}\left(\mathbf{f}^{j}, \mathbf{w}\right). \tag{8}$$

We define the bilinear form  $\mathbb{A}(\cdot, \cdot; \cdot)$  and the the linear continuous functional  $\mathbb{L}(.)$  by

$$\mathbb{A}(\varpi^{j},\mathbf{v}^{j};\mathbf{w}) = (\mathbf{v}^{j},\mathbf{w}) + \tau_{j}a(\varpi^{j},\mathbf{v}^{j};\mathbf{w}) \text{ and } \mathbb{L}(\mathbf{w}) = (\mathbf{v}^{j-1},\mathbf{w}) + \tau_{j}(\mathbf{f}^{j},\mathbf{w}).$$

Based on the positivity and the inf-sup condition of the bilinear form  $A(\cdot, \cdot; \cdot)$  stated in the following lemma we prove that problem (8) has a unique solution, see [6] and [11, Chap I, Lem 4.1] for the proof.

**Lemma 1** For  $1 \le j \le J$ ,  $\mathbb{A}(\cdot, \cdot; \cdot)$  satisfies:

$$\forall \mathbf{w} \in \mathbb{V} \setminus \{0\}, \quad \sup_{(\varpi^j, \mathbf{v}^j) \in \mathbb{U}} \mathbb{A}(\varpi^j, \mathbf{v}^j; \mathbf{w}) > 0$$

and

$$\forall \left(\varpi^{j}, \mathbf{v}^{j}\right) \in \mathbb{U}, \quad \sup_{\mathbf{w} \in \mathbb{V}} \frac{\mathbb{A}(\varpi^{j}, \mathbf{v}^{j}; \mathbf{w})}{\|\mathbf{w}\|_{L^{2}(\Omega)^{d}}} \geq \beta \left( \left\|\varpi^{j}\right\|_{L^{2}(\Omega)} \frac{d(d-1)}{2} + \left\|\mathbf{v}^{j}\right\|_{L^{2}(\Omega)^{d}} \right).$$

By combining Lemma 1 and the inf-sup condition on the bilinear form  $b(\cdot, \cdot)$  stated in (5), we conclude using arguments in [11], the well-posedness of problems (6) and (7).

# 4 The spectral element discrete problem

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In the following, we suppose that the domain  $\Omega$  has a partition without overlapping into a finite number of rectangles  $\Omega_k$  for d = 2 or a rectangular parallelepiped for d = 3

$$\Omega = \bigcup_{k=1}^{k=K} \Omega_k, \quad \text{such that} \quad \Omega_k \cap \Omega_l = \emptyset \quad \text{for } 1 \le k \ne l \le K.$$

We remark that the set of subdomain  $\Omega_k$  verifies that the intersection of two subdomains  $\overline{\Omega}_k$  and  $\overline{\Omega}_l$ ,  $1 \le k \ne l \le K$  is equal to a vertex, or a hole edge for d = 2 or a hole face for d = 3. We consider that these edges for d = 2 or a face for d = 3 are the cuts  $\Upsilon_i$  defined in Sect. 2.

The discretization is done by the spectral element method based on the idea of Nédélec's cubic three-dimensional meshes [9, Sect. 2]. We consider  $\mathbb{P}_{pq}(\Omega)$  (resp.,  $\mathbb{P}_{pqr}(\Omega)$ ) the polynomials space of degree p in the direction x and q in the direction y (resp., and r in the direction z). These spaces are just denoted  $\mathbb{P}_n(\Omega)$  if p = q = r = n.

Then, in concordance with these definitions and for an integer  $N \ge 2$ , we introduce the local discrete spaces:

$$\mathbb{D}_{N}^{k} = \begin{cases} \mathbb{P}_{N,N-1}(\Omega_{k}) \times \mathbb{P}_{N-1,N}(\Omega_{k}) & \text{if } d = 2, \\ \mathbb{P}_{N,N-1,N-1}(\Omega_{k}) \times \mathbb{P}_{N-1,N,N-1}(\Omega_{k}) \times \mathbb{P}_{N-1,N-1,N}(\Omega_{k}) & \text{if } d = 3, \end{cases}$$
$$\mathbb{C}_{N}^{k} = \begin{cases} \mathbb{P}_{N}(\Omega_{k}) & \text{if } d = 2, \\ \mathbb{P}_{N-1,N,N}(\Omega_{k}) \times \mathbb{P}_{N,N-1,N}(\Omega_{k}) \times \mathbb{P}_{N,N,N-1}(\Omega_{k}) & \text{if } d = 3, \end{cases}$$
$$\mathbb{M}_{N}^{k} = \mathbb{P}_{N-1}(\Omega_{k}). \end{cases}$$

Then, the space of the discrete velocity in  $H_0(\text{div}, \Omega)$  is

$$\mathbb{D}_N = \{\mathbf{v}_N \in \mathbb{K}(\Omega); \mathbf{v}_N |_{\Omega_k} \in \mathbb{D}_N^k \}.$$

The space that approximates the vorticity in  $H_0(\mathbf{curl}, \Omega)$  is defined as

$$\mathbb{C}_N = \left\{ \varphi_N \in H_0(\mathbf{curl}, \Omega); \varphi_N |_{\Omega_k} \in \mathbb{C}_N^k \right\}.$$

Finally, the space in which we approximate the pressure is defined by

$$\mathbb{M}_N = \left\{ p_N \in L^2_0(\Omega); p_N|_{\Omega_k} \in \mathbb{M}_N^k \right\}.$$

*Remark* 1 The functions in  $\mathbb{D}_N$  have a continuous normal trace through the interface  $\overline{\Omega}_k \cap \overline{\Omega}_l$ . The functions in  $\mathbb{C}_N$  have a continuous trace on the interface  $\overline{\Omega}_k \cap \overline{\Omega}_l$  in dimension d = 2 and a continuous tangent trace in dimension d = 3. Thanks to the previous choice, our proposed discretization is completely conforming.

For  $\xi_0 = -1$  and  $\xi_N = 1$ , we consider the N - 1 Gauss–Lobatto nodes  $\xi_i$ ,  $1 \le i \le N - 1$  on the interval ]–1, 1[ and the N + 1 weights  $\rho_i$ ,  $0 \le i \le N$ . The nodes  $\xi_i$ ,  $1 \le i \le N - 1$  are the roots of  $L'_N$ , where  $L_N$  is the Legendre polynomial. Hence, the Gauss–Lobatto quadrature formula is:

$$\forall \varphi_N \in \mathbb{P}_{2N-1}(-1,1), \quad \int_{-1}^1 \varphi_N(\zeta) \, d\zeta = \sum_{i=0}^N \varphi_N(\xi_i) \rho_i.$$
(9)

We also recall the property, see [10]:

$$\forall \varphi_N \in \mathbb{P}_N(-1,1), \quad \|\varphi_N\|_{L^2(-1,1)}^2 \le \sum_{i=0}^N \varphi_N^2(\xi_i)\rho_i \le 3\|\varphi_N\|_{L^2(-1,1)}^2.$$
(10)

Let  $F_k$  be the affine bijection from  $]-1, 1[^d$  into  $\Omega_k$ . Thus, based on formula (10), we introduce the local discrete scalar product defined by: for the continuous functions  $\varphi$  and  $\psi$  on  $\overline{\Omega}_k$ :

$$\begin{split} (\varphi,\psi)_N^k \\ &= \begin{cases} \frac{\operatorname{meas}(\Omega_k)}{4} \sum_{i=0}^N \sum_{j=0}^N (\varphi \circ F_k)(\xi_i,\xi_j)(\psi \circ F_k)(\xi_i,\xi_j)\rho_i\rho_j & \text{if } d=2, \\ \frac{\operatorname{meas}(\Omega_k)}{8} \sum_{i=0}^N \sum_{j=0}^N \sum_{k=0}^N (\varphi \circ F_k)(\xi_i,\xi_j,\xi_k)(\psi \circ F_k)(\xi_i,\xi_j,\xi_k)\rho_i\rho_j\rho_k & \text{if } d=3. \end{cases} \end{split}$$

The global scalar product is then defined on continuous functions  $\varphi$  and  $\psi$  on  $\overline{\Omega}$  as:

$$(\varphi,\psi)_N = \sum_{k=1}^K (\varphi,\psi)_N^k.$$

Then, if the data function **f** is continuous on  $\overline{\Omega} \times [0, T]$ , we deduce the discrete problem from (6), and (7). This problem is constructed by using a combination of Galerkin's method and numerical integration.

For  $\mathbf{v}_N^0 = I_N(\mathbf{v}_0)$ , and if  $\mathbf{v}_N^{j-1}$  is known, find  $(\varpi_N^j, \mathbf{v}_N^j, p_N^j)$  in  $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$  such that for  $1 \le j \le J$ ,

$$\forall \mathbf{w}_{N} \in \mathbb{D}_{N}, \quad (\mathbf{v}_{N}^{j}, \mathbf{w}_{N})_{N} + \tau_{j} \nu (\mathbf{curl} \varpi_{N}^{j}, \mathbf{w}_{N})_{N} - \tau_{j} (\operatorname{div} \mathbf{w}_{N}, p_{N}^{j})_{N}$$

$$= (\mathbf{v}_{N}^{j-1}, \mathbf{w}_{N})_{N} + \tau_{j} (\mathbf{f}^{j}, \mathbf{w}_{N})_{N},$$

$$\forall q_{N} \in \mathbb{M}_{N}, \quad (\operatorname{div} \mathbf{v}_{N}^{j}, q_{N})_{N} = 0,$$

$$\forall \vartheta_{N} \in \mathbb{C}_{N}, \quad (\varpi_{N}^{j}, \vartheta_{N})_{N} - (\mathbf{v}_{N}^{j}, \mathbf{curl} \vartheta_{N})_{N} = 0,$$

$$(11)$$

where  $I_N$  is the Lagrange interpolating operator with values in  $\mathbb{P}_N(\Omega)$ . We define the discrete bilinear forms  $\mathbb{A}_N(\cdot, \cdot; \cdot)$ ,  $b_N(\cdot, \cdot)$ , and  $c_N(\cdot, \cdot; \cdot)$  by

$$\mathbb{A}_{N}(\boldsymbol{\varpi}_{N}^{j}, \mathbf{v}_{N}^{j}; \mathbf{w}_{N}) = (\mathbf{v}_{N}^{j}, \mathbf{w}_{N})_{N} + \tau_{j}\nu (\mathbf{curl}\boldsymbol{\varpi}_{N}^{j}, \mathbf{w}_{N})_{N},$$
  

$$b_{N}(\mathbf{w}_{N}, p_{N}^{j}) = -(\operatorname{div}\mathbf{w}_{N}, p_{N}^{j})_{N} \quad \text{and}$$
  

$$c_{N}(\boldsymbol{\varpi}_{N}^{j}, \mathbf{v}_{N}^{j}; \boldsymbol{\vartheta}_{N}) = (\boldsymbol{\varpi}_{N}^{j}, \boldsymbol{\vartheta}_{N})_{N} - (\mathbf{v}_{N}^{j}, \mathbf{curl}\boldsymbol{\vartheta}_{N})_{N}.$$

We easily prove using (10) and Cauchy–Schwarz that the bilinear forms  $\mathbb{A}_N(\cdot, \cdot; \cdot)$ ,  $b_N(\cdot, \cdot)$ , and  $c_N(\cdot, \cdot; \cdot)$  are, respectively, continuous on the spaces  $(\mathbb{C}_N \times \mathbb{D}_N) \times \mathbb{D}_N$ ,  $\mathbb{D}_N \times \mathbb{M}_N$ , and  $(\mathbb{C}_N \times \mathbb{D}_N) \times \mathbb{C}_N$  with norms that do not depend of N. Based on the exactness of the Gauss–Lobatto quadrature formula (9), we show that the discrete bilinear form  $b_N(\cdot, \cdot)$ and the continuous form  $b(\cdot, \cdot)$  are equal on  $\mathbb{D}_N \times \mathbb{M}_N$ .

To prove the well-posedness of problem (11), we begin by introducing the following discrete kernels:

$$\begin{split} \mathbb{V}_N &= \left\{ \mathbf{v}_N \in \mathbb{D}_N; \forall q_N \in \mathbb{M}_N, b_N(\mathbf{v}_N, q_N) = 0 \right\} = \mathbb{D}_N \cap \mathbb{V}, \\ \mathbb{U}_N &= \left\{ (\boldsymbol{\vartheta}_N, \mathbf{v}_N) \in \mathbb{C}_N \times \mathbb{V}_N; \forall \chi_N \in \mathbb{C}_N, c_N(\boldsymbol{\vartheta}_N, \mathbf{v}_N; \chi_N) = 0 \right\}. \end{split}$$

This permits us to define the following reduced discrete problem:

For  $\mathbf{v}_N^0 = I_N(\mathbf{v}_0)$  and if  $\mathbf{v}_N^{j-1}$  is known, find  $(\varpi_N^j, \mathbf{v}_N^j) \in \mathbb{U}_N$ , such that for all  $1 \le j \le J$ ,

$$\forall \mathbf{w}_N \in V_N, \quad \mathbb{A}_N\left(\varpi_N^j, \mathbf{v}_N^j; \mathbf{w}_N\right) = \mathbb{L}_N(\mathbf{w}_N), \tag{12}$$

such that  $\mathbb{L}_N(\mathbf{w}_N) = (\mathbf{v}_N^{j-1}, \mathbf{w}_N)_N + \tau_N(\mathbf{f}^j, \mathbf{w}_N)_N$ , which is linear and continuous on  $\mathbb{V}_N$ .

Before proving the well-posedness of problem (12), we will now explore certain characteristics of the **curl** operator. According to [11] also referenced in [20, Thm 2.1], it can be deduced that the range of  $\mathbb{C}_N$  through the curl operator is included in  $\mathbb{D}_N$ . Additionally, we have a related result that necessitates introducing further notation.

Let  $\Gamma_i$  denote the connected components of  $\partial\Omega$ , with  $\Gamma_0$  representing the boundary of the only unbounded connected component of  $\mathbb{R}^3 | \overline{\Omega}$ . With this setup, we are now able to introduce the space

$$H^1_*(\Omega) = \left\{ \eta \in H^1(\Omega); \eta = 0 \text{ on } \Gamma_0 \text{ and } \eta = \text{constant on } \Gamma_i, 1 \le i \le I \right\}.$$

**Lemma 2** the kernel of the curl operator in  $\mathbb{C}_N$  is equal to  $\{0\}$  for d = 2, equal to the range of the space

$$\mathbb{Z}_N = \left\{ \zeta_N \in H^1_*(\Omega); \zeta_N |_{\Omega_k} \in \mathbb{P}_N(\Omega_k), 1 \le k \le K \right\},\$$

by the gradient operator for d = 3.

**Proof 1** In the case of d = 2, a curl-free function  $\psi_N$  within  $\mathbb{C}_N$  is constant on  $\Omega$ . Since it vanishes on  $\partial \Omega$ , it must be zero.

In dimension d = 3, let  $\psi_N$  be a curl-free function in  $\mathbb{C}_N$ . Utilizing [11, Chap. I, Thm 2.9], it follows that, given the simply connected domain  $\Omega^{\diamond}$  introduced in Sect. 2,  $\psi_N$  is equal on  $\Omega^{\diamond}$  to the gradient of a function  $\xi$  in  $H^1(\Omega^{\diamond})$ . This function  $\xi$  is defined up to an additive constant. The identity  $\psi_N = \operatorname{grad} \xi$  on  $\Omega_k$  implies that each  $\xi|_{\Omega_k}$  belongs to  $\mathbb{P}(\Omega_k)$ . Furthermore, considering the fact that  $\psi_N \cdot \mathbf{n}$  vanishes on  $\partial\Omega$ , it can be deduced that  $\psi_N$  has a zero tangential gradient on  $\partial\Omega$ , thus being constant on each  $\Gamma_i$ .

It is also observed that  $\psi_N \cdot \mathbf{n}$  is continuous through each  $\Upsilon_i$ , leading to the conclusion that the tangential gradient of the jump of  $\xi$  through each  $\Upsilon_i$  is zero. Consequently, the jump of  $\xi$  is constant. As  $\xi$  is constant on each  $\Gamma_i$ , the jump of  $\xi$  through each  $\overline{\Upsilon}_i \cap \Gamma_i$  is zero, implying that the jump of  $\xi$  through each  $\Upsilon_i$  is also zero. Consequently,  $\xi$  belongs to  $H^1(\Omega)$ . Finally, subtracting its value on  $\Gamma_0$  from  $\xi$  shows that  $\psi_N$  is the gradient of a function in  $\mathbb{Z}_N$ .

Conversely, it can be easily verified that the gradients of all functions in  $\mathbb{Z}_N$  belong to  $\mathbb{C}_N$  and are curl-free.

We are now ready to articulate and prove the central result of this section.

**Proposition 3** There exists a discrete operator  $B_N$  defined from the kernel  $\mathbb{V}_N$  into  $\mathbb{C}_N$  such that

- $\forall \mathbf{v}_N \in \mathbb{V}_N$ , curl  $B_N(\mathbf{v}_N) = \mathbf{v}_N$ ;
- $\forall \psi_N \in \mathbb{Z}_N$ ,  $(B_N(\mathbf{v}_N), \operatorname{\mathbf{grad}} \psi_N)_N = 0;$
- $\forall \mathbf{v}_N \in \mathbb{V}_N, \quad \left\| B_N(\mathbf{v}_N) \right\|_{H(\operatorname{curl},\Omega)} \le C \|\mathbf{v}_N\|_{L^2(\Omega)^d};$  (13)

where C is a constant independent of N.

**Proof 2** In the case of dimension d = 2

Consider any polynomial  $\mathbf{v}_N$  in  $\mathbb{V}_N$ . Suppose  $\Omega$  is included in a rectangle  $\Omega^* = ]a, a'[ \times ]b, b'[$ , and let  $\tilde{\mathbf{v}}_N$  be the extension of  $v_N$ , achieved by setting it to zero outside  $\Omega^*$ . Thus,  $\tilde{\mathbf{v}}_N$  remains divergence-free on  $\Omega^*$ . Representing its components as  $\tilde{\mathbf{v}}_{Nx}$  and  $\tilde{\mathbf{v}}_{Ny}$ , we examine the function defined on  $\Omega^*$  by the following integral:

$$\phi_N(x,y) = \int_b^y \tilde{\mathbf{v}}_{Nx}(x,\varsigma) \, d\varsigma. \tag{14}$$

It can be easily verified that each  $\phi_N|_{\Omega_k}$  belongs to  $\mathbb{P}_N(\Omega_k)$ . The continuity of  $\phi_N$  through each horizontal edge shared by two subdomains  $\Omega_k$  (where a horizontal edge is defined

as an edge contained in a line  $y = y_0$ ) follows directly from its definition. Furthermore, as  $\tilde{\mathbf{v}}_{Nx} = \mathbf{v}_N \cdot \mathbf{n}$  is continuous through all vertical edges shared by two subdomains  $\Omega_k$ , the same property holds for  $\phi_N$ . Therefore,  $\phi_N$  belongs to  $H(\mathbf{curl}, \Omega)$ .

On the other hand, we observe that given  $\mathbf{v}_N$  is divergence-free:

$$(\partial_x \phi_N)|_{\Omega_k}(x,y) = \int_b^y (\partial_x \tilde{\mathbf{v}}_{Nx})(x,\varsigma) \, d\varsigma = -\int_b^y (\partial_y \tilde{\mathbf{v}}_{Ny})(x,\varsigma) \, d\varsigma = -\tilde{\mathbf{v}}_{Ny}(x,y).$$

This equation implies that **curl**  $\phi_N$  is equal to  $\mathbf{v}_N$  on  $\Omega$ . Finally, considering that:

- $\partial_x \phi_N$  vanishes on the horizontal edges of  $\Omega$  and on  $\Omega^* | \overline{\Omega}$ ;
- $\partial_{\gamma}\phi_N$  vanishes on the vertical edges of  $\Omega$  and also on  $\Omega^*|\overline{\Omega}$ ;
- $\phi_N$  is zero at the point (a, b), it follows that  $\phi_N$  is zero on  $\Gamma_0$  and equal to a constant  $c_i$  on each  $\Gamma_i$ . Then, due to the condition  $\mathbf{v}_N \cdot \mathbf{n} = 0$  on  $\Upsilon_i$ , all these constants are determined to be zero. Therefore,  $\phi_N$  belongs to  $\mathbb{C}_N$  and satisfies  $\operatorname{curl} \phi_N = \mathbf{v}_N$  on  $\Omega$ . According to Lemma 2, the restriction of this  $\phi_N$  to  $\Omega$  coincides with  $B_N(\mathbf{v}_N)$ . Moreover, estimate (13) follows from a simple Poincaré–Friedrichs inequality applied to (14).

In the case of dimension d = 3

The construction of function  $\phi_N$  is done in four steps.

1- Similar to the case in dimension d = 2, we assume that  $\Omega$  is contained in a rectangular parallelepiped  $\Omega^* = ]a, a'[\times]b, b'[\times]c, c'[$ , and we extend  $\mathbf{v}_N$  to  $\Omega^*$  to  $\tilde{\mathbf{v}}_N$  by setting it to zero outside  $\Omega$ . Denoting its components as  $\tilde{\mathbf{v}}_{Nx}$ ,  $\tilde{\mathbf{v}}_{Ny}$ , and  $\tilde{\mathbf{v}}_{Nz}$ , we begin by defining a function  $\phi_N^{\sharp} = (\phi_{Nx}^{\sharp}, \phi_{Ny}^{\sharp}, \phi_{Nz}^{\sharp})$  by:

$$\phi_{Nx}^{\sharp}(x,y,z) = \int_{c}^{z} \tilde{\mathbf{v}}_{Ny}(x,y,\varsigma) d\varsigma, \qquad \phi_{Ny}^{\sharp}(x,y,z) = -\int_{c}^{z} \tilde{\mathbf{v}}_{Nx}(x,y,\varsigma) d\varsigma,$$
  
$$\phi_{Nz}^{\sharp} = 0.$$
 (15)

The first two components of  $\phi_N^{\sharp}|_{\Omega_k}$  are elements of  $\mathbb{P}_{N-1,N,N}(\Omega_k)$  and  $\mathbb{P}_{N,N-1,N}(\Omega_k)$ , respectively, indicating that  $\phi_N^{\sharp}|_{\Omega_k}$  belongs to  $\mathbb{C}_N^k$ . This function verifies the property where the first two components of its curl are equal to  $\tilde{\mathbf{v}}_{Nx}$  and  $\tilde{\mathbf{v}}_{Ny}$ . Additionally, given that  $\mathbf{v}_N$  is a member of  $\mathbb{V}_N$  and is divergence-free, it follows that  $\phi_N^{\sharp}$  satisfies

$$(\partial_x \phi_{Ny}^{\sharp} - \partial_y \phi_{Nx}^{\sharp})(x, y, z) = -\int_c^z (\partial_x \tilde{\mathbf{v}}_{Nx} + \partial_y \tilde{\mathbf{v}}_{Ny})(x, y, \zeta) \, d\zeta$$
$$= \int_c^z \partial_z \tilde{\mathbf{v}}_{Nz}(x, y, \zeta) \, d\zeta = \mathbf{v}_{Nz}(x, y, z).$$

Hence, **curl**  $\phi_N^{\sharp}$  is equal to  $v_N$  on each  $\Omega_k$ . Furthermore, the continuity of  $\phi_{Nx}^{\sharp}$  through each face of two  $\Omega_k$  contained in a plane  $y = y_0$  and  $z = z_0$  follows from its definition and the property of  $\mathbf{v}_N$ . Similarly,  $\phi_{Ny}^{\sharp}$  is continuous through each face of two  $\Omega_k$  contained in a plane  $x = x_0$  and  $z = z_0$ , demonstrating that  $\phi_N^{\sharp}$  belongs to  $H(cu, \Omega)$ . Additionally, the following inequality can be readily derived from (15):

$$\|\phi_N^{\sharp}\|_{H(\operatorname{curl},\Omega)} \le C \|\mathbf{v}_N\|_{L^2(\Omega)^3}.$$
(16)

2- Observing that  $\partial \Omega$  is within the union of a finite number of planes, we designate  $\gamma_j, 1 \le j \le J$  as the connected components of the intersections of  $\partial \Omega$  with these planes.

For each  $\gamma_j$ , depending on whether it is contained in a plane  $x = x_0$ ,  $y = y_0$ , or  $z = z_0$ , we establish

$$\begin{aligned} h_{Ny}^{j}(y,z) &= -\int_{c}^{z} \tilde{\mathbf{v}}_{Nx}(x_{0},y,\varsigma) \, d\varsigma, \qquad h_{Nz}^{j}(y,z) = 0, \quad \text{or} \\ h_{Nx}^{j}(x,z) &= -\int_{c}^{z} \tilde{\mathbf{v}}_{Ny}(x,y_{0},\varsigma) \, d\varsigma, \qquad h_{Nz}^{j}(x,z) = 0, \quad \text{or} \\ h_{Nx}^{j}(x,y) &= -\int_{c}^{z_{0}} \tilde{\mathbf{v}}_{Ny}(x,y,\varsigma) \, d\varsigma, \qquad h_{Ny}^{j}(x,y) = -\int_{c}^{z_{0}} \tilde{\mathbf{v}}_{Nx}(x,y,\varsigma) \, d\varsigma. \end{aligned}$$

We note that the vector  $h_N^j$  with these components is tangential to  $\gamma_j$ . Its restriction to each intersection  $\gamma_j \cap \partial \Omega_k$ , having a positive measure in  $\gamma_j$ , belongs to  $\mathbb{P}_{N-1,N}(\gamma_j \cap \partial \Omega_k) \times \mathbb{P}_{N,N-1}(\gamma_j \cap \partial \Omega_k)$ , using the appropriate notation for these new spaces. Moreover, the two-dimensional curl of these functions  $h_N^j$  is zero on each  $\gamma_j$  (specifically,  $\partial_z h_{Ny}^j$  vanishes on the faces contained in a plane  $x = x_0$ ,  $\partial_z h_{Nx}^j$  vanishes on the faces contained in a plane  $x = x_0$ ,  $\partial_z h_{Nx}^j$  vanishes on the faces contained in a plane  $z = z_0$ ). Additionally, the tangential components of  $h_N^j$  and  $h_N^l$  on each edge shared by  $\gamma_j$  and  $\gamma_l$  are equal.

As  $\partial \Omega | \bigcup_{i=1}^{l} \partial \Upsilon_{i}$  is simply connected, it follows from [21, Prop. 3.1], that there exists a function  $\chi_{N}$  in  $H^{1}(\partial \Omega | \bigcup_{i=1}^{l} \partial \Upsilon_{i})$ , vanishing at a corner of  $\Gamma_{0}$ , such that the tangential gradient of the restriction of  $\chi_{N}$  to each  $\gamma_{j}$  is equal to  $h_{N}^{j}$ . Furthermore, the following estimate can be derived from [21, Prop. 4.7], (a more comprehensive proof would involve complex notation, refer to [21] for more details):

$$\|\chi_N\|_{H^{\frac{1}{2}}(\partial\Omega|\bigcup_{i=1}^I\partial\Upsilon_i)} \le \left\|\phi_N^{\sharp} \times n\right\|_{H^{-\frac{1}{2}}(\partial\Omega)}.$$
(17)

Note that the restriction of  $\chi_N$  to each  $\gamma_j \cap \overline{\Omega}_k$ , with a positive measure, belongs to  $\mathbb{P}_N(\gamma_j \cap \overline{\Omega}_k)$ , and the jump of  $\chi_N$  through each  $\partial \Upsilon_i$  is constant.

3- We recall, according to [22, Chap. II, Thm 4.1], the existence of a lifting operator  $L_k$  from  $\mathbb{P}_N(\Omega)$  into  $\mathbb{P}_N(\Omega_k)$  if  $\gamma$  is a face of  $\Omega_k$  contained in  $\partial \Omega$ . For any  $\theta_N$  in  $\mathbb{P}_N(\Omega)$ , the trace of  $L_k(\theta_N)$  is such that it:

- Equals  $\theta_N$  on  $\gamma$ ;
- Is zero on the opposite face to γ;
- Also, when θ<sub>N</sub> is zero on an edge of γ, it is zero on the face that shares this edge with γ.

This operator is applied iteratively on the  $\Omega_k$  values, and on the faces  $\gamma$  of  $\Omega_k$  that are contained in  $\partial\Omega$ . At each step, we subtract the traces of the new function  $\chi_N$  from the remaining traces on  $\Omega_l$ . where l > k and  $\Omega_l$  shares a face or an edge with  $\Omega_k$  (for details, refer to [22, Chap. II]). This process leads to the existence of a  $\kappa_N$  in  $H^1(\Omega^\circ)$  such that  $\phi_N^{\sharp}$  – grad  $\kappa_N$  belongs to  $\mathbb{C}_N$  (here, grad denotes the gradient on  $\Omega^\circ$ ). Moreover, as per [22, Chap. II, Thm 4.1], this function satisfies

$$\|\operatorname{grad} \kappa_N\|_{L^2(\Omega)^3} \leq C \|\chi_N\|_{H^{\frac{1}{2}}(\partial\Omega|\bigcup_{i=1}^I \partial\Upsilon_i)}.$$

Hence, from (16) and (17) we conclude that

$$\|\operatorname{grad} \kappa_N\|_{L^2(\Omega)^3} \le C \|\mathbf{v}_N\|_{L^2(\Omega)^3}.$$
(18)

Ultimately, by combining the Lax–Milgram Lemma with (10) and a generalized Poincaré– Friedrichs inequality, it can be deduced that there exists a unique  $\bar{\kappa}_N$  in  $\mathbb{Z}_N$  satisfying,

$$\forall \eta_N \in \mathbb{Z}_N$$
,  $(\operatorname{grad} \bar{\kappa}_N, \operatorname{grad} \eta_N)_N = (\phi_N^{\sharp} - \operatorname{grad} \kappa_N, \operatorname{grad} \eta_N)_N$ .

Additionally, this function fulfills

$$\|\operatorname{grad}\bar{\kappa}_N\|_{L^2(\Omega)^3} \le C(\|\phi_N^{\mathfrak{p}}\|_{L^2(\Omega)^3} + \|\operatorname{grad}\kappa_N\|_{L^2(\Omega)^3}).$$
(19)

The selection of  $\bar{\kappa}_N$  ensures that the function  $\phi_N^{\sharp} - \operatorname{grad} \kappa_N - \operatorname{grad} \bar{\kappa}_N$  is equal to  $B_N(\mathbf{v}_N)$ . Therefore, the desired estimate follows from (16), (18), and (19).

Now, from proposition 3 we are able to prove the well-posedness of the reduced discrete problem (12). We begin by establishing the analogs of Lemma 1 for the discrete bilinear form  $\mathbb{A}_N(\cdot, \cdot; \cdot)$  (see [20, Thm 2.1] and [23] for similar results in the finite-element method).

**Lemma 3** For  $1 \le j \le J$ , the form  $\mathbb{A}_N(\cdot, \cdot; \cdot)$  satisfies

$$\forall \mathbf{w}_N \in \mathbb{V}_N \setminus \{\mathbf{0}\}, \quad \sup_{(\varpi_N^j, \mathbf{v}_N^j) \in \mathbb{U}_N} \mathbb{A}_N(\varpi_N^j, \mathbf{v}_N^j; \mathbf{w}_N) > 0.$$

**Proof 3** Let  $\mathbf{w}_N \in \mathbb{V}_N$  such that for all  $(\boldsymbol{\varpi}_N^j, \mathbf{v}_N^j) \in \mathbb{U}_N$ ,

$$\mathbb{A}_N(\boldsymbol{\varpi}_N', \mathbf{v}_N'; \mathbf{w}_N) = 0. \tag{20}$$

We consider  $B_N(\mathbf{w}_N) = \psi_N$ , and we examine the problem: Find  $\kappa_N \in \mathbb{V}_N$  such that

$$\forall \mu_N \in \mathbb{V}_N, \quad (\kappa_N, \mu_N)_N = \left(\psi_N, B_N(\mu_N)\right)_N. \tag{21}$$

As the two norms  $\|\cdot\|_{L^2(\Omega)^3}$  and  $\|\cdot\|_{H(\operatorname{div},\Omega)}$  are equivalent on  $\mathbb{V}_N$  thanks to (9), we show that the form  $(\cdot, \cdot)_N$  is elliptic and  $(\psi_N, B_N(\mu_N))_N$  is continuous considering the variable  $\mu_N$ . Thus, by using the Lax–Milgram Lemma, we conclude that problem (21) has a unique solution  $\kappa_N \in \mathbb{V}_N$ , which satisfies, for any  $\varphi_N \in \mathbb{C}_N$ ,

 $(\kappa_N, \operatorname{\mathbf{curl}} \varphi_N)_N = (\psi_N, B_N(\operatorname{\mathbf{curl}} \varphi_N))_N.$ 

If  $B_N(\operatorname{curl} \varphi_N) = \varphi_N + \operatorname{grad} \alpha_N$ , for  $\alpha_N \in H^1_0(\Omega) \cap \mathbb{P}_N(\Omega)$  we conclude that:

 $(\kappa_N, \operatorname{curl} \varphi_N)_N = (\psi_N, \varphi_N)_N.$ 

By considering  $\overline{\omega}_N = \psi_N$  and  $\mathbf{w}_N = \kappa_N$ , (20) is equivalent to

$$\|\psi_N\|_{L^2(\Omega)^3}^2 + \tau_j \nu \|\mathbf{curl}\,\psi_N\|_{L^2(\Omega)^3}^2 = 0.$$

This allows us to conclude that  $\mathbf{w}_N = \mathbf{0}$ .

The following lemma addresses the inf-sup condition satisfied by the bilinear form  $\mathbb{A}_N(\cdot,\cdot;\cdot)$ .

**Lemma 4** For  $1 \le j \le J$ , there exists a positive constant  $\delta$  independent of N and j such that:  $\forall (\varpi_N^j, \mathbf{v}_N^j) \in \mathbb{U}_N$ ,

$$\sup_{\mathbf{w}_{N}\in\mathbb{V}_{N}}\frac{\mathbb{A}_{N}(\varpi_{N}^{j},\mathbf{v}_{N}^{j};\mathbf{w}_{N})}{\|\mathbf{w}_{N}\|_{L^{2}(\Omega)^{d}}} \geq \delta\left(\left\|\varpi_{N}^{j}\right\|_{H(\mathbf{curl},\Omega)}+\left\|\mathbf{v}_{N}^{j}\right\|_{L^{2}(\Omega)^{d}}\right).$$
(22)

**Proof 4** For  $(\varpi_N^j, \mathbf{v}_N^j) \in \mathbb{U}_N$  and if  $\mathbf{w}_N = \mathbf{v}_N^j + \mathbf{curl} \varpi_N^j$  we have

$$\mathbb{A}_{N}(\varpi_{N}^{j}, \mathbf{v}_{N}^{j}; \mathbf{w}_{N}) \geq \left\| \mathbf{v}_{N}^{j} \right\|_{L^{2}(\Omega)^{d}}^{2} + \tau_{j} \nu \left\| \varpi_{N}^{j} \right\|_{H(\mathbf{curl}, \Omega)}^{2}.$$
(23)

By combining (23) and the fact that

$$\|\mathbf{w}_N\|_{L^2(\Omega)^d} \leq \sqrt{2} \left( \left\| \boldsymbol{\varpi}_N^j \right\|_{H(\mathbf{curl},\Omega)}^2 + \left\| \mathbf{v}_N^j \right\|_{L^2(\Omega)^d}^2 \right)^{\frac{1}{2}},$$

we conclude the desired inf-sup condition (22).

The subsequent result directly follows from Lemmas 3 and 4, as documented in [11, Chap. I, Lemma 4.1].

**Proposition 4** If data **f** belongs to  $L^2(0, T; L^2(\Omega)^d)$  and  $\mathbf{v}_0$  belongs to  $\mathbb{K}(\Omega)$ , problem (12) admits a unique solution  $(\varpi_N^j, \mathbf{v}_N^j)$  in  $\mathbb{U}_N$ , such that, for  $1 \le j \le J$ ,

$$\|\mathbf{v}_{N}^{j}\|_{L^{2}(\Omega)^{d}}^{2} \leq \left(\|\mathbf{v}_{N}^{0}\|_{L^{2}(\Omega)^{d}}^{2} + \frac{3^{d}c}{2\nu}\sum_{j=1}^{j}\tau_{j}\|\mathbf{I}_{N}(\mathbf{f}^{j})\|_{L^{2}(\Omega)^{d}}^{2}\right),\tag{24}$$

where c > 0 independent of N and j.

**Proof 5** Let  $\mathbf{w}_N = \mathbf{v}_N^j$  in (12), then:

$$\left(\mathbf{v}_{N}^{j}-\mathbf{v}_{N}^{j-1},\mathbf{v}_{N}^{j}\right)_{N}+\nu\tau_{j}\left(\operatorname{\mathbf{curl}}\boldsymbol{\varpi}_{N}^{j},\mathbf{v}_{N}^{j}\right)_{N}=\tau_{j}\left(\mathrm{I}_{N}\left(\mathbf{f}^{j}\right),\mathbf{v}_{N}^{j}\right)_{N}.$$

Hence, by (10) and the Cauchy–Schwarz inequality, we have:

$$\left(\mathbf{v}_{N}^{j}-\mathbf{v}_{N}^{j-1},\mathbf{v}_{N}^{j}\right)+\nu\tau_{j}\left(\mathbf{curl}\,\varpi_{N}^{j},\mathbf{v}_{N}^{j}\right)\leq 3^{d}\tau_{j}\left\|\mathbf{I}_{N}\left(\mathbf{f}^{j}\right)\right\|_{L^{2}\left(\Omega\right)^{d}}\left\|\mathbf{v}_{N}^{j}\right\|_{L^{2}\left(\Omega\right)^{d}}.$$

Hereafter, we integrate by parts and we use the identity

$$a(a-b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2,$$

we obtain

$$\begin{split} &\frac{1}{2} \left\| \mathbf{v}_{N}^{j} \right\|_{L^{2}(\Omega)^{d}}^{2} - \frac{1}{2} \left\| \mathbf{v}_{N}^{j-1} \right\|_{L^{2}(\Omega)^{d}}^{2} + \frac{1}{2} \left\| \mathbf{v}_{N}^{j} - \mathbf{v}_{N}^{j-1} \right\|_{L^{2}(\Omega)^{d}}^{2} \\ &+ \nu \tau_{j} \left\| \mathbf{curl} \mathbf{v}_{N}^{j} \right\|_{L^{2}(\Omega)}^{2} \frac{d(d-1)}{2} \leq 3^{d} \tau_{j} \left\| \mathbf{I}_{N} \left( \mathbf{f}^{j} \right) \right\|_{L^{2}(\Omega)^{d}} \left\| \mathbf{v}_{N}^{j} \right\|_{L^{2}(\Omega)^{d}}^{2}. \end{split}$$

Using Young's inequality and the following result proved in [12, Cor 3.16]:

$$\forall \varphi \in \mathbb{V}; \quad \|\varphi\|_{L^2(\Omega)^d} \le c \|\mathbf{curl}\,\varphi\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}},\tag{25}$$

we conclude that

$$\frac{1}{2} \|\mathbf{v}_{N}^{j}\|_{L^{2}(\Omega)^{d}}^{2} - \frac{1}{2} \|\mathbf{v}_{N}^{j-1}\|_{L^{2}(\Omega)^{d}}^{2} + \frac{\nu\tau_{j}}{c} \|\mathbf{v}_{N}^{j}\|_{L^{2}(\Omega)^{d}}^{2} \leq 3^{d}\tau_{j} \left(\frac{\varepsilon \|\mathbf{I}_{N}(\mathbf{f}^{j})\|_{L^{2}(\Omega)^{d}}^{2}}{2} + \frac{\|\mathbf{v}_{N}^{j}\|_{L^{2}(\Omega)^{d}}^{2}}{2\varepsilon}\right),$$

where c is the positive constant obtained in inequality (25).

Finally, if  $\varepsilon = \frac{3^d c}{2\nu}$  and we sum up over *j*, we obtain the desired result (24).

To proceed the well-posedness of problem (11), we establish an inf-sup condition of the form  $b_N(\cdot, \cdot)$ . This condition is based on the Boland and Nicolaides argument [24] and necessitates a standard finite-element result, which involves the Nédélec operator [9]. However, the situation is simpler in this context as the constant can depend on the size of the  $\Omega_k$  (provided that the  $\Upsilon_j$  are the union of faces of the subdomains). The first proof of this result can be found in [25].

**Lemma 5** There exists a positive constant  $\beta_{\natural}$  independent of N such that the form  $b_N(\cdot, \cdot)$  satisfies the following inf-sup condition:

$$\forall q_N \in \mathbb{M}_N, \quad \sup_{\mathbf{v}_N \in \mathbb{D}_N} \frac{b_N(\mathbf{v}_N, q_N)}{\|\mathbf{v}_N\|_{H(\operatorname{div},\Omega)}} \ge \beta_{\natural} \|q_N\|_{L^2(\Omega)}.$$
(26)

**Proof 6** It is worth noting that the forms  $b(\cdot, \cdot)$  and  $b_N(\cdot, \cdot)$  coincide on  $\mathbb{D}_N \times \mathbb{M}_N$ , and thus, we proceed with the form  $b(\cdot, \cdot)$ . Any  $q_N$  in  $\mathbb{M}_N$  can be expressed as the expansion

$$q_N = \tilde{q}_N + \bar{q}_N$$
 such that  $\bar{q}_N|_{\Omega_k} = \frac{1}{\operatorname{mes}(\Omega_k)} \int_{\Omega_k} q_N(x) \, dx$ .

Subsequently, each  $\tilde{q}_N|_{\Omega_k}$  belongs to  $\mathbb{M}_N^k \cap L_0^2(\Omega_k)$ . Employing a suitable mapping that transforms the reference domain  $]-1, 1[^d \text{ onto } \Omega_k$ , it can be deduced from [5, Lemma 3.9], that there exists a function  $\mathbf{v}_N^k$  in  $\mathbb{D}_N^k \cap H_0(\operatorname{div}, \Omega_k)$  such that

$$\operatorname{div}(\mathbf{v}_N^k) = -\tilde{q}_N|_{\Omega_k} \quad \text{and} \quad \left\|\mathbf{v}_N^k\right\|_{H(\operatorname{div},\Omega_k)} \leq \beta_k^{-1} \|\tilde{q}_N\|_{L^2(\Omega_k)},$$

where  $\beta_k$  is a constant dependent just on  $\Omega_k$ . Therefore, we define the function  $\tilde{\mathbf{v}}_N$  such that each  $\tilde{\mathbf{v}}_N|_{\omega_k}$  is equal to  $\mathbf{v}_N^k$  for  $1 \le k \le K$ . We observe that, as the  $\Upsilon_j$  are the union of faces of some  $\Omega_k$ ,  $\tilde{\mathbf{v}}_N \cdot \mathbf{n}$  vanishes on  $\Upsilon_j$ , indicating that  $\tilde{\mathbf{v}}_N$  belongs to  $\mathbb{D}_N$ . On the other hand, as  $\bar{q}_N$  belongs to  $L_0^2(\Omega)$  and is constant on each  $\Omega_k$ , it therefore belongs to  $\mathbb{M}_1$ . Then, from [25], we deduce the existence of a function  $\bar{\mathbf{v}}$  in  $\mathbb{D}_1$  such that

$$\operatorname{div}(\bar{\mathbf{v}}) = -\bar{q}_N|_{\Omega_k} \quad \text{and} \quad \|\bar{\mathbf{v}}\|_{H(\operatorname{div},\Omega)} \leq \beta_{\natural}^{-1} \|\bar{q}_N\|_{L^2(\Omega)}.$$

Thanks to the Boland and Nicolaides argument, we have  $\mathbf{v}_N = \tilde{\mathbf{v}}_N + \mu \bar{\mathbf{v}}$ , where  $\mu$  is a positive integer. It can be verified, through an integration by parts on each  $\Omega_k$ , that  $b(\tilde{\mathbf{v}}_N, q_N)$  is

equal to zero. Therefore, with the chosen  $\tilde{\mathbf{v}}_N$  and  $\bar{\mathbf{v}}$  we obtain that

$$b(\mathbf{v}_N, q_N) \ge \|\tilde{q}_N\|_{L^2(\Omega)} + \mu \|\bar{q}_N\|_{L^2(\Omega)} - \mu \|\bar{\mathbf{v}}\|_{H(\operatorname{div},\Omega)} \|\tilde{q}_N\|_{L^2(\Omega)}.$$

Thus, we have

$$\begin{split} b(\mathbf{v}_{N},q_{N}) &\geq \|\tilde{q}_{N}\|_{L^{2}(\Omega)}^{2} + \mu \|\bar{q}_{N}\|_{L^{2}(\Omega)}^{2} - \mu \beta_{\natural}^{-1} \|\bar{q}_{N}\|_{L^{2}(\Omega)} \|\tilde{q}_{N}\|_{L^{2}(\Omega)} \\ &\geq \frac{1}{2} \|\tilde{q}_{N}\|_{L^{2}(\Omega)}^{2} + \mu \left(1 - \frac{\mu}{2\beta_{\natural}^{2}}\right) \|\bar{q}_{N}\|_{L^{2}(\Omega)}^{2}. \end{split}$$

If  $\mu = \beta_{h}^{2}$  then we obtain

$$b(\mathbf{v}_{N}, q_{N}) \geq \frac{1}{2} \min(1, \beta_{\natural}^{2}) \left( \|\tilde{q}_{N}\|_{L^{2}(\Omega)}^{2} + \|\bar{q}_{N}\|_{L^{2}(\Omega)}^{2} \right).$$
(27)

We also know that

$$\|\mathbf{v}_{N}\|_{H(\operatorname{div},\Omega)}\| \leq \left(\max_{1 \leq k \leq K} \beta_{k}^{-1}\right) \|\tilde{q}_{N}\|_{L^{2}(\Omega)} + \beta_{\natural} \|\bar{q}_{N}\|_{L^{2}(\Omega)}.$$
(28)

Then, if we combine the two inequalities (27), (28) with the following orthogonality property

$$\|q_N\|_{L^2(\Omega)}^2 = \|\tilde{q}_N\|_{L^2(\Omega)}^2 + \|\bar{q}_N\|_{L^2(\Omega)}^2,$$

we conclude the desired inf-sub condition (26).

The proof of the ultimate theorem follows standard procedures, as detailed in [11, Chap. I, Lemma 4.1].

**Theorem 4.1** If data  $\mathbf{f}$  belongs to  $L^2(0, T; L^2(\Omega)^d)$  and  $\mathbf{v}_0$  belongs to  $\mathbb{K}(\Omega)$ , problem (11) has a unique solution  $(\varpi_N^j, \mathbf{v}_N^j, p_N^j)$  in  $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$ .

# 5 Conclusion

This work concerns the numerical analysis of the implicit Euler scheme in time and the spectral element discretization in space of the time-dependent Stokes problem with nonstandard boundary conditions. The proof of the well-posedness of the spectral full discrete problem is more technical, especially for the treatment of multiply connected domains. The numerical implementation of this results will be the subject of a forthcoming study.

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# Data Availability

No datasets were generated or analysed during the current study.

# **Declarations**

#### **Competing interests**

The authors declare no competing interests.

#### Author contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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