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Sign-changing solutions for Kirchhoff-type variable-order fractional Laplacian problems

Jianwen Zhou[1], Yueting Yang[1] and Wenbo Wang[1*]

*Correspondence:
wenbowangmath@163.com
[1]School of Mathematics and Statistics, Yunnan University, Kunming, Yunnan, P.R. China

Abstract

In this paper, we are concerned with the Kirchhoff-type variable-order fractional Laplacian problems involving critical exponents and logarithmic nonlinearity. By using the constraint variational method, we show the existence of one least energy sign-changing solution. Moreover, we show that this energy is strictly larger than twice the ground energy.

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Keywords: Kirchhoff-type problem; Variable-order fractional Laplacian; Sign-changing solution

1 Introduction and main results

In this paper, we are interested in the existence of the least energy sign-changing solution of the following Kirchhoff-type problem:

$$\begin{cases} (1+b[u]_{s(\cdot)}^2)(-\Delta)^{s(\cdot)}u + V(x)u = |u|^{q(x)-2}u\ln|u|^2 + \lambda|u|^{2^*(x)-2}u, & \text{in } \Omega, \\ u=0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where

$$[u]_{s(\cdot)}^2 := \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s(x,y)}}\,dx\,dy, \tag{1.2}$$

$b>0$, $s(\cdot):\mathbb{R}^N\times\mathbb{R}^N \to (0,1)$ is a continuous function, Ω is a bounded domain in $\mathbb{R}^N$ with regular boundary, $\lambda>0$ is a parameter, $N>2s(x,y)$ for all $(x,y)\in\Omega\times\Omega$, $(-\Delta)^{s(\cdot)}$ is the variable-order fractional Laplace operator, and $4<q(x)<2^*(x):=\frac{2N}{N-2s(x,x)}$ for all $x\in\Omega$. The variable-order fractional Laplace operator $(-\Delta)^{s(\cdot)}$ is defined as follows: for each $x\in\mathbb{R}^N$,

$$(-\Delta)^{s(\cdot)}\varphi(x) = 2P.V.\int_{\mathbb{R}^N} \frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2s(x,y)}}\,dy, \tag{1.3}$$

along any $\varphi\in C_0^\infty(\Omega)$, where $P.V.$ denotes the Cauchy principal values. As $s(\cdot)\equiv$ const., the variable-order fractional Laplace operator $(-\Delta)^{s(\cdot)}$ reduces to the usual fractional Laplace operator; see [4, 18] for a concise introduction to the fractional Laplace operator and related variational results. The other form of the fractional operator can be seen in [10] and the references therein.

In 1883, Kirchhoff [13] proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\left(\frac{\partial u}{\partial x}\right)^2 = 0. \tag{1.4}$$

The above Kirchhoff-type equations were also introduced by Lions [17]. The authors of [5] said the nonlocal Kirchhoff problems of parabolic type can model several biological systems, such as population density. For more physical backgrounds, we refer the reader to [3, 16]. Many interesting results on the existence of positive solutions, multiple solutions, bound state solutions, semiclassical state solutions, and sign-changing solutions for Kirchhoff-type equations can be found in [1, 19, 20, 23] and the references therein. For the fractional Kirchhoff problem, we mention that the authors of [9] used the finite-dimensional reduction method and perturbed arguments to study the singular perturbation fractional Kirchhoff equations with critical case

$$\left(a + b\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dx\right)(-\Delta)^s u = \big(1 + \varepsilon K(x)\big)u^{2_s^*-1}, \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

and the nondegenerate results are also given. See [30] for the fractional Kirchhoff problem with strong singularity, that is, the right-hand term is $f(x)u^{-\gamma}$, where $\gamma > 1$. Meanwhile, the fractional Kirchhoff-type p-Laplacian problem has attracted extensive attention. See, e.g., [4, 5, 7, 8, 14, 16, 24–29] for the existence, multiplicity, and concentration phenomena.

We mention that in 2019, Liang and Rădulescu [15] considered the following critical Kirchhoff problems with logarithmic nonlinearity:

$$\begin{cases} (a + b[u]_{s,p}^p)(-\Delta)_p^s u = \lambda |u|^{q-2} u \ln|u|^2 + |u|^{p_s^*-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.6}$$

Under suitable assumptions, they obtain a least energy sign-changing solution. There is a logarithmic term in the above problem; please see [6, 8, 11, 15, 24] for related results. The Schrödinger equation with logarithmic term appears in a lot of physical fields, such as quantum mechanics, quantum optics, and nuclear physics. We also quote the paper [22] for other singular integral equations and their physical background. In that paper, the boundary integral equation method is used.

We also mention that in 2022, Wang and Zhang [25] proved the existence of infinitely many solutions via Clark's theorem for the following problem:

$$\begin{cases} M([u]_{s(\cdot)}^2)(-\Delta)^{s(\cdot)} u + V(x)u = \lambda |u|^{p(x)-2} u + \mu |u|^{q(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.7}$$

Recently, Liang et al. [14] studied the following problem:

$$\begin{cases} (a + b[u]_{s(\cdot)}^2)(-\Delta)^{s(\cdot)} u = |u|^{q(x)-2} u + \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.8}$$

They used constraint variational methods and the quantitative deformation lemma to obtain the existence of one least energy sign-changing solution.

In this paper, motivated by the above paper, we are pursuing a sign-changing weak solution of problem (1.1). To the best of our knowledge, there is no work concerning this problem. To state our results, we make the following assumptions:

$(S1)$ $0 < s_- := \min_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x,y) \le s_+ := \max_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x,y) < 1$;

$(S2)$ $s(\cdot)$ is symmetric, that is, $s(x,y) = s(y,x)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$;

$(V1)$ $V(x)$ is a continuous function satisfying

$$\inf_{x\in\Omega} V(x) > V_0 > 0. \tag{1.9}$$

Now, we can state our results as follows.

Theorem 1.1 *Assume that* $(S1)$, $(S2)$, *and* $(V1)$ *hold. Then, for* $4 < q(x) < 2^*(x)$ *for all* $x \in \Omega$, *there exists* $\lambda_1 > 0$ *such that for all* $\lambda \ge \lambda_1$, *problem* (1.1) *has a least energy sign-changing solution* u_b.

Now, with regard to the property of double energy, according to the proof of the above theorem we can define

$$\mathcal{N}_\lambda = \left\{u \in E \setminus \{0\} : \left\langle J'_\lambda(u), u\right\rangle = 0\right\}, \tag{1.10}$$

and we have the following theorem.

Theorem 1.2 *Assume that* $(S1)$, $(S2)$, *and* $(V1)$ *hold. Then, for* $4 < q(x) < 2^*(x)$ *for all* $x \in \Omega$, *there exists* $\lambda^* > 0$ *such that for all* $\lambda \ge \lambda^*$,

$$c^* := \inf_{u\in\mathcal{N}_\lambda} J_\lambda(u) > 0 \tag{1.11}$$

is achieved and $J_\lambda(u_b) > 2c^*$.

It is worthy of pointing out that our results are different from those in [8] or [14]. From a technical point of view, we have three major difficulties. One is that both the fractional Laplacian and the Kirchhoff term are nonlocal. This makes the decomposition of the energy function much more complicated. The second is that the logarithmic term is sign-changing. The third is that our problem is Sobolev critical. In contrast to [14], our nonlinearity term contains a logarithmic term. Fortunately, for Ω is bounded, the functional I (see (2.4)) is C^1. Compared with [8], in our present paper, we add a perturbation parameter λ before the critical term in order to depress the energy value, so we can deal with the Sobolev critical problem. This is called a local P.S. condition. So far, in our opinion, adding a perturbation parameter λ before the logarithmic term is not effective to study the critical problem since this term is sign-changing. Our objective is to study the logarithmic term and the exponents are functions. We will use the variable-exponent Lebesgue space $L^{p(x)}(\Omega)$; see [12] for the generalized Orlicz space $L^\varphi(\Omega)$.

From now on, we always assume that $(S1)$, $(S2)$, and $(V1)$ hold unless otherwise stated. We need to find a sign-change minimizer of the corresponding minimization problem.

2 Proof of Theorem 1.1 and Theorem 1.2

We continue to use the notations and work space as in [14]. For a function $m : \Omega \to \mathbb{R}$, we set

$$\underline{m} = \operatorname*{ess\,inf}_{x\in\Omega} m(x), \qquad \overline{m} = \operatorname*{ess\,sup}_{x\in\Omega} m(x). \tag{2.1}$$

Since V is continuous, in $H_0^{s(\cdot)}(\Omega)$, we can choose the equivalent norm

$$\|u\|^2 = [u]_{s(\cdot)}^2 + \int_\Omega V(x)u^2\,dx. \tag{2.2}$$

For convenience, we denote $E := H_0^{s(\cdot)}(\Omega)$ with the norm $\|\cdot\|$, which is a Hilbert space with inner product $(\cdot,\cdot)_E$.

The corresponding energy functional of (1.1) is defined as

$$\begin{aligned} J_\lambda(u) =& \frac{1}{2}\|u\|^2 + \frac{b}{4}[u]_{s(\cdot)}^4 + 2\int_\Omega \frac{1}{q(x)^2}|u|^{q(x)}\,dx \\ & - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\ln|u|^2\,dx - \lambda\int_\Omega \frac{1}{2^*(x)}|u|^{2^*(x)}\,dx, \quad u\in E. \end{aligned} \tag{2.3}$$

We can verify that $J_\lambda \in C^1(E,\mathbb{R})$. Indeed, in our case Ω is a bounded domain with regular boundary. In virtue of the results in [2] or [21],

$$I(u) := \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\ln|u|^2\,dx \tag{2.4}$$

belongs to $C^1(E,\mathbb{R})$. And for $u, v\in E$,

$$\langle I'(u), v\rangle = \int_\Omega |u|^{q(x)-2}uv\ln|u|^2\,dx + 2\int_\Omega |u|^{q(x)-2}uv\,dx. \tag{2.5}$$

Our goal is to find a sign-changing critical point of J_λ. Although many words are similar to [14], we need to check our results word by word since our functional contains the logarithmic term $I(u)$.

Let us denote

$$u^+(x) := \max\{u(x),0\}, \qquad u^-(x) := \min\{u(x),0\}. \tag{2.6}$$

Clearly, $u = u^+ + u^-$. For convenience, for any $u\in E$, $u^\pm \neq 0$, let us define a function $\Psi_u :$ $[0,\infty)\times[0,\infty)$ by

$$\Psi_u(\alpha,\beta) := J_\lambda\bigl(\alpha u^+ + \beta u^-\bigr). \tag{2.7}$$

Furthermore,

$$H(u) := \int_\Omega\int_\Omega \frac{(u^+(x)-u^+(y))(u^-(x)-u^-(y))}{|x-y|^{N+2s(x,y)}}\,dx\,dy. \tag{2.8}$$

Obviously,

$$H(u) = -2\int_{\Omega}\int_{\Omega}\frac{u^+(x)u^-(y)}{|x-y|^{N+2s(x,y)}}\,dx\,dy > 0. \tag{2.9}$$

We define the sign-changing Nehari manifold

$$\mathcal{M}_\lambda = \left\{u \in E, u^{\pm} \neq 0 : \langle J'_\lambda(u), u^+\rangle = \langle J'_\lambda(u), u^-\rangle = 0\right\}. \tag{2.10}$$

We need to prove $\mathcal{M}_\lambda \neq \emptyset$. We have the following lemma. We remark that the last conclusion in Lemma 2.1 will be used later.

Lemma 2.1 *For $u \in E$, $u^{\pm} \neq 0$, there exists a unique (α_u, β_u) of positive numbers such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_\lambda$. Moreover, (α_u, β_u) is the unique maximum point of Ψ_u on $[0,\infty) \times [0,\infty)$. Furthermore, if $\langle J'_\lambda(u), u^{\pm}\rangle \leq 0$, then $0 < \alpha_u, \beta_u \leq 1$.*

Proof Since the proof is almost standard (see [8]), we just sketch the proof for the reader's convenience. For all $r(x) \in (q(x), 2^*(x))$, noting that $4 < q(x) < 2^*(x)$, choosing $\varepsilon > 0$ small, we can have

$$\left\langle J'_\lambda\left(\alpha u^+ + \beta u^-\right), \alpha u^+\right\rangle > 0, \quad \text{for any } \alpha > 0 \text{ small enough and all } \beta > 0. \tag{2.11}$$

Similarly, it yields

$$\left\langle J'_\lambda\left(\alpha u^+ + \beta u^-\right), \beta u^-\right\rangle > 0, \quad \text{for any } \beta > 0 \text{ small enough and all } \alpha > 0. \tag{2.12}$$

Therefore, there exists $\delta_1 > 0$ such that

$$\left\langle J'_\lambda\left(\delta_1 u^+ + \beta u^-\right), \delta_1 u^+\right\rangle > 0 \quad \text{and} \quad \left\langle J'_\lambda\left(\alpha u^+ + \delta_1 u^-\right), \delta_1 u^-\right\rangle > 0. \tag{2.13}$$

Like [8], we can choose $\delta_2^* > 0$ such that when $\beta \in [\delta_1, \delta_2^*]$, we have

$$\left\langle J'_\lambda\left(\delta_2^* u^+ + \beta u^-\right), \delta_2^* u^+\right\rangle \leq 0. \tag{2.14}$$

Similarly, we have

$$\left\langle J'_\lambda\left(\alpha u^+ + \delta_2^* u^-\right), \delta_2^* u^-\right\rangle \leq 0. \tag{2.15}$$

Letting $\delta_2 > \delta_2^*$ be large enough, we obtain

$$\left\langle J'_\lambda\left(\delta_2^* u^+ + \beta u^-\right), \delta_2^* u^+\right\rangle < 0 \quad \text{and} \quad \left\langle J'_\lambda\left(\alpha u^+ + \delta_2^* u^-\right), \delta_2^* u^-\right\rangle < 0 \tag{2.16}$$

for all $\alpha, \beta \in [\delta_1, \delta_2]$. Combining (2.13) with (2.16), there exists $(\alpha_u, \beta_u) \in (0,\infty) \times (0,\infty)$ such that $T_u(\alpha_u, \beta_u) = (0,0)$.

Secondly, we prove the uniqueness of the pair (α_u, β_u). It can be divided into two cases.

Case 1. $u \in \mathcal{M}_\lambda$. Let (α_u, β_u) be a pair of numbers such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_\lambda$. Next we show that $(\alpha_u, \beta_u) = (1,1)$.

For the case $0 < \alpha_u \le \beta_u$, if $\beta_u > 1$, $\langle J_\lambda(\alpha_u u^+ + \beta_u u^-), u^- \rangle = 0$ can lead to a contradiction. Therefore, we conclude that $\beta_u \le 1$. Similarly, $\langle J_\lambda(\alpha_u u^+ + \beta_u u^-), u^+ \rangle = 0$ implies that $\alpha_u \ge 1$. Consequently, $\alpha_u = \beta_u = 1$. For the other case, $0 < \beta_u \le \alpha_u$, we can adopt a similar argument as above to get $\alpha_u = \beta_u = 1$.

Case 2. $u \notin \mathcal{M}_\lambda$. Suppose that there exist $(\tilde{\alpha_1}, \tilde{\beta_1})$, $(\tilde{\alpha_2}, \tilde{\beta_2})$ such that

$$u_1 = \tilde{\alpha_1} u^+ + \tilde{\beta_1} u^- \in \mathcal{M}_\lambda, \qquad u_2 = \tilde{\alpha_2} u^+ + \tilde{\beta_2} u^- \in \mathcal{M}_\lambda. \tag{2.17}$$

Similar to [8], we obtain $\tilde{\alpha_2} = \tilde{\alpha_1}$, $\tilde{\beta_2} = \tilde{\beta_1}$.

Thirdly, we will prove that (α_u, β_u) is the unique maximum point of Ψ_u on $[0, +\infty) \times [0, +\infty)$. Clearly, (α_u, β_u) is a critical point of Ψ_u. Obviously,

$$2\rho^{q(x)} - q(x)\rho^{q(x)} \ln|\rho|^2 \le 2, \quad \forall \rho \in (0, \infty). \tag{2.18}$$

It follows that

$$\lim_{|(\alpha,\beta)| \to \infty} \Psi_u(\alpha, \beta) = -\infty. \tag{2.19}$$

Hence, (α_u, β_u) is the unique critical point of Ψ_u in $(0, +\infty) \times (0, +\infty)$. So it is sufficient to check that maximum point cannot be achieved on the boundary of $(0, +\infty) \times (0, +\infty)$. The boundary is

$$\{0, +\infty\} \times (0, +\infty) \cup (0, +\infty) \times \{0, +\infty\} \cup \{0, +\infty\} \times \{0, +\infty\}. \tag{2.20}$$

In view of (2.19), the maximum point of Ψ_u cannot be $+\infty \times (0, +\infty)$, $(0, +\infty) \times +\infty$, or $\{0, +\infty\} \times \{0, +\infty\}$ if $(0, \beta_u)$ is a maximum point of Ψ_u for some real positive number $0 < \beta_u < +\infty$. However, Ψ_u is an increasing function with respect to α if α is small enough. This is absurd. Similarly, Ψ_u cannot achieve its global maximum point at $(\alpha_u, 0)$.

The remaining part is to prove the last conclusion. We also divide this into two cases. For case 1, if $\beta_u \le \alpha_u$, and jointly $\langle J'_\lambda(u), u^\pm \rangle \le 0$ with $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_\lambda$, we obtain $0 < \alpha_u \le 1$. For case 2, if $\alpha_u \le \beta_u$, we can get $\beta_u \le 1$ as before. □

Now, consider the following minimization problem:

$$c_\lambda := \inf_{u \in \mathcal{M}_\lambda} J_\lambda(u). \tag{2.21}$$

We need to prove it is well defined.

Lemma 2.2 *We have* $c_\lambda > 0$.

Proof Since $\underline{r}, \overline{r}, \underline{2^*}, \overline{2^*} > 4$, similar to [8], there exists $\rho > 0$ such that

$$\|u^\pm\|^2 \ge \rho, \quad \text{for all } u \in \mathcal{M}_\lambda. \tag{2.22}$$

In light of $\langle J'_\lambda(u), u \rangle = 0$ and

$$2\frac{1}{q(x)^2}|u|^{q(x)} + \frac{1}{4}\left(1 - \frac{4}{q(x)}\right)|u|^{q(x)} \ln|u|^2 \ge 0, \tag{2.23}$$

we get

$$\begin{aligned} J_\lambda(u) &= J_\lambda(u) - \frac{1}{4}\langle J'_\lambda(u), u\rangle \\ &\geq \frac{1}{4}\|u\|^2. \end{aligned} \tag{2.24}$$

Thus, we have $c_\lambda > 0$. □

Next, we let $\lambda \to \infty$ to get the asymptotic property of $c_\lambda = \inf_{u\in\mathcal{M}_\lambda} J_\lambda(u)$.

Lemma 2.3 *We have* $\lim_{\lambda\to\infty} c_\lambda = 0$.

Proof For any $u \in E$ with $u^\pm \neq 0$, using Lemma 2.1, for each $\lambda > 0$, there exist $\alpha_\lambda, \beta_\lambda > 0$ such that $\alpha_\lambda u^+ + \beta_\lambda u^- \in \mathcal{M}_\lambda$. Similar to [8], $\{(\alpha_\lambda, \beta_\lambda)\}_\lambda$ can be bounded. So let $\{\lambda_n\} \subset (0, +\infty)$ be such that $\lambda_n \to \infty$ as $n \to \infty$, and we have $(\alpha_{\lambda_n}, \beta_{\lambda_n}) \to (\alpha_0, \beta_0)$. We have the following claim.

Claim 2.4 *We claim that* $\alpha_0 = \beta_0 = 0$.

If $\alpha_0 > 0$ or $\beta_0 > 0$, by $\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^- \in \mathcal{M}_{\lambda_n}$, we have

$$\begin{aligned} &\left\|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\right\|^2 + b\left[\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\right]^4_{s(\cdot)} \\ &\quad= \int_\Omega \left|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\right|^{q(x)} \ln\left|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\right|^2 dx \\ &\qquad+ \lambda_n \int_\Omega \left|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\right|^{2^*(x)} dx. \end{aligned} \tag{2.25}$$

Using the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega \left|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\right|^{2^*(x)} dx \to \int_\Omega \left|\alpha_0 u^+ + \beta_0 u^-\right|^{2^*(x)} dx > 0, \tag{2.26}$$

which is a contradiction. Consequently, we finish the proof. We point out that our parameter λ is before the critical term, which ensures that the corresponding energy is depressed. □

The next lemma shows that c_λ can be achieved when λ is large enough. We borrow the idea from [8] or [15]. However, our case is different from both of them since it appears the terms

$$\lambda\frac{\max\{\alpha^{\underline{2^*}}, \alpha^{\overline{2^*}}\}}{\underline{2^*}} \quad \text{and} \quad \lambda\frac{\max\{\beta^{\underline{2^*}}, \beta^{\overline{2^*}}\}}{\underline{2^*}}. \tag{2.27}$$

For strict logic, we check it word by word patiently.

Lemma 2.5 *There exists* $\lambda_1 > 0$ *such that for all* $\lambda > \lambda_1$, c_λ *is achieved*.

Proof Let $\{u_n\}$ be a minimization sequence. Obviously, $\{u_n\}$ is bounded in E. Up to a subsequence, we may assume that $u_n \rightharpoonup u$ in E. Thus, we have

$$\begin{aligned}
&\liminf_{n\to\infty} J_\lambda\left(\alpha u_n^+ + \beta u_n^-\right)\\
&\quad\geq J_\lambda\left(\alpha u^+ + \beta u^-\right)\\
&\qquad+ \frac{\alpha^2}{2}A_1 + \frac{\beta^2}{2}A_2 + \frac{b\alpha^4}{2}A_3\left[u^+\right]^2_{s(\cdot)} + \frac{b\alpha^4}{4}A_3^2 + \frac{b\beta^4}{2}A_4\left[u^-\right]^2_{s(\cdot)} + \frac{b\beta^4}{4}A_4^2 \\
&\qquad- \lambda\frac{\max\{\alpha^{\underline{2^*}},\alpha^{\overline{2^*}}\}}{\underline{2^*}}B_1 - \lambda\frac{\max\{\beta^{\underline{2^*}},\beta^{\overline{2^*}}\}}{\underline{2^*}}B_2,
\end{aligned} \tag{2.28}$$

where

$$\begin{aligned}
&A_1 = \lim_{n\to\infty}\left\|u_n^+ - u^+\right\|^2, \qquad A_2 = \lim_{n\to\infty}\left\|u_n^- - u^-\right\|^2,\\
&A_3 = \lim_{n\to\infty}\left[u_n^+ - u^+\right]^2_{s(\cdot)}, \qquad A_4 = \lim_{n\to\infty}\left[u_n^- - u^-\right]^2_{s(\cdot)},
\end{aligned} \tag{2.29}$$

and

$$B_1 = \lim_{n\to\infty}\left|u_n^+ - u^+\right|^{2^*(x)}_{2^*(x)}, \qquad B_2 = \lim_{n\to\infty}\left|u_n^- - u^-\right|^{2^*(x)}_{2^*(x)}. \tag{2.30}$$

Since our proof is too long, we present it in three steps.

Step 1: We want to prove that $u^\pm \neq 0$.

We only prove $u^+ \neq 0$ since $u^- \neq 0$ can be proven by an analogous method. If $u^+ = 0$. We will divide it into two cases.

Case 1: $B_1 = 0$. According to (2.28), for all $\alpha > 0$, we have

$$c_\lambda \geq J_\lambda\left(\alpha u^+\right) + \frac{\alpha^2}{2}A_1 + \frac{b\alpha^4}{2}A_3\left[u^+\right]^2_{s(\cdot)} + \frac{b\alpha^4}{4}A_3^2 - \lambda\frac{\max\{\alpha^{\underline{2^*}},\alpha^{\overline{2^*}}\}}{\underline{2^*}}B_1. \tag{2.31}$$

Subcase 1: $A_1 = 0$. This contradicts (2.22).

Subcase 2: $A_1 > 0$. By (2.31) and Lemma 2.3, we have

$$0 < \frac{\alpha^2}{2}A_1 \leq c_\lambda \to 0 \quad \text{for all } \alpha > 0 \text{ and } \lambda \to \infty. \tag{2.32}$$

This is absurd.

Case 2: $B_1 > 0$. This yields $A_1 > 0$. Let

$$f_1(\alpha) := \frac{\alpha^2}{2}A_1 - \lambda\frac{\alpha^{\underline{2^*}}}{\underline{2^*}}B_1, \qquad f_2(\alpha) := \frac{\alpha^2}{2}A_1 - \lambda\frac{\alpha^{\overline{2^*}}}{\underline{2^*}}B_1. \tag{2.33}$$

We can take $\delta_0 > 0$, independent of λ, such that

$$0 < \delta_0 \leq \min\left\{\max_{\alpha\geq 0} f_1(\alpha), \max_{\alpha\geq 0} f_2(\alpha)\right\}. \tag{2.34}$$

However, we have

$$
\begin{aligned}
&\max_{\alpha\geq 0}\left\{\frac{\alpha^2}{2}A_1+\frac{b\alpha^4}{2}A_3\big[u^+\big]^2_{s(\cdot)}+\frac{b\alpha^4}{4}A_3^2-\lambda\frac{\max\{\alpha^{\underline{2^*}},\alpha^{\overline{2^*}}\}}{\underline{2^*}}B_1\right\} \\
&\quad\leq c_\lambda\to 0 \quad \text{as } \lambda\to\infty,
\end{aligned} \tag{2.35}
$$

which is a contradiction.

Step 2: We shall prove $u_n\to u$ in $L^{2^*(x)}(\Omega)$.

We only prove $B_1=0$ since the proof for $B_2=0$ is similar. If $B_1>0$, we divide it into two cases to lead to a contradiction.

Case 1: $B_2>0$. For all $\alpha>0$, let

$$
\begin{aligned}
\varphi_1(\alpha):&=\frac{\alpha^2}{2}A_1+\frac{b\alpha^4}{2}A_3\big[u^+\big]^2_{s(\cdot)}+\frac{b\alpha^4}{4}A_3^2-\lambda\frac{\max\{\alpha^{\underline{2^*}},\alpha^{\overline{2^*}}\}}{\underline{2^*}}B_1, \\
\varphi_2(\beta):&=\frac{\beta^2}{2}A_2+\frac{b\beta^4}{2}A_4\big[u^-\big]^2_{s(\cdot)}+\frac{b\beta^4}{4}A_4^2-\lambda\frac{\max\{\beta^{\underline{2^*}},\beta^{\overline{2^*}}\}}{\underline{2^*}}B_2.
\end{aligned} \tag{2.36}
$$

We can choose $\widehat{\alpha},\widehat{\beta}>0$ such that

$$
\varphi(\widehat{\alpha})=\max_{\alpha\geq 0}\varphi_1(\alpha),\qquad \varphi(\widehat{\beta})=\max_{\alpha\geq 0}\varphi_2(\beta). \tag{2.37}
$$

Let

$$
\Psi_u(\overline{\alpha_u},\overline{\beta_u}):=\max_{(\alpha,\beta)\in[0,\widehat{\alpha}]\times[0,\widehat{\beta}]}\Psi_u(\alpha,\beta). \tag{2.38}
$$

We can prove that $(\overline{\alpha_u},\overline{\beta_u})\in(0,\widehat{\alpha})\times(0,\widehat{\beta})$, which ensures that $(\overline{\alpha_u},\overline{\beta_u})$ is the critical point of Ψ_u.

According to Lemma 2.1, $(\overline{\alpha_u},\overline{\beta_u})=(\alpha_u,\beta_u)$. Noting that (2.28), we have

$$
\begin{aligned}
c_\lambda &= \liminf_{n\to\infty} J_\lambda\big(\alpha u_n^+ +\beta u_n^-\big) \\
&\geq J_\lambda\big(\alpha_u u^+ +\beta_u u^-\big) \\
&\quad +\frac{\alpha_u^2}{2}A_1+\frac{\beta_u^2}{2}A_2+\frac{b\alpha_u^4}{2}A_3\big[u^+\big]^2_{s(\cdot)}+\frac{b\alpha_u^4}{4}A_3^2+\frac{b\beta_u^4}{2}A_4\big[u^-\big]^2_{s(\cdot)}+\frac{b\beta_u^4}{4}A_4^2 \\
&\quad -\frac{\max\{\alpha_u^{\underline{2^*}},\alpha_u^{\overline{2^*}}\}}{\underline{2^*}}B_1-\frac{\max\{\beta_u^{\underline{2^*}},\beta_u^{\overline{2^*}}\}}{\underline{2^*}}B_2 \\
&> J_\lambda\big(\alpha_u u^+ +\beta_u u^-\big)\geq c_\lambda,
\end{aligned}
$$

which is a contradiction.

Case 2: $B_2=0$. Clearly, there exists $\beta_0\in[0,\infty)$ such that

$$
J_\lambda\big(\alpha u^+ +\beta u^-\big)\leq 0,\quad \forall(\alpha,\beta)\in[0,\widehat{\alpha}]\times[\beta_0,\infty). \tag{2.39}
$$

Hence, there exists $(\overline{\alpha_u},\overline{\beta_u})\in[0,\widehat{\alpha}]\times[0,\infty)$ such that

$$
\Psi_u(\overline{\alpha_u},\overline{\beta_u})=\max_{(\alpha,\beta)\in[0,\widehat{\alpha}]\times[0,\infty)}\Psi_u(\alpha,\beta). \tag{2.40}
$$

We also can prove that $(\overline{\alpha_u}, \overline{\beta_u}) \in (0, \widehat{\alpha}) \times (0, \infty)$. This implies that $(\overline{\alpha_u}, \overline{\beta_u})$ is the critical point of Ψ_u. Then it follows that

$$\begin{aligned} c_\lambda &= \liminf_{n\to\infty} J_\lambda\left(\alpha u_n^+ + \beta u_n^-\right) \\ &\geq J_\lambda\left(\alpha_u u^+ + \beta_u u^-\right) \\ &\quad + \frac{\alpha_u^2}{2}A_1 + \frac{\beta_u^2}{2}A_2 + \frac{b\alpha_u^4}{2}A_3\left[u^+\right]_{s(\cdot)}^2 + \frac{b\alpha_u^4}{4}A_3^2 + \frac{b\beta_u^4}{2}A_4\left[u^-\right]_{s(\cdot)}^2 \\ &\quad + \frac{b\beta_u^4}{4}A_4^2 - \lambda\frac{\max\{\alpha_u^{2^*_-}, \alpha_u^{\overline{2^*}}\}}{\underline{2^*}}B_1 \\ &> J_\lambda\left(\alpha_u u^+ + \beta_u u^-\right) \\ &\geq c_\lambda. \end{aligned} \tag{2.41}$$

Step 3: We can prove that c_λ is achieved. Similar to [15], we omit it here. □

Proof of Theorem 1.1 With Lemmas~2.1–2.5 in hand, we only need to clarify that the minimizer u_b is a critical point of J_λ for $\lambda > \lambda_1$, where λ_1 is from Lemma 2.5. Our method used here is different from that used in [15] or [8]. If u_b is not a critical point of J_λ, we can choose a function $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\langle J_\lambda(u_b), \phi\rangle \leq -1$. We choose $\varepsilon > 0$ small enough such that

$$\left\langle J'_\lambda\left(su_b^+ + tu_b^- + \sigma\phi\right), \phi\right\rangle \leq -\frac{1}{2}, \quad \forall (s,t,\sigma) \in B_\varepsilon(1,1,0), \tag{2.42}$$

where $B_\varepsilon(1,1,0)$ is an open ball of radius ε centered at $(1,1,0)$. We introduce a smooth cut-off function $0 \leq \eta \leq 1$ such that

$$\eta(s,t) = \begin{cases} 1, & (s,t) \in \overline{B_{\frac{\varepsilon}{2}}(1,1)}, \\ 0, & (s,t) \in \overline{B_\varepsilon^c(1,1)}. \end{cases} \tag{2.43}$$

We make the following perturbation:

$$\gamma(s,t) = \begin{cases} su_b^+ + tu_b^-, & (s,t) \in B_\varepsilon^c(1,1), \\ su_b^+ + tu_b^- + \varepsilon\eta(s,t)\phi, & (s,t) \in B_\varepsilon(1,1). \end{cases} \tag{2.44}$$

Obviously, $\gamma(s,t)$ is continuous from $\mathbb{R} \times \mathbb{R}$ to $(E, \|\cdot\|)$. For $\varepsilon > 0$ small enough, we have $\gamma(s,t)^\pm \neq 0$. We have the following claim.

Claim 2.6 *We claim that* $\sup_{s,t\geq 0} J_\lambda(\gamma(s,t)) < c_\lambda$.

Indeed, if $(s,t) \in B_\varepsilon^c(1,1)$, by Lemma 2.1, we get $J_\lambda(\gamma(s,t)) < c_\lambda$. If $(s,t) \in B_\varepsilon(1,1)$, using the mean value theorem, there is $\overline{\sigma} \in (o, \varepsilon)$ such that

$$\begin{aligned} J_\lambda\left(\gamma(s,t)\right) &= J_\lambda\left(su_b^+ + tu_b^-\right) + \left\langle J'_\lambda\left(su_b^+ + tu_b^- + \overline{\sigma}\eta(s,t)\phi\right), \phi\right\rangle \\ &\leq J_\lambda\left(su_b^+ + tu_b^-\right) - \frac{1}{2}\eta(s,t) \\ &< c_\lambda. \end{aligned} \tag{2.45}$$

However, in view of Lemma 2.1, for $(s,t) \in (1-\frac{\varepsilon}{2},1) \times (1-\frac{\varepsilon}{2},1)$, we have

$$\big\langle J'_\lambda\big(su_b^+ + tu_b^- + \overline{\sigma}\eta(s,t)\phi\big), u_b^+\big\rangle > 0, \qquad \big\langle J'_\lambda\big(su_b^+ + tu_b^- + \overline{\sigma}\eta(s,t)\phi\big), u_b^-\big\rangle > 0. \tag{2.46}$$

Similarly, for $(s,t) \in (1,1+\frac{\varepsilon}{2}) \times (1,1+\frac{\varepsilon}{2})$, we have

$$\big\langle J'_\lambda\big(su_b^+ + tu_b^- + \overline{\sigma}\eta(s,t)\phi\big), u_b^+\big\rangle < 0, \qquad \big\langle J'_\lambda\big(su_b^+ + tu_b^- + \overline{\sigma}\eta(s,t)\phi\big), u_b^-\big\rangle < 0. \tag{2.47}$$

Therefore, there is $(s_0,t_0) \in (1-\frac{\varepsilon}{2},1+\frac{\varepsilon}{2}) \times (1-\frac{\varepsilon}{2},1+\frac{\varepsilon}{2})$ such that

$$s_0u_b^+ + t_0u_b^- + \overline{\sigma}\eta(s_0,t_0)\phi \in \mathcal{M}_\lambda. \tag{2.48}$$

This contradicts the above claim. □

Next, we want to prove the property of double energy of u_b.

Proof of Theorem 1.2 Based on Lemma 2.5 and standard arguments, there exists $\lambda_2 > 0$ such that for all $\lambda \geq \lambda_2$, the minimization problem

$$c^* := \inf_{u\in\mathcal{N}_\lambda} J_\lambda(u) \tag{2.49}$$

is well defined and it admits a minimizer which is a critical point of J_λ. It is called a ground state of (1.1).

According to Theorem 1.1, we know that problem (1.1) has a least energy sign-changing solution u_b when $\lambda \geq \lambda_1$.

Let $\lambda^* = \max\{\lambda_1,\lambda_2\}$. Let u_b be obtained in Theorem 1.1. A standard proof implies that there exist $\bar{s} > 0$ and $\bar{t} > 0$ such that $\bar{s}u_b^+ \in \mathcal{N}_\lambda$ and $\bar{t}u_b^- \in \mathcal{N}_\lambda$. If we define

$$g_1(s) := J_\lambda\big(su_b^+\big), \qquad g_2(t) := J_\lambda\big(tu_b^+\big), \tag{2.50}$$

we have $g_1(\bar{s}) = \max_{s\geq 0} g_1(s)$, $g_1(\bar{t}) = \max_{t\geq 0} g_1(t)$. So,

$$\begin{aligned}
c_\lambda &= \sup_{s,t\geq 0} J_\lambda\big(su_b^+ + tu_b^-\big) \\
&= \sup_{s,t\geq 0}\bigg[J_\lambda\big(su_b^+\big) + J_\lambda\big(tu_b^-\big) + \frac{st}{2}H(u_b) + \frac{b}{16}s^2t^2H(u_b) + \frac{b}{2}s^2t^2\big[u_b^+\big]_{s(\cdot)}^2\big[u_b^-\big]_{s(\cdot)}^2 \\
&\quad + \frac{b}{4}s^3t\big[u_b^+\big]_{s(\cdot)}^2H(u_b) + \frac{b}{4}st^3\big[u_b^-\big]_{s(\cdot)}^2H(u_b)\bigg] \\
&> \sup_{s\geq 0} J_\lambda\big(su_b^+\big) + \sup_{t\geq 0} J_\lambda\big(tu_b^-\big) \\
&\geq 2c^*.
\end{aligned}$$

□

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Yueting Yang wrote the manuscript, Jianwen Zhou provided the idea. Wenbo Wang read the paper and gave some good suggestions which improved the quality of this paper. All authors reviewed the manuscript.

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